Kirchhoff Graphs

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ABSTRACT

Kirchhoff's laws are well-studied for electrical networks with voltage and current sources, and edges marked by resistors. Kirchhoff's voltage law states that the sum of voltages around any circuit of the network graph is zero, while Kirchhoff's current law states that the sum of the currents along any cutset of the network graph is zero. Given a network, these requirements may be encoded by the circuit matrix and cutset matrix of the network graph. The columns of these matrices are indexed by the edges of the network graph, and their row spaces are orthogonal complements.

For (chemical or electrochemical) reaction networks, one must naturally study the opposite problem, beginning with the stoichiometric matrix rather than the network graph. This leads to the following question: given such a matrix, what is a suitable graphic rendering of a network that properly visualizes the underlying chemical reactions? Although we can not expect uniqueness, the goal is to prove existence of such a graph for any matrix. Specifically, we study Kirchhoff graphs, originally introduced by Fehribach. Mathematically, Kirchhoff graphs represent the orthocomplementarity of the row space and null space of integer-valued matrices. After introducing the definition of Kirchhoff graphs, we will survey Kirchhoff graphs in the context of several diverse branches of mathematics.

Beginning with combinatorial group theory, we consider Cayley graphs of the additive group of vector spaces, and resolve the existence problem for matrices over finite fields. Moving to linear algebra, we draw a number of conclusions based on a purely matrix-theoretic definition of Kirchhoff graphs, specifically regarding the number of edges in Kirchhoff graphs. Next we observe a geometric approach, reviewing James Clerk Maxwell's theory of reciprocal figures, and presenting a number of results on Kirchhoff duality. We then turn to algebraic combinatorics, where we study equitable partitions, quotients, and graph automorphisms. In addition to classifying the matrices that are the quotient of an equitable partition, we demonstrate that many Kirchhoff graphs arise from equitable edge-partitions of directed graphs. Finally we study matroids, where we review Tutte's algorithm for determining when a binary matroid is graphic, and extend this method to show that every binary matroid is $\mathbb{Z}_2$-Kirchhoff. The underlying theme throughout each of these investigations is determining new ways to both recognize and construct Kirchhoff graphs.

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For Grampa Del

Contents

List of Symbols

$A(G)$	The adjacency matrix of graph G
$A(G/\pi)$	The quotient matrix of equitable (vertex) partition π
$A_E(D)$	The signed edge adjacency matrix of digraph D
$A_E(D/\pi)$	The quotient matrix of equitable edge-partition π
$\mathscr{B}$	The set of bases of a matroid
$\mathcal{B}$	A base of a matroid
B_i	A bridge of a cocircuit of a matroid
C	A cycle
$c_{i,j}$	The number of neighbors in cell j of any vertex in cell i of an equitable (vertex) partition
C_n	The cycle graph on n vertices
$\mathscr{C}$	The set of circuits of a matroid
cl	The matroid closure operator
$\deg(v)$	The degree of vertex v
D	A directed multigraph
$\mathcal{D}, \overline{D}$	A vector graph
$\mathcal{D}^{/\boldsymbol{s}_n}$	A vector graph $\mathcal{D}$ with all $\boldsymbol{s}_n$ edges contracted
$\mathcal{D}^{\setminus\boldsymbol{s}_n}$	A vector graph $\mathcal{D}$ with all $\boldsymbol{s}_n$ edges deleted
$\Delta(\boldsymbol{x})$	A diagonal matrix with diagonal $\boldsymbol{x}$
$\mathscr{D}_i$	The distance partition cells of a rooted spanning tree
$d_{i,j}$	The net (signed) number of edges in cell j adjacent to any edge in cell i of an equitable edge-partition
$\mathrm{diag}(A)$	A diagonal matrix whose diagonal is equal to that of square matrix A
$\Delta(G)$	A diagonal matrix of vertex degrees
$\Delta_E(D)$	A diagonal matrix of edge adjacency values

e or e_i or ϵ	An edge
$E(G)$	The edge set of undirected graph G
$E(D)$	The edge set of directed graph D
$\mathcal{E}(D)$	The edge space of digraph D
E_i	The cells of an edge-partition
$\mathbb{F}$	A field
G	An undirected graph
$\overline{G}$	The complement of graph G
$G[V']$	The induced subgraph of G on vertex set V'
$G[E']$	The induced subgraph of G on edge set E'
Γ	A group
G/π	The quotient of graph G with respect to partition π
H	A subset of elements of a group (Chapter 2) An undirected graph (Chapters 5 and 9) A compact representation matrix (Chapter 8)
I_n	The $n \times n$ identity matrix
$\iota(e)$	The initial vertex of directed edge e
$\mathscr{I}$	The independent sets of a matroid
J_n	The $n \times n$ all-ones matrix
K_n	The complete graph on n vertices
κ	Regularity of a regular graph
$\lambda(v)$	The incidence vector of vertex v
$\lambda_p(v)$	The mod-p incidence vector of vertex v
Λ	A diagonal matrix

$L(G)$	The combinatorial Laplacian matrix of graph G
$L_E(D)$	The signed edge Laplacian matrix of digraph D
$\lambda_\pi(v)$	The incidence vector of vertex v with respect to partition π
$\mathcal{M}$	A matrix
$M(G)$	The cycle matroid of graph G
$M[A]$	The column matroid of matrix A
$M'(G)$	The bond matroid of graph G
M_n	The $n \times 2^n$ matrix with all binary n-tuples as its columns
$\mathrm{Null}(A)$	The null space of matrix A
N	A null space matrix
η_j	The number of edges assigned vector edge $\boldsymbol{s}_j$
O_i	Cells of an orbit partition
ω	A constant (Chapter 3) The weight of a path (Chapter 5)
Ω	A subgroup of permutations
P	A partition of $V(G)$ (Chapter 1) A permutation matrix (Chapter 3) The characteristic matrix of a partition (Chapters 5 and 7)
p	A prime number (Chapter 2 only)
φ	A vector assignment
$\mathcal{P}$	A polyhedron
π	A graph partition
ϕ or ψ	A graph automorphism
$\mathbb{Q}$	The rational numbers
$Q(D)$	The incidence matrix of digraph D

$\mathcal{Q}_3$	The cube graph
$\mathbb{R}$	The real numbers
$\mathrm{Row}(A)$	The row space of matrix A
r	A row function (Chapter 5) The rank function of a matroid (Chapter 8)
$\boldsymbol{s}_i$	Vector edges
S	A set of vectors (Chapter 3)
σ_C	Sign function of edges with respect to cycle C
$s_{i,j}$	Quantity identifying the adjacency of directed edges e_i and e_j
σ	The antipodal map
$S(M)$	The ground set of matroid M
T	A tree (Chapters 1, 5, and 8) A characteristic matrix (Chapters 3, 6, and 7)
$\tau(e)$	The terminal vertex of directed edge e
ϑ	A cut vector
$U(D)$	The cut space of digraph D
U_i	Cells of an orbit edge-partition
$U_{k,n}$	The uniform matroid on n elements
$\mathscr{U}$	Partition of non-overlapping bridges of a matroid cocircuit
Υ_X	The indicator (vector) of set X
v or v_i	A vertex
$V(G)$	The vertex set of undirected graph G
$V(D)$	The vertex set of directed graph D
$\mathcal{V}(D)$	The vertex space of digraph D
V_i	Subsets of a vertex partition

$V_{\mathcal{I}}$	An $\lvert E(D)\rvert \times \lvert E(D)\rvert$ matrix whose i^{th} row is the incidence vector of the initial vertex of edge e_i
$V_{\mathcal{T}}$	An $\lvert E(D)\rvert \times \lvert E(D)\rvert$ matrix whose i^{th} row is the incidence vector of the terminal vertex of edge e_i.
w	A weight function
$\chi(C)$	The cycle vector of cycle C
$\chi_p(C)$	The mod-p cycle vector of cycle C
$\chi_\pi(C)$	The cycle vector of cycle C with respect to partition π
Y_n	The prism graph on $2n$ vertices
$\mathbb{Z}$	The integers
$Z(D)$	The cycle space of digraph D
$\mathbf{0}$	The zero vector

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Introduction

Kirchhoff's laws are well-studied for electrical networks with voltage and current sources, and edges marked by resistors; see, for example, [92]. Kirchhoff's voltage law states that the sum of voltages around any circuit of the network graph is zero, while Kirchhoff's current law can be formulated as the requirement that the sum of the currents along any cutset of the network graph is zero. Given a network, these requirements may be encoded by the circuit matrix, R, and cutset matrix, X, of the network graph. The columns of R and X are both indexed by the edges of the network graph and $RX^t = 0$ over $\mathbb{Z}_2$. A network with positive resistors is known to be uniquely solvable if and only if the subset of edges of the network graph determined by the voltage sources does not contain a cycle, and the subset of edges determined by the current sources does not contain a cutset. It should be noted, however, that the network graph is typically not recoverable from the matrices R and X.

We study the opposite problem, where rather than beginning with a network graph we now begin with a matrix. For example, an electrochemical reaction network can be encoded by an integer-valued matrix, the stoichiometric matrix, the columns of which correspond to reaction steps. This leads to the following question: given this matrix, what is a suitable graphic rendering of a network that properly visualizes the underlying chemical reactions? Although we can not expect uniqueness, the goal is to prove existence of such a graph for any matrix.

Specifically, we will consider a relatively-new type of graph, Kirchhoff graphs. Kirchhoff graphs were introduced by Fehribach [28] as a mathematically precise graphic representation of electrochemical reaction networks. Kirchhoff graphs represent the orthocomplementarity of the row space and null space of the stoichiometric matrix of a reaction network [29]. As such, a chemical engineer can use a Kirchhoff graph in the way an electrical engineer uses a standard circuit diagram.

Our goal is to study the mathematical foundations of Kirchhoff graphs. While it is straightforward to introduce the notion of a Kirchhoff graph, as the reader will observe in the course of this text, these structures can be viewed from an array of mathematical perspectives. Therefore we will survey Kirchhoff graphs within the context of several diverse branches of mathematics. Each distinct viewpoint will present a unique interpretation of Kirchhoff graphs, and inspire a new set of results regarding properties of these graphical objects.

The body of this work will be presented as a sequence of studies that seek to understand Kirchhoff graphs within a variety of mathematical disciplines. Therefore, beyond the outline of chapters that follows, this section will forego a complete and formal introduction and literature review. Instead, each chapter will begin with a review of known results, including references to the relevant literature. Although we will be considering a number of different mathematical fields, these topics are not unrelated. Care will be taken throughout to draw parallels between the various interpretations of Kirchhoff graphs, and indeed many later definitions and results are inspired by those in earlier chapters.

After Chapter 1 reviews the fundamentals of graph theory, including our choice of notation, the remaining chapters can be broken-down into three parts: Chapters 2 through 4, Chapters 5 through 7, and Chapters 8 through 10.

The first part, comprised of Chapters 2 through 4, studies the relationship between matrices and Kirchhoff graphs. Chapter 2 begins by motivating our study of Kirchhoff graphs through examining a sample chemical reaction network. This leads to a formal definition of vector graphs and Kirchhoff graphs in Section 2.2, derived from finite subgraphs of Cayley color graphs (a classical object in combinatorial group theory). We then resolve the existence problem of Kirchhoff graphs for matrices over finite fields. Chapter 3 introduces the notion of a *vector assignment*, which is used to give an entirely matrix-based definition of Kirchhoff graphs. This allows us to draw a number of conclusions about the number of edges in a Kirchhoff graph, and show that identifying Kirchhoff graphs is closely related to studying permutation-invariance of linear subspaces. Lastly, Chapter 4 takes a more geometric approach to Kirchhoff graphs. After reviewing James Clerk Maxwell's theory of reciprocal figures, this leads us to study a notion of duality in Kirchhoff graphs.

At the heart of vector graphs and Kirchhoff graphs is a partitioning of the edges of a digraph into a finite set of (vector) classes. Therefore the second part, Chapters 5 through 7, studies graph partitioning in general; more specifically, equitable partitions. Chapter 5 introduces the classical theory of equitable (vertex) partitions of undirected graphs. Section 5.2 presents necessary and sufficient conditions for a square matrix to be the quotient of an equitable partition. This result, in turn, allows us to show that every equitable partition belongs to an infinite family of equitable partitions that all share the same quotient. Next, Chapter 6 extends the notion of equitable partitioning to the the edges of directed multigraphs. We demonstrate that any equitable edge-partition with a symmetric quotient matrix is Kirchhoff. The remainder of the chapter studies the exactness of that correspondence by considering uniform partitions and connected graphs. Lastly, Chapter 7 presents a collection of additional topics in graph partitioning. These include examining *almost* equitable edge partitions (and their relationship to Kirchhoff partitions), in addition to a number of methods of constructing equitable edge-partitions.

A common theme underlying Chapters 2 through 7 is the relationship between graphs and matrices. This naturally leads to a study of *matroids*, which were originally introduced to understand the abstract notion of dependence as it relates to graphs and matrices. This comprises the third and final part of this text, Chapters 8 through 10. Chapter 8 provides an introduction to matroid theory, including a number of the most classical definitions and results. Section 8.3 closes the chapter with an observation of the author on when a binary matroid is graphic. Along these lines, Chapter 9 presents an algorithm due to Tutte that determines if a binary matroid is graphic. In addition to translating Tutte's method into current terminology, we present an in-depth example of his algorithm. Lastly, Chapter 10 synthesizes the ideas of Chapters 2, 6, 8, and 9. Specifically, after proposing a natural definition of a $\mathbb{Z}_2$-Kirchhoff matroid, we demonstrate that every binary matroid is $\mathbb{Z}_2$-Kirchhoff.

The results presented in this text inspire a number of interesting questions for further research. Many of these open problems arise from a specific interpretation of Kirchhoff graphs. In order to present these questions in a natural context, we will not consolidate them in a concluding section. Much as we chose to distribute the introductory material, the reader will find a collection of open problems sprinkled throughout the text. None of these questions will be answered here, but instead are meant to suggest avenues of future research. We will use the heading "**Question:**" to draw attention to the most interesting of these open problems.

Notation

As this text surveys Kirchhoff graphs within the context of a number of mathematical disciplines, a variety of notation is introduced in the following chapters. We strive to remain consistent with the conventions of each respective field, while also minimizing overlap between chapters. A list of universal symbols is included immediately following the Table of Contents, which we hope is of assistance to the reader.

In general, large objects will always be denoted with capital letters, and sub-objects in lower case. For example, G will always be a graph, and D a digraph. Vertices and edges will be noted by v and e (or v_i and e_i) respectively. Matrices will always be denoted by capital letters, most often A and B, although also occasionally M, N, and R. The (i, j)-entry of matrix A will be denoted either by $A(i, j), A_{i,j}$, or $a_{i,j}$ depending on context. We will not deal with any complex-valued matrices, and so the *transpose* of matrix A will be denoted A^t. Vectors will implicitly be assumed to be *column* vectors (unless otherwise stated), and will be denoted with bold lower case letters. If $\boldsymbol{x}$ is a vector, then we will let x_i denote the i^{th} entry of vector $\boldsymbol{x}$. The two distinguished exceptions to these rules are cycle vectors and vertex incidence vectors. These will be denoted $\chi(C)$ and $\lambda(v)$, respectively, and will be assumed to be row vectors unless otherwise noted.

The letters i and j will be reserved as generic indices, and p, q, r, s, and t will also be used. The letters k, m and n are reserved throughout to represent cardinalities or dimensions. It will be clear from context which sets or subspaces these letters measure. Both functions and multiplicative scalars will be denoted with lower-case Greek letters; once again, the specific role will be clear in context.

Chapter 1

Fundamentals of Graph Theory

We begin, as is traditional in graph theory literature, with an introduction to the terminology and notation that will be used throughout this text. In general, we will follow the conventions of Bollobás' *Modern Graph Theory* [10]. This chapter is not intended as a complete review of basic graph theory–standard topics such as connectivity and coloring will not be discussed. Rather, emphasis will be placed on introducing topics that will be used in the chapters to come. This chapter will present primarily definitions and examples, as well as a few theorems that will be relevant in future sections. For a more comprehensive introduction to graph theory, the reader is referred to the standard texts of Harary [55], Bollobás [10], and Diestel [23].

1.1 Definitions

A *graph* G is a pair of disjoint sets (V, E) such that E is a subset of $[V]^2$. That is, the elements of E are unordered pairs of elements of V. The elements of V are the *vertices* of G, and the elements of E are its *edges*. Given a graph $G = (V, E)$, $V = V(G)$ is the *vertex set* of G and $E = E(G)$ is the *edge set*. We consider only *finite* graphs, meaning the sets V and E are always finite. The usual way to depict a graph is to draw a dot for each vertex, and to join two dots by a line if the two corresponding vertices form an edge. For example, Figure 1.1.1 illustrates a graph with vertex and edge sets

$$V = \Big\{1, 2, 3, 4, 5\Big\} \quad \text{and} \quad E = \Big\{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}\Big\}.$$

Although a graph is defined as an ordered pair, it is more usually thought of as a collection of vertices, some of which are joined by edges. Often the simplest way to describe small graphs is with a drawing, a technique that will be used repeatedly in this text.

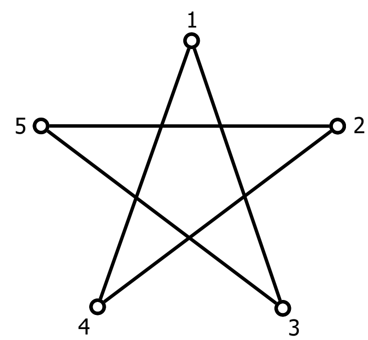


Figure 1.1.1: A graph G with vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$.

An edge $e = \{u, v\} \in E(G)$ is said to *join* the vertices u and v. Two vertices u and v are *adjacent* or *neighbors* if $\{u, v\} \in E(G)$. We will sometimes use the notation $u \sim v$ to denote that u and v are adjacent. Vertices u and v are the *endvertices* of edge e, and both u and v are *incident* to the edge e. Two edges are *adjacent* if they share a common endvertex.

A graph G with vertex set $V(G) = \{v_i\}$ can be entirely encoded by a matrix. The *adjacency matrix* $A = A(G)$ of G is a $|V(G)| \times |V(G)|$ matrix where

$$A_{i,j} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j \\ 0 & \text{Otherwise} \end{cases}.$$

For example, the graph G in Figure 1.1.1 has adjacency matrix

$$A(G) = \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 2} \\ {\scriptstyle 3} \\ {\scriptstyle 4} \\ {\scriptstyle 5} \end{array} \begin{array}{c} \begin{array}{ccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The *neighborhood* of a vertex v is the set of all vertices adjacent to v, and the *star* of v is the set of all edges incident to v. The *degree* of a vertex v, denoted $\deg(v)$, is the cardinality of its neighborhood. That is, $\deg(v)$ is the number of vertices adjacent to v or, equivalently, the number of edges incident to vertex v. As every edge has two endvertices, the sum of the vertex degrees in a graph must be twice the number of edges,

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|. \tag{1.1}$$

Justified by a standard double-counting argument, (1.1) is the classical "handshaking lemma." A graph is *regular* if all vertices have the same degree, and κ*-regular* if all vertices have degree κ.

A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A subgraph G' is an *induced subgraph* if it contains all edges of G that join two vertices of V'. We denote G', the subgraph *induced on* V', by $G[V']$. Figure 1.1.2 illustrates a subgraph and an induced subgraph of the graph G in Figure 1.1.1. If $V' = V$ we say that G' is a *spanning subgraph* of G.

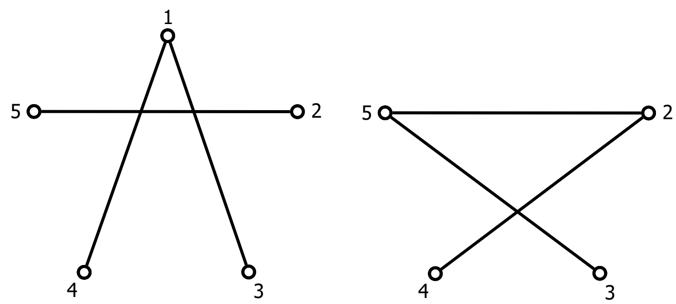


Figure 1.1.2: A subgraph and an induced subgraph of the graph G in Figure 1.1.1.

A graph is *complete* if every pair of vertices is joined by an edge. The complete graph on n vertices will be denoted by K_n. For any integer $r \geq 2$, a graph G is r*-partite* if $V(G)$ can be partitioned into r classes such that the endvertices of every edge lie in different classes. A 2-partite graph is called *bipartite*. A *complete* r*-partite graph* is one in which every pair of vertices in different classes are adjacent. The complete r-partite graph with partition classes of size $n_1, n_2, \ldots, n_r$ is denoted by $K_{n_1,n_2,\ldots,n_r}$. The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are illustrated in Figure 1.1.3. We will return to these two graphs repeatedly in the chapters to come. The *complement* $\overline{G}$ of a graph $G = (V, E)$ has vertex set V and edge set $[V]^2 \backslash E$. The graph G of Figure 1.1.1 and its complement $\overline{G}$ are illustrated in Figure 1.1.4.

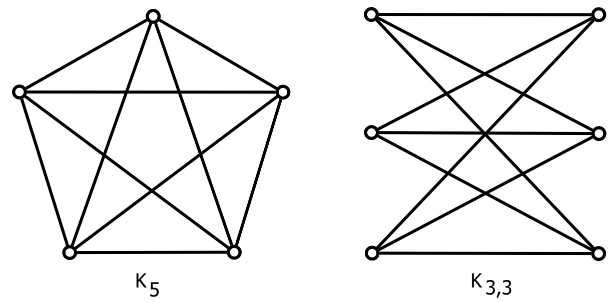


Figure 1.1.3: The complete graph K_5 and the complete bipartite graph $K_{3,3}$.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves both adjacencies and non-adjacencies. That is, $G = (V, E)$ is isomorphic to $G' = (V', E')$ if there exists a bijection $\phi : V \to V'$ such that $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(G')$. We use $G \cong G'$ to denote that G is isomorphic to G', and the bijection ϕ is called an *isomorphism*. If $V = V'$ we call ϕ an *automorphism*. For example, the graph G in Figure 1.1.4 is isomorphic to its complement $\overline{G}$. One automorphism from G to $\overline{G}$ is given by ϕ, where

$$\phi(1) = 1 \quad \phi(2) = 3 \quad \phi(3) = 5 \quad \phi(4) = 2 \quad \phi(5) = 4.$$

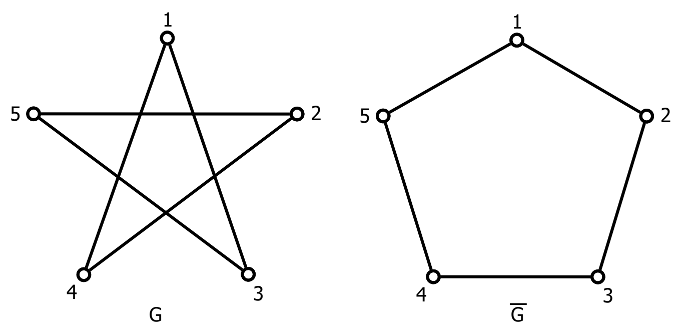


Figure 1.1.4: The graph G of Figure 1.1.1 and its complement $\overline{G}$.

We often construct new graphs from pre-existing ones. Given a graph $G = (V, E)$, for any $V' \subseteq V$ the graph $G \backslash V'$ is the subgraph of G obtained by deleting the vertices of V', and all edges incident with a vertex of V'. Equivalently, $G \backslash V' = G[V \backslash V']$. Similarly, if $E' \subseteq E(G)$ then $G \backslash E' = (V(G), E(G) \backslash E')$. If $V' = \{v\}$ or $E' = \{e\}$ this notation is often simplified to $G - v$ and $G - e$. On the other hand, for any $e \notin E(G)$, $G + e$ is used to denote the graph $(V, E \cup e)$. Given any edge $e = \{u, v\} \in E(G)$, the graph G/e is obtained from G by *contracting* the edge e. That is, we identify the vertices u and v, and remove any loops or duplicate edges. For example, given G as in Figure 1.1.1, the graph $G/\{3,5\}$ is illustrated in Figure 1.1.5. A *contraction* of G is any graph obtained from G by a sequence of edge contractions, and a *minor* of G is any subgraph of a contraction of G.

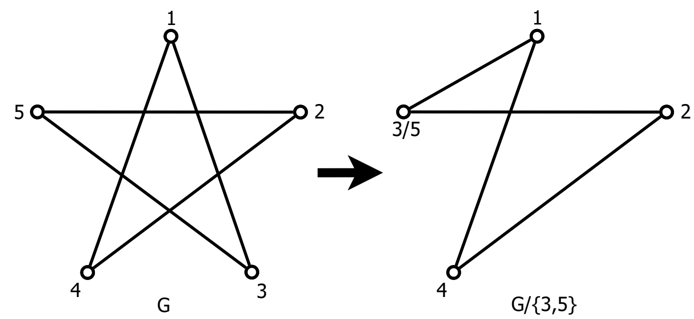


Figure 1.1.5: The contraction $G/\{3,5\}$ for the graph G as in Figure 1.1.1.

1.2 Walks, Cycles, and Connectedness

A *walk* W in a graph is an alternating sequence of vertices and edges

$$W = v_0, e_1, v_1, e_2, \ldots, e_k, v_k, \tag{1.2}$$

where $e_i = \{v_{i-1}, v_i\}$ $(0 < i \leq k)$. The vertices v_i and edges e_i of a walk need not be distinct. For each edge e_i in (1.2) we say that walk W *traverses* edge e_i, and the *length* of W is k. A walk with distinct vertices is a *path*. That is, a path P is a graph of the form

$$V(P) = \Big\{v_0, v_1, \ldots, v_k\Big\} \quad E(P) = \Big\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\Big\}, \tag{1.3}$$

where the vertices v_i are all distinct. The vertices v_0 and v_k are the *ends* of a path, and a *u-v path* is a path with ends u and v. A walk is *closed* if $v_0 = v_k$, and a closed walk with distinct vertices is a *cycle*. That is, a cycle C is a graph of the form

$$V(C) = \Big\{v_0, v_1, \ldots, v_k\Big\} \quad E(C) = \Big\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_0\}\Big\},$$

where the vertices v_i are all distinct. Equivalently, C can be written as $P + \{v_k, v_0\}$, where P is a path of the form (1.3). Figure 1.2.1 illustrates a path of length 3 and a cycle of length 5.

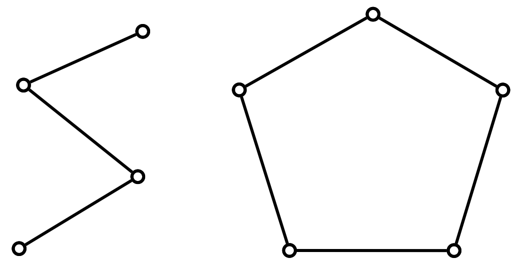


Figure 1.2.1: A path of length 3 and a cycle of length 5.

A graph G is *connected* if for every pair of distinct vertices u and v there is a u-v path in G. A *component* or *connected component* of G is a maximal connected subgraph (so, for instance, a connected graph has one component). For example, the first graph in Figure 1.1.2 is disconnected and has 2 components, whereas the second graph is connected. A *cutvertex* of G is any vertex whose deletion increases the number of components of G. Similarly a *bridge* is an edge whose deletion increases the number of components.

An *acyclic* graph is one that does not contain cycles, and we call any acyclic graph a *forest*. A connected forest is called a *tree*. The *leaves* of a tree are the vertices of degree 1, and every tree with at least two vertices always has a leaf. Theorem 1.1 is the classical characterization of trees, and can be found in most standard graph theory texts. We present it here without proof.

Theorem 1.1. *The following assertions are equivalent for a graph T.*

(i) T is a tree.

(ii) Any pair of vertices of T are linked by a unique path in T.

(iii) T is minimally connected: T is connected but for any edge e the graph $T - e$ is disconnected (that is, every edge of T is a bridge).

(iv) T is maximally acyclic: T contains no cycles, but for any pair of non-adjacent vertices u and v of T, $T + \{u, v\}$ contains a cycle.

Propositions 1.1 and 1.2 are also classical results that can be found, for example, in [10]. Each will be used at various points throughout this text.

Proposition 1.1. A tree on n vertices has $n - 1$ edges. A forest on n vertices with k components has $n - k$ edges.

Proposition 1.2. Every connected graph G has a spanning tree–that is, a connected acyclic subgraph containing every vertex of G.

Spanning trees can be constructed by a variety of methods, two of the most standard being *breadth-first search* and *depth-first search* [23]. Figure 1.2.2 depicts two examples of spanning trees, one for K_5 and one for $K_{3,3}$. Note that the spanning tree of K_5 has 4 edges ($|V(K_5)| = 5$), and the spanning tree of $K_{3,3}$ has 5 edges ($|V(K_{3,3}| = 6$).

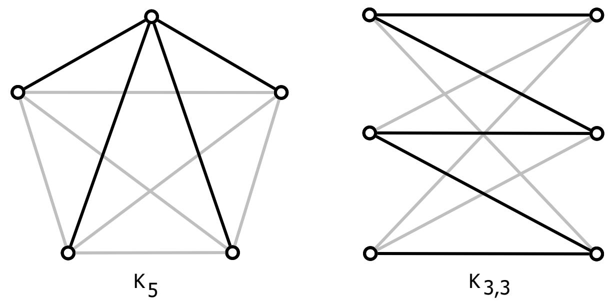


Figure 1.2.2: Two spanning trees, one of K_5 and one of $K_{3,3}$.

1.3 Multigraphs and Digraphs

By definition, a graph does not contain an edge joining a vertex to itself, known as a *loop*, nor does a graph contain several edges joining the same pair of endvertices, known as *multiple edges*. In a *multigraph*, both loops and multiple edges are permitted. Occasionally to avoid confusion, a graph without loops or multiple edges will be called a *simple graph*.

A *directed graph* (or *digraph*) consists of a set V of vertices and a set E of edges. Every element of E is an ordered pair (u, v) of vertices $u, v \in V$. A digraph is *finite* if both V and E are finite. An edge $e = (u, v)$ is said to be *directed from* u to v, and we often say that e *exits* vertex u and *enters* vertex v. The direction of an edge will also sometimes be referred to as its *orientation*. Two vertices u and v are *adjacent* if either $(u, v) \in E$ or $(v, u) \in E$. The vertex u is the *initial vertex* of the edge (u, v) and v is the *terminal vertex*. Digraphs are typically depicted by drawing a dot for each vertex, and adding an arrow from vertex u to vertex v for every edge (u, v). For example, figure 1.3.1 illustrates a directed graph with vertex and edge sets

$$V = \Big\{1, 2, 3, 4, 5\Big\} \quad \text{and} \quad E = \Big\{(1, 3), (3, 5), (5, 2), (2, 4), (4, 1)\Big\}.$$

A *multi-digraph* is a digraph that may have multiple edges with the same initial and terminal vertices. Digraphs, and more specifically multi-digraphs, will be the primary focus of this text. Occasionally to distinguish digraphs from graphs, a graph (i.e. one whose edges do not have a direction) will be called an *undirected graph*.

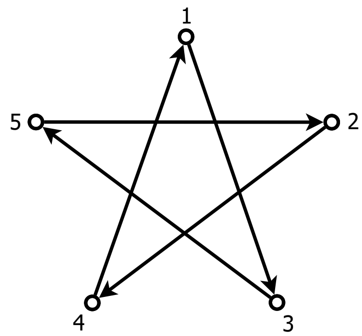


Figure 1.3.1: A directed graph D with vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{(1, 3), (3, 5), (5, 2), (2, 4), (4, 1)\}$.

Much as in the undirected case, a *subgraph* of a digraph $D = (V, E)$ is a pair $D' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. A digraph D' is an *induced subgraph* of D if it contains all edges of D between any two vertices of V'. We denote the subgraph induced on V' by $D[V']$. The notation for deletion of vertices and edges is analogous to that of the undirected case. Two digraphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency, non-adjacency, as well as initial and terminal vertices. That is, $D = (V, E)$ is isomorphic to $D' = (V', E')$ if there exists a bijection $\phi : V \to V'$ such that $(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in E'$. The bijection ϕ is called an *isomorphism*, and if $V = V'$ we call ϕ an *automorphism*.

A *walk* W in a digraph is an alternating sequence of vertices and edges

$$W = v_0, e_1, v_1, e_2, v_2 \ldots, e_k, v_k$$

that may traverse edges regardless of orientation. That is, either $e_i = (v_{i-1}, v_i)$ or $e_i = (v_i, v_{i-1})$. When $e_i = (v_{i-1}, v_i)$ we say that e_i is *oriented as* W. Under this definition of a walk, paths and cycles are defined analogously to the undirected case.

Remark 1.1. Under these definitions walks, paths, and cycles may all traverse edges regardless of their orientations. This is consistent with some modern graph theory texts (for example [10]), although it is a departure from more classical references. This convention has been chosen carefully. In this text we will study graphical objects that model physical systems. The edges of these graphs will model processes that are *reversible*. That is, these processes (specifically chemical reactions) can occur in either a forward *or* a backward direction. Allowing walks to traverse edges regardless of orientation will allow us to faithfully represent this inherent reversibility.

As before, a digraph D is *connected* if every pair of distinct vertices are joined by a path in D, and a *component* is a maximal connected subgraph. An *acyclic* digraph is one that does not contain cycles. When there is no danger of confusion, we will refer to an acyclic digraph as a *forest*, and a connected forest will still be a *tree*.

1.4 Vector Spaces Associated with Graphs

In this section we consider four classical vector spaces associated with graphs. Digraphs will be our primary focus, therefore we define these spaces for finite digraphs, and each is a vector space over the real numbers $\mathbb{R}$ (some authors, such as [10], prefer $\mathbb{C}$). These vector spaces are just as easily, and analogously, defined for undirected graphs. In that case, all vector spaces, vectors, and dot products are taken over $\mathbb{Z}_2$, the field of order two. These $\mathbb{Z}_2$ vector spaces–defined on undirected graphs–will be more familiar to the graph theorist, and can be found in any standard graph theory text.

Remark 1.2. To avoid confusion with this more classical case, we will always use the notation (mod 2) to indicate when arithmetic or vectors are taken over $\mathbb{Z}_2$. For the remainder of this section, all vectors and inner products will be over $\mathbb{R}$.

Let $D = (V(D), E(D))$ be a finite directed graph, with vertex set $V(D) = \{v_1, \ldots, v_n\}$ and edge set $E(D) = \{e_1, \ldots, e_m\}$.

Definition 1.1. The ***vertex space*** $\mathcal{V}(D)$ is the vector space of all functions from $V(D)$ into $\mathbb{R}$. Similarly, the ***edge space*** $\mathcal{E}(D)$ is the vector space of all functions from $E(D)$ into $\mathbb{R}$.

For each vertex v_i, let ν_i be the function $\nu_i : V(D) \to \mathbb{R}$ that is zero everywhere, except at vertex v_i where it is 1. The set $\{\nu_1, \ldots, \nu_n\}$ is the standard basis of $\mathcal{V}(D)$; any element $\boldsymbol{x} \in \mathcal{V}(D)$ can be expressed as the formal sum

$$\boldsymbol{x} = \sum_{i=1}^{n} x_i \nu_i$$

where $\boldsymbol{x}(v_i) = x_i \in \mathbb{R}$. Therefore $\mathcal{V}(D)$ has dimension $n = |V(D)|$, and any element $\boldsymbol{x}$ is often expressed as a vector of length n, $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$. The vertex space is endowed with an inner product under which the standard basis is orthonormal,

$$\left\langle \boldsymbol{x}, \boldsymbol{y} \right\rangle = \sum_{i=1}^{n} x_i y_i = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}.$$

Similarly, for each edge e_i, let ϵ_i be the function $\epsilon_i : E(D) \to \mathbb{R}$ that is zero everywhere, except at edge e_i where it is 1. The set $\{\epsilon_1, \ldots, \epsilon_m\}$ is the standard basis of $\mathcal{E}(D)$; any element $\boldsymbol{u} \in \mathcal{E}(D)$ can be expressed as the formal sum

$$\boldsymbol{u} = \sum_{i=1}^{m} u_i \epsilon_i$$

where $\boldsymbol{u}(e_i) = u_i \in \mathbb{R}$. Therefore $\mathcal{E}(D)$ has dimension $m = |E(D)|$, an any element $\boldsymbol{u}$ is often expressed as a vector of length m, $\boldsymbol{u} = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}$. The edge space is endowed with an inner product under which the standard basis is orthonormal,

$$\Big\langle \boldsymbol{u}, \boldsymbol{w} \Big\rangle = \sum_{i=1}^{m} u_i w_i = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix} \cdot \begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}.$$

1.4.1 The Cycle Space and the Cut Space

Let D be a multi-digraph with edge set $E(D) = \{e_i\}$, and let C be any cycle in D with cyclic orientation

$$C = v_1 \cdot e_1 \cdot v_2 \cdot e_2 \cdots v_l \cdot e_l \cdot v_{l+1}$$

where vertex $v_{l+1} = v_1$. For each $j \in \{1, \ldots, l\}$, edge $e_j \in E(D)$ has endpoints v_j and v_{j+1}. If edge e_j is oriented from v_j to v_{j+1} we say that e_j is *oriented as C*. That is, C traverses edge e_j consistent with its orientation. When cycle C traverses e_j opposite to its orientation–so e_j is oriented from v_{j+1} to v_j–edge e_j is *not oriented as C*.

The oriented cycle C can be identified with an element $\boldsymbol{z}_C \in \mathcal{E}(D)$. Namely,

$$\boldsymbol{z}_C(e_i) = \begin{cases} 1 & \text{if } e_i \in E(C) \text{ and } e_i \text{ is oriented as } C \\ -1 & \text{if } e_i \in E(C) \text{ and } e_i \text{ is not oriented as } C \\ 0 & \text{if cycle } C \text{ does not traverse edge } e_i \end{cases}$$

Definition 1.2. The ***cycle space*** of D, denoted $Z(D)$, is the subspace of $\mathcal{E}(D)$ spanned by the elements $\boldsymbol{z}_C$ for all cycles C in D.

As $\boldsymbol{z}_C \in \mathcal{E}(D)$, we can represent $\boldsymbol{z}_C$ as a vector in $\mathbb{R}^{|E(D)|}$. We call this vector the ***characteristic vector***, or the ***cycle vector*** of C, denoted $\chi(C)$. For example, the oriented cycle $C = v_1 v_4 v_2 v_1$ in Figure 1.4.1 has cycle vector

$$\chi(C) = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \end{array}.$$

In this case, $E(C) = \{e_1, e_4, e_6\}$ and edge e_1 is oriented as C, whereas edges e_4 and e_6 are not oriented as C.

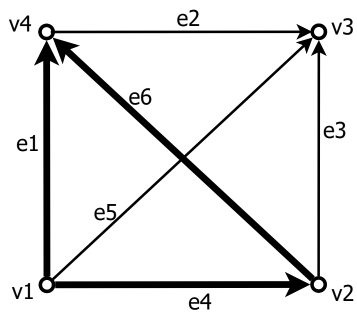


Figure 1.4.1: A cycle C with cycle vector $\chi(C) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$.

Now let $P = (V_1, V_2)$ be a partition of $V(D)$. That is, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V(D)$. Let $E(V_1, V_2)$ denote the set of all directed edges in $E(D)$ with one endpoint in V_1 and the other endpoint in V_2.

Definition 1.3. A partition $P = (V_1, V_2)$ of $V(D)$ as described above is called a ***cut*** of D, and $E(V_1, V_2)$ denotes the set of ***cut edges***.

Note that some authors refer to $E(V_1, V_2)$ itself as the cut as these are the edges that one would need to "cut" to separate the vertex sets V_1 and V_2 in D. Given any cut P of D, we can identify P with an element $\boldsymbol{u}_P \in \mathcal{E}(D)$. Namely,

$$\boldsymbol{u}_P(e_i) = \begin{cases} 1 & \text{if } e_i \text{ is directed from } V_1 \text{ to } V_2 \\ -1 & \text{if } e_i \text{ is directed from } V_2 \text{ to } V_1 \\ 0 & \text{if } e_i \notin E(V_1, V_2) \end{cases}$$

That is, $\boldsymbol{u}_P(e_i) = 0$ if and only if both endpoints of e_i are in V_1, or both are in V_2.

Definition 1.4. The ***cut space*** of D, denoted $U(D)$, is the subspace of $\mathcal{E}(D)$ spanned by the elements $\boldsymbol{u}_P$ for all cuts P of D.

Note that as $\boldsymbol{u}_P \in \mathcal{E}(D)$ we can represent $\boldsymbol{u}_P$ as a vector in $\mathbb{R}^{|E(D)|}$. We will refer to this vector as the ***cut vector*** corresponding to P, and denote it as $\vartheta(P)$. For example, let $P = (V_1, V_2)$ be the cut of the digraph in Figure 1.4.2 where $V_1 = \{v_1, v_4\}$ and $V_2 = \{v_2, v_3\}$. Then $E(V_1, V_2) = \{e_2, e_4, e_5, e_6\}$, and

$$\begin{array}{rc} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \vartheta(P) = & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix}. \end{array}$$

Remark 1.3. We can construct a cut in D by letting V_1 be a single vertex $v_i \in V(D)$. In the special case of such a cut $(v_i, V(D) - v_i)$, we will call this vector in $\mathbb{R}^{|E(D)|}$ the ***incidence vector*** of vertex v_i, and denote it by $\lambda(v_i)$. For example, in Figure 1.4.2,

$$\lambda(v_1) = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

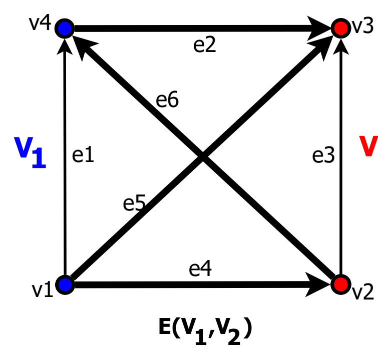


Figure 1.4.2: A cut P with cut vector $\vartheta(P) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix}$.

The cycle space and the cut space of a graph provide one definition of a natural duality between graphs. We mention it here for completeness, and will return to this idea in Chapter 4.

Definition 1.5. Two digraphs D and D' are ***abstract duals*** if $E(D) = E(D')$ and the cycle space of one is the cut space of the other. That is,

$$Z(D) = U(D') \quad \text{and} \quad U(D) = Z(D').$$

1.4.2 Orthogonality of $Z(D)$ and $U(D)$

The cut space and cycle space of a graph are intimately related, as demonstrated by Theorem 1.2.

Theorem 1.2. *Given a finite digraph D, the cycle space $Z(D)$ and cut space $U(D)$ are orthogonal subspaces.*

Proof. Let C be any cycle in D and let $P = (V_1, V_2)$ be any cut in D. To prove that

$Z(D) \perp U(D)$ we must show that

$$\Big\langle \boldsymbol{z}_C, \boldsymbol{u}_P \Big\rangle = \sum_{i=1}^{|E(D)|} \boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = 0.$$

However $\boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) \neq 0$ if and only if both $\boldsymbol{z}_C(e_i)$ and $\boldsymbol{u}_P(e_i)$ are nonzero. For each $e_i \in E(D)$ let $\iota(e_i) \in V(D)$ denote the initial vertex of e_i, and let $\tau(e_i) \in E(D)$ denote the terminal vertex. There are four cases in which $\boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) \neq 0$.

Case 1. If $\iota(e_i) \in V_1$ and $\tau(e_i) \in V_2$, and e_i is oriented as C, then $\boldsymbol{u}_P(e_i) = 1$ and $\boldsymbol{z}_C(e_i) = 1$, so

$$\boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = 1.$$

Case 2. If $\tau(e_i) \in V_1$ and $\iota(e_i) \in V_2$, and e_i is *not* oriented as C, then $\boldsymbol{u}_P(e_i) = -1$ and $\boldsymbol{z}_C(e_i) = -1$, so

$$\boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = 1.$$

Remark 1.4. In both Cases 1 and 2, C traverses edge e_i beginning at its V_1 vertex, and ending at its V_2 vertex.

Case 3. If $\iota(e_i) \in V_1$ and $\tau(e_i) \in V_2$, and e_i is *not* oriented as C, then $\boldsymbol{u}_P(e_i) = 1$ and $\boldsymbol{z}_C(e_i) = -1$, so

$$\boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = -1.$$

Case 4. If $\tau(e_i) \in V_1$ and $\iota(e_i) \in\in V_2$, and e_i is oriented as C, then $\boldsymbol{u}_P(e_i) = -1$ and $\boldsymbol{z}_C(e_i) = 1$, so

$$\boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = -1.$$

Remark 1.5. In both Cases 3 and 4, C traverses edge e_i beginning at its V_2 vertex, and ending at its V_1 vertex.

As C is a cycle in D, the number times C moves from a V_1 vertex to a V_2 vertex must be *equal to* the number of times it moves from a V_2 vertex to a V_1 vertex. That is, the number of times Cases 1 and 2 occur (combined) must be equal to the number of times Cases 3 and 4 occur. Therefore,

$$\Big|\{i : \boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = 1\}\Big| = \Big|\{i : \boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = -1\}\Big|$$

Furthermore,

$$\sum_{i=1}^{|E(D)|} \boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = \Big|\{i : \boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = 1\}\Big| - \Big|\{i : \boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = -1\}\Big|.$$

Therefore,

$$\left\langle \boldsymbol{z}_C, \boldsymbol{u}_P \right\rangle = \sum_{i=1}^{|E(D)|} \boldsymbol{z}_C(e_i)\boldsymbol{u}_P(e_i) = 0.$$

Given that C was an arbitrary cycle in D and $P = (V_1, V_2)$ was an arbitrary cut in D it follows that $Z(D) \perp U(D)$. □

Corollary 1.1. *Given any vertex $v \in V(D)$ and any cycle C in D,*

$$\lambda(v) \cdot \chi(C) = 0.$$

Theorem 1.3. *Let D be a finite digraph with k components. As before, let $n = |V(D)|$ and $m = |E(D)|$. Then*

$$dim\ Z(D) = m - (n - k) \text{ and } dim\ U(D) = n - k.$$

In other words, $\mathcal{E}(D)$ is the orthogonal direct sum of $Z(D)$ and $U(D)$,

$$\mathcal{E}(D) = Z(D) \oplus U(D).$$

Remark 1.6. Note that it is enough to prove this theorem for the case k =1. If D has k distinct components $D_1, D_2, \ldots D_k$, then $\mathcal{E}(D)$ is the orthogonal direct sum

$$\mathcal{E}(D) = \mathcal{E}(D_1) \oplus \mathcal{E}(D_2) \oplus \cdots \oplus \mathcal{E}(D_k).$$

Moreover for each i, $Z(D_i) = Z(D) \cap \mathcal{E}(D_i)$ and $U(D_i) = U(D)\mathcal{E}(D_i)$.

Henceforth we consider only the case k = 1, so that D is connected. The following proof can be found in Bollobás [10].

Proof. As $Z(D) \perp U(D)$ it follows that $Z(D) \cap U(D) = \mathbf{0} \in \mathcal{E}(D)$. Therefore because dim $\mathcal{E}(D) = m$, it suffices to show that:

(i) dim$Z(G) \geq m - (n - 1)$

(ii) dim$U(G) \geq n - 1$

Let T be a spanning tree of digraph D. Note that directed edges may be oriented in any manner in T, and as T has n vertices, it has $n - 1$ edges. If necessary, relabel the edges of D (and T) so that $E(T) = \{e_1, \ldots e_{n-1}\}$. We call $e_1, \ldots, e_{n-1}$ the *tree edges* of T. The remaining edges, $e_n, e_{n+1} \ldots, e_m$ are the *chords* of T. We will use T to find $m - n + 1$ independent vectors in $Z(D)$ and $n - 1$ independent vectors in $U(D)$.

Proof of (i). Fix some $j \geq n$. Adding chord e_j to T introduces a (unique) cycle C_j in $T + e_j$ containing edge e_j. Moreover, C_j does not contain any other chord e_i. Therefore as C_j is a cycle in D,

$$\boldsymbol{z}_{C_j}(e_j) = 1 \text{ and } \boldsymbol{z}_{C_j}(e_i) = 0 \text{ for } i \neq j \quad (i \geq n).$$

We call the cycle C_j the ***fundamental cycle*** of e_j with respect to T. Letting j range over all chords e_j ($n \leq j \leq m$), we find $m-n+1$ fundamental cycles $C_n, \ldots, C_m$. This gives $m-n+1$ elements $\boldsymbol{z}_{C_n}, \boldsymbol{z}_{C_{n+1}}, \ldots \boldsymbol{z}_{C_m} \in \mathcal{E}(D)$ such that

$$\boldsymbol{z}_{C_j}(e_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \geq n \text{ and } i \neq j \end{cases}.$$

Now let $\mathbf{0}$ be the zero element of $\mathcal{E}(D)$. If there exists constants $\gamma_n, \gamma_{n+1} \ldots \gamma_m$ such that $\mathbf{0} = \sum_{i=n}^{m} \gamma_i \boldsymbol{z}_{C_i}$, then for all $j \geq n$,

$$0 = \mathbf{0}(e_j) = \sum_{i=n}^{m} \gamma_i z_{C_i}(e_j) = \gamma_j.$$

That is, every coefficient γ_j must be 0, so $\boldsymbol{z}_{C_n}, \ldots, \boldsymbol{z}_{C_m}$ are independent in $\mathcal{E}(D)$. As $\boldsymbol{z}_{C_i(D)} \in Z(D)$ for all i, it follows that $\dim Z(D) \geq m - n + 1 = m - (n-1)$.

Proof of (ii). For $1 \leq i \leq n-1$, as before let $\iota(e_i)$ and $\tau(e_i)$ denote the initial and terminal vertices of edge e_i respectively. We see that by deleting e_i from T, the spanning tree is separated into two components, denoted T_1^i and T_2^i. Assume, without loss of generality, that $\iota(e_i) \in T_1^i$ and $\tau(e_i) \in T_2^i$. Let $V_1^i = V(T_1^i)$ and $V_2^i = V(T_2^i)$, so that $P_i = (V_1^i, V_2^i)$ is a cut of D. We call this cut (V_1^i, V_2^i) the ***fundamental cut*** of e_i with respect to T.

As defined, $\iota(e_i) \in V_1^i$ and $\tau(e_i) \in V_2^i$, so $\boldsymbol{u}_{P_i}(e_i) = 1$. For any tree edge e_j such that $j \neq i$, the endpoints of the e_j either both lie in T_1^i or in T_2^i. Therefore, the endpoints of e_j are either both contained in V_1^i or both in V_2^i. As a result, for any $j \neq i$ ($1 \leq j \leq n-1$), we must have $\boldsymbol{u}_{P_i}(e_j) = 0$. Therefore letting i range over all tree edges, we find $n-1$ fundamental cuts, $P_1, \ldots, P_{n-1}$. This gives $n-1$ elements $\{\boldsymbol{u}_{P_1}, \ldots \boldsymbol{u}_{P_{n-1}}\} \in \mathcal{E}(D)$ such that

$$u_{P_j}(e_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \leq n-1 \text{ and } i \neq j \end{cases}.$$

If there exists constants $\mu_1 \dots \mu_{n-1}$ such that $\mathbf{0} = \sum_{j=1}^{n-1} \mu_j u_{P_j}$, then for all $1 \leq i \leq n-1$,

$$0 = \mathbf{0}(e_i) = \sum_{j=1}^{n-1} \mu_j u_{P_j}(e_i) = \mu_i.$$

That is, every coefficient μ_i is 0, so $\boldsymbol{u}_{P_1}, \dots, \boldsymbol{u}_{P_{n-1}}$ are independent in $\mathcal{E}(D)$. As $\boldsymbol{u}_{P_j} \in U(D)$ for all $1 \leq j \leq n-1$, it follows that dim $U(D) \geq n-1$. $\square$

This completes our review of classical graph theory. The primary purpose of this chapter was to establish the notation and terminology conventions used in the chapters that follow. While we presented mainly definitions and examples, complete proofs of Theorems 1.2 and 1.3 were given. These should be kept in mind, as the orthogonality of cut vectors and cycle vectors is a theme that we will return to repeatedly throughout this text. Chapter 2 now begins our study of Kirchhoff graphs. After starting with a motivating example, we will introduce definitions and prove a number of fundamental results.

Chapter 2

Vector Graphs and Kirchhoff Graphs

Kirchhoff graphs were introduced by Fehribach [28] as a mathematically precise version of the reaction route graphs discussed by Fishtik, Datta *et al.* [31], [32], [33], [34], to represent electrochemical reaction networks. In the context of these networks, Kirchhoff graphs represent the orthocomplementarity of the row space and null space of the stoichiometric matrix for a given reaction network [29]. A number of graphs have been used to study of reaction networks; a summary can be found in [20] or [27]. A more thorough treatment can be found in [103], and the interested reader can find a wealth of references in both [103] and the survey in [99]. One noteworthy sequence of work on graphs and reaction networks is that of species-complex-linkage graphs. Beginning in 1973 with [62] and [63], these graphs have been studied in a variety of contexts in the following decades, including [17], [19], [89], and [97]. As noted in [29], while these graphs are useful in studying reaction kinetics and stability of equilibria, they are unrelated to the fundamental spaces of any matrix.

Section 2.1.1 begins with an electrochemical reaction network view of Kirchhoff graphs. Specifically, the details of how Kirchhoff graphs depict Kirchhoff's current and potential laws as applied to a sample network are explained through an illustrative example. Section 2.1 builds upon the work of [28],[29] and [94], presenting definitions, examples, and fundamental results on Kirchhoff graphs. Motivated by reaction networks, Section 2.1.2 abstracts the ideas presented in Section 2.1.1, and gives an example of how to construct a Kirchhoff graph for a rational-valued matrix. In the process, connections of Kirchhoff graphs with graph theory, linear algebra, and group theory are revealed. We strive to present this material in such a way that constructing or identifying a Kirchhoff graph will follow intuitively from basic principles. After Section 2.2 develops mathematically precise definitions of

Kirchhoff graphs, 2.2.1 presents a number of fundamental results, including a generalization of the Kirchhoff property and a natural notion of graph operations on Kirchhoff graphs.

The definitions presented in Section 2.2 are mathematically equivalent to those of [29]; Section 2.3.1 now generalizes these to $\mathbb{Z}_p$-valued matrices and $\mathbb{Z}_p$-Kirchhoff graphs. The primary open mathematical problem in the study of Kirchhoff graphs is the following conjecture, due to Fehribach [28] [29].

Conjecture 2.1. *Every rational-valued matrix has a Kirchhoff graph.*

This is an unsolved (and probably difficult) problem in general, and no universal construction methods are known. Specific examples are considered in [29], and [28] proves that a matrix of either rank or nullity 1 always has a Kirchhoff graph. Section 2.3.2 now addresses this problem for $\mathbb{Z}_p$-valued matrices. Theorem 2.1 will show that for any integer-valued matrix A, there exists a nontrivial $\mathbb{Z}_p$-Kirchhoff graph for the matrix $A \pmod p$ for sufficiently large prime p. Moreover, nonzero $\mathbb{Z}_p$-Kirchhoff graphs are explicitly constructed for $\mathbb{Z}_p$-valued matrices with an entry-wise nonzero vector in the row space.

Remark 2.1. Much of this material was presented by the author in [93].

2.1 Introduction to Kirchhoff Graphs

This section will present a general introduction to Kirchhoff graphs, before formal definitions are given in Section 2.2. Section 2.1.1 explains how Kirchhoff graphs represent Kirchhoff's current and potential laws, and Section 2.1.2 begins to consider constructing Kirchhoff graphs for integer-valued matrices.

2.1.1 Simplified Hydrogen Evolution Reaction Network

As motivation for the study of Kirchhoff graphs, let us first consider a simplified version of the hydrogen evolution reaction (HER) network. The overall reaction is

$$\boldsymbol{b}: \qquad 2\mathrm{H_2O} + 2\mathrm{e^-} \rightleftharpoons \mathrm{H_2} + 2\mathrm{OH^-} \tag{2.1}$$

but this reaction is not achieved directly. It only occurs as the result of two elementary reaction steps,

$$\begin{aligned} \boldsymbol{s}_1: &\quad 2\mathrm{H{\cdot}S} &\rightleftharpoons&\quad 2\mathrm{S} + \mathrm{H_2}\,, \\ \boldsymbol{s}_2: &\quad \mathrm{H{\cdot}S} + \mathrm{H_2O} + \mathrm{e^-} &\rightleftharpoons&\quad \mathrm{S} + \mathrm{H_2} + \mathrm{OH^-}\,. \end{aligned} \tag{2.2}$$

To be clear, only the elementary steps occur in nature; the overall reaction is achieved *only* as a result of the elementary steps. This reaction network consists of six reacting species: molecular water (H_2O) and hydrogen (H_2), hydroxide ions (OH^-), electrons (e^-), and two types of reaction sites on a catalyst, free sites (S) and sites occupied by hydrogen (H·S).

Each of the reactions $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, and $\boldsymbol{b}$ are *reversible*. Dependent upon external factors such as temperature, pH, or component concentrations, the reactions as written in (2.1) and (2.2) can proceed either from left to right, or from right to left. The reactions proceeding from left to right in (2.1) are (2.2) are the *forward* reactions, and the *backward* reactions proceed from right to left.

This network can be studied using its stoichiometric matrix A, given in (2.3). In this matrix, the columns correspond to the reactions (elementary steps or the overall reaction), while the rows correspond to the reacting species.

$$A = \begin{array}{c} \\ \mathrm{H_2} \\ \mathrm{OH^-} \\ \mathrm{H_2O} \\ \mathrm{e^-} \\ \mathrm{S} \\ \mathrm{H{\cdot}S} \end{array} \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{b} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \\ 2 & 1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \end{array} \tag{2.3}$$

Positive entries in this matrix are stoichiometric coefficients for species on the right side of the reactions (2.1) and (2.2), the reaction products, while negative entries are stoichiometric coefficients for species on the left, the reactants. If a species is absent from a reaction, the corresponding entry in the stoichiometric matrix is zero.

Each reaction step can be viewed as a vector in the stoichiometric space; the stoichiometric coefficient of each molecule, ion or reaction site is an entry in the corresponding vector. Products (species on the right of (2.1) and (2.2)) have positive entries, whereas reactants are negative. In the context of this reaction network, the Kirchhoff potential law requires that the overall reaction must be a linear combination of the two elementary steps. Clearly from the stoichiometry,

$$\boldsymbol{b} = 2\boldsymbol{s}_2 - \boldsymbol{s}_1. \tag{2.4}$$

This vector equation (2.4) also implies that the net change in the electrochemical potential across this combination of elementary steps is the same as the net change across the overall reaction. Note that electrochemical potential for reversible electrochemical reactions is the equivalent of electrical voltage for electrical circuits; we refer the reader to Newman [82] for further details. A more thorough discussion of electrochemical potentials and reaction networks can be found in [26] or [30]. This net-change equivalence makes clear why (2.4) is a version of the Kirchhoff potential law. In fact, beyond the balance of electrochemical potential, equation (2.4) also requires that the production or consumption of each of the reacting species across these elementary steps is the same as the production or consumption across the overall reaction.

Now, let r_1, r_2, and r_b denote the relative reaction rates of steps $\boldsymbol{s}_1, \boldsymbol{s}_2$, and $\boldsymbol{b}$ respectively. These rates must satisfy a version of the Kirchhoff current law to prevent any of the species from accumulating within the network (recall that current is the time rate of change in electrical charge). For example, all of the open reaction sites (S) produced by the network must be consumed by the network, and any H_2 produced by the elementary steps must be accounted for by the overall reaction. To balance the creation and consumption of both S and H $\cdot$ S, the rate for $\boldsymbol{s}_2$ must be twice the rate of $\boldsymbol{s}_1$. That is,

$$r_2 = 2r_1. \tag{2.5}$$

To balance the creation and consumption of OH^-, e^- and H_2O, the rate for $\boldsymbol{s}_2$ must also be twice the overall rate:

$$r_2 = 2r_b. \tag{2.6}$$

The vector equation relating the elementary steps to the overall reaction (2.4) may seem unrelated to the rate balance equations (2.5) and (2.6). In fact they can all be described in terms of the stoichiometric matrix, A. Looking at the coefficients in (2.4), one sees that this relationship can be encoded by the vector

$$\begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{b} \end{array} \\ \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \end{array}.$$

On the other hand, the coefficients of the two rate-balance equations (2.5) and (2.6) yield the vectors

$$\begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{b} \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{b} \end{array} \\ \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \end{array}.$$

More importantly, observe that

$$\mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^t \right\} \quad \text{and} \quad \mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \right\},$$

where $\mathrm{Null}(A)$ and $\mathrm{Row}(A)$ are used to denote the null space and row space of A, respectively. Again, the electrochemistry dictates these relationships, while of course the fundamental theorem of linear algebra guarantees that $\mathrm{Null}(A)$ and $\mathrm{Row}(A)$ are orthogonal complements.

Now, the Kirchhoff graph (if it exists) is defined to depict this orthogonality graphically [27][28][29]. Speaking generally, a Kirchhoff graph is a finite digraph with labeled edges. The notion of labeling edges, and how those labels are chosen, will be of paramount importance in what is to follow. In this context, the labels are used to denote reaction steps and can be repeated within the network. So, for example, the HER network will have three types of edges: $\boldsymbol{s}_1, \boldsymbol{s}_2$, and $\boldsymbol{b}$.

Remark 2.2. The orientation of each edge will represent the direction of each reaction. An edge traversed with its orientation indicates a *forward* reaction, whereas a backwards reaction is achieved by crossing an edge against its orientation.

The structure of this graph will be considered with respect to the labeling. While a more concrete definition will be given Section 2.2, each cycle in the Kirchhoff graph must correspond to a vector in Null(A), while each vertex must correspond to a vector in Row(A). In this example, Figure 2.1.1 is a Kirchhoff graph for the simplified HER reaction network. Methods of recognizing and constructing this graph will be explored further in the following section.

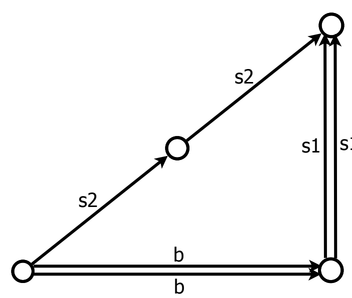


Figure 2.1.1: A Kirchhoff graph for the simplified HER network

Observe that only the stoichiometric matrix was needed to identify a Kirchhoff graph for the HER network. Building on well-known group theoretic techniques, Section 2.2 presents a rigorous definition of Kirchhoff graphs for a matrix.

2.1.2 A Matrix and its Kirchhoff Graph

Section 2.1.1 motivates the study of Kirchhoff graphs by presenting a simplified electrochemical reaction network. Observe, however, that the stoichiometric matrix was all that was needed to construct the associated graph. In this section, we will begin with a matrix and demonstrate how the properties of that matrix can lead to the construction of a Kirchhoff graph.

Consider the matrix A, as in (2.7).

$$A = \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \\ \end{array}$$
$$A = \left[\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \tag{2.7}$$

A consists of 4 columns, $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$ and $\boldsymbol{s}_4$, each lying in the vector space $\mathbb{Q}^2$.

$$\boldsymbol{s}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{s}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{s}_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{2.8}$$

Continuing as in Section 2.1.1, each cycle in a Kirchhoff graph for A must correspond to a vector in Null(A); this correspondence will be made exact in Proposition 2.1.

Remark 2.3. Observe that the null space of any rational-valued matrix has a basis composed of integer-valued vectors. We will use this fact implicitly in this and all future examples. Clearly a cycle may only traverse edges an integer multiple of times, thus we specifically compare cycles to *integer-valued* vectors in Null(A).

For example,

$$\begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \\ \end{array}$$
$$\left[\begin{array}{cccc} 1 & 1 & -1 & 0 \end{array}\right]^t \in \mathrm{Null}(A).$$

That is, in light of (2.8),

$$\boldsymbol{s}_1 + \boldsymbol{s}_2 - \boldsymbol{s}_3 = \mathbf{0},$$

where $\mathbf{0}$ denotes the all-zero vector. Therefore a labeled cycle of the form shown in Figure 2.1.2 is permitted in a Kirchhoff graph for A.

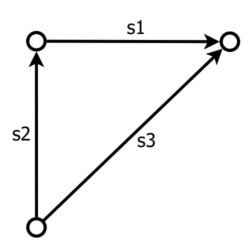


Figure 2.1.2: A cycle permitted in a Kirchhoff graph for matrix A, as $[1, 1, -1, 0]^t \in \mathrm{Null}(A)$.

On the other hand, as

$$\begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \\ \end{array}$$
$$\left[\begin{array}{cccc} 1 & 1 & 0 & -1 \end{array}\right]^t \notin \mathrm{Null}(A),$$

the label $\boldsymbol{s}_3$ in Figure 2.1.2 cannot be replaced by the label $\boldsymbol{s}_4$.

The requirement that all cycles correspond to elements of Null(A) necessitates that the labeled edges will satisfy precisely the same integer dependencies (including coefficients) as the column vectors of A. Therefore in drawing a Kirchhoff graph, each labeled edge *can be drawn as* the column by which it is labeled. Observe that the edges $\boldsymbol{s}_1, \boldsymbol{s}_2$ and $\boldsymbol{s}_3$ in Figure 2.1.2 are each drawn as the vectors given in (2.8). On the other hand, with edges drawn as vectors it is easy to recognize that as $\boldsymbol{s}_1 + \boldsymbol{s}_2 - \boldsymbol{s}_4 \neq \mathbf{0}$, a cycle can ever have the form

$$\begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array}$$
$$\left[\begin{array}{cccc} 1 & 1 & 0 & -1 \end{array}\right].$$

This is illustrated in Figure 2.1.3.

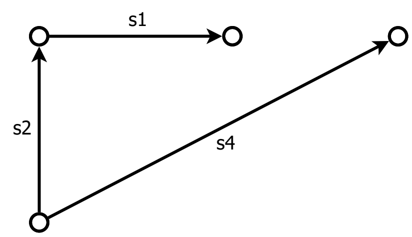


Figure 2.1.3: By drawing edges as vectors, because $\boldsymbol{s}_1 + \boldsymbol{s}_2 - \boldsymbol{s}_4 \neq 0$, no cycle in a Kirchhoff graph for A can have the form $[1, 1, 0, -1]$.

Consider the digraph D presented in Figure 2.1.4. We embed the vertices in the plane and label their coordinates. (Note parentheses denote points in space, and we reserve square brackets for vectors). All labeled edges now correspond to vectors, which are precisely:

$$\boldsymbol{s}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{s}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{s}_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

If the vector edges in the graph form a cycle, it means the corresponding sum of vectors $-$with coefficients to respect orientations and multiplicities$-$is zero. That is, the vector of coefficients lies in Null(A).

Therefore, by constructing a graph on edges that are precisely the column vectors of A, we are *guaranteed* that the cycles will correspond to vectors in Null(A). While a precise definition of this coefficient vector will be given in Definition 2.3, this is easy to check. For example, the cycle on vertices $(0,0) - (1,0) - (2,1) - (0,0)$ traverses an $\boldsymbol{s}_1$ edge and an $\boldsymbol{s}_3$ edge each with the correct orientation, followed by an $\boldsymbol{s}_4$ edge with the opposite orientation.

This cycle corresponds to the vector

$$\begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix}^t \in \mathrm{Null}(A). \end{array}$$

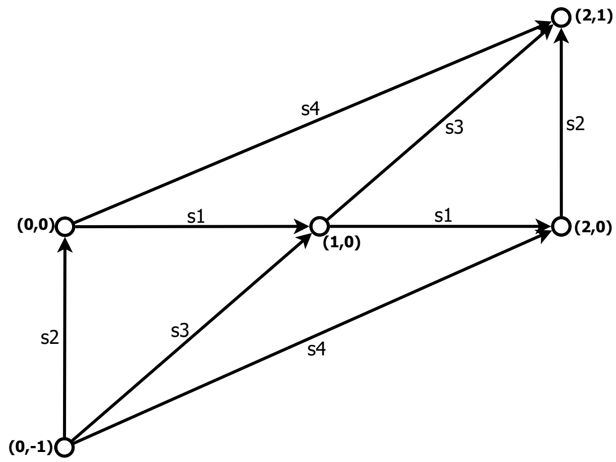


Figure 2.1.4: A Kirchhoff graph for matrix A.

Remark 2.4. Using vectors as edges will play an important role in constructing Kirchhoff graphs. One could conceivably begin with an arbitrary digraph and begin assigning edge labels. However, finding a set of labels that reflects the desired structure of matrix A may be difficult or even impossible. Constructing graphs with edges that are column vectors of A guarantees that the cycles capture the desired structure. [1]

Returning to Figure 2.1.4, the labeled edges incident at each vertex can be encoded by

$$\begin{array}{rl}
 & \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\
(0,0)/(2,0): & \pm\begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix} \\
 & \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\
(0,-1)/(2,1): & \pm\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \\
 & \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\
(1,0): & \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}$$

where a 1 in entry $\boldsymbol{s}_i$ denotes an outgoing edge labeled $\boldsymbol{s}_i$, and a -1 denotes an incoming edge. A zero indicates that the same number of $\boldsymbol{s}_i$ edges both enter and exit a vertex. Observe that all of these vectors lie in the row space of A. In fact, D is a Kirchhoff graph for matrix A.

[1]The columns of any matrix row equivalent to A also present a set of edge vectors.

2.2 Vector and Kirchhoff Graphs for a Matrix

Let Γ be any group, written additively, and let H be any subset of the elements of Γ.

Definition 2.1. The ***Cayley color graph*** (Γ, H) is a directed graph with vertex set Γ and edge set

$$E = \{(g, g+h) : g \in \Gamma, h \in H\}$$

[90][96]. That is, (Γ, H) has one vertex associated with each group element of Γ, and every edge is colored (or labeled) by the difference vector $h = (g+h) - g$.

For each $h \in H$, every vertex is incident with exactly two h edges: one entering and one exiting. The Cayley color graph is finite if and only if the group Γ is finite, and connected if and only if H is a set of generators of the group Γ. If $\Gamma = \mathbb{Z}_p^m$, any Cayley color graph for Γ is finite, with p^m vertices and $|H|p^m$ edges. For example, the Cayley color graph

$$\left(\mathbb{Z}_3^2, \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right\}\right)$$

is illustrated in Figure 2.2.1, drawn on the flat torus.

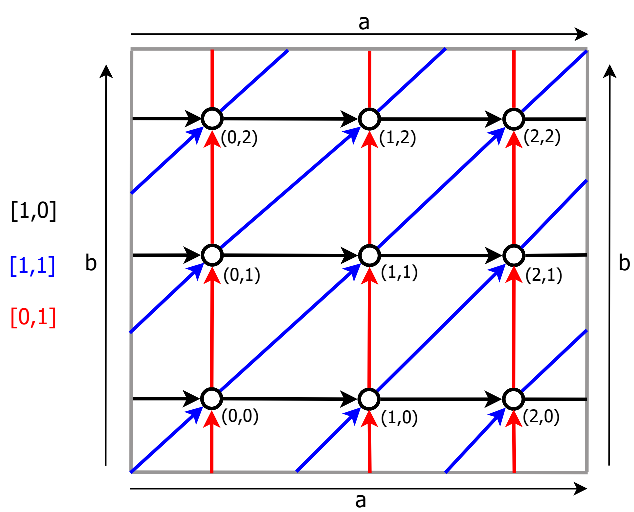


Figure 2.2.1: The Cayley color graph $(\mathbb{Z}_3^2, \{[1,0],[1,1],[0,1]\})$, drawn on the flat torus.

Cayley color graphs for vector spaces over the finite field $\mathbb{Z}_p$ (for p a prime) will be considered further in Section 2.3.1.

Now, let A be any $m \times n$ matrix with entries in $\mathbb{Q}$, and let $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n \in \mathbb{Q}^m$ denote the column vectors of A. The Cayley color graph $(\mathbb{Q}^m, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\})$ is an infinite directed graph with $\{\boldsymbol{s}_1, \ldots \boldsymbol{s}_n\}$ as vector edges.

Definition 2.2. Any graph obtainable from a finite subgraph of the Cayley color graph $(\mathbb{Q}^m, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\})$ by assigning (positive integer) multiplicities to the edge vectors is a ***vector graph for*** A.

That is, a vector graph for A has a finite number of vertices and vector edges, each of which is a column vector of matrix A. Let $\mathcal{D}$ be any vector graph for matrix A. A *cycle* in $\mathcal{D}$ is a closed walk with no repeated vertices, that may traverse edges regardless of orientation.

Remark 2.5. Classically, the cycles of a digraph must always traverse an edge in the direction of its orientation. However the vector edges in our graphs are meant to represent reversible processes (reactions), which have both a forward *and* a backward direction. Null processes (reactions) can include these backward steps, therefore cycles must be allowed to traverse edges regardless of orientation. Note that this matches the conventions of [10].

Definition 2.3. For each cycle C the ***cycle vector***, denoted $\chi(C)$, is a row vector with entries indexed by $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$. For each i, entry $\boldsymbol{s}_i$ is the net number of times cycle C traverses an $\boldsymbol{s}_i$ edge. Add 1 to the i^{th} component each time C traverses an $\boldsymbol{s}_i$ with the correct orientation, and subtract 1 for each $\boldsymbol{s}_i$ traversed in the opposite orientation.

Proposition 2.1. For any cycle C of vector graph $\mathcal{D}$, $\chi(C)^t \in \mathrm{Null}(A)$.

Proposition 2.1 is the "correspondence" between cycles and null space vectors alluded to in Section 2.1.1. It is guaranteed by choosing a vector graph for A. The proof is straightforward given Definition 2.3. The entries of $\chi(C)$ are the coefficients of vectors traversed in C. As C is a cycle, the corresponding sum of vectors must be zero. Therefore, the vector of coefficients, $\chi(C)^t$, must lie in $\mathrm{Null}(A)$.

Example 2.1. Consider the matrix A in (2.9). A consists of 4 columns, $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$ and $\boldsymbol{s}_4$, each lying in the vector space $\mathbb{Q}^2$.

$$A = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{2.9}$$

Figure 2.2.2 presents a vector graph for A. Cross-hatches are used to denote multiple edges with the same endpoints and vector label. The cycle $C = (0,0) - (1,0) - (1,1) - (0,0)$, for example, has cycle vector:

$$\chi(C)^t = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & -1 & 0 \end{array}\right]^t \end{array} \in \mathrm{Null}(A).$$

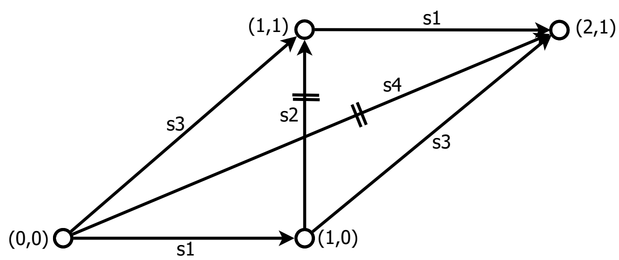


Figure 2.2.2: A vector graph for matrix A. Hash marks indicate multiplicity and vertex coordinates are given by ordered pairs.

Next, consider the vector edges incident at each vertex.

Definition 2.4. For each vertex v the ***incidence vector***, denoted $\lambda(v)$, is a row vector with entries indexed by $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$. For each i, entry $\boldsymbol{s}_i$ is the net number of $\boldsymbol{s}_i$ that exit vertex v. Equivalently, it is the number of $\boldsymbol{s}_i$ edges exiting v minus the number of $\boldsymbol{s}_i$ entering.

For example, the vertex v at $(1,1)$ in Figure 2.2.2 has incidence vector

$$\lambda(v) = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \left[\begin{array}{cccc} 1 & -2 & -1 & 0 \end{array} \right] \end{array}.$$

Recall that in Section 2.1.1, Kirchhoff's current law translated graphically to vertices that correspond to elements of Row(A). This can be rephrased by saying that $\lambda(v) \in \text{Row}(A)$ for all vertices v. This leads to a precise definition of Kirchhoff graphs.

Definition 2.5. Let $\mathcal{D}$ be any vector graph for matrix A. If $\lambda(v) \in \text{Row}(A)$ for all vertices v of $\mathcal{D}$, and the cycle vectors of $\mathcal{D}$ span Null(A), then $\mathcal{D}$ is a ***Kirchhoff graph for*** A.

Take a moment to consider this definition with respect to the sample network in Section 2.1.1. As $\mathcal{D}$ is a vector graph for A, all cycle vectors lie in Null(A), and Kirchhoff's potential law is satisfied. The requirement that all incidence vectors lie in Row(A) ensures satisfaction of Kirchhoff's current law as well. Finally, closed circuits are vital to this network: the added requirement that the cycle vectors span Null(A) ensures that all dependency relations are captured.

Remark 2.6. Given the stoichiometric matrix A of a reaction network, a Kirchhoff graph for that matrix serves as a network diagram. With edge vectors corresponding to reaction steps, Kirchhoff's current law is guaranteed by the requirement that all vertex incidence vectors lie in the row space of A. In order to satisfy Kirchhoff's potential law, the labeled edges must satisfy the same dependencies (including coefficients) as the columns of A. By constructing a graph on edges that are precisely the column vectors of A, we are *guaranteed* that all cycle vectors will lie in Null(A). Therefore using vectors as edges plays an important role in constructing Kirchhoff graphs.

Example 2.2. Full Hydrogen Evolution Reaction Network The full hydrogen evolution reaction network has the same overall reaction as the simplified network in Section 2.1.1, with one additional elementary step,

$$\boldsymbol{s}_3 : \quad \mathrm{S} + \mathrm{H_2O} + \mathrm{e^-} \rightleftharpoons \mathrm{H{\cdot}S} + \mathrm{OH^-}. \tag{2.10}$$

Thus the complete reaction network is

$$\begin{array}{rrcl} \boldsymbol{s}_1 : & 2\mathrm{H{\cdot}S} & \rightleftharpoons & 2\mathrm{S} + \mathrm{H_2} \\ \boldsymbol{s}_2 : & \mathrm{H{\cdot}S} + \mathrm{H_2O} + \mathrm{e^-} & \rightleftharpoons & \mathrm{S} + \mathrm{H_2} + \mathrm{OH^-} \\ \boldsymbol{s}_3 : & \mathrm{S} + \mathrm{H_2O} + \mathrm{e^-} & \rightleftharpoons & \mathrm{H{\cdot}S} + \mathrm{OH^-} \\ \boldsymbol{b} : & 2\mathrm{H_2O} + 2\mathrm{e^-} & \rightleftharpoons & \mathrm{H_2} + 2\mathrm{OH^-} \end{array} \tag{2.11}$$

In light of (2.11), the stoichiometric matrix of the full network is

$$A = \begin{array}{c} \\ \mathrm{H_2} \\ \mathrm{OH^-} \\ \mathrm{H_2O} \\ \mathrm{e^-} \\ \mathrm{S} \\ \mathrm{H{\cdot}S} \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{b} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & -1 & -1 & -2 \\ 2 & 1 & -1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \end{array}. \tag{2.12}$$

Both Null(A) and Row(A) are two dimensional, and each of the two vector graphs in Figure 2.2.3 are Kirchhoff graphs for this full hydrogen evolution reaction network. A more thorough study of this network was given by Fishtik, Callaghan, Fehribach & Datta [31].

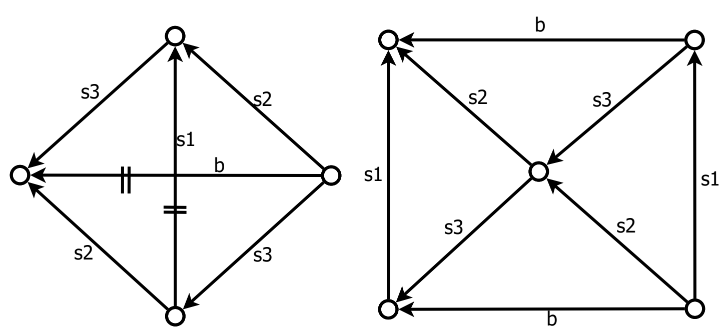


Figure 2.2.3: Two Kirchhoff graphs for the HER network.

Note that the column vectors of A lie in $\mathbb{Q}^6$ while any drawing necessarily lies in $\mathbb{R}^2$. We choose 2-dimensional projections that make clear which vector edges are different. In this case, projection onto the 4^{th} and 5^{th} entries of each column vector was chosen. Projection onto the 2^{nd} and 5^{th} entries of the column vectors of the stoichiometric matrix A in (2.3) was used to construct Figure 2.1.1.

A natural first question is whether every rational-valued matrix has a Kirchhoff graph. This question is likely difficult to answer in general [29] and is, in fact, the primary open question in studying Kirchhoff graphs. Section 2.3 will partially answer this question for matrices over finite fields. On the other hand, it is already clear from Figure 2.2.3 that in the case that a Kirchhoff graph exists, it need not be *unique*. This raises a number of interesting questions, which are open for future research. For example, given a rational-valued matrix A,

Question: If there exists a Kirchhoff graph for matrix A, what is a *minimal* example? Is there a suitable notion of minimality (such as number of vertices or number of directed edges) that leads to a well-defined *canonical* Kirchhoff graph for a given matrix?

Clearly both Kirchhoff graphs in Figure 2.2.3 have 8 vector edges. However the first graph has only 4 vertices whereas the second has 5. Let v be the central vertex in the second graph of Figure 2.2.3. Observe that

$$\lambda(v) = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

We call any such vertex with a zero-incidence vector a ***null vertex***. This raises a number of additional questions.

Question: If there exists a Kirchhoff graph for matrix A, does there exist an example with no null vertices? If the answer is "no," can we determine the minimum number of null vertices in a Kirchhoff graph for A?

Before moving on to $\mathbb{Z}_p$-valued matrices, we first observe some fundamental properties of Kirchhoff graphs.

2.2.1 Elementary Properties of Kirchhoff Graphs

Sections 2.1.1 and 2.2 presented motivation and precise definitions of Kirchhoff graphs. This section will examine elementary properties and graph operations on Kirchhoff graphs.

Let A and A' be two rational-valued matrices that are row equivalent. Let $\{\boldsymbol{s}_1, \dots, \boldsymbol{s}_n\}$ denote the columns of A and $\{\boldsymbol{s}'_1, \dots, \boldsymbol{s}'_n\}$ the columns of A'.

Proposition 2.2. If matrix A has a Kirchhoff graph, then so does A' [28].

Proof. Let $\mathcal{D}$ be any Kirchhoff graph for A. Relabel every $\boldsymbol{s}_i$ edge in $\mathcal{D}$ by the vector $\boldsymbol{s}'_i$. As A and A' are row equivalent, the result is a vector graph $\mathcal{D}'$ for A'. Moreover, the incidence and cycle vectors of $\mathcal{D}'$ are precisely those of $\mathcal{D}$. Therefore, every incidence vector lies in $\mathrm{Row}(A) = \mathrm{Row}\ (A')$, and the cycle vectors span $\mathrm{Null}(A) = \mathrm{Null}(A')$. Thus $\mathcal{D}'$ is a Kirchhoff graph for $\mathcal{D}'$. $\square$

The Kirchhoff property also presents a simple relationship between cycle and incidence vectors.

Proposition 2.3. Let $\mathcal{D}$ be a Kirchhoff graph for matrix some A. For any vertex v and any cycle C,

$$\lambda(v) \cdot \chi(C) = 0. \tag{2.13}$$

This condition suggests a natural way to generalize the concept of a Kirchhoff graph independent of a matrix, which was proposed in [94]. We will call any vector graph that satisfies (2.13) for all v and C a ***Kirchhoff graph***.

Remark 2.7. Given a rational-valued matrix, it is always easy to construct an *infinite* object with the desired properties. Let A be any matrix in $\mathbb{Q}^{m\times n}$ with column vectors $\boldsymbol{s}_1, \dots, \boldsymbol{s}_n$. The Cayley color graph $(\mathbb{Q}^m, \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_n\})$ is itself an infinite graph with vector edges satisfying the desired cycle condition. Moreover, each vertex is incident to exactly one incoming and one outgoing edge labeled $\boldsymbol{s}_i$, for each i. That is, every vertex v is a null vertex with $\lambda(v) = \mathbf{0}$, and the desired orthogonality is trivially satisfied. This infinite graph is none too revealing for modeling real-world networks, however, hence the requirement that a Kirchhoff graph be derived from a finite vector graph.

It is important to realize that the structure of Kirchhoff graphs is considered with respect to the edges' vector labels. Bearing this in mind, one can consider graph operations on Kirchhoff graphs. Operations applied to an edge of a standard graph–now applied to a set of vector edges in a Kirchhoff graph–maintain the Kirchhoff property.

Let $\mathcal{D}$ be a vector graph with edge vectors $\{\boldsymbol{s}_1, \dots, \boldsymbol{s}_n\}$. Let $\mathcal{D}^{\setminus \boldsymbol{s}_n}$ be obtained from $\mathcal{D}$ by deleting all $\boldsymbol{s}_n$ edges.

Proposition 2.4. If $\mathcal{D}$ is a Kirchhoff graph, then $\mathcal{D}^{\backslash \boldsymbol{s}_n}$ is also a Kirchhoff graph.

Proof. Let C be any cycle of $\mathcal{D}^{\backslash \boldsymbol{s}_n}$ with cycle vector

$$\chi(C) = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} \end{array} \\ \left[\begin{array}{ccc} x_1 & \cdots & x_{n-1} \end{array} \right] \end{array}.$$

Clearly the edges of C are also the edges of a cycle $\widetilde{C}$ in $\mathcal{D}$, with cycle vector

$$\chi(\widetilde{C}) = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} & \boldsymbol{s}_n \end{array} \\ \left[\begin{array}{cccc} x_1 & \cdots & x_{n-1} & 0 \end{array} \right] \end{array} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} & \boldsymbol{s}_n \end{array} \\ \left[\begin{array}{cccc} - & \chi(C) & - & 0 \end{array} \right] \end{array}.$$

Now let v be any vertex of $\mathcal{D}^{\backslash s_n}$, with incidence vector

$$\lambda(v) = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} \end{array} \\ \left[\begin{array}{ccc} y_1 & \cdots & y_{n-1} \end{array} \right] \end{array}.$$

Let $\widetilde{v}$ be the corresponding vertex in $\mathcal{D}$, so that

$$\lambda(\widetilde{v}) = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} & \boldsymbol{s}_n \end{array} \\ \left[\begin{array}{cccc} y_1 & \cdots & y_{n-1} & * \end{array} \right] \end{array} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} & \boldsymbol{s}_n \end{array} \\ \left[\begin{array}{cccc} - & \lambda(v) & - & * \end{array} \right] \end{array},$$

where $*$ may be any integer including 0. Then since $\mathcal{D}$ is a Kirchhoff graph,

$$\lambda(v) \cdot \chi(C) = \sum_{i=1}^{n-1} x_i y_i = \sum_{i=1}^{n-1} x_i y_i + 0 = \lambda(\widetilde{v}) \cdot \chi(\widetilde{C}) = 0.$$

As C and v were arbitrary, $\mathcal{D}^{\backslash \boldsymbol{s}_n}$ is a Kirchhoff graph. □

Label $\boldsymbol{s}_n$ was chosen in Proposition 2.4 simply for ease of notation. It is clear from the proof that for any label $\boldsymbol{s}_k$, the graph $\mathcal{D}^{\backslash \boldsymbol{s}_k}$ obtained by deleting all $\boldsymbol{s}_k$ edges is also Kirchhoff.

Corollary 2.1. *Let S_0 be any subset of $\{\boldsymbol{s}_1, \cdots, \boldsymbol{s}_n\}$ and let $\mathcal{D}^{\backslash S_0}$ be obtained from $\mathcal{D}$ by deleting all edge vectors contained in S_0. Then $\mathcal{D}^{\backslash S_0}$ is a Kirchhoff graph.*

Alternatively, let $\mathcal{D}^{/\boldsymbol{s}_n}$ be the graph obtained from $\mathcal{D}$ by contracting all $\boldsymbol{s}_n$ edges.

Proposition 2.5. If $\mathcal{D}$ is a Kirchhoff graph, then $\mathcal{D}^{/\boldsymbol{s}_n}$ is also a Kirchhoff graph.

Proof. Let C be any cycle of $\mathcal{D}^{/\boldsymbol{s}_n}$ with cycle vector

$$\chi(C) = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} \end{array} \\ \left[\begin{array}{ccc} x_1 & \cdots & x_{n-1} \end{array} \right] \end{array}.$$

Then there exists a cycle $\widetilde{C}$ in $\mathcal{D}$, with cycle vector

$$\chi(\widetilde{C}) = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} & \boldsymbol{s}_n \\ \end{array} \quad \begin{array}{cccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} & \boldsymbol{s}_n \end{array}$$

$$\chi(\widetilde{C}) = \left[\begin{array}{cccc} x_1 & \cdots & x_{n-1} & * \end{array}\right] = \left[\begin{array}{cccc} - & \chi(C) & - & * \end{array}\right],$$

where $*$ may be any integer, including 0. Let v be any vertex of $\mathcal{D}^{/\boldsymbol{s}_n}$, with incidence vector

$$\begin{array}{rc} & \begin{array}{ccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} \end{array} \\ \lambda(v) = & \left[\begin{array}{ccc} y_1 & \cdots & y_{n-1} \end{array}\right]. \end{array}$$

Suppose that in contracting all $\boldsymbol{s}_n$ edges vertices $v^{(1)}, \ldots, v^{(k)}$ of $\mathcal{D}$ are coalesced to form vertex v in $\mathcal{D}^{/\boldsymbol{s}_n}$ (where it may be that k = 1). Then

$$\begin{array}{rc} & \begin{array}{cccc} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{n-1} & \boldsymbol{s}_n \end{array} \\ \lambda(v^{(1)}) + \lambda(v^{(2)}) + \cdots \lambda(v^{(k)}) = & \left[\begin{array}{cccc} y_1 & \cdots & y_{n-1} & 0 \end{array}\right]. \end{array}$$

Because $\mathcal{D}$ is a Kirchhoff graph,

$$\begin{aligned} \lambda(v) \cdot \chi(C) &= \sum_{i=1}^{n-1} x_i y_i = \sum_{i=1}^{n-1} x_i y_i + 0 \\ &= \left[\begin{array}{cccc} x_1 & \cdots & x_{n-1} & * \end{array}\right] \cdot \left[\begin{array}{cccc} y_1 & \cdots & y_{n-1} & 0 \end{array}\right] \\ &= \chi(\widetilde{C}) \cdot \left(\lambda(v^{(1)}) + \lambda(v^{(2)}) + \cdots \lambda(v^{(k)})\right) \\ &= \chi(\widetilde{C}) \cdot \lambda(v^{(1)}) + \chi(\widetilde{C}) \cdot \lambda(v^{(2)}) + \cdots + \chi(\widetilde{C}) \cdot \lambda(v^{(k)}) \\ &= 0 + 0 \cdots + 0 = 0. \end{aligned}$$

As C and v were arbitrary, $\mathcal{D}^{/\boldsymbol{s}_n}$ is a Kirchhoff graph. □

Once again, it is clear from the proof that for any vector $\boldsymbol{s}_k$, the graph $\mathcal{D}^{/\boldsymbol{s}_k}$ obtained by contracting all $\boldsymbol{s}_k$ edges is also Kirchhoff.

Corollary 2.2. *Let S_0 be any subset of $\{\boldsymbol{s}_1, \cdots, \boldsymbol{s}_n\}$ and let $\mathcal{D}^{/S_0}$ be obtained from $\mathcal{D}$ by contracting all edge vectors contained in S_0. Then $\mathcal{D}^{/S_0}$ is a Kirchhoff graph.*

Propositions 2.4 and 2.5 demonstrate that the operations of edge deletion and edge contraction, now applied to a set of vector edges in a Kirchhoff graph, maintain the Kirchhoff property. These two results (and more specifically their converses) inspire a number of interesting questions regarding the construction of Kirchhoff graphs. For example, let A be any rational valued matrix with n columns. Now let

$$A' = \left[\begin{array}{c|c} A & \boldsymbol{x} \end{array}\right]$$

be any rational valued matrix with $n+1$ columns, obtained from A by adjoining a new column. Then Proposition 2.4 ensures that any Kirchhoff graph for A' can be used to derive a Kirchhoff graph for the matrix A. On the other hand,

Question: Can a Kirchhoff graph for matrix A be used to construct a Kirchhoff graph for the matrix A'?

Determining a method that answers "yes" would facilitate iterative construction of Kirchhoff graphs. In particular, beginning with a matrix of rank m and nullity 1, one could then sequentially construct Kirchhoff graphs for any matrix of rank m.

2.3 Kirchhoff Graphs for $\mathbb{Z}_p$-valued Matrices

It should be clear thus far that Kirchhoff graphs combine elements of linear algebra, graph theory, and group theory. As section 2.3.1 will demonstrate, the definition of Kirchhoff graphs can easily be generalized to matrices over number fields other than $\mathbb{Q}$. Moreover, the conjecture that every $\mathbb{Q}$-valued matrix has a Kirchhoff graph is an overarching open problem. Section 2.3 will demonstrate that in most cases if $\mathbb{Q}$ is replaced by $\mathbb{Z}_p$, this conjecture becomes a theorem. More importantly, Lemma 2.1 will show that Kirchhoff graphs for $\mathbb{Z}_p$-matrices are closely related to finding Kirchhoff graphs for rational-valued matrices.

2.3.1 $\mathbb{Z}_p$-Kirchhoff Graphs

For p a prime, let A_p be any $m \times n$ matrix with entries in $\mathbb{Z}_p$ (i.e. the finite field of integers mod p). Let $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$ denote the column vectors of A_p, now in $\mathbb{Z}_p^m$.

Definition 2.6. Any graph obtainable from the Cayley color graph $(\mathbb{Z}_p^m, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\})$ by assigning (positive integer) multiplicities to the vector edges is a $\mathbb{Z}_p$***-vector graph for*** A_p.

Observe that finiteness is no longer a requirement of Definition 2.6, as $(\mathbb{Z}_p^m, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\})$ is itself finite. Let $\mathcal{D}$ be a $\mathbb{Z}_p$-vector graph for A_p, and C any cycle of $\mathcal{D}$ (again, cycles may traverse edges regardless of orientation). The cycle vector for C, now denoted $\chi_p(C)$, is analogous to Definition 2.3, except all entries are taken as integers mod p.

Proposition 2.6. For any cycle C of $\mathbb{Z}_p$-vector graph $\mathcal{D}$, $\chi_p(C)^t \in \mathrm{Null}(A_p)$.

Similarly, for each vertex v the definition of the incidence vector, now denoted $\lambda_p(v)$, is analogous to Definition 2.4, with all entries taken as integers mod p. This allows the notion of Kirchhoff graphs to be extended to $\mathbb{Z}_p$ matrices.

Definition 2.7. Let $\mathcal{D}$ be any $\mathbb{Z}_p$-vector graph for matrix A_p. If $\lambda_p(v) \in \mathrm{Row}(A_p)$ for all vertices v of $\mathcal{D}$, and the cycle vectors of $\mathcal{D}$ span $\mathrm{Null}(A_p)$, then $\mathcal{D}$ is a $\mathbb{Z}_p$***-Kirchhoff graph for*** A_p.

Example 2.3. Consider the matrix A_3 in (2.14), with entries in $\mathbb{Z}_3$.

$$A_3 = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}. \tag{2.14}$$

Figure 2.3.1 presents a $\mathbb{Z}_3$-Kirchhoff graph for A_3. All incidence vectors (taken mod 3) lie in $\mathrm{Row}(A_3)$, and all cycle vectors (taken mod 3) lie in $\mathrm{Null}(A_3)$. Including Figure 2.3.1, this section will present three vector graphs over $\mathbb{Z}_3^2$ (c.f. Figures 2.3.4 and 2.3.5). Each will be drawn on the flat torus, allowing the occurrences of a given vector to have the same slope.

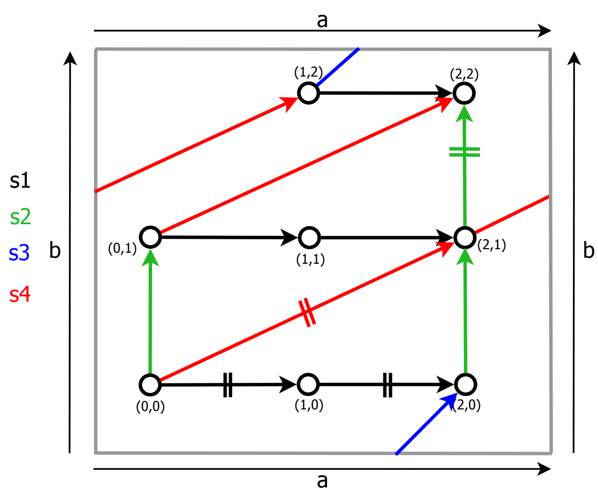


Figure 2.3.1: A $\mathbb{Z}_3$-Kirchhoff graph for matrix A_3, drawn on the flat torus. Again, hash marks indicate multiplicity; vertex coordinates are given as ordered pairs.

The $\mathbb{Z}_p$-Kirchhoff property also translates to cycle and incidence vectors. Let $\mathcal{D}$ be a $\mathbb{Z}_p$-Kirchhoff graph for matrix A_p. For any vertex v and any cycle C,

$$\lambda_p(v) \cdot \chi_p(C) \equiv 0 \pmod{p}. \tag{2.15}$$

Any vector graph that satisfies (2.15) for all v and C is a $\mathbb{Z}_p$***-Kirchhoff graph***.

Remark 2.8. Propositions 2.2, 2.4, and 2.5, and Corollaries 2.1 and 2.2 all remain true if "Kirchhoff" is replaced by "$\mathbb{Z}_p$-Kirchhoff" for any prime p.

Interest in studying $\mathbb{Z}_p$-Kirchhoff graphs is motivated by more than simply extending definitions. The primary focus for applications is understanding Kirchhoff graphs of rational-valued matrices [28]. However, the rows of any $\mathbb{Q}$-valued matrix can be scaled to give an integer-valued matrix that is row equivalent. Thus, by Proposition 2.2 it is sufficient to consider matrices with integer entries. As Lemma 2.1 demonstrates, proving existence (or nonexistence) of Kirchhoff graphs for integer matrices is closely tied to existence of $\mathbb{Z}_p$-Kirchhoff graphs.

Lemma 2.1. *Let A be an integer-valued matrix, and let A_p denote the matrix A (mod p) for any prime p. Let $\mathcal{D}$ be a vector graph for A. $\mathcal{D}$ is a Kirchhoff graph for A if and only if for all prime p, with all edge vectors taken mod p, it is a $\mathbb{Z}_p$-Kirchhoff graph for A_p.*

Proof. For each p, let $\mathcal{D}_p$ be the vector graph obtained by the mod p-reduction of all edge vectors of $\mathcal{D}$. Note that in the process of mod p-reduction, some vertices may be identified. For example, the vertices $(0,0)$ and $(6,3)$ in $\mathbb{Q}^2$ become the same vertex in $\mathbb{Z}_3^2$ [2]. As $\mathcal{D}$ is a vector graph for A, $\mathcal{D}_p$ must be a vector graph for A_p, and the cycle vectors of $\mathcal{D}$ span Null(A) if and only if the (mod p) cycle vectors of $\mathcal{D}_p$ span Null(A_p). Finally, $\mathcal{D}$ is a Kirchhoff graph if for all vertices v and all cycles C, $\lambda(v) \cdot \chi(C) = 0$. This holds if and only if $\lambda(v) \cdot \chi(C) \equiv 0 \pmod{p}$ for all prime p. However, this is true if and only if for all vertices v' and all cycles C' of $\mathcal{D}_p$, $\lambda_p(v') \cdot \lambda_p(C') \equiv 0 \pmod{p}$ for all p. That is, if and only if for all prime p, vector graph $\mathcal{D}_p$ is a $\mathbb{Z}_p$-Kirchhoff graph. $\square$

Although constructing finite Kirchhoff graphs for integer matrices can be challenging [29], the finite nature of the group $\mathbb{Z}_p^m$ makes the construction of $\mathbb{Z}_p$-Kirchhoff graphs less difficult. Indeed one observation is straight-forward.

Proposition 2.7. For p a prime, let A_p be any $m \times n$ matrix with entries in $\mathbb{Z}_p$ and columns $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$. The Cayley color graph $\mathcal{D}_{A_p} := (\mathbb{Z}_p^m, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\})$ is (trivially) Kirchhoff.

Proof. $\mathcal{D}_{A_p}$ is clearly a $\mathbb{Z}_p$-vector graph for A_p with cycle vectors that span Null(A_p). As $\mathcal{D}_{A_p}$ is a Cayley color graph, every incidence vector is the zero vector, and so $\mathcal{D}_{A_p}$ is (trivially) Kirchhoff. $\square$

While Cayley color graphs over $\mathbb{Z}_p$ are trivially Kirchhoff, in the sense that all incidence vectors are zero, they are the starting point for constructing or showing the existence of graphs that are non-trivially Kirchhoff. Section 2.3.2 will demonstrate that such Kirchhoff graphs exist for many $\mathbb{Z}_p$-valued matrices. Before moving on, we emphasize that although Cayley color graphs are trivially Kirchhoff, they are not trivial examples. Example 2.4 shows that a Kirchhoff graph with nonzero incidence vectors, when reduced mod p, can lead to the Cayley color graph as a $\mathbb{Z}_p$-Kirchhoff graph.

[2] A more detailed example of mod p reduction is given in Example 2.4.

Example 2.4. Recall the rational-valued matrix A introduced in Example 2.1.

$$A = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{2.16}$$

Vector graph $\mathcal{D}$ in Figure 2.3.2 is a Kirchhoff graph for A. By Lemma 2.1, reducing all edge vectors of $\mathcal{D}$ mod 3 leads to a $\mathbb{Z}_3$-Kirchhoff graph for $A_3 = A \pmod 3$. Observe that through mod 3 reduction of edge vectors, some vertices must be identified as well. For example, the vertices $(0,0), (3,0), (3,3)$, and $(6,3)$ in $\mathcal{D}$ must become a single vertex in the $\mathbb{Z}_3$-Kirchhoff graph. After mod 3 reduction of all vector edges of $\mathcal{D}$, the result is precisely the Cayley color graph $(\mathbb{Z}_3^2, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_4\})$, with all edges doubled. In particular, although $\mathcal{D}$ has nonzero incidence vectors, when reduced mod 3 the resulting $\mathbb{Z}_3$-Kirchhoff graph has only zero incidence vectors. That is, although Cayley color graphs trivially satisfy the Kirchhoff property, these $\mathbb{Z}_p$-Kirchhoff graphs (with all zero incidence vectors) can arise from Kirchhoff graphs with nonzero incidence vectors.

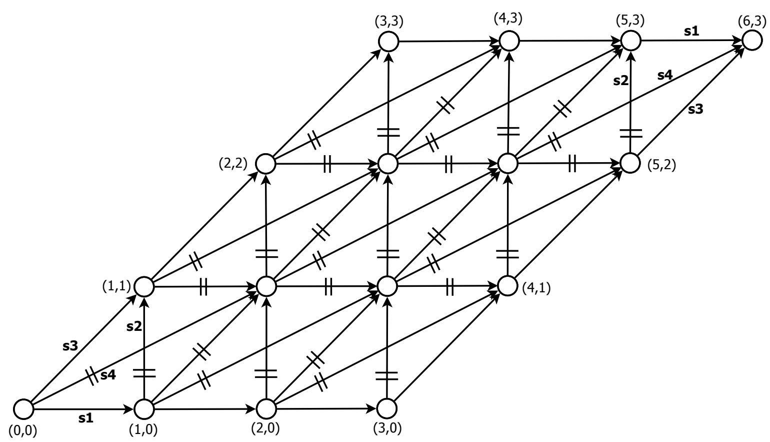


Figure 2.3.2: $\mathcal{D}$, a Kirchhoff graph for matrix A. When all vectors are reduced mod 3, the result is the Cayley color graph $(\mathbb{Z}_3^2, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_4\})$ with all edges doubled.

2.3.2 Nonzero $\mathbb{Z}_p$-Kirchhoff Graphs

Lemma 2.1 demonstrated that every $\mathbb{Z}_p$ matrix has a trivial $\mathbb{Z}_p$-Kirchhoff graph, with all zero incidence vectors. This section considers $\mathbb{Z}_p$ matrices that have a $\mathbb{Z}_p$-Kirchhoff graph with at least one nonzero incidence vector.

Definition 2.8. A $\mathbb{Z}_p$-Kirchhoff graph is ***nonzero*** if it has at least one nonzero (mod p) incidence vector.

Theorem 2.1 establishes the existence of nonzero $\mathbb{Z}_p$-Kirchhoff graphs when p is sufficiently large compared to the matrix dimensions. For any prime p, let A_p be an $m \times n$ matrix with entries in $\mathbb{Z}_p$, and columns $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$. Recall by Proposition 2.2 that it is sufficient to consider matrices A_p with full row rank.

Theorem 2.1. *If $p > n/m$, then A_p has a nonzero $\mathbb{Z}_p$-Kirchhoff graph.*

Proof. Beginning with the Cayley color graph $\mathcal{D}_{A_p} = (\mathbb{Z}_p^m, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\})$, assign to each edge an undetermined multiplicity, w_i. As $\mathcal{D}_{A_p}$ has p^m vertices, and each vertex is incident to $2n$ vector edges, there are np^m unknown multiplicities. Finding a nonzero Kirchhoff graph will be equivalent to finding a nontrivial solution to a homogeneous system of linear equations in these np^m unknowns. First, observe that vertex incidence vectors can be written in terms of the multiplicities w_i. For example, a vertex with the vector labels and multiplicities given in Figure 2.3.3 has incidence vector

$$\begin{array}{ccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 \end{array}$$
$$\begin{bmatrix} (w_1 - w_2) & (w_3 - w_4) & (w_5 - w_6) \end{bmatrix}. \tag{2.17}$$

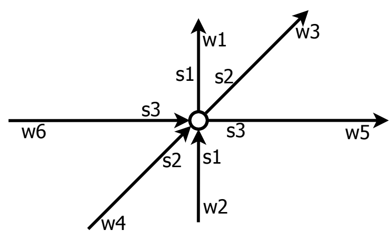


Figure 2.3.3: An example of edge vectors with unknown multiplicities.

Therefore, the p^m vertex incidence vectors can all be written in terms of the np^m undetermined multiplicities. A $\mathbb{Z}_p$-Kirchhoff graph arises whenever $\lambda(v) \cdot \boldsymbol{b} \equiv 0 \pmod{p}$ for all vertices v and any $\boldsymbol{b} \in \mathrm{Null}(A_p)$. As $\mathrm{Null}(A_p)$ has dimension $(n - m)$, this gives a homogeneous system of $(n - m)p^m$ equations in np^m unknowns. We will demonstrate that when $mp > n$ there exists a solution that gives a nonzero $\mathbb{Z}_p$-Kirchhoff graph.

Choose some column vector $\boldsymbol{s}_i$. As each vector in $\mathbb{Z}_p^m$ has order p, the $\boldsymbol{s}_i$ edges of $\mathcal{D}_{A_p}$ are partitioned into p^{m-1} cycles each of length p. (for example, consider vector $\boldsymbol{s}_2$ and the cycle $v_1 - v_4 - v_7 - v_1$ in Figure 2.3.4). Assigning multiplicity 1 to each edge of one such cycle–and multiplicity 0 to all other edges–gives a solution to the system. Moreover, all incidence vectors of the resulting vector graph are the zero vector. By considering all such cycles for each edge vector $\boldsymbol{s}_i$, we find that there are $np^m/p = np^{m-1}$ solutions of this form, that are linearly independent. Moreover, any $\mathbb{Z}_p$-Kirchhoff graph with only zero incidence vectors must assign the same multiplicity to the edges in each cycle of this form. Therefore any solution of the system that leads to a trivial Kirchhoff graph can be written as a linear combination of these solutions. That is, the system has an np^{m-1} dimensional solution space that leads to *all* $\mathbb{Z}_p$-Kirchhoff graphs with only zero incidence vectors.

Any solution to the system outside of this np^{m-1} dimensional space gives a set of multiplicities for a nonzero $\mathbb{Z}_p$-Kirchhoff graph (note we may always choose the multiplicities so that $0 \leq w_i < p$). Therefore, any system with a solution space of dimension greater than np^{m-1} has a nonzero Kirchhoff graph. Recalling that the homogeneous system has $(n-m)p^m$ equations in np^m unknowns, the solution space has dimension at least

$$np^m - (n-m)p^m = mp^n = (mp)p^{m-1} > np^{m-1} \tag{2.18}$$

as $mp > n$. Therefore, A_p has a nonzero Kirchhoff graph. $\square$

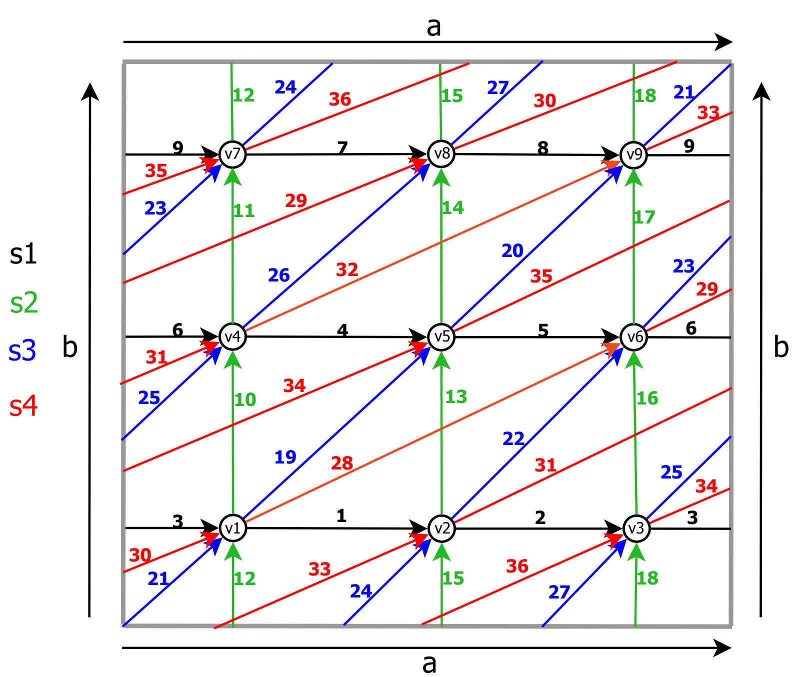


Figure 2.3.4: The Cayley color graph $(\mathbb{Z}_3^2, \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4\})$ with unknown multiplicities w_i assigned to the edges. Weights are indicated only by their integer subscripts

Example 2.5. Once again, we return to the matrix A_3 introduced in Example 2.3.

$$A_3 = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{2.19}$$

The Cayley color graph $(\mathbb{Z}_3^2, \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4\})$, with unknown multiplicities $w_1, \ldots, w_{36}$ assigned to each edge, is shown in Figure 2.3.4 (note weights are denoted simply by the integer subscripts). Given that

$$\left\{ \left[\begin{array}{cccc} 1 & 1 & -1 & 0 \end{array}\right]^t, \left[\begin{array}{cccc} 2 & 1 & 0 & -1 \end{array}\right]^t \right\}$$

is a basis for Null(A_3), the system of equations can be summarized by the matrix equation (2.20).

$$\begin{array}{c} \\ \lambda(v_1) \\ \lambda(v_2) \\ \lambda(v_3) \\ \lambda(v_4) \\ \lambda(v_5) \\ \lambda(v_6) \\ \lambda(v_7) \\ \lambda(v_8) \\ \lambda(v_9) \end{array} \begin{array}{c} \begin{array}{cccc} \quad\boldsymbol{s}_1\quad & \quad\boldsymbol{s}_2\quad & \quad\boldsymbol{s}_3\quad & \quad\boldsymbol{s}_4\quad \end{array} \\ \left[\begin{array}{cccc} (w_1 - w_3) & (w_{10} - w_{12}) & (w_{19} - w_{21}) & (w_{28} - w_{30}) \\ (w_2 - w_1) & (w_{13} - w_{15}) & (w_{22} - w_{24}) & (w_{31} - w_{33}) \\ (w_3 - w_2) & (w_{16} - w_{18}) & (w_{25} - w_{27}) & (w_{34} - w_{36}) \\ (w_4 - w_6) & (w_{11} - w_{10}) & (w_{26} - w_{25}) & (w_{32} - w_{31}) \\ (w_5 - w_4) & (w_{14} - w_{13}) & (w_{20} - w_{19}) & (w_{35} - w_{34}) \\ (w_6 - w_5) & (w_{17} - w_{16}) & (w_{23} - w_{22}) & (w_{29} - w_{28}) \\ (w_7 - w_9) & (w_{12} - w_{11}) & (w_{24} - w_{23}) & (w_{36} - w_{35}) \\ (w_8 - w_7) & (w_{15} - w_{14}) & (w_{27} - w_{26}) & (w_{30} - w_{29}) \\ (w_9 - w_8) & (w_{18} - w_{17}) & (w_{21} - w_{20}) & (w_{33} - w_{32}) \end{array}\right] \end{array} \left[\begin{array}{cc} 1 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{array}\right] \equiv \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right] \pmod 3 \tag{2.20}$$

With $m = 2$, $n = 4$, and $p = 3$, clearly $p > n/m$ and so by Theorem 2.1, A_3 has a nonzero $\mathbb{Z}_3$-Kirchhoff graph. One solution to the system (2.20) that leads to a nonzero Kirchhoff graph is

$$\{w_1, w_2, w_{17}, w_{28}\} = 2 \quad \{w_4, w_5, w_8, w_{10}, w_{16}, w_{27}, w_{29}, w_{32}\} = 1$$

and all other multiplicities zero. Removing all edges of multiplicity zero gives the nonzero $\mathbb{Z}_3$-Kirchhoff graph originally presented in Figure 2.3.1.

Theorem 2.1 guarantees the existence of nonzero $\mathbb{Z}_p$-Kirchhoff graphs for sufficiently large p, but it does not indicate how such graphs are constructed. The remaining results are constructive. Theorem 2.2 shows how to construct a nonzero $\mathbb{Z}_p$-Kirchhoff graph in many cases, and Theorem 2.3 deals with binary matrices and the construction of $\mathbb{Z}_2$-Kirchhoff graphs. For p a prime, let A_p be an $m \times n$ matrix with entries in $\mathbb{Z}_p$, and columns $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$.

Theorem 2.2. *If there exists a vector $\boldsymbol{x} \in$ Row(A_p) with all entries nonzero (mod p), then A_p has a nonzero Z_p-Kirchhoff graph.*

Proof. Let $x_1, \ldots, x_n$ denote the entries of $\boldsymbol{x}$, where each x_i may be chosen so that $0 < x_i < p$. Beginning with the Cayley color graph $\mathcal{D}_{A_p} = (\mathbb{Z}_p^m, \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\})$, we will use $\boldsymbol{x}$ to assign multiplicities to each edge of $\mathcal{D}_{A_p}$. Choose some vector $\boldsymbol{s}_i$. As before, the $\boldsymbol{s}_i$ edges of $\mathcal{D}_{A_p}$ can be partitioned into p^{m-1} cycles each of length p. Assign multiplicity x_i to some edge of each cycle. Following edge orientations, traverse each cycle and successively assign the edges multiplicity $2x_i, 3x_i, \ldots (p-1)x_i$, and px_i respectively. The edges assigned multiplicity px_i may be deleted, and all other multiplicities may be reduced mod p to lie between 1 and $p-1$. Repeating this process for all vectors $\boldsymbol{s}_i$, the result is a vector graph for A_p in which $\lambda_p(v) \equiv \boldsymbol{x} \pmod{p}$ for all vertices v. As $\boldsymbol{x} \in \mathrm{Row}(A_p)$, it follows that this vector graph is, in fact, a nonzero $\mathbb{Z}_p$-Kirchhoff graph for A_p. $\square$

Corollary 2.3. *Let A be an integer-valued matrix with no zero columns. Then for sufficiently large prime p, the matrix $A_p = A \pmod{p}$ has a nonzero $\mathbb{Z}_p$-Kirchhoff graph.*

Example 2.6. Consider the matrix A_3' given in (2.21).

$$A_3' = \begin{array}{c} \begin{matrix} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \tag{2.21}$$

Summing the two rows, observe $\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \in \mathrm{Row}(A_3')$. Figure 2.3.5 shows the multiplicities assigned in the proof of Theorem 2.2, and the resulting $\mathbb{Z}_3$-Kirchhoff graph.

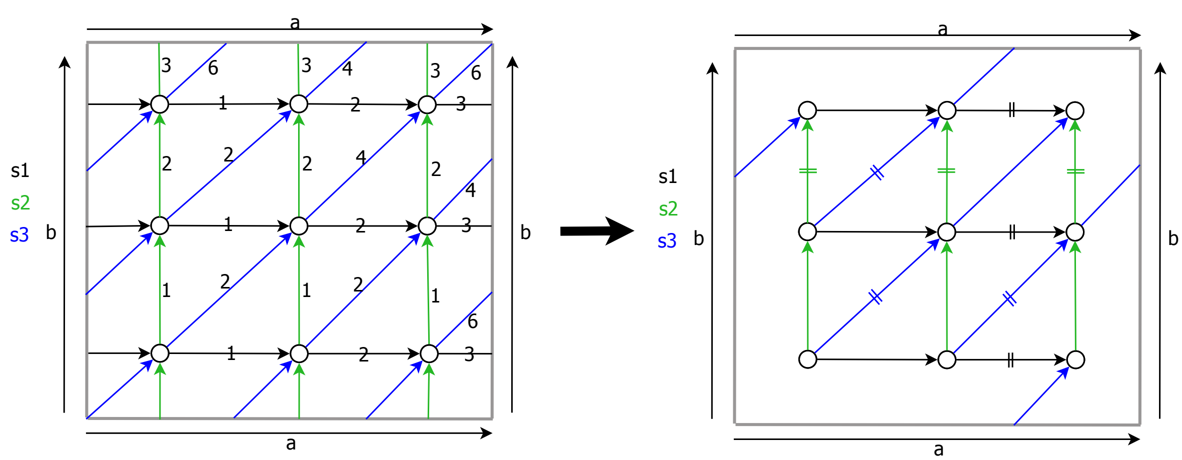


Figure 2.3.5: Multiplicities assigned to the Cayley color graph $(\mathbb{Z}_3^2, \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\})$, and the resulting $\mathbb{Z}_3$-Kirchhoff graph for matrix A_3'.

2.3.3 Binary Matrices and $\mathbb{Z}_2$-Kirchhoff Graphs

Finally, let us turn our attention to binary matrices. The relationship between binary matrices and $\mathbb{Z}_2$-Kirchhoff graphs will be considered much more thoroughly in Chapter 10.

Theorem 2.3. *Every nonzero binary matrix has a nonzero $\mathbb{Z}_2$-Kirchhoff graph.*

Proof. Let A_2 be any $m \times n$ binary matrix with columns $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$, and let $\mathcal{D}_{A_2}$ be the Cayley color graph $(\mathbb{Z}_2^m, \{\boldsymbol{s}_1, \ldots \boldsymbol{s}_n\})$. As each element of the additive group $\mathbb{Z}_2^m$ is its own inverse, the edges of $\mathcal{D}_{A_2}$ occur in pairs: for each vector edge, there is a second copy of that vector edge with the same endpoints and opposite orientation. Let $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$ be any nonzero row of A_2. For each $1 \leq k \leq n$, if $x_k = 1$, remove one vector edge from every pair of $\boldsymbol{s}_k$ vectors in the graph $\mathcal{D}_{A_2}$. The result is a $\mathbb{Z}_2$-vector graph for A_2 with the same cycles as $\mathcal{D}_{A_2}$. However, for all vertices v in the newly constructed graph, $\lambda_2(v) \equiv \boldsymbol{x} \pmod 2$, and the result is a nonzero $\mathbb{Z}_2$-Kirchhoff graph for A_2. □

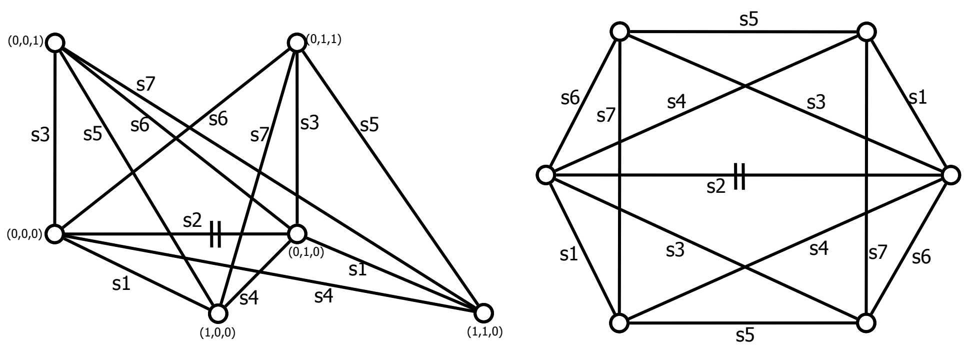


Figure 2.3.6: Two drawings of a nonzero $\mathbb{Z}_2$-Kirchhoff graph for A_2. The first emphasizes that this graph is a subgraph of the complete Cayley color graph, whereas the second uses symmetry to help indicate which edges are the same vector. Note that as $1 \equiv -1 \pmod 2$, edge orientations have been omitted.

Given a binary matrix A_2, there are often many nonzero $\mathbb{Z}_2$-Kirchhoff graphs for A beyond those constructed in Theorem 2.3. For example, consider the binary matrix A_2 given in (2.22).

$$A_2 = \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \tag{2.22}$$

Figure 2.3.6 presents two drawings of a nonzero $\mathbb{Z}_2-$Kirchhoff graph for the matrix A_2. Note that as $1 \equiv -1 \pmod 2$ edge orientations have been omitted. Moreover, this graph was not constructed using any of the previous proof techniques. Instead, it is a proper subgraph of the complete Cayley color graph, obtained by deleting two vertices (in this case $(1,0,1)$ and $(1,1,1)$), and some additional edges to maintain the desired vertex conditions. Note the distinguishing of element $\boldsymbol{s}_2$ as the only doubled edge results from the choice of deleted vertices; deleting other pairs of vertices would result in other vector edges remaining doubled. Beginning with the complete Cayley color graph, six vertices, as shown here, is the minimum number of vertices in a proper subgraph with cycle vectors that span the null space of A_2.

2.3.4 Discussion

The primary open problem in the study of Kirchhoff graphs is the conjecture of Fehribach that every rational-valued matrix has a Kirchhoff graph. As Lemma 2.1 demonstrated, studying $\mathbb{Z}_p$-Kirchhoff graphs could well lead to proving this conjecture. In particular, Proposition 2.7 demonstrated that every $\mathbb{Z}_p$-valued matrix has a $\mathbb{Z}_p$-Kirchhoff graph. This graph–the Cayley color graph–has all zero incidence vectors. Although this structure may appear trivial, Example 2.4 demonstrated that a Kirchhoff graph for an integer matrix A, when reduced mod p, leads to the Cayley color graph for the matrix $A \pmod p$. The reverse of this procedure poses an interesting question.

Question: Given an integer-valued matrix A, can the full Cayley color graph of the matrix $A \pmod p$ be "unfolded" into a Kirchhoff graph for A?

The definition of nonzero Kirchhoff graphs requires that these graphs have cycle vectors that span the null space of the corresponding matrix, but only that some nonzero element of the row space be represented by a vertex cut. One might ask that a basis for the row space be present in the vertex cuts; indeed this is often possible in the construction of specific graphs. The electrochemistry application which inspires this definition, however, suggests that the nonzero Kirchhoff graph definition is best. As was discussed in Section 2.1.1, one wishes all reaction pathways (i.e. null space vectors) are present in the graph as cycles, but only that the vertices preserve the rate balances (i.e. lie in the row space).

Theorem 2.1 shows that for any $m \times n$ matrix with entries in $\mathbb{Z}_p$, there exists a nonzero $\mathbb{Z}_p$-Kirchhoff graph whenever $p > n/m$. The proof of this theorem was of a different nature than most others presented here. First, an unknown multiplicity was assigned to each edge of the Cayley color graph. Writing each vertex incidence vector in terms of these unknown multiplicities lead to a homogeneous system of linear equations. Finding a nonzero $\mathbb{Z}_p$-Kirchhoff graph was then equivalent to finding an appropriate solution to this system. A similar method could be applied to rational-valued matrices. Beginning with a finite subgraph of the Cayley color graph, assign an unknown multiplicity to each edge and derive a system of linear equations. The problem then becomes,

Question: Can we always find a finite subgraph of the Cayley color graph that guarantees that the corresponding system has a *nonnegative integer* solution?

Answering this question as “yes” would prove existence of a Kirchhoff graph for every integer-valued matrix. A word of caution, however. This method of solution is an example of integer linear programming: we derive an integer-valued system of linear equations and then attempt to find a nonnegative integer-valued solution. The general problem of integer linear programming is well-known to be NP-complete [38]. Thus as we consider progressively larger vector graphs, this could become a computationally difficult problem. On the other hand,

Question: Is there an efficient (i.e. polynomial-time) method of finding a nonnegative integer solution to these vector-graph based integer linear systems?

Alternatively, based on the dimensions and entries of the matrix, one may be able to determine an upper bound on the number of vertices or edges required to guarantee a solution to the system. These upper bounds could then be used to constrain a computer search algorithm.

Chapter 3

Kirchhoff Graphs and Matrices

Vector graphs were introduced in Section 2.2 as finite subgraphs of Cayley color graphs. Chapter 3 will now consider an alternative interpretation of vector graphs. Specifically, a vector graph can be viewed as a directed graph with elements of some vector space assigned to each edge. Studying the structure of the underlying digraph−in conjunction with the structure of the vector assignments−leads to a re-interpretation of the Kirchhoff property entirely in terms of matrices, as well as a number of interesting results.

This chapter will study vector graphs through the lens of *vector assignments*: functions that assign elements of a vector space to the edges of a digraph (Definition 3.1). In fact, these functions can be viewed as a special case of *voltage assignments*, which assign elements of a group to the edges of a directed graph. A digraph together with a voltage assignment is known as a *voltage graph*, originally introduced by Gross [48]. In that context, voltage graphs were used to construct imbeddings of graphs in surfaces, using branched or unbranched coverings of simpler imbeddings [53]. Gross and Tucker later generalized this theory to show that so-called permutation voltage assignments generate all graph coverings [52]. The theory of voltage graphs was built upon the foundations of *current graphs*, which were introduced combinatorially by Gustin [54]. Gustin's method was later used to solve the Heawood map-coloring problem by Ringel, Youngs, Terry, and Welch [95], a problem that was revisited by Gross and Tucker using voltage graphs [51]. The theory of current graphs was developed into topological generality by Gross and Alpert [1], [49], and [50].

That being said, this text takes a primarily combinatorial approach to Kirchhoff graphs. Although a topological interpretation of Kirchhoff graphs will not be considered here, it presents an interesting research question for the topologically-inclined.

Section 3.1 begins by reinterpreting vector graphs in terms of vector assignments on digraphs. This facilitates an equivalent definition of Kirchhoff graphs, now in terms of the incidence matrix of a digraph and the characteristic matrix of the vector assignment (Theorem 3.1). Section 3.2 then examines how this definition can be used to count vector edges. In particular, Theorem 3.2 shows that the vector edges *must* occur the same number of times in Kirchhoff graphs for most matrices of nullity 1. This leads to the notion of uniform Kirchhoff graphs, those in which all edge vectors occur the same number of times, introduced in Section 3.3. After Section 3.2.1 determines the equivalence classes of matrices that share the same Kirchhoff graphs, Theorem 3.3 and Lemma 3.6 demonstrate that large classes of matrices have exclusively uniform Kirchhoff graphs. Conversely, Theorem 3.4 shows that any matrix with a Kirchhoff graph (and in particular, a non-uniform Kirchhoff graph) always has a uniform Kirchhoff graph. Next, Section 3.4 demonstrates that finding Kirchhoff graphs is closely related to studying permutation-invariance of subspaces. Finally, Section 3.5 presents a few examples that examine the relationship between matrix properties and Kirchhoff graph properties. One in particular, Lemma 3.9, will be needed in Chapter 4.

3.1 A Matrix Definition of Kirchhoff Graphs

Let $D = (V(D), E(D))$ be a finite directed multi-graph with vertex set $V(D)$ and edge set $E(D)$. Let $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$ be a finite set of n non-zero vectors in some vector space.

Definition 3.1. A ***vector assignment*** (or a (D, S)-vector assignment), φ, is any surjective function $\varphi : E(D) \to S$.

As before let a cycle of D be any closed walk, with no repeated vertices, that may traverse edges regardless of orientation. A vector assignment φ is ***consistent*** if for any cycle C of D,

$$\sum_{e \in E(C)} \sigma_C(e)\varphi(e) = \mathbf{0},$$

where $\sigma_C(e) = 1$ if C traverses e in the direction of its orientation, and $\sigma_C(e) = -1$ if C traverses e against its orientation. That is, φ is consistent with D if the signed sum of vectors around any cycle is the zero vector.

Corollary 3.1. *A vector graph, as defined in Section 2.2, consists of a multi-digraph D together with a consistent vector assignment φ.*

Remark 3.1. Throughout this chapter, we will be studying both a vector graph and its underlying digraph in parallel. To distinguish between the two structures, we will use a bar to denote corresponding parts of a vector graph. Suppose D is a digraph with vertices $\{v_1, \ldots, v_p\}$ and edges $\{e_1, \ldots, e_m\}$. If φ is a consistent vector assignment on D, we let $\overline{D} = (D, \varphi)$ denote the associated vector graph. Then $\overline{D}$ has vertices $V(\overline{D}) = \{\overline{v_1}, \ldots \overline{v_p}\}$. This distinction will be of importance, for example, when studying incidence vectors. $\lambda(v_i)$ is the incidence vector of vertex v_i in digraph D, and is indexed by the edges $e_1, \ldots e_m$. $\lambda(\overline{v_i})$, on the other hand, is the incidence vector of vertex $\overline{v_i}$ in vector graph $\overline{D}$ and, as in Section 2.2, is indexed by the set of *vector* edges, $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$. The relationship between $\lambda(v_i)$ and $\lambda(\overline{v_i})$ will be considered in Lemma 3.1.

Example 3.1. Consider the digraph D with vertices $\{v_1, \ldots, v_4\}$ and edges $\{e_1, \ldots, e_6\}$ as illustrated in Figure 3.1.1.

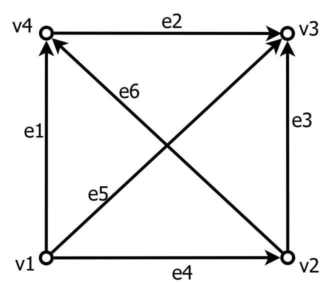


Figure 3.1.1: A digraph D with 4 vertices and 6 edges.

Let $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_4\} \subseteq \mathbb{Q}^2$ be a set of 4 vectors,

$$S = \left\{ \boldsymbol{s}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \boldsymbol{s}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \boldsymbol{s}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \boldsymbol{s}_4 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}.$$

Moreover, let φ be the (D,S)-vector assignment

$$\varphi : \{e_1, e_3\} \mapsto \boldsymbol{s}_1 \qquad \varphi : \{e_2, e_4\} \mapsto \boldsymbol{s}_2 \qquad \varphi : e_5 \mapsto \boldsymbol{s}_3 \qquad \varphi : e_6 \mapsto \boldsymbol{s}_4.$$

One may verify that φ is consistent with D. For example, $(e_1)(-e_6)(-e_4)$ is a cycle in D, and

$$\varphi(e_1) - \varphi(e_6) - \varphi(e_4) = \boldsymbol{s}_1 - \boldsymbol{s}_4 - \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore $\overline{D} = (D, \varphi)$ is a vector graph. Moreover, as $S \subseteq \mathbb{Q}^2$, vector graph $\overline{D}$ can be drawn in the plane where each edge is drawn as (and labeled by) its assigned vector $\boldsymbol{s}_j$, as shown in Figure 3.1.2.

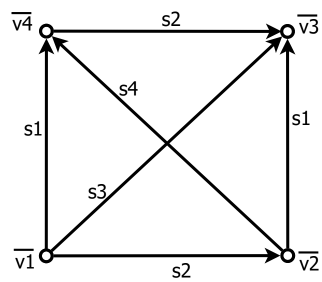


Figure 3.1.2: Vector Graph $\overline{D} = (D, \varphi)$.

The action of a vector assignment φ can be represented in matrix form. For any digraph D with vector assignment φ, let T be the $|E(D)| \times |S|$ matrix where

$$T_{i,j} = \begin{cases} 1 & \text{if } \varphi(e_i) = \boldsymbol{s}_j \\ 0 & \text{Otherwise} \end{cases}.$$

We will call T the ***characteristic matrix*** of φ. For example, the vector assignment φ in Example 3.1 has characteristic matrix

$$T = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Remark 3.2. Let T be the characteristic matrix of some vector assignment φ.

(1.) Each row of T has exactly one entry 1 and all other entries 0.

(2.) Each column of T has at least one nonzero entry.

(3.) T has full column rank and, under left multiplication, represents a surjective map $\mathbb{R}^{|E(D)|} \to \mathbb{R}^{|S|}$.

For simplicity, we will often call any zero-one matrix satisfying (1.)-(3.) a ***characteristic matrix***. Moreover, let M be any matrix with $|E(D)|$ columns. Observe that the the j^{th} column of the product MT is the sum of all columns of M with index i such that $\varphi(e_i) = \boldsymbol{s}_j$.

For any digraph D, let $Q = Q(D)$ be the ***incidence matrix*** of digraph D. That is, Q is a

$|V(D)| \times |E(D)|$ matrix where

$$Q_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of edge } e_j \\ -1 & \text{if } v_i \text{ is the terminal vertex of edge } e_j \\ 0 & \text{if vertex } v_i \text{ is not an endpoint of edge } e_j \end{cases}.$$

For example, the digraph in Figure 3.1.1 has incidence matrix

$$Q(D) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 \end{array} \right] \end{array}.$$

For any vertex v of digraph D, recall that $\lambda(v)$ denotes the *incidence vector* of v. That is, $\lambda(v)$ is a vector with $|E(D)|$ entries, and j^{th} entry 1 if v is the initial vertex of e_j, -1 if it is the terminal vertex, and 0 otherwise. For example, the vertex v_1 in Figure 3.1.1 has incidence vector

$$\lambda(v_1) = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \end{array} \right] \end{array}.$$

Observe that $\lambda(v_j)$ is the j^{th} row of the incidence matrix Q.

Proposition 3.1. The rows of $Q = Q(D)$ lie in and span the cut space of D.

Proof. For any vertex v_i of D, consider the cut $(v_i, V(D)\backslash v_i)$. Clearly the cut vector $\vartheta(v_i, V(D)\backslash v_i) = \lambda(v_i)$, and so the rows of Q lie in the cut space of D. On the other hand, let (V_1, V_2) be any cut of D, where $V_1 = \{v_{i_1}, \ldots, v_{i_k}\}$. Then the cut vector $\vartheta(V_1, V_2)$ satisfies

$$\vartheta(V_1, V_2) = \lambda(v_{i_1}) + \cdots + \lambda(v_{i_k}).$$

Therefore since the rows of Q are $\lambda(v_i)$ $(1 \leq i \leq |V(D)|)$, the rows of Q span the cut space of D. $\square$

On the other hand, let C be any cycle of digraph D. Recall that $\chi(C)$ denotes the *cycle vector* of C. That is, $\chi(C)$ has $|E(D)|$ entries and

$$\chi(C)_j = \begin{cases} 1 & \text{if } C \text{ traverses edge } e_j \text{ in the direction of its orientation} \\ -1 & \text{if } C \text{ traverses } e_j \text{ against its orientation} \\ 0 & \text{Otherwise} \end{cases}.$$

For example, the cycle $C = v_1 - v_4 - v_3 - v_1$ in Figure 3.1.1 has cycle vector

$$\chi(C) = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & -1 & 0 \end{array}\right] \end{array}.$$

Proposition 3.2. The vectors $\{\chi(C)^t : C \text{ is a cycle of } D\}$ lie in and span Null(Q). That is, the cycle space of D is the null space of Q.

Proof. Clearly the vectors $\{\chi(C)^t : C \text{ is a cycle of } D\}$ lie in and span the cycle space of D, which is the orthogonal complement of the cut space. The result then follows immediately from Proposition 3.1. □

Now let $\overline{D}$ be any vector graph, arising from digraph D and consistent vector assignment φ. Recall from Definition 2.4 for each vertex $\overline{v}$ of $\overline{D}$ the incidence vector $\lambda(\overline{v})$ has entries indexed by $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$, where the $\boldsymbol{s}_i$ entry is the net number of $\boldsymbol{s}_i$ that exit vertex $\overline{v}$. As Lemma 3.1 illustrates, characteristic matrix T provides the link between the incidence vectors of digraph D and vector graph $\overline{D}$.

Lemma 3.1. *For any corresponding vertices v of D, and $\overline{v}$ of $\overline{D}$, then (using row vectors)*

$$\lambda(\overline{v}) = \lambda(v)T.$$

Proof. Given a vertex v of D, the incidence vector $\lambda(v)$ has i^{th} entry

$$\lambda(v)_i = \begin{cases} 1 & \text{if edge } e_i \text{ exits vertex } v \\ -1 & \text{if edge } e_i \text{ enters vertex } v \\ 0 & \text{Otherwise} \end{cases}.$$

Similarly, for corresponding vertex $\overline{v}$ of $\overline{D}$, the j^{th} entry of $\lambda(\overline{v})$ is

$$\lambda(\overline{v})_j = \Big|\{i : e_i \text{ exits } v \text{ and } \varphi(e_i) = \boldsymbol{s}_j\}\Big| - \Big|\{i : e_i \text{ enters } v \text{ and } \varphi(e_i) = \boldsymbol{s}_j\}\Big|.$$

Now consider the matrix product $\lambda(v)T$. $\lambda(v)T$ and has j^{th} entry

$$(\lambda(v)T)_j = \sum_{i=1}^{n} \lambda(v)_i T_{i,j}. \tag{3.1}$$

However, $\lambda(v)_i \in \{-1, 0, 1\}$ and $T_{i,j} \in \{0, 1\}$. Therefore,

$$\lambda(v)_i T_{i,j} = \begin{cases} 1 & \text{if } \lambda(v)_i = T_{i,j} = 1 \\ -1 & \text{if } \lambda(v)_i = -1 \text{ and } T_{i,j} = 1 \\ 0 & \text{Otherwise} \end{cases}.$$

Equivalently,

$$\lambda(v)_i T_{i,j} = \begin{cases} 1 & \text{if } e_i \text{ exits } v \text{ and } \varphi(e_i) = \boldsymbol{s}_j \\ -1 & \text{if } e_i \text{ enters } v \text{ and } \varphi(e_i) = \boldsymbol{s}_j \\ 0 & \text{Otherwise} \end{cases}.$$

Therefore by (3.1),

$$(\lambda(v)T)_j = \Big|\{i : e_i \text{ exits } v \text{ and } \varphi(e_i) = \boldsymbol{s}_j\}\Big| - \Big|\{i : e_i \text{ enters } v \text{ and } \varphi(e_i) = \boldsymbol{s}_j\}\Big| = \lambda(\overline{v})_j.$$

Thus for each j, $(\lambda(v)T)_j = \lambda(\overline{v})_j$ and so for each vertex $\overline{v}$ of $\overline{D}$,

$$\lambda(\overline{v}) = \lambda(v)T.$$

□

Similarly, recall from Definition 2.3 that for each cycle $\overline{C}$ of vector graph $\overline{D}$, the cycle vector $\chi(\overline{C})$ has entries indexed by $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$, and entry $\boldsymbol{s}_i$ is the net number of times cycle $\overline{C}$ traverses an $\boldsymbol{s}_i$ edge. Once again, characteristic matrix T provides the relationship between cycle vectors of digraph D and vector graph $\overline{D}$. The proof of Lemma 3.2 is analogous to that of Lemma 3.1, and is omitted here.

Lemma 3.2. *For any cycle $\overline{C}$ of $\overline{D}$, and its corresponding cycle C in D,*

$$\chi(\overline{C}) = \chi(C)T.$$

Now let D be any digraph with incidence matrix Q, and suppose φ is a consistent vector assignment on D, with edge vectors $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$. Let N be a matrix whose columns, $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$, form a basis for Null($Q$).

Theorem 3.1. *The vector graph $\overline{D} = (D, \varphi)$ is a Kirchhoff graph if and only if the columns of $(N^tT)^t$ lie in and span Null(QT). Put another way, $\overline{D}$ is Kirchhoff if and only if*

$$\left(\mathit{Null}(Q)^t T\right)^t = \mathit{Null}(QT). \tag{3.2}$$

Remark 3.3. Note that we use superscript t to denote the matrix transpose in order to avoid confusion with matrix T.

Proof. ($\Leftarrow$) Let $\overline{v}$ and $\overline{C}$ be any vertex and cycle of vector graph $\overline{D}$. Then given the corresponding vertex v of D, $\lambda(v) \in \text{Row}(Q)$, meaning

$$\lambda(\overline{v}) = \lambda(v)T \in \text{Row}(QT).$$

Similarly, for the corresponding cycle C in digraph D, $\chi(C) \in$Row(N^t), meaning

$$\chi(\overline{C}) = \chi(C)T \in \text{Row}(N^tT).$$

Then since the rows of $(N^tT)^t$ lie in and span Null(QT), $\chi(\overline{C})^t \in$ Null(QT). That is, $\chi(\overline{C}) \cdot \lambda(\overline{v}) = 0$. Therefore, as $\overline{v}$ and $\overline{C}$ were arbitrary, $\overline{D}$ is a Kirchhoff graph.

($\Rightarrow$) As $\overline{D}$ is a Kirchhoff graph, for any vertex $\overline{v_i}$ and cycle $\overline{C}$ of $\overline{D}$,

$$\lambda(\overline{v_i}) \cdot \chi(\overline{C}) = 0. \tag{3.3}$$

For any column $\boldsymbol{b}_j$ of N, by Proposition (3.2), there exists cycles $C_1, \ldots, C_k$ in D such that

$$\boldsymbol{b}_j^t = a_1\chi(C_1) + \cdots + a_k\chi(C_k)$$

for some scalars $a_1, \ldots, a_k$. Then for any row $\lambda(v_i)$ of incidence matrix Q,

$$\begin{aligned}
\lambda(v_i)T \cdot \boldsymbol{b}_j^tT &= \lambda(v_i)T \cdot \left(\sum_{r=1}^{k} a_r\chi(C_r)\right)T \\
&= \lambda(v_i)T \cdot \left(\sum_{r=1}^{k} a_r\chi(C_r)T\right) \\
&= \sum_{r=1}^{k} a_r\left(\lambda(v_i)T \cdot \chi(C_r)T\right) \\
&= \sum_{r=1}^{k} a_r\left(\lambda(\overline{v_i}) \cdot \chi(\overline{C_r})\right) \\
&= \sum_{r=1}^{k} a_r(0) && \text{by (3.3)} \\
&= 0.
\end{aligned}$$

Therefore for each column $\boldsymbol{b}_j$ of N, $(\boldsymbol{b}_j^tT)^t \in$ Null(QT), and so the columns of $(N^tT)^t$ lie in Null(QT).

Next we show that the columns of $(N^tT)^t$ span Null(QT). Let $\boldsymbol{x}$ be any vector in Null(QT). In particular, $\boldsymbol{x} \in \mathbb{R}^{|S|}$. However,

$$\mathbb{R}^{|E(D)|} = \text{Row}(Q) \oplus \text{Row}(N^t).$$

Furthermore, under left-multiplication characteristic matrix T is a surjective map $\mathbb{R}^{|E(D)|} \rightarrow$

$\mathbb{R}^{|S|}$. Therefore $\boldsymbol{x}$ can be written as:

$$\boldsymbol{x}^t = \beta_1 \boldsymbol{b}_1^t T + \cdots + \beta_k \boldsymbol{b}_k^t T + \alpha \boldsymbol{a} T$$

for some constants $\alpha, \beta_1, \ldots, \beta_k$ and some vector $\boldsymbol{a} \in$Row(Q). In particular, $\boldsymbol{a}T \in$Row(QT), so that

$$\boldsymbol{a}T \cdot \boldsymbol{x}^t = 0,$$

as $\boldsymbol{x}$ was chosen in Null(QT). However, we have already shown that for each i $(1 \leq i \leq k)$, $(\boldsymbol{b}_i^t T)^t \in$Null$(QT)$. Thus for each i $(1 \leq i \leq k)$,

$$\boldsymbol{a}T \cdot \boldsymbol{b}_i^t T = 0.$$

Therefore,

$$\begin{aligned} 0 &= \boldsymbol{a}T \cdot \boldsymbol{x}^t \\ &= \boldsymbol{a}T \cdot \left(\beta_1 \boldsymbol{b}_1^t T + \cdots + \beta_k \boldsymbol{b}_k^t T + \alpha \boldsymbol{a} T\right) \\ &= \beta_1 (\boldsymbol{a}T \cdot \boldsymbol{b}_1^t T) + \cdots + \beta_k (\boldsymbol{a}T \cdot \boldsymbol{b}_k^t T) + \alpha (\boldsymbol{a}T \cdot \boldsymbol{a}T) \\ &= 0 + \cdots + 0 + \alpha (\boldsymbol{a}T \cdot \boldsymbol{a}T) \\ &= \alpha \|\boldsymbol{a}T\|^2. \end{aligned}$$

Therefore either $\alpha = 0$ or $\boldsymbol{a}T$ is the zero vector. In either case, $\boldsymbol{x}^t \in \text{span}\{\boldsymbol{b}_1^t T, \ldots, \boldsymbol{b}_k^t T\}$. Since $\boldsymbol{x} \in$Null(QT) was arbitrary, it follows that the columns of $(N^t T)^t$ span Null(QT). □

Corollary 3.2. *A vector graph $\overline{D}$ is Kirchhoff if and only if for all vertices $\overline{v}$ and all cycles $\overline{C}$,*

$$\lambda(\overline{v}) \cdot \chi(\overline{C}) = 0.$$

That is, if and only if for all vertices v and all cycles C of digraph D,

$$\lambda(v)T \cdot \chi(C)T = 0.$$

Theorem 3.1 is significant in that it re-interprets the Kirchhoff property in terms of two matrices–incidence matrix Q and characteristic matrix T. Moreover, neither Q nor T capture any information about the *vectors* $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$ of vector graph $\overline{D}$. Matrix Q indexes how the vertices of D are interconnected, and matrix T only indexes which edges are assigned to the same vector. As it turns out, given any two matrices Q and T satisfying (3.2), a set of vector edges can be determined *based on* Q and T. This is demonstrated by Corollary 3.3, which is straightforward given Theorem 3.1.

Let D be any digraph with incidence matrix Q. Let T be any zero-one characteristic matrix. Moreover, suppose Q and T satisfy (3.2),

$$\left(\text{Null}(Q)^t T\right)^t = \text{Null}(QT).$$

Then let $\boldsymbol{s}_1, \dots, \boldsymbol{s}_n$ be the columns of the product QT, and let $\varphi : E(D) \to \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_n\}$ be the vector assignment from $E(D)$ to S with characteristic matrix T.

Corollary 3.3. *The vector graph $\overline{D} = (D, \varphi)$ is a Kirchhoff graph for the matrix QT.*

Corollary 3.3 is a significant observation, in that it means that one need not *start* with a set of vectors, or even a matrix, when studying Kirchhoff graphs. Instead, one can begin simply with a multi-digraph D and ask,

Question: For which zero-one characteristic matrices T do $Q(D)$ and T satisfy (3.2)?

In the case that such a T exists, a set of suitable edge vectors, namely the columns of the product QT, can be extracted *after the fact*. This suggests a number of interesting computational problems. For any digraph D, taking T to be the identity matrix satisfies (3.2). However, should any other zero-one characteristic matrix T satisfy (3.2), this means that D is the underlying digraph of some nontrivial Kirchhoff graph. While there are only a finite number of possible T's to be checked for a given digraph, the number of such T's is exponential in the number of edges of D. Thus an interesting question is,

Question: Given the incidence matrix of a digraph, is there an efficient way of determining which characteristic matrices T satisfy (3.2)?

Alternatively, one could consider the opposite problem, which is one of matrix decomposition.

Question: Given an integer-valued matrix A, can A be decomposed as a product QT where Q is the incidence matrix of a digraph, and T is the characteristic matrix of a vector assignment?

A deterministic algorithm answering either of these questions would be a solution to the existence problem of Kirchhoff graphs.

3.2 Vector Assignments and Edge Vector Counting

Corollary 3.3 allows us to begin considering the number of times each vector edge occurs in a Kirchhoff graph. Let $\overline{D}$ be a vector graph with n vector edges, $\boldsymbol{s}_n, \dots, \boldsymbol{s}_n$. Let D and φ be the underlying digraph and vector assignment of $\overline{D}$. For each $1 \leq j \leq n$, define η_j to be the number of edges of D to which vector $\boldsymbol{s}_j$ is assigned under φ. Then η_j is the number of nonzero entries (i.e. the number of 1's) in the j^{th} column of matrix T. Moreover, in view of

Remark 3.2, it follows that

$$\Lambda = T^tT = \begin{bmatrix} \eta_1 & & & & \\ & \eta_2 & & & \\ & & \ddots & & \\ & & & \eta_{m-1} & \\ & & & & \eta_n \end{bmatrix}$$

is a diagonal matrix with $(j,j)-$entry η_j. Now suppose $\overline{D} = (D,\varphi)$ is a Kirchhoff graph, and let matrices Q, N and T be as in Section 3.1.

Lemma 3.3. *For any vector $\boldsymbol{x} \in$ Null(QT),*

$$\Lambda\boldsymbol{x} \in \mathit{Null}(QT).$$

Proof. Let $\boldsymbol{x}$ be any column vector $\boldsymbol{x} \in \text{Null}(QT)$, so

$$QT\boldsymbol{x} = \mathbf{0}.$$

Characteristic matrix T is an $|E(D)| \times |S|$ matrix where $|S| \leq |E(D)|$ and T has full column rank. Therefore,

$$T\boldsymbol{x} \neq \mathbf{0}.$$

Given that $QT\boldsymbol{x} = \mathbf{0}$, we must have $T\boldsymbol{x} \in \text{Null}(Q)$. That is,

$$(T\boldsymbol{x})^t \in \text{Row}(N^t).$$

Therefore there exists some row vector $\boldsymbol{y}$ such that

$$\boldsymbol{x}^tT^t = (T\boldsymbol{x})^t = \boldsymbol{y}N^t.$$

That is,

$$\boldsymbol{x}^tT^tT = \boldsymbol{y}N^tT.$$

Equivalently,

$$\boldsymbol{x}^t\Lambda = \boldsymbol{y}N^tT. \tag{3.4}$$

Now since $\overline{D}$ is a Kirchhoff graph, by Theorem 3.1,

$$(\boldsymbol{y}N^tT)^t \in \text{Null}(QT).$$

Thus by (3.4),

$$\Lambda\boldsymbol{x} \in \text{Null}(QT).$$

$\square$

Let $\overline{D} = (D, \varphi)$ be a Kirchhoff graph with vector edges $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$. As usual let Q be the incidence matrix of D and let T be the characteristic matrix of φ. Suppose that the matrix QT has a 1-dimensional null space, say

$$\mathrm{Null}(QT) = \mathrm{span}\{\boldsymbol{b}\}$$

for some $\boldsymbol{b}^t = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$ with integer entries.

Theorem 3.2. *There exists a constant ω independent of j such that if $b_j \neq 0$, then*

$$\eta_j = \omega.$$

Proof. As QT has a 1-dimensional null space,

$$\mathrm{Null}(QT) = \{\beta \boldsymbol{b} : \beta \in \mathbb{R}\}.$$

$\overline{D}$ is Kirchhoff and $\boldsymbol{b} \in \mathrm{Null}(QT)$, so by Lemma 3.3

$$\Lambda \boldsymbol{b} = \omega \boldsymbol{b} \tag{3.5}$$

for some constant ω. Then equating the i^{th} entries of (3.5),

$$\eta_i b_i = \omega b_i$$

for all i $(1 \leq i \leq n)$. Therefore for any j such that $b_j \neq 0$,

$$\eta_j = \omega.$$

That is, there exists a constant ω such that if $b_j \neq 0$, vector edge $\boldsymbol{s}_j$ occurs exactly ω times in Kirchhoff graph $\overline{D}$. Moreover, if $\boldsymbol{b}$ has all nonzero entries, then all vector edges of $\overline{D}$ occur the same number of times. We call such a Kirchhoff graph *uniform*, and will return to this idea in Section 3.3.

□

Corollary 3.4. *If the entries of $\boldsymbol{b}$ are all nonzero, then $|E(D)|$ is divisible by $|S|$.*

Proof. As $b_j \neq 0$ for all j, by Theorem 3.2,

$$\eta_1 = \eta_2 = \cdots = \eta_n.$$

However,

$$|E(D)| = \eta_1 + \eta_2 + \cdots + \eta_n$$

and so $|E(D)|$ must be divisible by $n = |S|$. □

3.2.1 Kirchhoff Graphs and Matrix Equivalence

While Section 2.2 defined a Kirchhoff graph for a single matrix, Theorem 3.1 and Definition 3.2 can be used to show that if there exists a Kirchhoff graph for some matrix A, there exists a Kirchhoff graph for a number of other matrices. This will allow us, in turn, to extend the results of Theorem 3.2. Definition 3.2 gives a more general definition of a Kirchhoff graph for a matrix, equivalent to that in the existing literature [27], [29].

Definition 3.2. For a matrix $A \in \mathbb{Z}^{m\times n}$, a vector graph $\overline{D}$ is a **Kirchhoff graph for** A if and only if:

(i) $\overline{D}$ has n vector edges, $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$.

(ii) The cycle vectors of $\overline{D}$ lie in and span Null(A).

(iii) The vertex incidence vectors of $\overline{D}$ lie in Row(A).

Proposition 3.3. If $\overline{D}$ is any Kirchhoff graph for some matrix A, $\overline{D}$ is a Kirchhoff graph for another matrix A' if and only if A and A' are row-equivalent.

Proof. $\overline{D}$ is a Kirchhoff graph for A if and only if the cycle vectors of $\overline{D}$ lie in and span Null(A), and the incidence vectors of $\overline{D}$ lie in Row(A). Similarly, $\overline{D}$ is a Kirchhoff graph for A' if and only if the cycle vectors of $\overline{D}$ lie in and span Null(A'), and the incidence vectors of $\overline{D}$ lie in Row(A'). Thus $\overline{D}$ is a Kirchhoff graph for both A and A' if and only if Null$(A) =$ Null(A') and Row$(A) =$ Row(A'). That is, if and only if A and A' are row-equivalent. □

Let $A \in \mathbb{Z}^{m\times n}$ be an integer matrix with column vectors $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$. Suppose $\overline{D}$ is a Kirchhoff graph for A. Then $\overline{D} = (D, \varphi)$ for some digraph D and vector assignment φ with characteristic matrix T. As before, let Q be the incidence matrix of digraph D.

Corollary 3.5. *Matrices A and QT are row-equivalent.*

Proof. By Corollary 3.3 $\overline{D}$ is a Kirchhoff graph for matrix QT. Therefore by Proposition 3.3, A and QT must be row-equivalent. □

Now let P be any $n \times n$ permutation matrix, so that $A' = AP$ arises by re-ordering the columns of A. Similarly, the matrix TP arises by re-ordering the columns of matrix T, and thus is also a characteristic matrix. Recall that T is the matrix representation of a vector assignment from digraph D to vector set $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$. Let S' be a re-indexed set of these vectors, $S' = \{\boldsymbol{s}'_1, \ldots, \boldsymbol{s}'_n\}$, where

$$\boldsymbol{s}'_j = \boldsymbol{s}_i \text{ if and only if } P_{i,j} = 1.$$

Then matrix TP is the characteristic matrix of a (D, S')-vector assignment, φ'.

Proposition 3.4. Vector assignment φ' is consistent.

Proof. Let e_k be any edge of digraph D. Let i be the index such that $\varphi(e_k) = \boldsymbol{s}_i$ and j be the index such that $\varphi'(e_k) = \boldsymbol{s}'_j$. Then $T_{k,i} = 1$ and $(TP)_{k,j} = 1$. However as $T_{k,i} = 1$, $(TP)_{k,j} = 1$ if and only if $P_{i,j} = 1$. Thus since $P_{i,j} = 1$, $\boldsymbol{s}_i = \boldsymbol{s}'_j$ by definition of φ', and consistency of φ' follows from consistency of φ. □

Knowing consistency of vector assignment φ', $\overline{D}' = (D, \varphi')$ is thus a vector graph. Moreover,

Lemma 3.4. *$\overline{D}' = (D, \varphi')$ is a Kirchhoff graph for the matrix $A' = AP$.*

Proof. Recall that (TP) is the characteristic matrix of vector assignment φ'. Let Q be the incidence matrix of digraph D. Then for any $\boldsymbol{a} \in \text{Row}(Q)$ and any column $\boldsymbol{b} \in \text{Null}(Q)$,

$$\boldsymbol{a}(TP) \cdot \boldsymbol{b}^t(TP) = \boldsymbol{a}T \cdot \boldsymbol{b}^tT,$$

because P is a permutation matrix, and thus unitary. However, because $\overline{D} = (D, \varphi)$ is a Kirchhoff graph,

$$\boldsymbol{a}T \cdot \boldsymbol{b}^tT = 0.$$

Thus for all $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{a}(TP) \cdot \boldsymbol{b}^t(TP) = 0,$$

and it follows by Theorem 3.1 that $\overline{D}' = (D, \varphi')$ is a Kirchhoff graph.

Moreover, since $\overline{D}$ is a Kirchhoff graph for A, by Corollary 3.5, A and QT are row-equivalent. As P is a permutation matrix, $A' = AP$ and QTP are row-equivalent. Given that TP is the matrix representation of φ', $\overline{D}'$ is a Kirchhoff graph for $Q(TP)$. Therefore since QTP is row equivalent to A', by Proposition 3.3 $\overline{D}'$ is a Kirchhoff graph for matrix A'. □

This result, presented in full formality, is not at all surprising. Observe that right-multiplying by a permutation matrix simply re-orders the columns of matrices. This column-reordering essentially corresponds to re-labeling the vector edges used in a vector graph. Intuitively, if A has a Kirchhoff graph $\overline{D}$, and A' arises by reordering the columns of A, a suitable relabeling of the vectors in $\overline{D}$ should give a Kirchhoff graph for A'. The details presented above verify that our definitions hold under this natural equivalence. Moreover, this demonstrates that if some matrix A has a Kirchhoff graph, then so does a class of matrices related to A. In particular,

Definition 3.3. Let A, A' be two rational-valued matrices with n columns. We will say that A and A' are ***K - equivalent*** if there exists some matrix A_0 with n columns such that:

(i) A and A_0 are row-equivalent.

(ii) $A' = A_0P$ for some $n \times n$ permutation matrix P.

Lemma 3.5. *If $\overline{D}$ is a Kirchhoff graph for matrix A, then up to vector re-labeling it is also a Kirchhoff graph for any matrix K - equivalent to A.*

Proof. This result follows from the preceding results. □

3.3 Uniform Kirchhoff Graphs

Definition 3.2 and the natural notion of K-equivalence now allow us to extend the edge-counting results of Theorem 3.2 to matrices with larger null spaces.

Definition 3.4. A Kirchhoff graph $\overline{D}$ is ***uniform*** if all edge vectors occur the same number of times. That is, $\overline{D}$ is uniform if

$$\eta_1 = \eta_2 = \cdots = \eta_n.$$

Before Theorem 3.3 presents one of the main results of this section, we first introduce one piece of notation. Let N be any matrix with n rows and k columns ($k \leq n$), and let I_k denote the $k \times k$ identity matrix. We will say that matrix N is in ***I-M Form*** if N is of the form:

$$N = \left[\begin{array}{c} I_k \\ \hline \mathcal{M}_{(n-k)\times k} \end{array} \right]. \tag{3.6}$$

Now let A' be any rational-valued matrix with n columns. There exist rational-valued matrices A and N such that

(i) A' is K-equivalent to A.

(ii) Matrix N is in I-M form (3.6).

(iii) The columns of N are a basis for Null(A).

Theorem 3.3. *If $\mathcal{M}$ has no zero entries, then any Kirchhoff graph $\overline{D} = (D, \varphi)$ for A is uniform. That is, there exists a constant ω such that φ assigns each vector $\boldsymbol{s}_i$ to exactly ω edges of $E(D)$. Equivalently,*

$$\eta_1 = \eta_2 = \cdots = \eta_n. \tag{3.7}$$

Theorem 3.3 states that *most* Kirchhoff graphs must be uniform. Later, Theorem 3.4 will show that if a matrix has a Kirchhoff graph, it must always have a uniform Kirchhoff graph.

Proof. Let Q be the incidence matrix of digraph D and, as before let $\Lambda = T^tT$ be the diagonal matrix with (j, j)-entry η_j. Now let $\boldsymbol{b_1}, \ldots, \boldsymbol{b_k}$ denote the columns of N, so that $\{\boldsymbol{b_1}, \ldots, \boldsymbol{b_k}\}$ are a basis for Null(A). In particular, since $\overline{D}$ is a Kirchhoff graph for A, $\{\boldsymbol{b_1}, \ldots, \boldsymbol{b_k}\}$ are a basis for Null(QT). By Lemma 3.3,

$$\Lambda \boldsymbol{b_1} \in \text{Null}(QT).$$

Therefore as $\{\boldsymbol{b_1}, \ldots, \boldsymbol{b_k}\}$ are a basis for Null(QT), there exist scalars $\gamma_1, \ldots, \gamma_k$ such that

$$\Lambda \boldsymbol{b_1} = \gamma_1 \boldsymbol{b_1} + \cdots + \gamma_k \boldsymbol{b_k}. \tag{3.8}$$

For each index $2 \leq j \leq k$, consider the j^{th} entry of (3.8). As N is in I-M form, $(\boldsymbol{b_1})_j = 0$. Therefore since Λ is a diagonal matrix, the j^{th} entry on the left hand side of (3.8) is 0. The only column of N with a nonzero j^{th} entry is $\boldsymbol{b_j}$, which has j^{th} entry 1. Therefore the right hand side of (3.8) has j^{th} entry γ_j, and it follows that for all $2 \leq j \leq k$,

$$\gamma_j = 0.$$

Therefore,

$$\Lambda \boldsymbol{b_1} = \gamma_1 \boldsymbol{b_1}. \tag{3.9}$$

Let $\omega_1 = \gamma_1$. Comparing the j^{th} entries of (3.9), it follows that for any j $(1 \leq j \leq n)$,

$$\omega_1(\boldsymbol{b_1})_j = \eta_j(\boldsymbol{b_1})_j.$$

Therefore if $(\boldsymbol{b_1})_j \neq 0$, it must be that $\eta_j = \omega_1$. That is, for all indices j such that $(\boldsymbol{b_1})_j \neq 0$, $\eta_j = \omega_1$. For each i $(2 \leq i \leq k)$ repeat the above argument with index i in the place of 1. Then we find k constants $\omega_1, \ldots, \omega_k$ such that for each j,

$$\text{if } (\boldsymbol{b_i})_j \neq 0, \text{ then } \eta_j = \omega_i. \tag{3.10}$$

However, as sub-matrix $\mathcal{M}$ had no zero entries,

$$(\boldsymbol{b_i})_n \neq 0 \text{ for all } 1 \leq i \leq k.$$

Therefore it follows that

$$\omega_1 = \omega_2 = \cdots = \omega_k,$$

and moreover,

$$\eta_1 = \eta_2 = \cdots = \eta_n.$$

□

Corollary 3.6. *Any Kirchhoff graph for matrix A' is uniform.*

Proof. As A' is K-equivalent to A, this follows directly from Lemma 3.5. □

When the hypotheses of Theorem 3.3 are met, because $\eta_1 = \eta_2 = \cdots = \eta_n$ it follows that each column of T sums to the same value. Given that T has all entries either 0 or 1 (and no zero-row), T must have $\omega \cdot n$ rows for some integer ω. Recalling that vector assignment matrix T has $|E(D)|$ rows,

Corollary 3.7. *If $\mathcal{M}$ has no zero entries, then in any Kirchhoff graph $\overline{D}' = (D, \varphi')$ for A', digraph D must satisfy*

$$|E(D)| = \omega \cdot n$$

for some positive integer ω.

It is clear from the proof of Theorem 3.3, that the assumption that $\mathcal{M}$ have no zero entries is stronger than necessary. In fact, a number of stronger results can be obtained directly from (3.10).

Corollary 3.8. *If $\mathcal{M}$ has a row with all nonzero entries, then $\overline{D}$ is uniform.*

Corollary 3.9. *For any two distinct columns $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$ of N, if there exists some index r such that*

$$(\boldsymbol{b}_i)_r \neq 0 \text{ and } (\boldsymbol{b}_j)_r \neq 0,$$

then there exists a constant $\omega_{i,j}$ where for any indices s such that either $(\boldsymbol{b}_i)_s \neq 0$ or $(\boldsymbol{b}_j)_s \neq 0$,

$$\eta_s = \omega_{i,j}.$$

Careful consideration of Corollary 3.9 reveals that the only case in which (3.7) does not hold is if matrix N as in (3.6), up to row and column reordering, is of the form

$$N = \left[\begin{array}{cc} I_{k_1} & 0 \\ 0 & I_{k_2} \\ \hline A_{n_1 \times k_1} & 0 \\ 0 & B_{n_2 \times k_2} \end{array}\right], \tag{3.11}$$

where $k_1 + k_2 = k$ and $n_1 + n_2 = (n - k)$. We will say a matrix N in I-M form is ***block decomposable*** if it is K-equivalent to a matrix of the form (3.11). Let A', A, and N be as in Theorem 3.3.

Lemma 3.6. *If matrix N is not block decomposable, then any Kirchhoff graph for A (and thus for A') is uniform. That is, each edge vector occurs the same number of times.*

Proof. This is a direct consequence of Corollary 3.9. □

On the other hand, more can be said if matrix N *is* block decomposable. Suppose that N is block decomposable with decomposition

$$N = \left[\begin{array}{cccc} I_{k_1} & 0 & 0 & 0 \\ 0 & I_{k_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & I_{k_r} \\ \hline A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_r \end{array}\right], \tag{3.12}$$

where $k_1 + \cdots + k_r = k$ and each A_i is $n_i \times k_i$ (so $n_1 + \cdots + n_r = (n-k)$). Observe that for each j from 1 up to r, the induced submatrix

$$\left[\begin{array}{c} I_{k_j} \\ \hline A_j \end{array}\right] \tag{3.13}$$

is in I-M form. A block decomposition is ***minimal*** if for each j the induced submatrix (3.13) is not itself block decomposable.

Now let A' be any integer-valued matrix, and let A and N be as in Theorem 3.3. Suppose that N is block decomposable. By Lemma 3.5, we may suppose that N has a minimal block decomposition of the form (3.12). Observe that a minimal decomposition of this form induces a partition $(C_1, \ldots C_r)$ of the column indices, and a partition $(R_1, \ldots R_r)$ of the row indices such that

$$N_{i,j} \neq 0 \text{ if and only if } i \in R_m \text{ and } j \in C_m \text{ for some } m \ (1 \leq m \leq r).$$

That is, a row with index in R_m has nonzero entries only in columns indexed by C_m, and vice versa.

Lemma 3.7. *In any Kirchhoff graph for matrix A, for each $j \in \{1, \ldots, r\}$, there exists a constant ω_j such that*

$$\eta_i = \omega_j \text{ for all } i \text{ in index set } R_j.$$

That is, if N is block decomposable, the vector edges can be partitioned into r classes such that the vectors within each class must all occur the same number of times.

Proof. As the block decomposition (3.12) of N is minimal, each induced submatrix

$$\begin{array}{cc} & C_j \\ R_j & \left[\begin{array}{c} I_{k_j} \\ -- \\ A_j \end{array}\right] \end{array}$$

is not block decomposable. Therefore if $R_j = \{j_1, \ldots, j_M\}$, by Lemma 3.6

$$\eta_{j_1} = \cdots = \eta_{j_M}.$$

□

Lemma 3.7 can now be used to prove Theorem 3.4.

Theorem 3.4. *Let A' be any integer valued matrix. If there exists a Kirchhoff graph for A', then there exists a uniform Kirchhoff graph for A'.*

Proof. As before, there exist rational-valued matrices A and N satisfying properties (i)-(iii) as in Theorem 3.3. If N is not block decomposable, then by Lemma 3.6, any Kirchhoff graph for A (and thus for A') must be uniform. Otherwise N is block decomposable. We may assume N has a minimal block decomposition as in (3.12). Let $(C_1, \ldots, C_r)$ and $(R_1, \ldots, R_r)$ be the resulting partitions of the column and row indices of N.

For each $j \in \{1, \ldots, r\}$, let M_j be a rational matrix with column indices R_j such that the columns of

$$N_j = \begin{array}{cc} & \cdots C_j \cdots \\ \begin{array}{c} \vdots \\ R_j \\ \vdots \end{array} & \left[\begin{array}{c} I_{k_j} \\ -- \\ A_j \end{array}\right] \end{array} \tag{3.14}$$

form a basis for $\mathrm{Null}(M_j)$. Then by (3.12), because there exists a Kirchhoff graph for A' and thus A, there must exist a Kirchhoff graph for each M_j.

For each $j \in \{1, \ldots, r\}$, let $\overline{D_j}$ be a Kirchhoff graph for M_j. As the block decomposition of N is minimal, the matrix N_j is not block decomposable. Because N_j is in I-M form, it follows by Lemma 3.6 that $\overline{D_j}$ is a uniform Kirchhoff graph. That is, there exists a constant ω_j such that all vector edges of $\overline{D_j}$ occur ω_j times. Now let $\omega = \mathrm{lcm}(\omega_1, \ldots, \omega_r)$. For each $j \in \{1, \ldots, r\}$, let $\overline{D_j}'$ be a Kirchhoff graph obtained from $\overline{D_j}$ by scaling all edge multiplicities by ω/ω_j. In particular, $\overline{D_j}'$ is a uniform Kirchhoff graph for M_j in which all vector edges occur ω times. That is, $\overline{D_j}'$ is a Kirchhoff graph for any matrix for which the columns of

$$\begin{array}{c c} & \cdots C_j \cdots \\ \begin{matrix} \vdots \\ R_j \\ \vdots \end{matrix} & \begin{bmatrix} I_{k_j} \\ -- \\ A_j \end{bmatrix} \end{array}$$

are a basis for the null space. However, $(R_1, \ldots, R_r)$ was a partition of the row indices of N. Therefore the vector edges among the $\overline{D}'_j$ are *distinct*. So for example, the disjoint union $\overline{D}'_1 \cup \overline{D}'_2$ is a Kirchhoff graph for any matrix for which the columns of

$$\begin{array}{c c} & \begin{matrix} \cdots C_1 \cdots & \cdots C_2 \cdots \end{matrix} \\ \begin{matrix} \vdots \\ R_1 \\ \vdots \\ \vdots \\ R_2 \\ \vdots \end{matrix} & \left[\begin{array}{c|c} I_{k_1} & 0 \\ -- & 0 \\ A_1 & 0 \\ \hline 0 & I_{k_2} \\ 0 & -- \\ 0 & A_2 \end{array}\right] \end{array}$$

are a basis for the null space. More importantly, the disjoint union $\overline{D} = \overline{D}'_1 \cup \cdots \cup \overline{D}'_r$ is a Kirchhoff graph for any matrix for which the columns of (3.15) are a basis for the null space.

$$\begin{array}{c c} & \begin{matrix} C_1 & C_2 & \cdots & C_r \end{matrix} \\ \begin{matrix} \vdots \\ R_1 \\ \vdots \\ \vdots \\ R_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ R_r \\ \vdots \end{matrix} & \left[\begin{array}{c|c|c|c} I_{k_1} & 0 & 0 & 0 \\ -- & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ \hline 0 & I_{k_2} & 0 & 0 \\ 0 & -- & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & I_{k_r} \\ 0 & 0 & 0 & -- \\ 0 & 0 & 0 & A_r \end{array}\right] \end{array} \tag{3.15}$$

Observe, however, that (3.15) is a row-reordering of matrix N. Therefore $\overline{D}$ is a Kirchhoff graph for matrix A. Moreover, the vector edges are distinct among the $\overline{D}'_j$, and each $\overline{D}'_j$ is uniform with ω occurrences of each edge. Therefore every vector edge of $\overline{D}$ occurs ω times, and $\overline{D}$ is uniform. □

Example 3.2. Let A be the full row-rank matrix given in (3.16).

$$A = \begin{array}{c} \begin{array}{cccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 \end{array} \\ \begin{bmatrix} 2 & 2 & -1 & 0 & 2 & 0 \\ 3 & 4 & 0 & -1 & 0 & -2 \\ 2 & -1 & -1 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array} \tag{3.16}$$

The columns of N, as in (3.17), are a basis for Null(A). Clearly N is in I-M form, and is block decomposable.

$$N = \begin{array}{c|cc} & 1 & 2 \\ \boldsymbol{s}_1 & 1 & 0 \\ \boldsymbol{s}_2 & 0 & 1 \\ \hline \boldsymbol{s}_3 & 2 & 0 \\ \boldsymbol{s}_4 & 3 & 0 \\ \boldsymbol{s}_5 & 0 & -1 \\ \boldsymbol{s}_6 & 0 & 2 \end{array} \tag{3.17}$$

This block decomposition of N is minimal, with column partition

$$\Big\{C_1 = \{1\}, C_2 = \{2\}\Big\}$$

and row partition

$$\Big\{R_1 = \{\boldsymbol{s}_1, \boldsymbol{s}_3, \boldsymbol{s}_4\}, R_2 = \{\boldsymbol{s}_2, \boldsymbol{s}_5, \boldsymbol{s}_6\}\Big\}.$$

These partitions decompose N into the induced submatrices

$$N_1 = \begin{array}{c|c} & 1 \\ \boldsymbol{s}_1 & 1 \\ \hline \boldsymbol{s}_3 & 2 \\ \boldsymbol{s}_4 & 3 \end{array} \quad N_2 = \begin{array}{c|c} & 2 \\ \boldsymbol{s}_2 & 1 \\ \hline \boldsymbol{s}_5 & -1 \\ \boldsymbol{s}_6 & 2 \end{array}.$$

Moreover, as in the proof of Theorem 3.4, take M_1 and M_2 to be the matrices

$$M_1 = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{array} \\ \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \end{array} \quad M_2 = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_2 & \boldsymbol{s}_5 & \boldsymbol{s}_6 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \end{array}.$$

That is, for $j = 1, 2$, the columns of N_j form a basis for $\mathrm{Null}(M_j)$. A uniform Kirchhoff graph for A can be constructed from Kirchhoff graphs for M_1 and M_2. Figure 3.3.1 illustrates two such Kirchhoff graphs: $\overline{D_1}$ is a Kirchhoff graph for matrix M_1, and $\overline{D_2}$ is a Kirchhoff graph for matrix M_2.

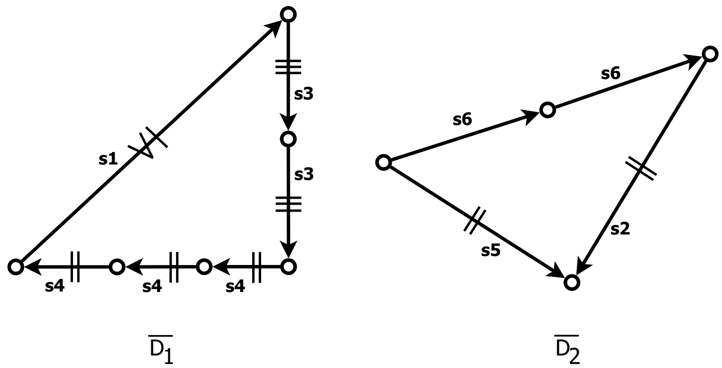


Figure 3.3.1: A Kirchhoff graph $\overline{D_1}$ for matrix M_1, and a Kirchhoff graph $\overline{D_2}$ for matrix M_2.

Observe that as neither N_1 nor N_2 are block decomposable, both $\overline{D_1}$ and $\overline{D_2}$ are uniform. That is, all vector edges of $\overline{D_1}$ occur $\omega_1 = 6$ times, and those of $\overline{D_2}$ occur $\omega_2 = 2$ times. Moreover, $\overline{D_1}$ has vector edges $\{\boldsymbol{s}_1, \boldsymbol{s}_3, \boldsymbol{s}_4\}$, which are distinct from the vector edges $\{\boldsymbol{s}_2, \boldsymbol{s}_5, \boldsymbol{s}_6\}$ of $\overline{D_2}$. Therefore, let $\omega = \mathrm{lcm}(\omega_1, \omega_2) = 6$. Then, as in the proof of Theorem 3.4, take $\overline{D}_1' = \overline{D_1}$, and let $\overline{D}_2'$ be obtained from $\overline{D_2}$ by scaling all edge multiplicities by $\omega/\omega_2 = 3$. Finally, let $\overline{D}$ be the disjoint union $\overline{D} = \overline{D}_1' \cup \overline{D}_2'$, as illustrated in Figure 3.3.2. By construction, $\overline{D}$ is a Kirchhoff graph for matrix A. Moreover, every edge vector of $\overline{D}$ occurs $\omega = 6$ times, and thus $\overline{D}$ is uniform.

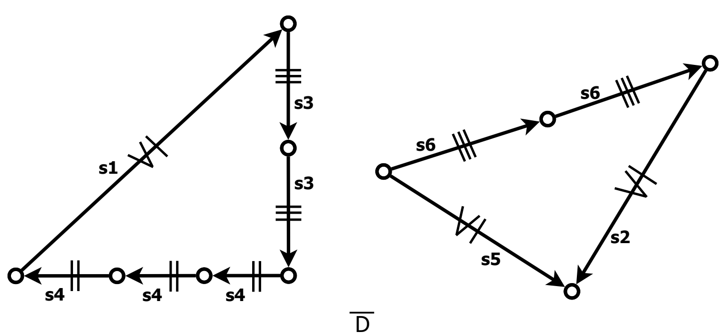


Figure 3.3.2: The disjoint union $\overline{D} = \overline{D}_1' \cup \overline{D}_2'$. $\overline{D}$ is a uniform Kirchhoff graph for matrix A.

3.4 Kirchhoff Graphs and Permutation Invariance

Theorem 3.1 is significant in that it realizes the Kirchhoff property based solely on the incidence matrix Q of a digraph, and the characteristic matrix T of a vector assignment. A number of applications of this result, such as Theorem 3.3 have been illustrated, and this section will present one more.

Let D be any digraph with incidence matrix Q, and let T be any $|E(D)| \times n$ characteristic matrix of a vector assignment on D.

Lemma 3.8. *There exists a Kirchhoff graph with characteristic matrix T and digraph D if and only if the row space of Q is invariant under right multiplication by TT^t.*

Proof. Let N be a rational matrix with $|E(D)|$ rows whose columns are a basis for Null(Q). Then by Theorem 3.3, there exists a Kirchhoff graph with digraph D and characteristic matrix T if and only if $QT(N^tT)^t = 0$. That is, if and only if

$$Q(TT^t)N = 0. \tag{3.18}$$

But (3.18) holds if and only if the rows of $Q(TT^t)$ are orthogonal to the columns of N, i.e. if and only if the row space of Q is invariant under right multiplication by TT^t. □

As before, for each $j \in \{1, \ldots, n\}$, let η_j be the sum of the j^{th} column of T. The product T^tT is then an $n \times n$ diagonal matrix with (j, j)-entry η_j. Lemma 3.8 instead considers the product TT^t. Without loss of generality, the columns of Q can be reordered so that T has the form

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \vdots & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.19}$$

That is, the first η_1 rows of T have a 1 in column 1, the next η_2 rows have a 1 in column 2, and in general, rows $\sum_{i=1}^{j-1} \eta_i + 1$ through $\sum_{i=1}^{j-1} \eta_i + \eta_j$ have a 1 in column j. As a result,

TT^t is a block-diagonal matrix

$$TT^t = \begin{bmatrix} J_{\eta_1} & 0 & 0 & 0 \\ 0 & J_{\eta_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{\eta_n} \end{bmatrix}, \tag{3.20}$$

where J_M denotes the $M \times M$ all-ones matrix. Therefore one can think of Lemma 3.8 as saying that the row space of Q is invariant under right-multiplication by a suitable block-diagonal 0-1 matrix. Matrices of the form (3.20) can be constructed in a variety of manners and, in particular, arise out of a specific class of permutation matrices.

Let P be an $m \times m$ permutation matrix, and let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ denote the standard basis vectors of $\mathbb{R}^m$, written as row vectors. Permutation matrix P defines an equivalence relation $\sim$ on $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$, where

$$\boldsymbol{e}_i \sim \boldsymbol{e}_j \text{ if and only if } \boldsymbol{e}_i P^k = \boldsymbol{e}_j \text{ for some } k.$$

We call the equivalence classes under $\sim$ the ***orbits*** of permutation matrix P. It is easy to verify the following.

Proposition 3.5. Let P be an $m \times m$ permutation matrix with orbits of equal cardinality k,

$$\{\boldsymbol{e}_{i_1}, \ldots \boldsymbol{e}_{i_k}\}, \{\boldsymbol{e}_{i_{(k+1)}}, \ldots, \boldsymbol{e}_{i_{2k}}\}, \{\boldsymbol{e}_{i_{(2k+1)}}, \ldots, \boldsymbol{e}_{i_{3k}}\}, \{\boldsymbol{e}_{i_{(m-k+1)}}, \ldots, \boldsymbol{e}_{i_m}\}.$$

Then $P^{k+1} = P$ and, moreover,

$$P + P^2 + \cdots + P^k = \begin{bmatrix} J_k & 0 & 0 & 0 \\ 0 & J_k & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_k \end{bmatrix}.$$

Lemma 3.8 and Proposition 3.5 combine to prove Theorem 3.5.

Let D be any multi-digraph with m edges and incidence matrix Q, and suppose there exists a permutation matrix P such that the row space of Q is invariant under right-multiplication by P. Without loss of generality, we may re-order the columns of Q so that the orbits of P are

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{k_1}\}, \{\boldsymbol{e}_{k_1+1}, \boldsymbol{e}_{k_1+2}, \ldots, \boldsymbol{e}_{k_1+k_2}\}, \ldots, \{\ldots, \boldsymbol{e}_{m-1}, \boldsymbol{e}_m\}.$$

Theorem 3.5. *If the orbits of P are all of equal cardinality k, then D is the underlying digraph of a Kirchhoff graph in which all edge vectors occur k times.*

Proof. As Row(Q) is invariant under right multiplication by P, it is also invariant under right multiplication by any power of P. In particular, Row(Q) is invariant under right-multiplication by

$$P + P^2 + \cdots + P^k = \begin{bmatrix} J_k & 0 & 0 & 0 \\ 0 & J_k & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_k \end{bmatrix}.$$

Now let T be the $|E(D)| \times |E(D)|/k$ matrix of the form (3.19), where the first k rows of T have a 1 in column 1, the next k rows have a 1 in column 2, and in general, rows $j(k-1)+1$ through jk have a 1 in column j. Then

$$TT^t = \begin{bmatrix} J_k & 0 & 0 & 0 \\ 0 & J_k & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_k \end{bmatrix}.$$

Therefore the row space of Q is invariant under right multiplication by TT^t, and the result follows by Lemma 3.8. $\square$

Observe from the proof of Theorem 3.5 that if a permutation has all orbits of size k, the resulting characteristic matrix T has all columns that sum to k. That is, permutations of this form lead to *uniform* Kirchhoff graphs. In light of Theorems 3.3 and 3.4, studying permutation invariance and incidence matrices may be a worthwhile step in understanding existence of Kirchhoff graphs. Specifically, Theorem 3.5 provides yet another way to search for Kirchhoff graphs. Beginning with a digraph D,

Question: Can we determine permutation matrices P under which the row space of $Q(D)$ is invariant?

Note that when we say "determine" in this question, we mean "devise an efficient method of recognizing." While one could certainly begin with brute-force computation, for a digraph D with m edges, there are $m!$ possible permutation matrices that must be checked. While this is a plausible approach for relatively small examples, studying large examples raises,

Question: Can properties of digraph D or incidence matrix $Q(D)$ be used to narrow the search for permutation matrices under which Row($Q(D)$) may be invariant? Alternatively, can we rule-out permutations under which it is *not* invariant?

On the other hand, one could begin with a permutation matrix P and study the $-1/0/1$-valued matrices invariant under post multiplication by P. Linear algebra problems of this form have been considered previously. For example, [15] characterized the subspaces of $\mathbb{R}^n$ that are invariant under P^tWP for all permutation matrices P, where W is a real symmetric matrix. More generally, permutation invariance is of interest in many computationally-driven disciplines. For example, in quantum information theory permutation-invariant *vectors* (and operators) are used to study invariant states of registers, and can be useful in performing calculations [107]. As noted in [107], permutation-invariant vectors are also commonly studied in multilinear algebra, for example [47], [69], and [70]. Furthermore, [107] notes that generalizations of these notions are relevant in representation theory, as explained in [45].

3.5 Properties of Matrices and Kirchhoff Graphs

Definition 3.2, which provided the link between matrices and Kirchhoff graphs, inspires many very natural questions. Specifically, one can inquire how the structure of a matrix A informs the structure of its Kirchhoff graphs. Problems of this sort are plentiful, and are accessible even to undergraduate researchers with a background in linear algebra and graph theory. For completeness, this section includes two results of this kind.

Example 3.3. Let D be a digraph with $n+1$ edges, and no multiple edges. Let φ be a vector assignment on D that assigns a unique vector to $n-1$ of the edges, and assigns the same vector to the remaining two edges. Can (D,φ) ever be Kirchhoff?

Let $E(D) = \{e_1, \ldots, e_{n+1}\}$, and let $\varphi_2 : E(D) \to \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$ be any vector assignment as described above. Without loss of generality, we may re-label the vectors $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$ and the edges of $E(D)$ so that $\varphi_2(e_1) = \varphi_2(e_{n+1}) = \boldsymbol{s}_1$, and φ has characteristic matrix

$$T = \begin{array}{c} \\ e_1 \\ e_2 \\ \vdots \\ e_n \\ e_{n+1} \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \cdots & \boldsymbol{s}_n \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & & \cdots & \\ 0 & \vdots & I_{n-1} & \vdots \\ 0 & & \cdots & \\ 1 & 0 & 0 & 0 \end{array} \right] \end{array}. \tag{3.21}$$

Lemma 3.9. *$\overline{D}_2 = (D, \varphi_2)$ is a Kirchhoff graph if and only if e_1 and e_{n+1} are either both bridges in D, or form a directed (oriented) cycle that is also a set of cut-edges in D.*

Proof. As D is a simple digraph, for any vertex v and any cycle C of D,

$$\lambda(v) \cdot \chi(C) = \sum_{i=1}^{n+1} \lambda(v)_i \chi(C)_i = 0. \tag{3.22}$$

For any vertex $\overline{v}$ of $\overline{D_2}$, $\lambda(\overline{v}) = \lambda(v)T$. Thus by (3.21),

$$\lambda(\overline{v})_i = \begin{cases} \lambda(v)_i & \text{for } 2 \leq i \leq n \\ \lambda(v)_1 + \lambda(v)_{n+1} & \text{when } i = 1 \end{cases}. \tag{3.23}$$

Similarly, for any cycle $\overline{C}$ in $\overline{D}_2$ with corresponding cycle C in D, $\chi(\overline{C}) = \chi(C)T$. Thus

$$\chi(\overline{C})_i = \begin{cases} \chi(C)_i & \text{for } 2 \leq i \leq n \\ \chi(C)_1 + \chi(C)_{n+1} & \text{when } i = 1 \end{cases}. \tag{3.24}$$

Therefore for any vertex $\overline{v}$ and any cycle $\overline{C}$ of $\overline{D}_2$,

$$\begin{aligned}
\lambda(\overline{v}) \cdot \chi(\overline{C}) &= \sum_{i=1}^{n} \lambda(\overline{v})_i \chi(\overline{C})_i \\
&= \lambda(\overline{v})_1 \chi(\overline{C})_1 + \sum_{i=2}^{n} \lambda(\overline{v})_i \chi(\overline{C})_i \\
&= \Big(\lambda(v)_1 + \lambda(v)_{n+1}\Big)\Big(\chi(C)_1 + \chi(C)_{n+1}\Big) + \sum_{i=2}^{n} \lambda(v)_i \chi(C)_i \text{ by (3.23) and (3.24)} \\
&= \Big(\lambda(v)_1 \chi(C)_{n+1} + \lambda(v)_{n+1} \chi(l)_1\Big) + \sum_{i=1}^{n+1} \lambda(v)_i \chi(C)_i \\
&= \Big(\lambda(v)_1 \chi(C)_{n+1} + \lambda(v)_{n+1} \chi(C)_1\Big) + 0 \text{ by (3.22)}
\end{aligned}$$

That is,

$$\lambda(\overline{v}) \cdot \chi(\overline{C}) = \lambda(v)_1 \chi(C)_{n+1} + \lambda(v)_{n+1} \chi(C)_1. \tag{3.25}$$

First, assume that $e_1 e_{n+1}$ is a directed cycle and set of cut edges in D. Then every vertex of D is either incident with neither, or both, of e_1 and e_{n+1}. If v is incident with neither, $\lambda(v)_1 = \lambda(v)_{n+1} = 0$ (and notably, $\lambda(v)_1 = -\lambda(v)_{n+1}$). If incident with both, $\lambda(v)_1 = -\lambda(v)_{n+1}$. Moreover as directed cycle $e_1 e_{n+1}$ is a set of cut edges, any cycle C in D must traverse e_1 and e_{n+1} the same net number of times. In particular, $\chi(C)_1 = \chi(C)_{n+1}$. Therefore for all v and C,

$$\lambda(v)_1 \chi(C)_{n+1} + \lambda(v)_{n+1} \chi(C)_1 = \lambda(v)_1 \chi(C)_1 - \lambda(v)_1 \chi(C)_1 = 0.$$

By (3.25) $\lambda(\overline{v}) \cdot \chi(\overline{C}) = 0$, and $\overline{D}_2$ is a Kirchhoff graph. On the other hand, assume that both e_1 and e_{n+1} are bridges in D. Then given *any* cycle C in D,

$$\chi(C)_1 = \chi(C)_{n+1} = 0. \tag{3.26}$$

Thus for any vertex $\overline{v}$ and any cycle $\overline{C}$ in $\overline{D}_2$, by (3.25) and (3.26):

$$\begin{aligned}\lambda(\overline{v}) \cdot \chi(\overline{C}) &= \lambda(v)_1\chi(C)_{n+1} + \lambda(v)_{n+1}\chi(C)_1 \\ &= \lambda(v)_1(0) + \lambda(v)_{n+1}(0) = 0.\end{aligned}$$

Therefore, $\overline{D}_2$ is a Kirchhoff graph.

Conversely, assume that at least one of e_1 and e_{n+1} is not a bridge in D, and either e_1e_{n+1} is not a directed cycle in D, or *is* a directed cycle but not a cutset. Assume, without loss of generality, e_1 is not a bridge in D. Then there exists some cycle C_0 in D such that $\chi(C_0)_1 \neq 0$.

Case 1. **e_1e_{n+1} is not a directed cycle in D.** As D was a simple digraph, e_1 and e_{n+1} cannot have the same initial and terminal vertices. Moreover, because e_1e_{n+1} is not directed cycle in D, there must exists a vertex v_0 that is incident to edge e_{n+1} but *not* edge e_1. In particular,

$$\lambda(v_0)_{n+1} \neq 0 \text{ and } \lambda(v_0)_1 = 0.$$

Then taking the corresponding vertex $\overline{v}_0$ and cycle $\overline{C}_0$ in $\overline{D}_2$, by (3.25),

$$\begin{aligned}(\overline{v}_0) \cdot \chi(\overline{C}_0) &= \lambda(v_0)_1\chi(C_0)_{m+1} + \lambda(v_0)_{n+1}\chi(C_0)_1 \\ &= 0 + \lambda(v_0)_{n+1}\chi(C_0)_1 \\ &= \lambda(v_0)_{n+1}\chi(C_0)_1 \neq 0.\end{aligned}$$

Therefore, $\overline{D}_2$ is not a Kirchhoff graph.

Case 2. **e_1e_{n+1} is a directed cycle but not a cut-set in D.** Let v_i and v_j denote the end-vertices of e_1 and e_{n+1}, so that $v_ie_1v_je_{n+1}v_i$ is the directed cycle in D. In particular,

$$\lambda(v_i)_1 = 1 \qquad \text{and} \qquad \lambda(v_i)_{n+1} = -1$$

As $\{e_1, e_n\}$ is not a cut-set of D, there exists a $v_j - v_i$ path P that does not traverse e_1 or e_{n+1}. Now consider the cycle:

$$C_P = v_ie_1v_jPv_i.$$

Then, in particular,

$$\chi(C_P)_1 = 1 \qquad \text{and} \qquad \chi(C_P)_{n+1} = 0.$$

Therefore by (3.25):

$$\begin{aligned}\lambda(\overline{v}_i)\cdot\chi(\overline{C}_P) &= \lambda(v_i)_1\chi(C_P)_{n+1}+\lambda(v_i)_{n+1}\chi(C_P)_1\\ &= 0+\lambda(v_i)_{n+1}\chi(C_P)_1\\ &= (-1)(1) = -1 \neq 0.\end{aligned}$$

Therefore, $\overline{D}_2$ is not a Kirchhoff graph. This completes the proof of Lemma 3.9. □

Example 3.4. Let $A \in \mathbb{Z}^{m\times n}$ be a $m \times n$ matrix with integer entries, and suppose column i of A is a positive integer multiple of column j. What configurations of $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$ edges are permitted in Kirchhoff graphs for A?

Letting $\boldsymbol{e}_1, \dots \boldsymbol{e}_n$ represent the standard basis elements of $\mathbb{Z}^n$. This means that

$$\boldsymbol{e}_i - p\boldsymbol{e}_j \in \mathrm{Null}(A)$$

for some integer $p \in \mathbb{N}$.

Lemma 3.10. *In any Kirchhoff graph for matrix A, any occurrence of edge $\boldsymbol{s}_i$ must occur as part of the configuration illustrated in Figure 3.5.1, where q is some integer $q \geq 1$.*

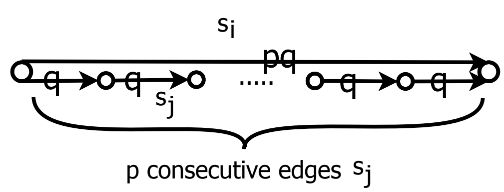


Figure 3.5.1: The permitted configuration for $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$. Note that in this drawing, we only represent the edges $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$, all other edge vectors $\boldsymbol{s}_k$ for $k \neq i, j$ can be incident to the vertices depicted.

Proof. Let $\overline{D}$ be any Kirchhoff graph for A. As $\boldsymbol{e}_i - p\boldsymbol{e}_j \in \mathrm{Null}(A)$, for all vertices $\overline{v}$ of $\overline{D}$, we must have that

$$\lambda(\overline{v})(\boldsymbol{e}_i - p\boldsymbol{e}_j) = 0.$$

That is,

$$\lambda(\overline{v})_i = p\lambda(\overline{v}_j). \tag{3.27}$$

In particular, $\lambda(\overline{v})_i$ and $\lambda(\overline{v})_j$ are either both zero, or both nonzero. This leads to the cycle configuration illustrated in Figure 3.5.1, with multiplicities chosen to ensure that (3.27) is satisfied. □

Along the lines of understanding the relationship between graphs and matrices, another reasonable question is the following. Per Definition 3.2, a vector graph $\overline{D}$ is a Kirchhoff graph for a matrix A if $\lambda(v) \in \text{Row}(A)$ for all vertices v, and $\text{span}\{\chi(C)^t : C \text{ is a cycle of } \overline{D}\} = \text{Null}(A)$.

Question: What can be said of a Kirchhoff graph in which the roles of $\text{Row}(A)$ and $\text{Null}(A)$ are reversed?

That is, suppose $\overline{D'}$ is a vector vector graph for which $\lambda(v) \in \text{Null}(A)$ for all vertices v, and $\text{span}\{\chi(C)^t : C \text{ is a cycle of } \overline{D'}\} = \text{Row}(A)$. What is the relationship between $\overline{D'}$ and $\overline{D}$? Can one be used to construct the other? This question leads to a natural notion of *duality* in Kirchhoff graphs, which is the topic of Chapter 4.

Chapter 4

Kirchhoff Graphs, Duality, and Maxwell Reciprocal Figures

In this chapter we propose a notion of duality with respect to Kirchhoff graphs. We will compare these new ideas to more classical notions of duality, including planarity and James Clerk Maxwell's theory of reciprocal figures, both of which utilize graphs embedded in $\mathbb{R}^2$. In order to aid these comparisons, we distinguish those vector graphs that are embeddable in $\mathbb{R}^2$. An $\mathbb{R}^2$***-vector graph*** will be any vector graph whose vector edges lie in $\mathbb{R}^2$, and an $\mathbb{R}^2$***-Kirchhoff graph*** is defined analogously. Notice that any $\mathbb{R}^2$-vector graph can be embedded in the plane where every edge vector *is drawn as* the assigned vector $\boldsymbol{s}_j$ (this is guaranteed by consistency of vector assignments). In this chapter we take a more geometric approach to Kirchhoff graphs, which are geometric in the sense that the edges are vectors. For more general discussion of geometric graphs, see [85], [86],[87], and [88].

Recall from Definition 3.2 that for a matrix $A \in \mathbb{Z}^{m\times n}$, a vector graph $\overline{D}$ is a Kirchhoff graph for A if and only if

(i) $\overline{D}$ has n vector edges, $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$.

(ii) The cycle vectors of $\overline{D}$ lie in and span Null(A).

(iii) The vertex incidence vectors of $\overline{D}$ lie in Row(A).

Moreover, recall from Definition 1.5 that two digraphs are (abstract) duals if the cycle space of one is the cut space of the other, and vice versa. Building upon these definitions, and in light of Theorem 1.2, we propose the following.

Definition 4.1. Let $A \in \mathbb{Z}^{m \times n}$ be an integer-valued matrix, and suppose Null(A) is the column span of B^t, for some $B \in \mathbb{Z}^{k \times n}$. Any Kirchhoff graph for matrix B will be called a ***Kirchhoff graph for Null(A)***.

Definition 4.2. Two Kirchhoff graphs $\overline{D}$ and $\overline{D'}$ are ***dual Kirchhoff graphs*** if there exists some matrix A such that $\overline{D}$ is a Kirchhoff graph for A, and $\overline{D'}$ is a Kirchhoff graph for Null(A). We say each vector graph is a ***Kirchhoff dual*** of the other.

One question we will begin to explore in this section is,

Question: Given a Kirchhoff graph $\overline{D}$ for a matrix A, can we use vector graph $\overline{D}$ to construct a vector graph $\overline{D'}$ that is a Kirchhoff graph for Null(A)? That is, can a Kirchhoff graph be used to construct its Kirchhoff dual?

Answering this question as "yes" would be a useful step towards proving the existence of a Kirchhoff graph for any integer-valued matrix (the primary open problem in the theory of Kirchhoff graphs [28, 29]). This would mean that given any integer matrix A, if one can construct a Kirchhoff graph for either A or Null(A), then there exists a Kirchhoff graph for A.

A natural place to begin exploring this idea of dual Kirchhoff graphs, and in particular dual $\mathbb{R}^2$-Kirchhoff graphs, is the well-developed theory of reciprocal figures of James Clerk Maxwell. Given a geometric graph satisfying certain properties, Maxwell's work provides a concrete and *universal* method of constructing a reciprocal diagram, where the vertex cuts and cycles in one diagram correspond exactly with the cycles and vertex cuts in the other. Section 4.1 presents a re-description of Maxwell's original work in [75] using modern mathematical terms. Given these foundations, Section 4.2 will describe commonalities between Maxwell figures and Kirchhoff graphs. Cases in which Maxwell's theory is no longer applicable lead us to consider other methods of finding $\mathbb{R}^2$-Kirchhoff duals in Section 4.3.

4.1 Maxwell Reciprocal Figures

James Clerk Maxwell introduced his idea of reciprocal figures in the mid 1860's [74][75][76]. He describes a type of geometric reciprocity, which has applications to mechanical problems. A "frame" is a geometric system of points connected by straight lines. In order to study equilibrium within these frames, he suggests we think of the points as mutually acting on each other, with forces in the direction of the lines adjoining them. Each line is thus endowed with one of two types of forces: if the force acting between its endpoints draws the points closer together, we call the force a *tension*, and if the force tends to separate the points, we call the force a *pressure*. Maxwell outlines a method for drawing a diagram of force corresponding to this frame, where every line of the force diagram represents, in magnitude and direction, the force acting on a line in the frame. Thus, the frame and its forces are presented in two separate diagrams, henceforth referred to as the ***frame diagram*** and the ***force diagram*** . In representing the forces in a separate diagram, Maxwell recognizes the loss of some interpretability: it is not immediately clear which forces act on which points in the frame. On the other hand, this method reduces finding the equilibrium of forces to examining whether the force diagram has closed polygons.

We draw a force diagram where each line in the force diagram is parallel to the line on which it acts in the frame. The lengths of these lines are proportional to the forces acting on the frame. For a frame that is in equilibrium, when any number of lines meet at a point in the frame, the corresponding lines in the force diagram form a closed polygon. In certain cases, Maxwell observes, for any lines that meet in the force diagram, the corresponding lines in the frame form a closed polygon. This is the type of geometric reciprocity Maxwell desires: two figures are considered "‘reciprocal" if either diagram can be taken as a frame diagram, and the other represents the system of forces that keeps that frame in equilibrium. More precisely, in [75], he gives,

Definition 4.3. Two plane rectilinear figures are ***reciprocal*** when they consist of an equal number of straight lines, so that the corresponding lines that meet in a point in one figure form a closed polygon in the other, and vice versa.

The two diagrams in Figure 4.1.1 are the original example of reciprocal figures as presented by Maxwell [75]. Note that the same letters are used to label corresponding (parallel) lines in the frame diagram and the force diagram.

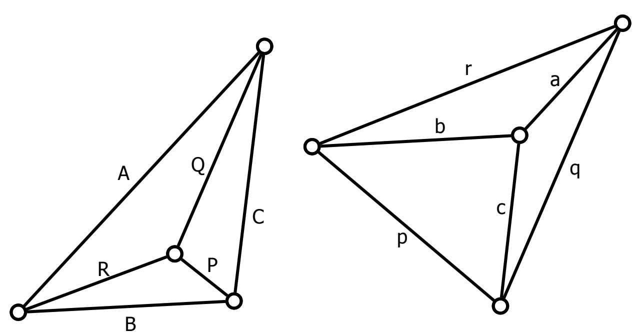


Figure 4.1.1: Maxwell's reciprocal figures as presented in [75]. Corresponding parallel edges are labeled with the same letter; capitalized in the frame diagram and lower-case in the force diagram.

Given that the frame is in equilibrium, we may draw the force diagram as outlined below.

For convenience, refer to points of the frame by the labels of the lines incident to it. For example, the center point in the frame diagram is PQR.

(1) Begin by drawing line p in the force diagram, parallel to line P in the frame. Choose a length for this line that represents the magnitude of the force acting along P. Although the length of this first edge is an arbitrary choice, once the length of line p has been fixed, the lengths of all other lines in the force diagram are determined.

(2) The forces acting along lines P, Q and R in the frame are in equilibrium. Therefore, in the force diagram, draw from one endpoint of p a line parallel to Q and from the other endpoint of p a line parallel to R. Thus, form a (closed) triangle pqr in the force diagram, representing the equilibrium of forces acting on point PQR. Steps (1) and (2) are illustrated in Figure 4.1.2. Note that once we fixed the length of edge p, the lengths of q and r are therefore predetermined and represent the magnitude of the forces acting on Q and R.

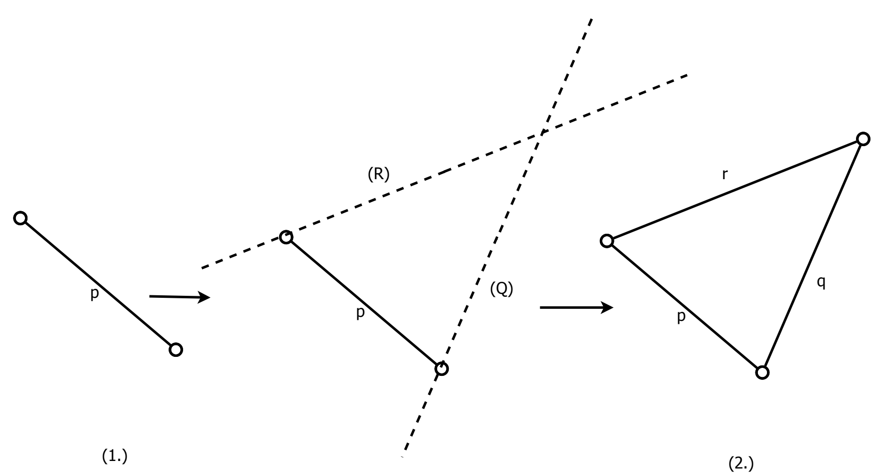


Figure 4.1.2: Steps (1) and (2).

(3) The other end of line P in the frame diagram meets lines B and C at a point. As the forces at point PBC are in equilibrium, draw a triangle in the force diagram with p as one side and lines parallel to B and C as the others. This can be done in one of two ways, illustrated in Figure 4.1.3.

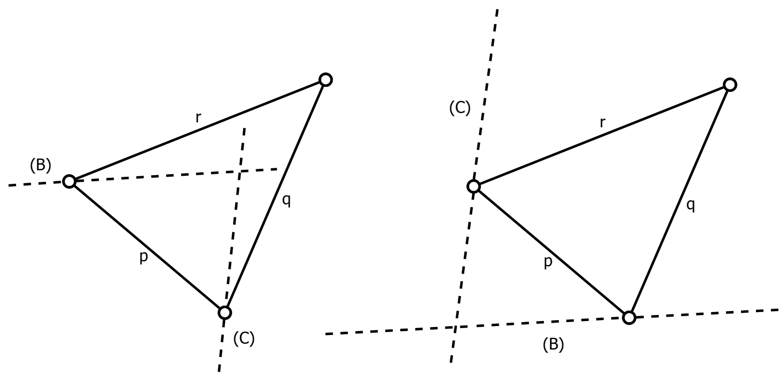


Figure 4.1.3: Two possible choices for Step (3).

Only one of these triangles belongs in the force diagram. As the force diagram is the reciprocal figure of the frame, look to the frame diagram to determine which of these two choices is correct. The endpoints of line p in the force diagram correspond to closed polygons in the frame diagram: namely, those polygons that contain P as an edge. That is, the endpoints of p in the force diagram correspond to polygons PRB and PQC. As PRB is a closed polygon in the frame diagram, line b in the force diagram (parallel to B) must meet lines r and p in a point. Similarly, since PQC is a closed polygon in the frame diagram, line c in the force diagram (parallel to C) must meet lines q and c in a point. This corresponds to choosing the first case shown in Figure 4.1.3.

(4) Consider the equilibrium of forces at point QCA in the frame diagram. Two of the corresponding lines of force, q and c, have already been determined in the force diagram. Therefore, the only choice for line a (parallel to A) in the force diagram must complete triangle qca. Step (4) is illustrated in Figure 4.1.4.

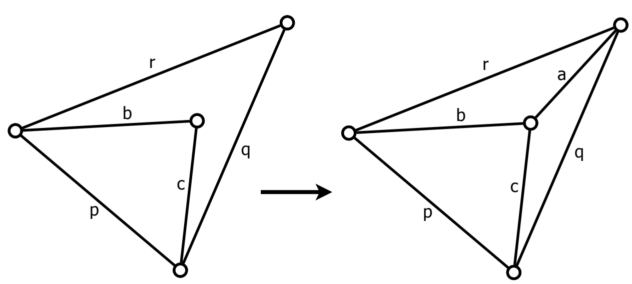


Figure 4.1.4: Step (4).

Remark 4.1. We can use an alternate notation to label the frame and force diagrams, known as *Bow's notation* or *interval notation*. In this method, we label regions of the frame diagram and the corresponding vertices in the force diagram with the same label. A full description of this method can be found in [6].

Maxwell's reciprocal figures are, in fact, geometric graphs that are dual to one another. Therefore of particular interest to us is Maxwell's description of diagrams that have reciprocal figures; namely, plane projections of (closed) polyhedra. Let $\mathcal{P}$ be any closed polyhedron in $\mathbb{R}^3$, i.e. a region bounded by some (finite) number of intersecting planes, $\{M_i\}$.

(1) Let $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_n$ be the normal vectors to planes $M_1, \ldots M_n$. Choose a plane of projection, M, that satisfies:

> (a) M does not intersect the closed polyhedron $\mathcal{P}$.
>
> (b) For all $1 \leq i \leq n$, a line through normal vector $\boldsymbol{n}_i$ of plane M_i intersects plane M.

The standard plane projection of polyhedron $\mathcal{P}$ onto plane M gives one of the two figures. Now, construct a *second* polyhedron $\mathcal{P}'$, which, with respect to some paraboloid of revolution, is a geometric reciprocal to the first polyhedron. The projection of $\mathcal{P}'$ onto plane of projection M is then a reciprocal figure of the projection of $\mathcal{P}$ onto M. These two figures are reciprocal in the sense that corresponding lines are perpendicular to each other. One may obtain reciprocal figures with parallel orientation simply by rotating one figure by 90 degrees in the plane. Construct $\mathcal{P}'$ as follows.

(2) Let z_0 be a fixed-point in three-space that does not lie in M, which we will call the *origin*. The line perpendicular to plane of projection M that passes through point z_0 will be called the *axis*, denoted by z. Lastly, let z_M be the point of intersection of line z with plane M.

(3) For each $1 \leq k \leq n$, draw a line that is normal to plane face M_k of $\mathcal{P}$ and passes through point z_0. This line will intersect (projection) plane M at a unique point, call it m_k. Each point m_k may be thought of as the "projection" of face M_k onto plane M.

(4) For each $1 \leq k \leq n$, let z_k be the point of intersection of axis z with face M_k. This intersection always exists, since we chose a plane M that is not parallel to any of the normal vectors $\boldsymbol{n}_k$ (and z is a normal to M). Let $\boldsymbol{d}_k$ be the (three-space) vector from z_M to z_k. That is, d_k is the vector between the intersection points of z with M and M_k. Ultimately, let

$$p_k = m_k - \boldsymbol{d}_k$$

We will call this point in space p_k the "point corresponding to face M_k of the polyhedron $\mathcal{P}$". Steps (2)–(4) of finding a point p_k corresponding to a particular face M_k are illustrated in Figure 4.1.5.

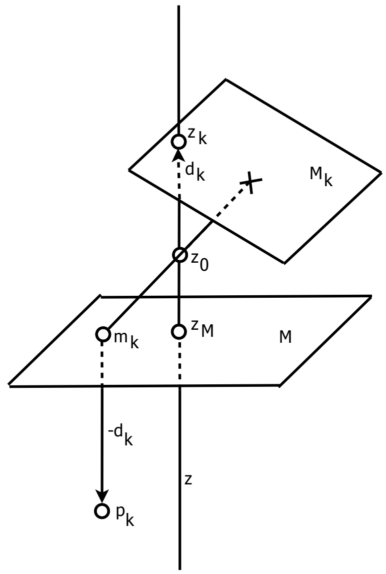


Figure 4.1.5: Finding a point p_k corresponding to a particular face M_k.

In this way, we determine n points $p_1, \cdots, p_n$ corresponding to the faces $M_1, \cdots M_n$ of $\mathcal{P}$. These points form the vertices of polyhedron $\mathcal{P}'$.

(5) For any two faces M_i and M_j that meet in an edge in polyhedron $\mathcal{P}$, draw a line between the corresponding points p_i and p_j. These new lines give us the edges of polyhedron $\mathcal{P}'$.

If we let M_iM_j denote the edge at the intersection of planes M_i and M_j and p_ip_j denote the line joining the corresponding points, one may check geometrically that the projection of line M_iM_j onto M is perpendicular to the projection of line p_ip_j onto M. In this way, the projection of $\mathcal{P}'$ onto M produces a figure, every line of which is perpendicular to the projection of the corresponding line of $\mathcal{P}$ onto M. What is more, for any lines that meet at a point in one projection, the corresponding lines form a closed polygon in the other projection. That is, the projection onto M of $\mathcal{P}$ and $\mathcal{P}'$ forms a pair of reciprocal figures in the sense of Maxwell.

4.2 Maxwell Reciprocal Figures and $\mathbb{R}^2$-Kirchhoff Graphs

Through his theory of reciprocal figures, Maxwell deals with a pair of geometric graphs in which the vertex-cuts of one graph correspond to the cycles in the other, and *vice versa*. In this section, we discuss the relationship between Maxwell's figures, $\mathbb{R}^2$-Kirchhoff graphs, and $\mathbb{R}^2$-Kirchhoff duals. To make a connection between these Maxwell figures and Kirchhoff graphs, we define an $\mathbb{R}^2$ vector graph corresponding to a given Maxwell diagram. First, we define a pair of embedded digraphs corresponding to Maxwell figures and then derive vector graphs accordingly.

Label the edges of the frame diagram as e_j and the corresponding (parallel) edges of the force diagram as f_j. We will adopt notation that indexes both vertices and cycles by these edge labels. Let $\{e_i, e_j, e_k\}$ denote the vertex incident to edges e_i, e_j and e_k, and use standard cycle notation for simple graphs, $f_i f_j f_k$. Observe that if $f_i f_j f_k$ is a cycle in the force diagram, there is a vertex of the form $\{e_i, e_j, e_k\}$ in the frame diagram, and *vice versa.* By arbitrarily assigning directions to each edge in the frame diagram, construct a corresponding plane-embedded digraph, D. We can then construct a plane-embedded digraph corresponding to the force diagram, D', by carefully assigning directions to each edge f_i. In particular, assign directions, such that for each cycle $f_i f_j f_k \dots$ traversed clockwise in the in the force diagram,

$$\chi(f_i f_j f_k \dots) = \lambda(\{e_i, e_j, e_k, \dots\}). \tag{4.1}$$

Finally, derive a pair of vector graphs, $\overline{D}$ and $\overline{D'}$, from D and D' by taking the directed edges, embedded in $\mathbb{R}^2$, as vectors. Let $\{\boldsymbol{s}_i\}$ denote the edge vectors of $\overline{D}$ and $\{\boldsymbol{t}_i\}$ denote the edge vectors of $\overline{D'}$.

Definition 4.4. Let $\overline{D}$ and $\overline{D'}$ be the pair of $\mathbb{R}^2$-vector graphs defined above. We say that $\overline{D}$ and $\overline{D'}$ are a pair of $\mathbb{R}^2$***-vector graphs corresponding to*** the Maxwell figures. Note that we say "a pair", as there are many such sets of vector graphs, depending on the arbitrary assignment of directions to edges in the frame diagram.

Example 4.1. Figure 4.2.1 displays the Maxwell reciprocal figures as in Figure 4.1.1, now re-labeled as described above.

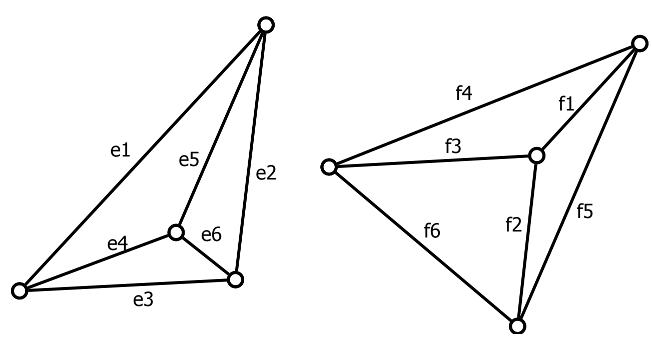


Figure 4.2.1: Maxwell reciprocal figures.

Figure 4.2.2 illustrates one pair of $\mathbb{R}^2$-vector graphs corresponding to these Maxwell reciprocals. In this example, every vector edge occurs exactly once. An example with repeated edge vectors appears at the end of this section. Taking

$$A = \begin{array}{c} \begin{array}{cccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \end{array} \quad \text{and} \quad B = \begin{array}{c} \begin{array}{cccccc} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \boldsymbol{t}_3 & \boldsymbol{t}_4 & \boldsymbol{t}_5 & \boldsymbol{t}_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right] \end{array},$$

It is easy to verify that $\overline{D}$ is a Kirchhoff graph for A. Moreover, Null(A) is the column span of B^t, and most importantly, $\overline{D'}$ is a Kirchhoff graph for B. Therefore beginning with Maxwell reciprocal diagrams, we have arbitrarily assigned directions to one diagram, then consistently assigned directions to the other diagram. The result is two $\mathbb{R}^2$-vector graphs that are both Kirchhoff and indeed are each others' Kirchhoff dual.

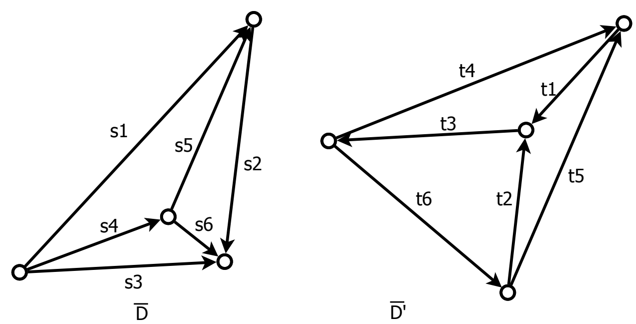


Figure 4.2.2: A pair of $\mathbb{R}^2$-vector graphs corresponding to Maxwell reciprocals.

Example 4.1 is but one example of a more general result. Let G be a graph embedded in the plane, with no multiple edges, that has a reciprocal figure in the sense of Maxwell.

Lemma 4.1. *If no pair of edges of G is parallel in the plane, then any vector graph corresponding to G is a Kirchhoff graph.*

Proof. Let $\overline{D}$ be any vector graph derived by assigning directions to each edge in G, and let $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$ be the resulting edge vectors. Because no edges of G were parallel in the plane, every edge vector $\boldsymbol{s}_i$ occurs exactly once, and therefore $\overline{D}$ is a Kirchhoff graph. □

Now let G and G' be a pair of Maxwell reciprocal figures and $\overline{D}$ and $\overline{D'}$ be any pair of $\mathbb{R}^2$-vector graphs corresponding to G and G'.

Theorem 4.1. *If G has no parallel edges in the plane, both $\overline{D}$ and $\overline{D'}$ are Kirchhoff graphs. Moreover, for any matrix A such that $\overline{D}$ is a Kirchhoff graph for A, $\overline{D'}$ is a Kirchhoff graph for Null(A). That is, $\overline{D}$ and $\overline{D'}$ are Kirchhoff duals.*

Proof. As G has no parallel edges in the plane, its Maxwell reciprocal G' also has no pair of parallel edges. Therefore, both $\overline{D}$ and $\overline{D'}$ are Kirchhoff graphs by Lemma 4.1. If $\overline{D}$ is a Kirchhoff graph for some matrix A, then $\overline{D'}$ is a Kirchhoff graph for Null(A) by (4.1). □

Remark 4.2. Lemma 4.1 and Theorem 4.1 above demonstrate that any Maxwell frame diagram without parallel edges leads to a number of pairs of dual $\mathbb{R}^2$-Kirchhoff graphs. This means that *most* Maxwell diagrams will lead directly to pairs of Kirchhoff duals: given a randomly-chosen closed polyhedron, a randomly-chosen plane projection will lead to a frame diagram with no parallel edges.

Though Remark 4.2 shows that there is significant overlap between Maxwell reciprocals and $\mathbb{R}^2$-Kirchhoff duals, arguably, the cases that make vector graphs interesting are those in which vector edges occur more than once. The connection between Maxwell figures and $\mathbb{R}^2$-Kirchhoff graphs becomes considerably more delicate when considering geometric graphs with parallel edges. The care with which the polyhedron and projection must be constructed in Example 4.2, however, demonstrates that these situations arise as particularly special cases of Maxwell diagrams.

Example 4.2. In this example, we explicitly construct a graph G with the following properties:

(1) When considered as a geometric graph, G has a Maxwell reciprocal, G'.

(2) There exists an $\mathbb{R}^2$-vector graph $\overline{G}$ corresponding to G that is not Kirchhoff.

(3) Any $\mathbb{R}^2$-vector graph $\overline{G'}$ corresponding to G' *is* Kirchhoff.

Consider the system of five planes in $\mathbb{R}^3$ given in (4.2).

$$-x + z = 0 \qquad -x + 3y - 2z = 0 \qquad y + z = 0 \qquad 5x - y + 4z = 0 \qquad z = 0 \tag{4.2}$$

The three-dimensional region bounded by these five planes forms a closed, bounded polyhedron in $\mathbb{R}^3$, as illustrated in Figure 4.2.3.

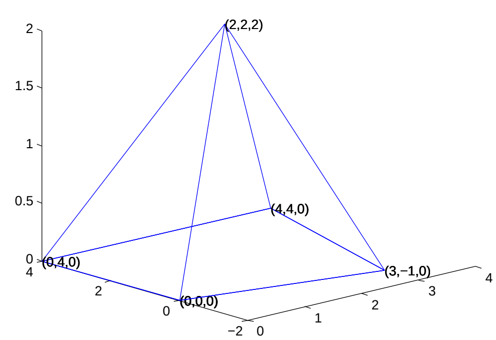


Figure 4.2.3: Closed polyhedron defined by (4.2).

The projection of this figure onto the xy-plane has a reciprocal figure in the sense of Maxwell. The projection of this polyhedron is shown in Figure 4.2.4.

Remark 4.3. This was a very specifically chosen embedding of this polyhedron, that forces two of the projected edges to be congruent. Note that these two edges of the polyhedron were *not* parallel in three-space, so a randomly chosen plane projection of this polyhedron would, in general, result in a geometric graph having no pair of congruent edges.

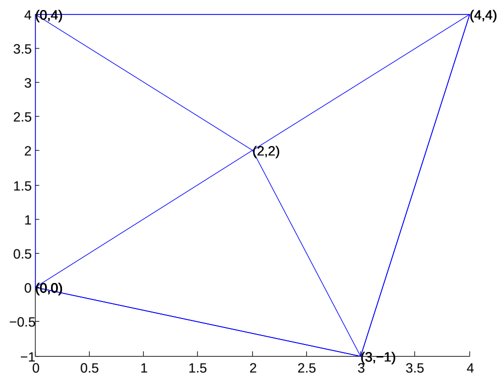


Figure 4.2.4: Projection on the xy-plane of the polyhedron in Figure 4.2.3.

Now, consider the projection in Figure 4.2.4 as a frame diagram. Being the plane projection of a convex polyhedron, construct the associated force diagram. as shown in Figure 4.2.5

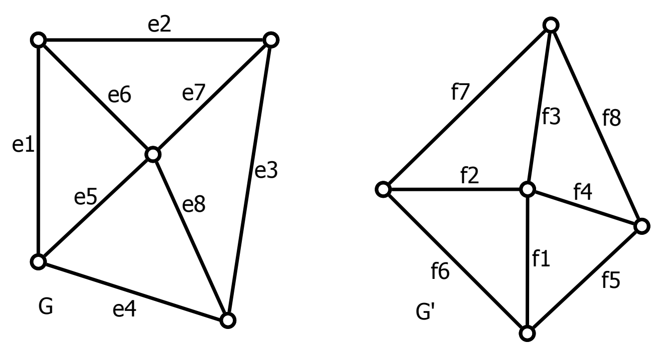


Figure 4.2.5: G and G', Maxwell reciprocals.

The vector graph $\overline{G}$ shown in Figure 4.2.6 is one vector graph corresponding to the frame diagram G in Figure 4.2.5. $\overline{G}$ has no multiple edges, and each vector edge occurs exactly once, except for edge $\boldsymbol{s}_1$, which occurs exactly twice. Given that both occurrences of $\boldsymbol{s}_1$ are not bridges (nor a directed cycle), it follows from Lemma 3.9 that $\overline{G}$ is *not* a Kirchhoff graph. On the other hand, observe that the force diagram G' in Figure 4.2.5 has no pair of congruent edges, and therefore *any* vector graph $\overline{G'}$ derived from G' is a Kirchhoff graph.

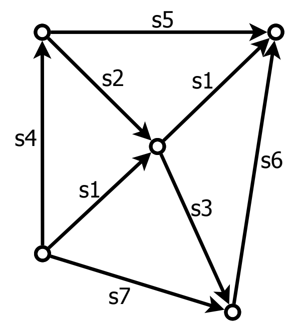


Figure 4.2.6: $\overline{G}$, an $\mathbb{R}^2$-vector graph corresponding to G.

Remark 4.4. This example illustrates a few important differences. First, not every $\mathbb{R}^2$-vector graph corresponding to a Maxwell figure is a Kirchhoff graph. Second, it is possible that a pair of Maxwell reciprocals generates a pair of corresponding $\mathbb{R}^2$-vector graphs, one of which is Kirchhoff, while the other is not. These discrepancies arise because the corresponding vector graphs may have differing numbers of distinct vector edges. When constructing Maxwell's reciprocal figures, we are guaranteed that every pair of corresponding edges is *parallel* to each other. The *length* of each edge in the reciprocal figure is prescribed in Maxwell's construction. However, if two parallel edges in the frame diagram have the same length, the corresponding edges in the force diagram are parallel, but need not have the same length. For example, in Figure 4.2.5, edges e_5 and e_7 are parallel *and* have the same length. In the reciprocal diagram, edges f_5 and f_7 are parallel but have *different* lengths.

This is not to suggest that the *only* Maxwell figures that correspond to dual Kirchhoff graphs are those with no parallel edges. Symmetry in the frame diagram can lead to symmetry in the force diagram, resulting in pairs of edges that are congruent in the plane, as illustrated in Example 4.3.

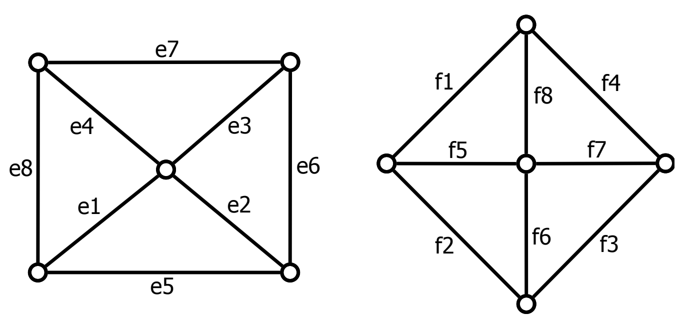


Figure 4.2.7: A pair of Maxwell reciprocal figures.

Example 4.3. Consider the pair of Maxwell reciprocal figures in Figure 4.2.7. Figure 4.2.8 illustrates a pair of $\mathbb{R}^2$-vector graphs corresponding to the reciprocal figures of Figure 4.2.7.

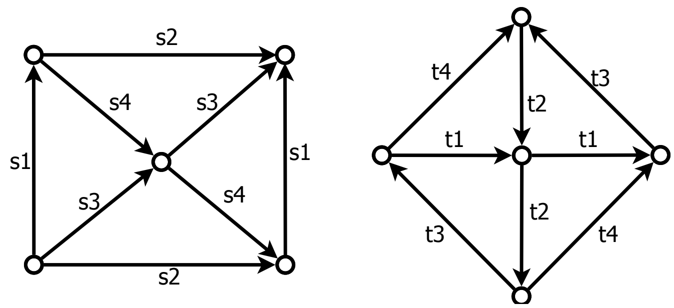


Figure 4.2.8: A pair of vector graphs corresponding to the Maxwell reciprocals in Figure 4.2.7.

One may verify that these two $\mathbb{R}^2$-vector graphs are Kirchhoff graphs for the matrices A and B respectively, where

$$A = \begin{array}{c} \begin{matrix} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad B = \begin{array}{c} \begin{matrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \boldsymbol{t}_3 & \boldsymbol{t}_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Moreover, Null(A) is the column span of B^t and these Kirchhoff graphs are, in fact, a pair of Kirchhoff duals.

4.3 Constructing Kirchhoff Duals

Remark 4.2 and Example 4.3 illustrate that in many cases, Maxwell reciprocal figures can lead to pairs of $\mathbb{R}^2$-Kirchhoff duals. Example 4.2, however, demonstrates that these two theories do not always agree: $\mathbb{R}^2$ vector graphs corresponding to a Maxwell figure are not necessarily Kirchhoff when edges are identified as vectors. Even if the vector graph corresponding to a Maxwell figure *is* Kirchhoff, the vector graph corresponding to the Maxwell reciprocal *need not* be Kirchhoff. Moreover, there are a number of classes of vector graphs that will never correspond to a Maxwell figure; for instance, vector graphs with multiple edges and vector graphs that are not the projection of a polyhedron. This leads us to begin exploring other methods of constructing dual vector graphs, specifically dual $\mathbb{R}^2$-vector graphs.

4.3.1 Planar $\mathbb{R}^2$-Kirchhoff Graphs

Before considering Kirchhoff graphs with multiple edges, we first illustrate an alternative method of constructing the pair of $\mathbb{R}^2$-Kirchhoff duals given in Figure 4.2.8, based on planarity. Rather than Maxwell reciprocals, we consider very specific embeddings of digraphs, that are dual in the standard sense.

Example 4.4. Begin with the first vector graph in Figure 4.2.8, $\overline{D}$. Construct a dual $\mathbb{R}^2$-vector graph for $\overline{D}$, $\overline{D'}$, as follows. First, view $\overline{D}$ as a directed graph D embedded in $\mathbb{R}^2$. Clearly no edges of D intersect in this embedding, meaning D is *planar*. Next, form the standard dual graph D' of D. The *faces* of D are the regions of $\mathbb{R}^2$ that contain no vertex or edge of D, and are bounded by a cycle of D. The *facial cycles* are those that bound a face. The faces of D form the vertices of dual graph D'. For any edge e separating two faces of D there is an edge e' connecting the corresponding vertices of D'. In order to assign directions to the edges of D we choose the following convention. For each face F, traverse the facial cycle in the clockwise direction. If edge e is traversed in the direction of its orientation, let F be the initial vertex of edge e' in D'. Otherwise when e is traversed against its orientation, F is the terminal vertex of e'.

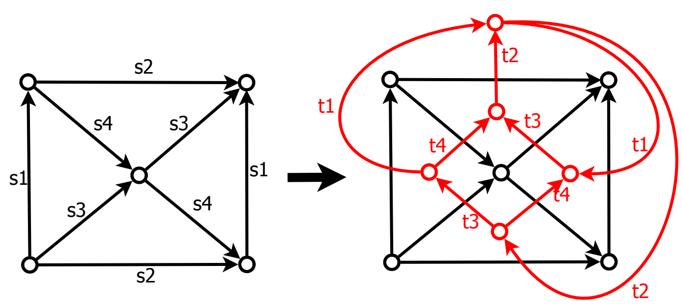


Figure 4.3.1: The standard dual of digraph D.

Now for each edge labeled $\boldsymbol{s}_i$ in $\overline{D}$, assign label $\boldsymbol{t}_i$ to the corresponding edge of D'. This process is illustrated in Figure 4.3.1, and leads to the following,

Question: Can we embed the vertices of D' in $\mathbb{R}^2$, such that for each j, all copies of every edge with label $\boldsymbol{t}_j$ are identical vectors in $\mathbb{R}^2$?

In the case of vector graph $\overline{D}$, the answer to this question is "yes". In particular, we can embed D' in $\mathbb{R}^2$ to give the vector graph $\overline{D'}$ as shown in Figures 4.3.2. Observe that this is precisely the $\mathbb{R}^2$-vector graph $\overline{D'}$ corresponding to Maxwell reciprocal G' in Example 4.3.

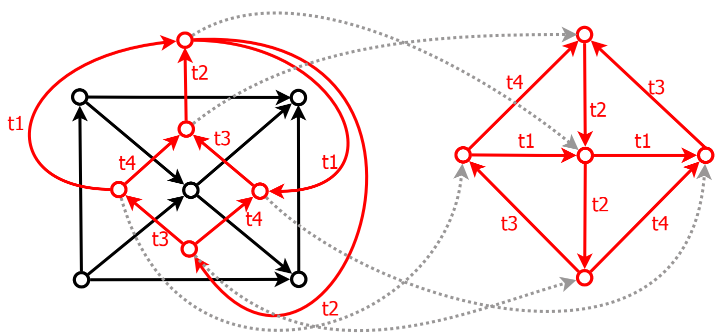


Figure 4.3.2: Arranging vector graph $\overline{D'}$.

Example 4.4 shows the roles of faces and vertices can be interchanged to find Kirchhoff duals. Moreover, this method can be used to address Kirchhoff graphs with multiple edges. Namely, multiple vector edges are replaced by consecutive (identical) vector edges in the dual $\mathbb{R}^2$-vector graph, and *vice versa*. This will be illustrated in Example 4.5.

Remark 4.5. Non-unit entries in an incidence vector indicate multiple edges, whereas non-unit entries in a cycle vector indicate consecutive edges. Therefore, replacement of multiple edges by consecutive edges (and *vice versa*) in the construction of dual vector graphs causes the cut space and cycle space to interchange roles.

Example 4.5. Consider the matrix

$$A = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 \end{array} \\ \left[\begin{array}{ccc} 2 & 0 & -1 \\ 0 & 2 & 1 \end{array} \right] \end{array} \text{ where } \mathrm{Null}(A) = \ \mathrm{span}\left\{ \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^t \right\}$$

The first $\mathbb{R}^2$-vector graph in Figure 4.3.3 is a Kirchhoff graph $\overline{D}$ for A. We also re-draw this vector graph in order to better illustrate the method of addressing multiple edges.

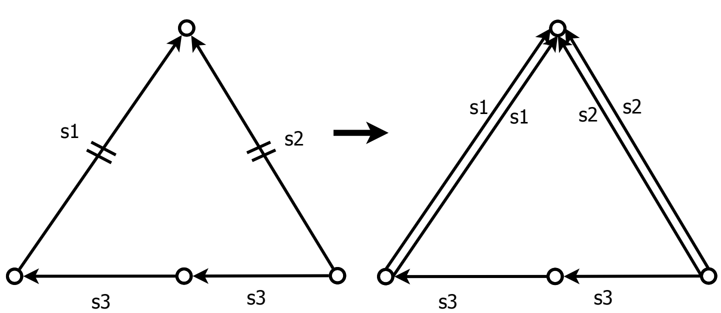


Figure 4.3.3: Kirchhoff graph $\overline{D}$ for matrix A.

As before, construct a dual directed graph and arrange the vertices to give a dual $\mathbb{R}^2$ vector graph, shown in Figure 4.3.4. Observe that multiple edges in the $\mathbb{R}^2$-vector graph translate consecutive edges in the dual $\mathbb{R}^2$-vector graph, and *vice versa*. One can easily verify that $\overline{D'}$ is an $\mathbb{R}^2$-Kirchhoff graph for Null(A).

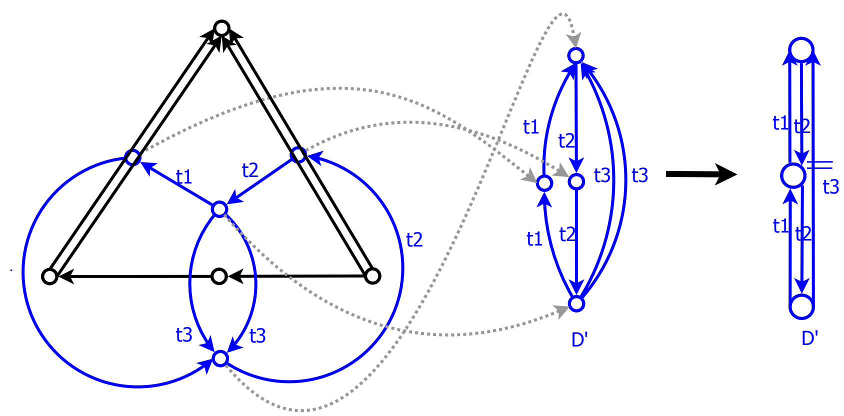


Figure 4.3.4: Arranging vector graph $\overline{D'}$.

One advantage of using dual methods is graph construction. In some cases, given a matrix A, it may be relatively straightforward to construct an $\mathbb{R}^2$-Kirchhoff graph for Null(A). Using a dual vector graph method may then lead to an $\mathbb{R}^2$-Kirchhoff graph for A without constructing one directly. In the example above it may not be intuitively clear how to construct an $\mathbb{R}^2$-Kirchhoff graph for the matrix $[1, -1, 2]$. On the other hand, it is a fairly easy task to construct an $\mathbb{R}^2$-Kirchhoff graph with *cycle space* spanned by $[1, -1, 2]$, and an $\mathbb{R}^2$-Kirchhoff graph for $[1, -1, 2]$ was then obtained via a Kirchhoff dual.

4.3.2 Non-Planar $\mathbb{R}^2$-Kirchhoff Graphs

The planar $\mathbb{R}^2$-Kirchhoff graphs given above are not representative of $\mathbb{R}^2$-Kirchhoff graphs in general, which need neither be planar nor have a geometric dual. One advantage of Definition 4.2 is the opportunity to construct dual $\mathbb{R}^2$-Kirchhoff graphs in cases where no dual graph exists under standard notions. In Example 4.6, we consider $\overline{H} = (K_{3,3}, \varphi)$, an $\mathbb{R}^2$-vector graph representation of $K_{3,3}$, and present a Kirchhoff dual, $\overline{H'}$. On its own, this result is significant: $K_{3,3}$ is non-planar, and has no dual under any classical definitions of duality. However, the Kirchhoff dual that we exhibit leads to a few further observations. Kirchhoff graph $\overline{H'}$ is an embedding of a digraph, that *has* a geometric dual, and embedding this geometric dual to create an $\mathbb{R}^2$-vector graph $\overline{H''}$ results in a Kirchhoff graph that is–nearly–the original $\overline{H}$.

Example 4.6. Consider the vector graphs $\overline{H}$ shown in Figure 4.3.5, and $\overline{H'}$ in Figure 4.3.6.

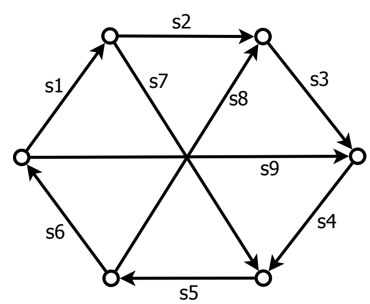


Figure 4.3.5: An $\mathbb{R}^2$-Kirchhoff graph, $\overline{H} = (K_{3,3}, \varphi)$.

Every vector edge of $\overline{H}$ occurs exactly once, and therefore $\overline{H}$ is an $\mathbb{R}^2$-Kirchhoff graph. Moreover, observe that $\overline{H}$ is an embedding of $K_{3,3}$ (with edge orientations). $K_{3,3}$ is non-planar, and has no geometric dual. However, $\overline{H}$ is an $\mathbb{R}^2$-Kirchhoff graph for the matrix A, as in (4.3).

$$A = \begin{array}{c} \begin{array}{ccccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 & \boldsymbol{s}_9 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \tag{4.3}$$

On the other hand, the null space Null(A) is the column span of B^t, where

$$B = \begin{array}{c} \begin{array}{ccccccccc} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \boldsymbol{t}_3 & \boldsymbol{t}_4 & \boldsymbol{t}_5 & \boldsymbol{t}_6 & \boldsymbol{t}_7 & \boldsymbol{t}_8 & \boldsymbol{t}_9 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

One may verify that vector graph $\overline{H'}$ as in Figure 4.3.6 is an $\mathbb{R}^2$-Kirchhoff for matrix B. That is, $\overline{H'}$ is a Kirchhoff dual of $\overline{H}$.

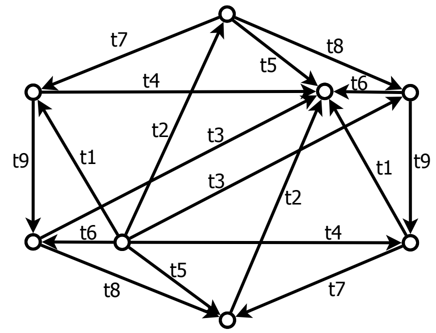


Figure 4.3.6: Dual $\mathbb{R}^2$-Kirchhoff graph, $\overline{H'}$.

Therefore, although $\overline{H}$ (i.e. $K_{3,3}$) does not have a dual graph under any standard notions, when taken as an $\mathbb{R}^2$-vector graph, it has a dual in the Kirchhoff sense.

Interestingly, this Kirchhoff dual $\overline{H'}$ can be viewed as the plane projection of a polyhedron in $\mathbb{R}^3$, meaning $\overline{H'}$ has a geometric dual. This polyhedron and its geometric dual are illustrated in Figure 4.3.7. For each edge labeled $\boldsymbol{t}_j$ in $\overline{H'}$, we label the corresponding edge as $\boldsymbol{s}_j$ in the geometric dual.

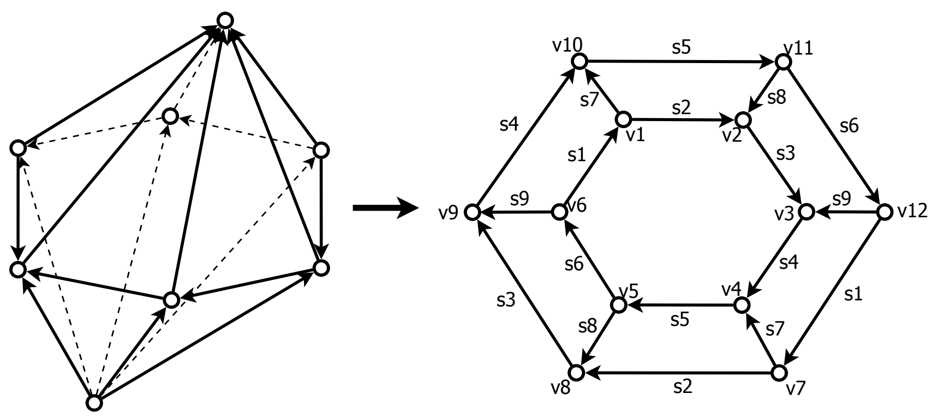


Figure 4.3.7: The polyhedron of $\overline{H'}$ and its geometric dual.

In order to construct this geometric dual as an $\mathbb{R}^2$-vector graph, $\overline{H''}$, we must embed the vertices in the plane so that all vector edges with label $\boldsymbol{s}_i$ are identical. This can be accomplished by, for each $1 \leq j \leq 6$, mapping the pair of vertices v_j and v_{j+6} to the same point in the plane. This embedding is illustrated in Figure 4.3.8.

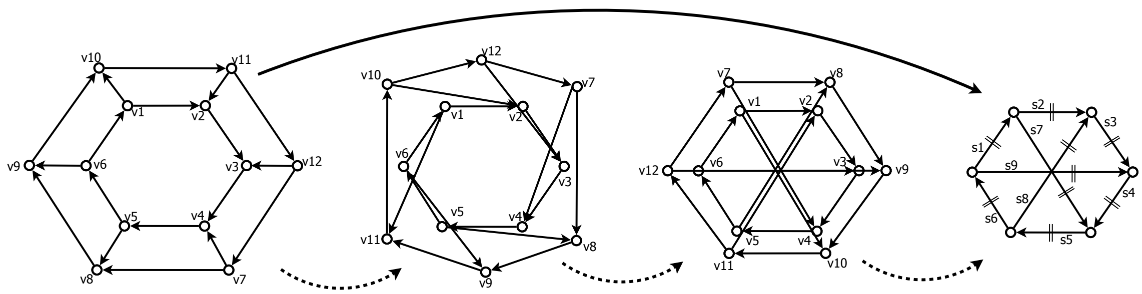


Figure 4.3.8: Constructing $\mathbb{R}^2$-vector graph $\overline{H''}$.

This $\mathbb{R}^2$-vector graph is vector graph $\overline{H}$, with every edge doubled. As a result, $\overline{H''}$ is both an $\mathbb{R}^2$-Kirchhoff graph for the matrix A and a Kirchhoff dual for $\overline{H'}$.

By following essentially the reverse process as that outlined in Example 4.6, we can illustrate how to directly construct an $\mathbb{R}^2$-Kirchhoff dual for K_5 (which is also non-planar).

Example 4.7. Consider the $\mathbb{R}^2$-vector graph in Figure 4.3.9, $\overline{K} = (K_5, \varphi)$, a vector graph constructed from a directed complete graph on 5 vertices.

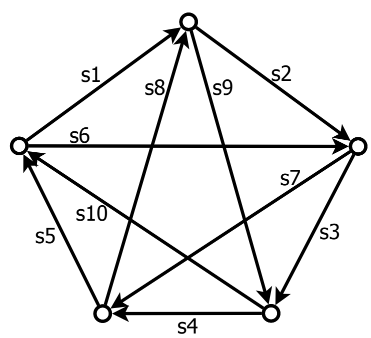


Figure 4.3.9: $\mathbb{R}^2$-Kirchhoff graph $\overline{K} = (K_5, \varphi)$.

Every vector edge of $\overline{K}$ occurs exactly once, so $\overline{K}$ is Kirchhoff. Moreover, it is a Kirchhoff graph for matrix A, where

$$A = \begin{array}{c} \begin{array}{cccccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 & \boldsymbol{s}_9 & \boldsymbol{s}_{10} \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The null space Null(A) is spanned by the columns of B^t, where

$$B = \begin{array}{c} \begin{array}{cccccccccc} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \boldsymbol{t}_3 & \boldsymbol{t}_4 & \boldsymbol{t}_5 & \boldsymbol{t}_6 & \boldsymbol{t}_7 & \boldsymbol{t}_8 & \boldsymbol{t}_9 & \boldsymbol{t}_{10} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Our goal is to construct an $\mathbb{R}^2$-Kirchhoff graph for matrix B. Rather than attempting to do so directly from the matrix, we can utilize any $\mathbb{R}^2$-Kirchhoff graph for A. Clearly, $\mathbb{R}^2$-vector graph $\overline{K''}$, shown in Figure 4.3.10, is vector graph $\overline{K}$ with each vector edge doubled.

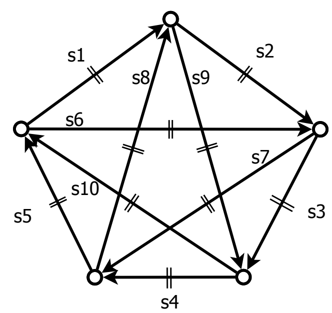


Figure 4.3.10: $\mathbb{R}^2$-vector graph $\overline{K''}$.

Therefore, $\overline{K''}$ is also a Kirchhoff graph for A, and any $\mathbb{R}^2$-Kirchhoff dual for $\overline{K''}$ is also a Kirchhoff dual for $\overline{K}$. We will demonstrate that we can use $\mathbb{R}^2$-vector graph $\overline{K''}$ to construct such a Kirchhoff dual, $\overline{K'}$. $\overline{K''}$ arises from a plane embedding of a digraph with 20 edges. For a moment, however, suppose that this digraph had more than five vertices; some of these vertices were then embedded at the same point when forming $\mathbb{R}^2$-vector graph $\overline{K''}$. Specifically, consider digraph D as in Figure 4.3.11, on **10** vertices and 20 edges.

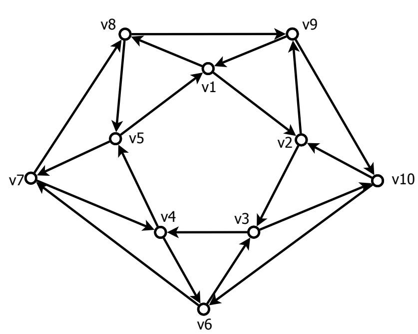


Figure 4.3.11: A digraph D with 10 vertices and 20 edges.

An embedding of D, that maps the pair of vertices v_j and v_{j+5} $(1 \leq j \leq 5)$ to the same point in the plane will produce $\mathbb{R}^2$-vector graph $\overline{K''}$. We alternately render digraph D in Figure 4.3.12, labeling edges to indicate those that are assigned the same vector.

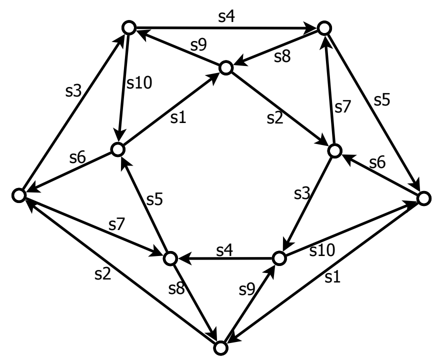


Figure 4.3.12: Labeling the edges of D to reflect which are assigned the same vector.

This digraph D is clearly planar, so we may construct its standard dual, D'. For each edge labeled $\boldsymbol{s}_j$ in D, label the corresponding edge as $\boldsymbol{t}_j$ in D'. Given dual D', shown in Figure 4.3.13, we have returned to the previous question: can the vertices of D' be embedded in $\mathbb{R}^2$, such that all directed edges with label $\boldsymbol{t}_j$ are identical vectors?

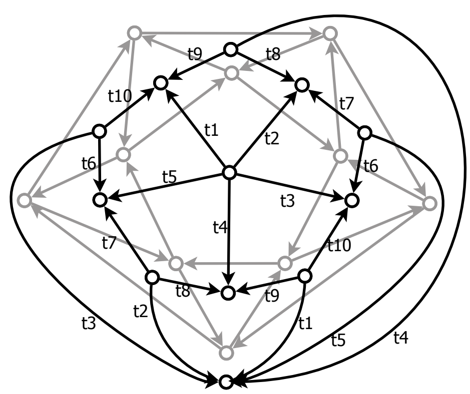


Figure 4.3.13: Constructing the dual D' of digraph D.

One such embedding of D' is illustrated in Figure 4.3.14, forming an $\mathbb{R}^2$-vector graph $\overline{K'}$. More importantly, one may verify that this vector graph $\overline{K'}$ is a Kirchhoff graph for matrix B. That is, $\overline{K'}$ is a Kirchhoff dual of $\overline{K} = (K_5, \varphi)$, and we have constructed $\overline{K'}$ directly from one of its $\mathbb{R}^2$-Kirchhoff duals, $\overline{K''}$.

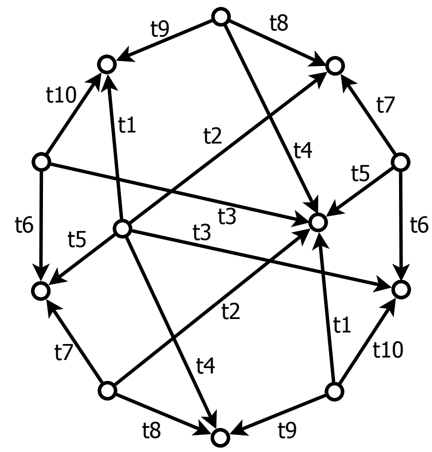


Figure 4.3.14: $\mathbb{R}^2$-vector graph $\overline{K'}$, a Kirchhoff dual of $\overline{K}$.

On this note we conclude our discussion of Kirchhoff duality, and thus the first part of this text. We will return to non-planar graphs K_5 and $K_{3,3}$ in Chapters 9 and 10. Specifically, in Chapter 10 we will construct *undirected* graphs that are dual to K_5 and $K_{3,3}$ in the Kirchhoff sense. This will rely on developing so-called *Kirchhoff edge partitions*. Observe that any vector graph can be viewed as a directed graph, the edges of which are partitioned into a finite set of classes, where each vector edge $\boldsymbol{s}_i$ defines a class. Using this (slightly abstracted) perspective, understanding the properties of vector assignments as *edge partitions* will help us to gain further insights into Kirchhoff graphs. Therefore the next part of this text, Chapters 5 through 7, is dedicated to studying partitions of graphs.

Chapter 5

Equitable Partitions of Graphs

Chapter 3 studied Kirchhoff graphs in the context of matrices. One key component in that theory was the characteristic matrix, T, of a vector assignment φ. Taking a closer look at the structure of matrix T, observe that it indexes which edges of D are assigned to the same vector under φ, regardless of the actual vectors being assigned. Therefore T can be viewed as the characteristic matrix of an edge *partitioning*. That is, a vector assignment φ acts to group the edges of a digraph into k disjoint subsets, one corresponding to each vector.

Therefore the next three chapters will study graph partitions, specifically so-called *equitable partitions*. Chapter 5 first introduces the classical theory of equitable vertex partitions, and then presents some new results. Later, Chapter 6 will extend the notion of equitable partitions to equitable edge-partitions of multi-digraphs. After demonstrating that these edge partitions satisfy the same properties of their vertex counterparts, we show that equitable edge-partitions can give rise to Kirchhoff graphs.

In this chapter we study equitable vertex partitions of graphs. For a classical introduction, see Godsil [39], Chapter 5. The term "equitable partition" was introduced by Haynsworth, in the context of matrix partitions [57]. Equitable partitions were then extended to graphs by Schwenk [98], to aid in calculating the characteristic polynomial of a graph. This theory was further developed by [21], and later McKay used equitable partitions in an algorithm to determine graph isomorphism [77][78]. Equitable graph partitions have been shown to be significant in the study of distance-regular graphs–see for example [43] and [41]–and walk-regular graphs, a combinatorial generalization of vertex-transitive graphs [42]. They are closely related to completely regular subsets, introduced by Delsarte [22], and later used by

Martin to study association schemes [71]. More recently, Godsil used equitable partitions to study compact graphs–those for which all doubly-stochastic matrices commute with the adjacency matrix [40].

One classical advantage of equitable partitions of graphs is that they allow one to study the eigenvectors and eigenvalues of a graph by considering those of a smaller quotient graph. The notion of a quotient graph was first developed by Sachs, using the term "divisor" rather than "quotient." An introduction to this theory can be found in [21], Chapter 4, in addition to a wealth of additional resources. Since then, graphs in the context of combinatorial group theory have been an active area of research. Excellent introductions can be found in both Stillwell [102] and Lyndon [68]. Recently lifts of graphs, specifically random lifts (for example, [3],[4], [9], and [25]) and covering graphs (for example, [2], [7], [66], and [83]) have been of particular interest. Perhaps most notably, in 2015 Friedman used a graph-based approach to prove the well-known Hanna Neumann conjecture on the intersection of finitely generated subgroups of free groups [35].

Section 5.1 begins with an introduction to equitable (vertex) partitions, including the notion of a quotient, and quotient matrix. Section 5.1.1 then defines an orbit partition and proves that every orbit partition is equitable, in addition to considering cases in which equitable–but not orbit–partitions can be constructed from orbit partitions. Section 5.2 then presents necessary and sufficient conditions for an integer-valued matrix to be the quotient of an equitable partition, Theorem 5.1. Next, Section 5.2.1 studies the relationship between partitions, quotients, and matrices. Theorem 5.2 shows that any equitable partition is part of an infinite family of equitable partitions, all having the same quotient matrix. Finally, Section 5.2.2 presents an efficient graph-based algorithm for checking the necessary and sufficient conditions of Theorem 5.1. To the author's knowledge, the results of Section 5.2 and its subsections have not appeared in the literature.

5.1 Equitable Partitions

Let G be a simple undirected graph with vertices $V(G) = \{v_i\}$. A ***partition*** of $V(G)$ is a set whose elements are disjoint, nonempty subsets of $V(G)$ whose union is $V(G)$. The elements of a partition π will be called *cells* and occasionally *classes*. A partition $\pi = (V_1, \ldots, V_k)$ of $V(G)$ is ***equitable*** if, for all i and j, the number of neighbors a vertex in V_i has in V_j is independent of the choice of vertex in V_i. Put another way, let $A(G)$ be the $|V(G)| \times |V(G)|$ adjacency matrix of G. Then partition π is equitable if for any $i, j \in \{1, \ldots, k\}$ and for any

$v_p \in V_i$, the number

$$c_{i,j} = \sum_{q:v_q \in V_j} A(G)_{p,q} \tag{5.1}$$

depends only on i and j, and not on the choice of vertex $v_p \in V_i$. For example, the graph G_M in Figure 5.1.1, known as the McKay graph, has equitable partition $\pi = (V_1, V_2)$ where

$$V_1 = \{v_1, v_2, v_4, v_5, v_7, v_8\} \text{ and } V_2 = \{v_3, v_6\}.$$

Observe that $c_{1,1} = c_{1,2} = 1$, $c_{2,1} = 3$, and $c_{2,2} = 0$.

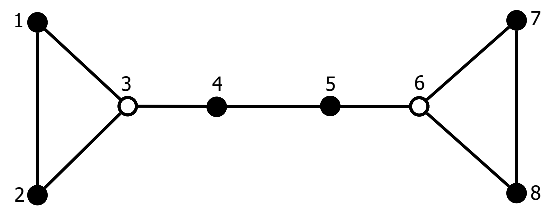


Figure 5.1.1: An equitable partition of the McKay graph.

Every graph has a partition that is trivially equitable, known as the discrete partition, in which each cell contains exactly one vertex. On the other hand, the trivial partition, consisting of exactly one cell, is equitable if and only if the graph is regular. Given an equitable partition $\pi = (V_1, \dots, V_k)$, any vertex in cell V_i has the same number of neighbors in cell V_i. Therefore for each $i \in \{1, \dots, k\}$, the induced subgraph $G[V_i]$ is a regular graph (of degree $c_{i,i}$). Moreover, if a graph G has an equitable partition, then so does its complement, $\overline{G}$.

Proposition 5.1. If $\pi = (V_1, \dots, V_k)$ is an equitable partition of G, then π is also an equitable partition of $\overline{G}$.

Proof. Given equitable partition $\pi = (V_1, \dots, V_k)$, let $c_{i,j}$ be defined as in (5.1). Then define

$$\overline{c}_{i,j} = \begin{cases} |V_j| - c_{i,j} & \text{if } i \neq j \\ |V_i| - c_{i,i} - 1 & \text{if } i = j \end{cases}. \tag{5.2}$$

For any cell V_i, let u be any vertex $u \in V_i$. Then for $j \neq i$, in graph G vertex u has $c_{i,j}$ neighbors in cell V_j. Therefore in $\overline{G}$, u has $\overline{c}_{i,j} = |V_j| - c_{i,j}$ neighbors in V_j. Similarly, in G vertex u has $c_{i,i}$ neighbors in cell V_i, so in $\overline{G}$ vertex u has $\overline{c}_{i,i} = |V_i| - c_{i,i} - 1$ neighbors in V_i. As $\overline{c}_{i,j}$ is independent of u, it follows that π is an equitable partition of $\overline{G}$. $\square$

For example, Figure 5.1.2 shows the complement of the McKay graph, $\overline{G_M}$, with the same partition as in Figure 5.1.1. This partition is equitable with

$$\overline{c}_{1,1} = 4, \overline{c}_{1,2} = 1, \overline{c}_{2,1} = 3, \overline{c}_{2,2} = 1.$$

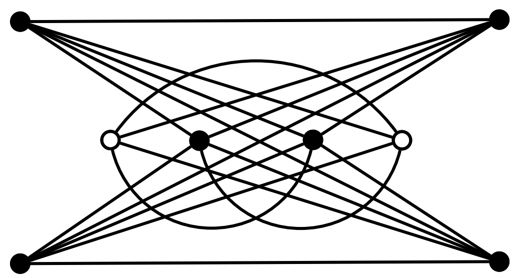


Figure 5.1.2: An equitable partition of the complement of the McKay graph.

Let G be a graph with equitable partition $\pi = (V_1, \ldots, V_k)$. As before, let $c_{i,j}$ denote the number of edges between any vertex in V_i and the vertices of V_j. We can define the ***quotient*** of G with respect to π, denoted G/π. G/π is a directed graph with one vertex for each partition cell $V_1, \ldots, V_k$, and $c_{i,j}$ edges directed from V_i to V_j for all $i, j \in \{1, \ldots, k\}$. Note that in general G/π has both loops and multiple edges. The quotients G_M/π and $\overline{G}_M/\pi$ are illustrated in Figure 5.1.3.

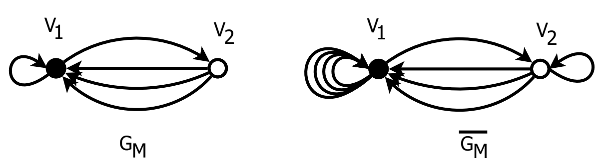


Figure 5.1.3: The quotients of G_M and $\overline{G_M}$ with respect to π.

The adjacency matrix of G/π, $A(G/\pi)$, is the $k \times k$ matrix with (i, j)-entry $c_{i,j}$. For example, the graphs G_M and $\overline{G_M}$ under partition π have

$$A(G_M/\pi) = \begin{array}{c c} & \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad A(\overline{G_M}/\pi) = \begin{array}{c c} & \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \end{array}.$$

Observe that in light of (5.2), for any graph G with an equitable partition, the matrix $A(\overline{G}/\pi)$ can be computed from $A(G/\pi)$. In particular, letting J_k be the $k \times k$ all-ones matrix, and

Ω be a $k \times k$ diagonal matrix with (i,i)-entry $|V_i|$,

$$A(G/\pi) + A(\overline{G}/\pi) = J_{k \times k}\Omega - I_k.$$

We will also call matrix $A(G/\pi)$ the ***quotient matrix*** of G with respect to π. Any such matrix A must be square and have nonnegative integer entries, and may be the quotient matrix of any number of graph/equitable partition pairs. For example, figure 5.1.4 shows two equitable partitions, one on a 15-vertex graph and one on a 20-vertex graph, both having quotient matrix

$$\begin{array}{c c} & \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \end{array} & \left[\begin{array}{ccc} 2 & 4 & 2 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{array}\right] \end{array}.$$

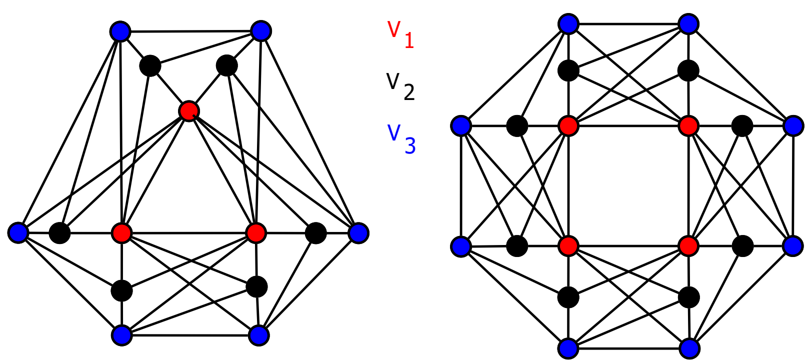


Figure 5.1.4: Two graphs, one with 15 vertices and one with 20, each having an equitable partition with the same quotient matrix.

More importantly, for any graph G with equitable partition π, the quotient matrix $A(G/\pi)$ is closely related to the adjacency matrix $A(G)$ of G. The ***characteristic matrix*** P of partition $\pi = (V_1, \ldots, V_k)$ is a $|V(G)| \times k$ matrix whose j^{th} column is the characteristic vector of set V_j. That is, $P_{i,j} = 1$ if $e_i \in V_j$ and is zero otherwise. Characteristic matrix P provides the relationship between $A(G)$ and $A(G/\pi)$.

Lemma 5.1. *Let π be a partition of $V(G)$ with characteristic matrix P. If π is equitable then*

$$A(G)P = PA(G/\pi).$$

Proof. For any $v_p \in V(G)$, suppose that $v_p \in V_i$, and consider $[A(G)P]_{p,j}$. Noting that for all q, $P_{q,j} = 1$ if and only if $v_q \in V_j$,

$$[A(G)P]_{p,j} = \sum_{q=1}^{|V(G)|} A(G)_{p,q} P_{q,j} = \sum_{q:v_q \in V_j} A(G)_{p,q} = c_{i,j}$$

by (5.1). On the other hand, as $v_p \in V_i$, row p of P is zero except for a 1 in column i. Therefore row p of $PA(G/\pi)$ is row i of $A(G/\pi)$, and $[PA(G/\pi)]_{p,j} = A(G/\pi)_{i,j} = c_{i,j}$. Therefore $A(G)P = PA(G/\pi)$. □

The reverse implication of Lemmas 5.1 is true as well.

Proposition 5.2. Let π be a partition of some graph G, with characteristic matrix P, and let $A = A(G)$. If

$$AP = PB$$

for some $k \times k$ matrix B, then π is equitable.

Proof. If $AP = PB$, then the columns of AP are linear combinations of the columns of P, and are thus constant on the cells of π. That is, if v_p and v_q belong to the same partition class of π, then for any j,

$$[AP]_{p,j} = [AP]_{q,j}.$$

However for any i and j,

$$[AP]_{i,j} = \sum_{q=1}^{n} A_{i,q} P_{q,j} = \sum_{q:v_q \sim v_i} P_{q,j}.$$

The last sum is the number of vertices in cell j that are adjacent to vertex v_i. However as the columns of AP are constant on the cells of π, this value depends only on the cell of π containing v_i, and not on the choice v_i. Therefore π is equitable. □

Building upon these well-known ideas, Section 6.1 will consider edge-partitions of directed graphs, and introduces criteria for such a partition to be equitable. First, Section 5.1.1 demonstrates the existence of large classes of equitable partitions. Then Section 5.2 will examine necessary and sufficient conditions for a matrix to be the quotient of an equitable partition.

5.1.1 Orbit Partitions

Let G be a simple undirected graph with vertices $V(G) = \{v_i\}$. Recall that an *automorphism* of G is a bijection $\phi : V(G) \to V(G)$ that preserves both adjacencies and non-adjacencies.

That is, $v_i \sim v_j$ if and only if $\phi(v_i) \sim \phi(v_j)$, and any automorphism ϕ acts as a permutation of $V(G)$. It is well-known that the set of all automorphisms of G forms a group under the operation of function composition. Denoted $\mathrm{Aut}(G)$, this is the *automorphism group* of G. Note that the identity map, which maps each vertex to itself, is an automorphism of every graph, and is sometimes called the trivial automorphism. Any graph whose automorphism group consists of only the trivial automorphism is called *asymmetric*.

For any subgroup Π of $\mathrm{Aut}(G)$, the action of Π on $V(G)$ naturally partitions the vertices into equivalence classes, where two vertices u and v are in the same class if and only if there is some $\phi \in \Pi$ such that $\phi(u) = v$. These equivalence classes are called the *orbits* of $V(G)$ under Π, and any partition of $V(G)$ arising as orbits of some $\Pi \leq \mathrm{Aut}(G)$ is called an ***orbit partition***.

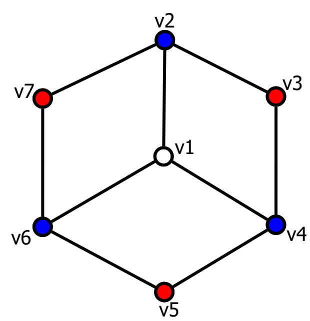


Figure 5.1.5: A graph G with a 3-cell orbit partition. This partition arises from the subgroup of $\mathrm{Aut}(G)$ generated by the permutation $\phi = (v_1)(v_2v_4v_6)(v_3v_5v_7)$.

Example 5.1. Consider the graph G on 7 vertices presented in Figure 5.1.5. Let ϕ be a permutation of the vertices, with disjoint cycle form

$$\phi = (v_1)(v_2v_4v_6)(v_3v_5v_7).$$

One may verify that ϕ is a graph automorphism. What is more, ϕ is a permutation of order 3 that generates a subgroup $\langle\phi\rangle \leq \mathrm{Aut}(G)$ of order 3. The vertex orbits under $\langle\phi\rangle$ are O_1, O_2, O_3, where

$$O_1 = \{v_1\} \quad O_2 = \{v_2, v_4, v_6\} \quad O_3 = \{v_3, v_5, v_7\}.$$

That is, $\pi = (O_1, O_2, O_3)$ is an orbit partition of G. Moreover, π is an equitable partition of

$V(G)$, where

$$A(G/\pi) = \begin{array}{c} \\ O_1 \\ O_2 \\ O_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} O_1 & O_2 & O_3 \end{array} \\ \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}.$$

This is, in fact, a more general result.

Proposition 5.3. Every orbit partition is an equitable partition.

Proof. Let Π be any subgroup of $\mathrm{Aut}(G)$ with orbits $O_1, \ldots, O_p$. For any orbit O_i, let $u, v \in O_i$. In particular, there exists some automorphism $\phi \in \Pi$ such that $\phi(u) = v$. Now for any $j \in \{1, \ldots, p\}$, let $v_1, \ldots, v_{|O_j|}$ denote the vertices of orbit O_j. As ϕ is an automorphism, each v_l is adjacent to u if and only if $\phi(v_l)$ is adjacent to $\phi(u) = v$. However, as O_j is an orbit of Π, $\{\phi(v_1), \ldots, \phi(v_{|O_j|})\} = \{v_1, \ldots, v_{|O_j|}\} = O_j$. Therefore u and v have the same number of neighbors in cell O_j. As i and j were arbitrary, the partition $(O_1, \ldots, O_p)$ is equitable. $\square$

On the other hand, not all equitable partitions are orbit partitions. The McKay graph (shown in Figure 5.1.1) is the smallest graph with an equitable partition that is not an orbit partition. The partition given in Figure 5.1.1 is equitable, yet there is no graph automorphism that maps, for example, v_4 to v_1. Other examples of equitable partitions that are not orbit partitions are illustrated in Figure 5.1.6.

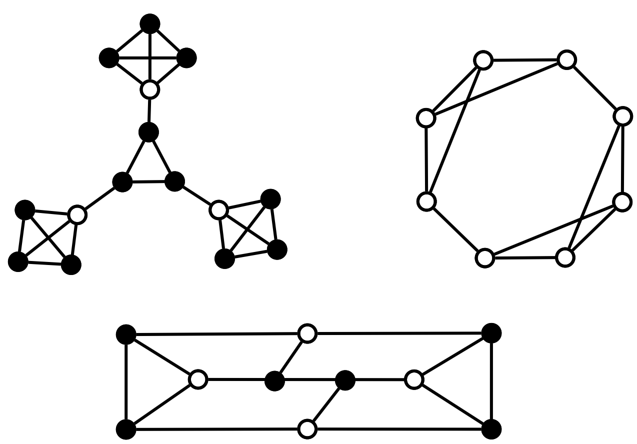


Figure 5.1.6: Three graphs with equitable partitions that are not orbit partitions

The drawings of the four graphs in Figure 5.1.1 and 5.1.6 combined include many of the standard examples of equitable partitions that are not orbit partitions. However, the drawings presented make it clear that each of these four graphs have symmetries, and therefore each has a nontrivial automorphism group. Interestingly, the cells of these equitable partitions can all be formed from unions of cells of suitable nontrivial orbit partitions. By nontrivial, we mean orbit partitions arising from nontrivial automorphisms. Every graph has the trivial automorphism, and as all orbits are singleton vertices, certainly any partition cell can be built up as a union of these orbits. Instead, we are interested in constructing an equitable partition through unions of non-singleton cells of orbit partitions.

Example 5.2. Consider the vertex enumeration of the G_M given in Figure 5.1.1. Let ϕ and ψ be two permutations on $V(G_M)$ with disjoint cycle form

$$\begin{aligned}\phi &= (v_1v_2)(v_7v_8)\\ \psi &= (v_1v_7)(v_2v_8)(v_3v_6)(v_4v_5)\end{aligned}$$

One may verify that both ϕ and ψ are automorphisms of G_M. Let $\langle\phi,\psi\rangle$ be the subgroup of $\mathrm{Aut}(G_M)$ generated by ϕ and ψ. The orbits of $V(G_M)$ under $\langle\phi,\psi\rangle$ are

$$O_1 = \{v_1, v_2, v_7, v_8\} \quad O_2 = \{v_4, v_5\} \quad O_3 = \{v_3, v_6\}.$$

The partition $\pi = (O_1, O_2, O_3)$ is equitable, and is illustrated in Figure 5.1.7.

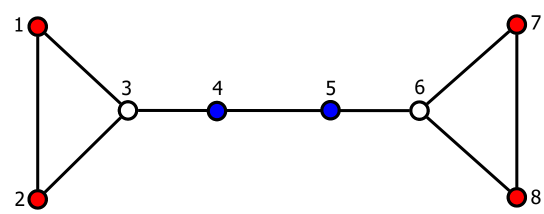


Figure 5.1.7: An equitable partition of the McKay graph.

The partition $\pi = (O_1, O_2, O_3)$ has quotient matrix

$$A(G_M/\pi) = \begin{array}{c} \\ O_1 \\ O_2 \\ O_3 \end{array} \begin{array}{c} \begin{array}{ccc} O_1 & O_2 & O_3 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{array}\right] \end{array}.$$

Observe that $c_{1,1} = c_{2,2}$, $c_{1,3} = c_{2,3}$, and $c_{1,2} = c_{2,1} = 0$. In particular, we may combine cells O_1 and O_2 without destroying equitability of the partition. That is, $(O_1 \cup O_2, O_3)$ is an equitable partition of G_M. In fact, $(O_1 \cup O_2, O_3)$ is precisely the equitable partition presented in Figure 5.1.1.

Similar arguments are true for the equitable (although not orbit) partitions in Figure 5.1.6. It is not unreasonable to wonder if this is a more general result. Can the cells of *any* equitable partition be formed from unions of cells of nontrivial orbit partitions? Example 5.3 will answer this question in the negative.

Example 5.3. Figure 5.1.8 illustrates G_F, a 4-regular graph on 24 vertices. Let $\pi = (V_1, V_2)$ be the partition of $V(G_F)$ illustrated in Figure 5.1.8, where the vertices of V_1 are colored blue, and the vertices of V_2 are red.

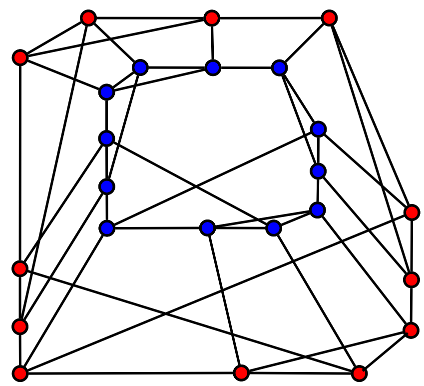


Figure 5.1.8: G_F, a 4-regular graph on 24 vertices, with a 2-cell equitable partition.

Observe that in addition to G_F being 4-regular, partition π is equitable, with quotient

$$A(G_F/\pi) = \begin{array}{c c} & \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \left[\begin{array}{cc} 3 & 1 \\ 1 & 3 \end{array} \right] \end{array}.$$

However, the induced subgraphs $G_F[V_1]$ and $G_F[V_2]$ are isomorphic, and each is the Frucht graph [36][37]. The Frucht graph is a well-known 3-regular graph on 12 vertices that is asymmetric. In particular, $\mathrm{Aut}(G_F[V_1])$ and $\mathrm{Aut}(G_F[V_2])$ are both trivial. Therefore there is no automorphism of G_F mapping any blue vertex to any other blue vertex, nor any red vertex to any other red vertex. That is, V_1 and V_2 cannot be constructed as unions of cells of nontrivial orbit partitions.

5.2 Necessary and Sufficient Conditions for the Quotient Matrix of an Equitable Partition

In this section, we present necessary and sufficient conditions for a matrix to be the quotient matrix of some equitable partition. Clearly any such matrix must be square, and have non-negative entries. Moreover, for any $i \neq j$, if a vertex in V_i has neighbors in V_j then clearly the vertices of V_j must have neighbors in V_i as well. Therefore for any $i \neq j$,

$$c_{i,j} = 0 \text{ if and only if } c_{j,i} = 0. \tag{5.3}$$

We will say that any matrix satisfying (5.3) has ***symmetric zeros***. Before presenting the main results, we first introduce some notation.

For any column vector $\boldsymbol{b}$ let $\Delta(\boldsymbol{b})$ be the diagonal matrix with (i,i)-entry b_i. Let A be any $k \times k$ matrix with (i,j) entry $a_{i,j}$. For each $1 \leq i \leq k-1$ let $\boldsymbol{a}_i$ be the $1 \times (k-i)$ row vector

$$\boldsymbol{a}_i = \begin{bmatrix} a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,k-1} & a_{i,k} \end{bmatrix}.$$

Similarly, for each $1 \leq j \leq k-1$ let $\boldsymbol{b}_j$ be the $(k-j) \times 1$ column vector

$$\boldsymbol{b}_j = \begin{bmatrix} a_{j+1,j} & a_{j+2,j} & \cdots & a_{k-1,j} & a_{k,j} \end{bmatrix}^t.$$

So, for example, if A is 4×4, then A has the form:

$$\begin{bmatrix} a_{1,1} & \cdots & \boldsymbol{a}_1 & \cdots \\ \vdots & a_{2,2} & \boldsymbol{a}_2 & \cdots \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & a_{3,3} & \boldsymbol{a}_3 \\ \vdots & \vdots & \boldsymbol{b}_3 & a_{4,4} \end{bmatrix}$$

Now let r be a function on $\{0, \ldots, k-1\}$ defined by

$$r(i) = \begin{cases} 0 & \text{if } i = 0 \\ \sum_{l=1}^{i}(k-l) & \text{if } 1 \leq i \leq k-1 \end{cases}.$$

Define $R(A)$ to be the $\frac{k(k+1)}{2} \times k$ matrix where for each i between 1 and $k-1$, the $(k-i) \times (k-i+1)$ submatrix found in rows $r(i-1)+1$ through $r(i)$ and columns $k-i+1$ through k is:

$$\left[\begin{array}{c|c} \boldsymbol{a}_i^t & \quad -\Delta(\boldsymbol{b}_i) \quad \end{array}\right].$$

For example, when A is 4×4, $R(A)$ is:

$$\left[\begin{array}{cccc}
\boxed{\begin{array}{c|ccc} a_{1,2} & -a_{2,1} & 0 & 0 \\ a_{1,3} & 0 & -a_{3,1} & 0 \\ a_{1,4} & 0 & 0 & -a_{4,1} \end{array}} \\
\begin{array}{c} 0 \\ 0 \end{array} \boxed{\begin{array}{c|cc} a_{2,3} & -a_{3,2} & 0 \\ a_{2,4} & 0 & -a_{4,2} \end{array}} \\
\begin{array}{cc} 0 & 0 \end{array} \boxed{\begin{array}{c|c} a_{3,4} & -a_{4,3} \end{array}}
\end{array}\right]$$

While the formal definition of matrix $R(A)$ is rather dense, the solutions of the matrix equation $R(A)\boldsymbol{x} = \boldsymbol{0}$ can be described simply and transparently. In particular, given a column vector $\boldsymbol{x} = \left[\begin{array}{ccc} x_1 & \dots & x_k \end{array}\right]^t$, the rows of the matrix product $R(A)\boldsymbol{x}$ are

$$\left\{ x_i\left(a_{i,j}\right) - x_j\left(a_{j,i}\right) : 1 \le i, j \le k \text{ and } i \ne j \right\}. \tag{5.4}$$

Therefore,

Lemma 5.2. *The vector $\boldsymbol{x}$ is a solution to the matrix equation $R(A)\boldsymbol{x} = \boldsymbol{0}$ if and only if for all $i \neq j$,*

$$x_i a_{i,j} - x_j a_{j,i} = 0 \quad \textit{that is,} \quad x_i a_{i,j} = x_j a_{j,i} \tag{5.5}$$

Matrix $R(A)$ now provides the necessary and sufficient conditions for a matrix A to be the quotient matrix of an equitable partition.

Theorem 5.1. *Let A be a $k \times k$ matrix with non-negative integer entries. Then there exists a graph G with equitable partition π such that $A = A(G/\pi)$ if and only if $R(A)\boldsymbol{x} = \boldsymbol{0}$ has a positive integer solution $\boldsymbol{x}$.*

Remark 5.1. Note that although we know that any quotient matrix *must* have symmetric zeros, this property is not included in the assumptions of Theorem 5.1. $R(A)\boldsymbol{x} = \mathbf{0}$ has a positive integer solution $\boldsymbol{x}$ if and only if for $i \neq j$,

$$x_i a_{i,j} = x_j a_{j,i}.$$

As all x_i, x_j are positive, clearly $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

Before presenting the proof of Theorem 5.1, we first present a few intermediate results. Recall that a graph is *κ-regular* if all vertices have degree κ.

Proposition 5.4. There exists a κ-regular graph on n vertices if and only if $n > \kappa$ and $n\kappa$ is even.

Proof. ($\Rightarrow$) As every vertex has degree κ, $n - 1 \geq \kappa$ and therefore $n > \kappa$. Moreover by the handshaking lemma,

$$n\kappa = \sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

so $n\kappa$ is even.

($\Leftarrow$) Suppose that $n > \kappa$ and $n\kappa$ is even. Let G be a graph with n vertices, $v_1, \ldots, v_n$.

> **Case 1.** If κ is even, for each i add an edge from v_i to v_j for each
>
> $$j \in \{i - \kappa/2, \ldots, i - 1, i + 1, \ldots, i + \kappa/2\} \pmod{n}.$$
>
> Observe that if $j \in \{i - \kappa/2, \ldots, i - 1, i + 1, \ldots, i + \kappa/2\} \pmod{n}$, then necessarily $i \in \{j - \kappa/2, \ldots, j - 1, j + 1, \ldots, j + \kappa/2\} \pmod{n}$ so this edge relation is symmetric. As $n > \kappa$, each vertex is assigned κ unique neighbors, meaning G is a κ-regular graph.
>
> **Case 2.** Otherwise if κ is odd, then n must be even. For each i add an edge from v_i to v_j for each
>
> $$j \in \{i - (\kappa - 1)/2, \ldots, i - 1, i + 1, \ldots, i + (\kappa - 1)/2\} \cup \{i + n/2\} \pmod{n}.$$
>
> Observe that if $j \in \{i - \kappa/2, \ldots, i - 1, i + 1, \ldots, i + \kappa/2\} \cup \{i + n/2\} \pmod{n}$, then necessarily $i \in \{j - (\kappa - 1)/2, \ldots, j - 1, j + 1, \ldots, j + (\kappa - 1)/2\} \cup \{j + n/2\} \pmod{n}$ so this edge relation is symmetric. As $n > \kappa$, each vertex is assigned κ unique neighbors, meaning G is a κ-regular graph.

The cases of $n = 7, \kappa = 4$ and $n = 8, \kappa = 5$ are illustrated in Figure 5.2.1. $\square$

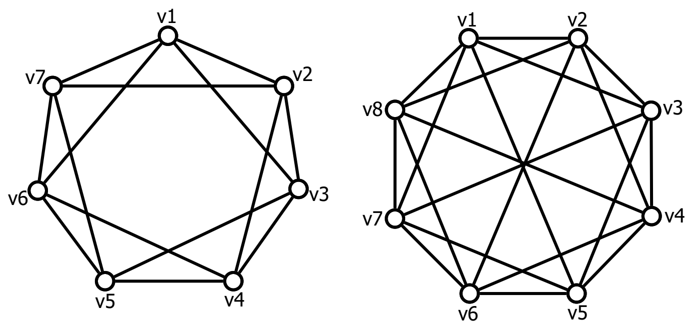


Figure 5.2.1: Two k-regular graphs on n vertices. The first, when $n = 7$ and $k = 4$, the second when $n = 8$ and $k = 5$.

On the other hand, recall that a simple graph G is *bipartite* if $V(G)$ can be partitioned into two components, U and W, such that all edges of G have one end in U and one end in W. A graph is called *biregular* if it is bipartite, and all vertices of U have the same degree, as do all vertices of W. If $a = \deg(u)$ for all $u \in U$ and $b = \deg(w)$ for all $w \in W$, we say that G is (a, b)-*biregular.*

Lemma 5.3. *There exists an (a, b)-biregular with vertex bipartition (U, W) if and only if $|W| \geq a$, and*

$$a|U| = b|W|.$$

Proof. ($\Rightarrow$) As every vertex in U has a neighbors in W, necessarily $|W| \geq a$. By direct computation, $a|U| = |E(G)| = b|W|$.

($\Leftarrow$) Given a and b, let m and n be any two integers such that $n \geq a$ and $am = bn$. Let U be a set of m vertices $\{u_1, \dots, u_m\}$ and let W be a set of n vertices $\{w_1, \dots, w_n\}$. Now let G be a graph with vertex set $V(G) = U \cup W$. We will iteratively add edges to G to construct an (a, b)-biregular graph with vertex bipartition (U, W). Observe that in each of the following steps, distinctness of the chosen vertices in W is guaranteed by $n \geq a$.

Step 1. Add edges between u_1 and a distinct vertices in W.

Step 2. Add edges between u_2 and a distinct vertices of W, beginning with vertices of degree 0 and moving on to vertices of degree 1 if necessary.

$\vdots$

Step p. Continuing in this manner, at Step p add edges between u_p and a distinct vertices of W. First add these edges to vertices of W with minimum degree after Step (p -1), and continue on to vertices of degree 1 greater if necessary.

It is easy to verify that after Step p, G contains $p \cdot a$ edges. U contains p vertices with degree a, and $(m-p)$ vertices of degree 0. On the other hand, if

$$ap = qn + r \quad (0 \leq r < n)$$

then all vertices of W have at least q neighbors in U, with r of them having exactly $q+1$. That is, W contains $(n-r)$ vertices of degree q and r vertices of degree $q+1$.

Step m. Following step $(m-1)$, G contains $a(m-1)$ edges and every vertex of U has degree a, other than u_m. Moreover as $am = bn$,

$$a(m-1) = (b-1)n + (n-a)$$

Therefore W has a vertices of degree $(b-1)$ and $(n-a)$ vertices of degree b. Add an edge from u_m to each vertex of W having degree $(b-1)$.

After step m, G contains $a \cdot m$ edges, and every vertex of U has degree a. Moreover,

$$am = bn + 0.$$

Therefore W has n vertices of degree b and 0 vertices of degree $b+1$. That is, G is an (a,b)-biregular graph with vertex partition (U,W). □

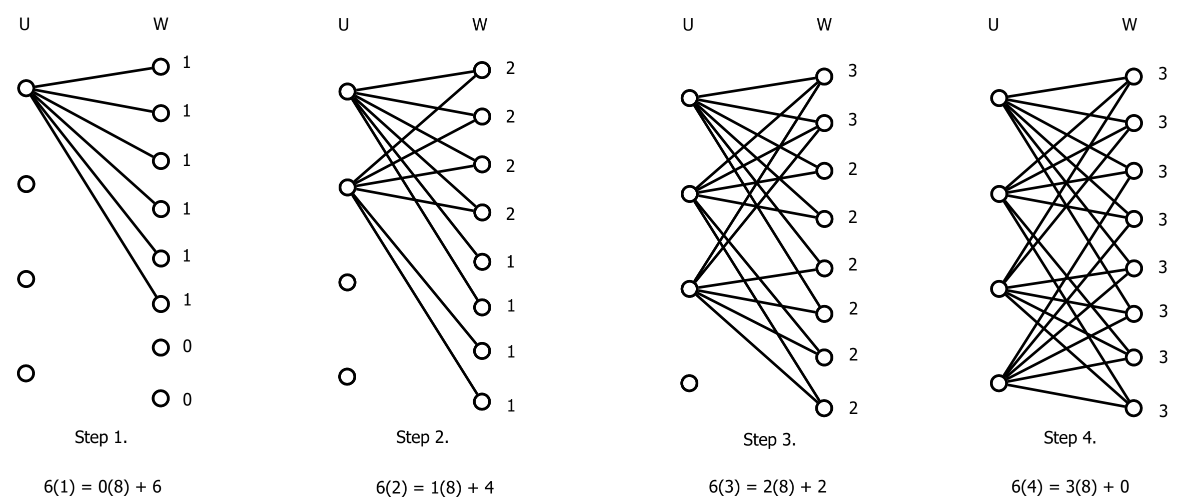


Figure 5.2.2: Constructing a $(6,3)$-biregular graph with $|U| = 4$ and $|W| = 8$

For example, the case of constructing a $(6,3)$-biregular graph with $|U| = 4$ and $|W| = 8$ is presented in 5.2.2. The degree of each vertex of W is displayed, and the equation $ap = q|W| + r$

is given beneath each step. One may check that at Step p, W contains $|W| - r$ vertices of degree q and r vertices of degree $q + 1$.

We are now prepared to present the proof of Theorem 5.1.

Proof. ($\Rightarrow$). Let G be a graph with equitable partition $\pi = (V_1, \ldots, V_k)$. Then clearly $A(G/\pi)$ is a $k \times k$ matrix with non-negative integer entries $c_{i,j}$, and symmetric zeros. By definition of equitability, for each i the induced subgraph $G[V_i]$ must be a $c_{i,i}$ regular graph. Moreover, for any $i \neq j$, the graph induced by taking the vertices of V_i and V_j and all $V_i - V_j$ edges must be a $(c_{i,j}, c_{j,i})$-biregular graph. Therefore by Lemma 5.3, for all $i \neq j$,

$$c_{i,j}|V_i| = c_{j,i}|V_j|.$$

That is, for all $i \neq j$,

$$c_{i,j}|V_i| - cj, i|V_j| = 0.$$

Thus by (5.5), the column vector

$$\boldsymbol{x}_v = \begin{bmatrix} |V_1| & \cdots & |V_k| \end{bmatrix}^t$$

is a solution to the equation

$$R(A(G/\pi))\boldsymbol{x}_v = \mathbf{0}.$$

As the entries of $\boldsymbol{x}_v$ are $\{|V_i|\}$, which are positive integers, the result follows.

($\Leftarrow$). Conversely, let A be any $k \times k$ matrix with non-negative integer entries $a_{i,j}$. Suppose $\boldsymbol{x}$ is a vector with positive integer entries solving $R(A)\boldsymbol{x} = \mathbf{0}$. As any scalar multiple of $\boldsymbol{x}$ remains in the null space of $R(A)$, if necessary scale $\boldsymbol{x}$ by a positive integer so that for each j,

$$\begin{cases} x_j > a_{i,j} & \text{for all } i \in \{1, \ldots, k\} \\ x_j a_{j,j} & \text{is even} \end{cases} \tag{5.6}$$

In particular, $R(A)\boldsymbol{x} = \mathbf{0}$ so by (5.5), for all $1 \leq i, j \leq k$ such that $i \neq j$,

$$x_i(a_{i,j}) = x_j(a_{j,i}). \tag{5.7}$$

Now for each i, let V_i be a set of x_i vertices, and let G be a graph with vertex set $V(G) = V_1 \cup \cdots \cup V_k$. Add edges to G so that for each i, the induced subgraph $G[V_i]$ is $a_{i,i}$-regular. This is guaranteed by Proposition 5.4 as $|V_i|a_{i,i}$ is even and $|V_i| > a_{i,i}$ by (5.6). For each $i \neq j$, add edges so that the edges between V_i and V_j form an $(a_{i,j}, a_{j,i})$-biregular graph. This is guaranteed by (5.6), (5.7), and Lemma 5.3. By construction, the resulting graph G has an equitable partition $\pi = (V_1, \ldots, V_k)$ with quotient matrix $A(G/\pi) = A$. $\square$

Remark 5.2. Note that the sufficiency conditions for existence of regular and biregular graphs contain constraints on the number of vertices. Specifically, a k-regular graph must have at least $k+1$ vertices, and an (a,b)-biregular graph must have at least a vertices in one vertex bipartition, and at least b in the other. Although Theorem 5.1 constructs an equitable partition by assembling regular and biregular graphs, no such constraints are included: it is sufficient to prove only that $R(A)\boldsymbol{x} = \mathbf{0}$ has a positive integer solution. As seen in the proof, any such positive integer solution can always be scaled to find another solution satisfying all the constraints required for regular and biregular graphs. Therefore in what follows we will only be concerned with finding some positive integer solution to $R(A)\boldsymbol{x} = \mathbf{0}$.

Example 5.4. Consider the matrix A in (5.8). A is a square matrix with nonnegative integer entries. Although A is not symmetric, A has symmetric zeros.

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 & 2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 3 \\ 0 & 0 & 3 & 2 & 0 & 3 \\ 1 & 0 & 0 & 4 & 2 & 0 \end{bmatrix} \tag{5.8}$$

We will use Theorem 5.1 to show that there exists a graph G with equitable partition π such that $A = A(G/\pi)$. In particular, $R(A)$ is a 15×5 matrix, as shown in (5.9).

$$R(A) = \begin{bmatrix} 2 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 & 3 & -2 \end{bmatrix} \tag{5.9}$$

The matrix equation $R(A)\boldsymbol{x} = \mathbf{0}$ has a positive integer solution $\boldsymbol{x}$, where

$$\boldsymbol{x}^t = \begin{bmatrix} 3 & 2 & 3 & 4 & 2 & 3 \end{bmatrix}. \tag{5.10}$$

Note that for a general integer-valued matrix B, finding a positive integer solution to the matrix equation $B\boldsymbol{x} = \mathbf{0}$ may be a computationally difficult problem (this is a case of integer linear programming). In the special case of $R(A)\boldsymbol{x} = \mathbf{0}$, however, Section 5.2.2 will demonstrate an efficient method that either finds a solution $\boldsymbol{x}$ or determines that none exist.

Given that the vector $\boldsymbol{x}$ in 5.10 is a positive integer solution to $R(A)\boldsymbol{x} = \mathbf{0}$, by Theorem 5.1 it follows that A is the quotient matrix of some equitable partition with 6 classes. Moreover, observe that for all j,

$$\begin{cases} x_j > a_{i,j} & \text{for all } i \in \{1, \dots, k\} \\ x_j a_{j,j} & \text{is even} \end{cases}$$

Therefore no scaling of $\boldsymbol{x}$ is needed, and this equitable partition can be achieved on a graph with $3 + 2 + 3 + 4 + 2 + 3 = 17$ vertices. In this case, $|V_i| = x_i$ for each $1 \leq i \leq 6$,

$$|V_1| = 3 \quad |V_2| = 2 \quad |V_3| = 3 \quad |V_4| = 4 \quad |V_5| = 2 \quad |V_6| = 3$$

One such equitable partition is illustrated in Figure 5.2.3.

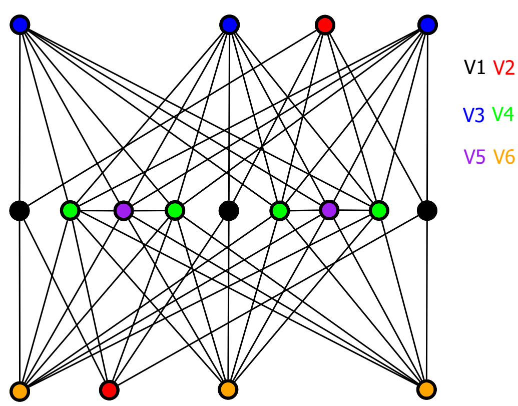


Figure 5.2.3: An equitable partition having (5.8) as its quotient matrix.

5.2.1 Partitions, Quotients, and Matrices

Theorem 5.1 provided necessary and sufficient conditions for an integer matrix to be the quotient matrix of an equitable partition. These conditions easily extend to determining whether a digraph is the quotient of an equitable partition.

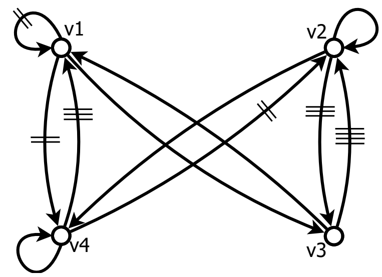


Figure 5.2.4: A multi-digraph D. Hash marks are used to indicate multiplicity of edges having the same initial and terminal vertices. Corollary 5.1 will be used to determine if D is the quotient of an equitable partition.

Let D be any digraph, which may have loops or multiple edges. Let $A = A(D)$ be the adjacency matrix of digraph D. That is, A is a $|V(D)| \times |V(D)|$ matrix whose (i, j)-entry is the number of edges directed from vertex v_i to vertex v_j.

Corollary 5.1. *D is the quotient of some equitable partition if and only if $R(A)\boldsymbol{x} = \mathbf{0}$ has a positive integer solution.*

Example 5.5. Consider the multi-digraph D in Figure 5.2.4. D has 4 vertices, meaning *if* D is the quotient of an equitable partition, that partition must have 4 classes. D has adjacency matrix

$$A(D) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 4 & 0 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \end{array}.$$

Then by Corollary 5.1, D is the quotient of an equitable partition if and only if $R(A)\boldsymbol{x} = \mathbf{0}$ has a positive integer solution $\boldsymbol{x}$. One may verify that

$$\boldsymbol{x}^t = \begin{bmatrix} 3 & 4 & 3 & 2 \end{bmatrix}$$

is one such solution. Moreover, for all j,

$$\begin{cases} x_j > a_{i,j} & \text{for all } i \in \{1, \dots, k\} \\ x_j a_{j,j} & \text{is even} \end{cases}$$

Therefore there exists a graph G with equitable partition $\pi = (V_1, V_2, V_3, V_4)$ such that

$|V_i| = x_i$. Moreover, $A(G/\pi) = A$, and therefore quotient G/π is precisely the digraph D. One example of such a G and π is illustrated in Figure 5.2.5.

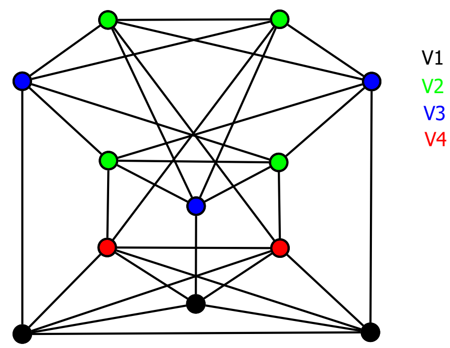


Figure 5.2.5: An equitable partition with quotient D.

To this point, we have presented a few examples of constructing an equitable partition with a given quotient matrix. In each case, we first find a solution $\boldsymbol{x}$ to $R(A)\boldsymbol{x} = \mathbf{0}$, and then scale this solution to determine the size of each vertex class. That is, we determine a vector $\boldsymbol{x}$ satisfying

$$R(A)\boldsymbol{x} = \mathbf{0} \quad \text{and} \quad \begin{cases} x_j > a_{i,j} & \text{for all } i \in \{1, \dots, k\} \\ x_j a_{j,j} & \text{is even.} \end{cases} \tag{5.11}$$

However, if $\boldsymbol{x}$ satisfies (5.11), then so does the vector $\beta\boldsymbol{x}$ for any $\beta \in \mathbb{Z}_+$. Therefore for any $\beta \in \mathbb{Z}_+$, there exists an equitable partition, having quotient matrix A, with βx_i vertices in class V_i. In particular, let G be a graph with equitable partition $\pi = (V_1, \dots, V_k)$ and quotient matrix A. Then the vector

$$\boldsymbol{x}^t = \begin{bmatrix} |V_1| & \dots & |V_k| \end{bmatrix}$$

satisfies (5.11), as does $\beta\boldsymbol{x}$ for any $\beta \in \mathbb{Z}_+$. Thus for every $\beta \in \mathbb{Z}_+$ there exists a graph G_β on $\beta|V(G)|$ vertices, with equitable vertex partition π_β such that

$$A\big(G/\pi\big) = A\big(G_\beta/\pi_\beta\big) \text{ for all } \beta \in \mathbb{Z}_+,$$

and the i^{th} partition class of π_β has cardinality $\beta|V_i|$. Thus we have proven,

Theorem 5.2. *Any equitable partition π lies in an infinite family of equitable partitions, all having the same quotient matrix. That is, if A is the quotient matrix of an equitable partition, it is the quotient matrix of an infinite family of graphs with equitable partitions.*

Within these infinite families, the equitable partition on a graph of a given size graph need not be unique. Moreover, orbit partitions and non-orbit partitions can belong to the same infinite family. This will be illustrated in the following examples.

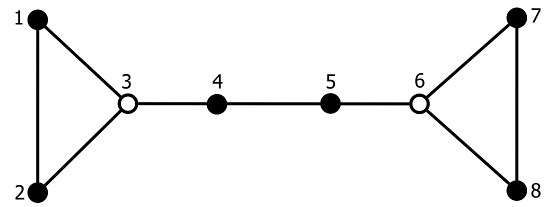


Figure 5.2.6: An equitable partition of the McKay graph.

Recall the McKay graph, G_M, shown in Figure 5.2.6. As labeled, G_M has equitable partition $\pi = \big(\{v_1, v_2, v_4, v_5, v_7, v_8\}, \{v_3, v_6\}\big)$ and quotient matrix

$$A_M = A(G_M/\pi) = \begin{array}{c} \\ V_1 \\ V_2 \end{array} \begin{array}{c} \begin{array}{cc} V_1 & V_2 \end{array} \\ \left[\begin{array}{cc} 1 & 1 \\ 3 & 0 \end{array}\right] \end{array}.$$

By Theorem 5.2, A_M is the quotient matrix of an infinite family of graphs with equitable partitions.

Example 5.6. Within this infinite family, there may be more than one graph of a given size. For example, the McKay graph has 8 vertices and an equitable partition with $|V_1| = 6$ and $|V_2| = 2$. Figure 5.2.7 illustrates another 8-vertex graph, with vertex partition satisfying $|V_1| = 6$ and $|V_2| = 2$. This partition is an *orbit* partition, and therefore equitable. Moreover, A_M is the quotient matrix of this partition.

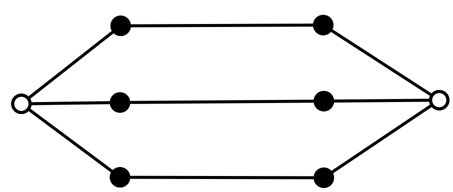


Figure 5.2.7: An *orbit* partition with $|V_1| = 6$ and $|V_2| = 2$, and quotient matrix A_M.

That is, there are *two* 8-vertex graphs, each having an equitable partition with $|V_1| = 6$ and $|V_2| = 2$, with quotient matrix A_M. Interestingly, only one of these partitions is an orbit partition.

Example 5.7. G_M has an equitable partition with $|V_1| = 6$ and $|V_2| = 2$, so

$$\boldsymbol{x}^t = \left[\ |V_1| \quad |V_2|\ \right] = \left[\ 6 \quad 2\ \right]$$

is a solution to $R(A_M)\boldsymbol{x} = \mathbf{0}$ satisfying (5.11). Therefore for any positive integer N, the vector $N\boldsymbol{x}$ satisfies (5.11) as well. For example, $2\boldsymbol{x} = \left[\ 12 \quad 4\ \right]$ satisfies (5.11), and there exists a graph G' with equitable partition $\pi' = (V_1', V_2')$ where

$$|V_1'| = 12 \quad |V_2'| = 4 \qquad \text{and } A(G'/\pi') = A_M.$$

In fact, there are *many* such graphs. Two examples are presented in Figure 5.2.8. Each graph has 16 vertices, partitioned into 2 classes of sizes 12 and 4 respectively. One may verify that each partition is equitable, and has quotient matrix A_M.

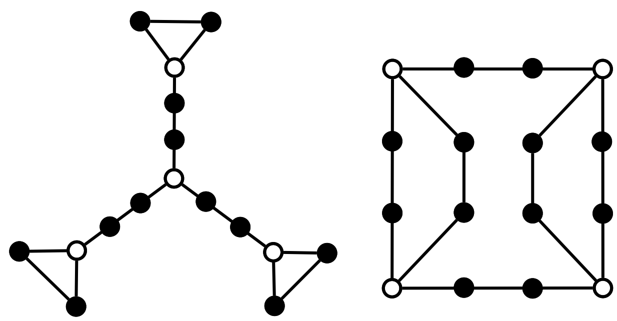


Figure 5.2.8: Two equitable partitions having $|V_1| = 12$, $|V_2| = 4$, and quotient matrix A_M. Neither is an orbit partition.

Neither of the partitions in Figure 5.2.8 is an orbit partition, though as suggested by the drawings, each of these graphs have a number of automorphisms. This need not be the case: Figure 5.2.9 illustrates a graph with very few nontrivial automorphisms, all of which are permutations of order 2. However, much like the previous examples, this graph has 16 vertices and an equitable partition with quotient matrix A_M.

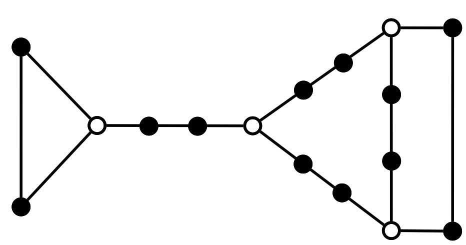


Figure 5.2.9: Another equitable partition with quotient A_M.

There are a number of other 16-vertex graphs having A_M as the quotient matrix of an equitable partition. If one considers disconnected graphs, 2 disjoint copies of the McKay graph, the disjoint union of the McKay graph and the graph in Figure 5.2.7, and 2 disjoint copies of the graph in Figure 5.2.7 are all examples. Of course the last of these examples is an orbit partition, however, an orbit partition of a 16-vertex graph with quotient matrix A_M need not be disconnected. Figure 5.2.10 illustrates a connected graph and an *orbit* partition, with quotient matrix A_M, satisfying $|V_1| = 12$ and $|V_2| = 4$.

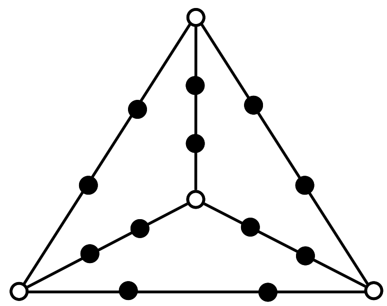


Figure 5.2.10: An orbit partition with $|V_1| = 12$, $|V_2| = 4$, and quotient matrix A_M.

Finally, Figure 5.2.11 illustrates two more members of the infinite family of equitable partitions having quotient matrix A_M. The first satisfies $|V_1| = 18$ and $|V_2| = 6$, and the second $|V_1| = 24$ and $|V_2| = 8$. More generally, members of this family can be derived as follows. First construct a 3-regular graph on $2n$ vertices, all in class V_2. Subdivide every edge with two new vertices, all of which belong to class V_1. The result is an equitable partition having quotient matrix A_M.

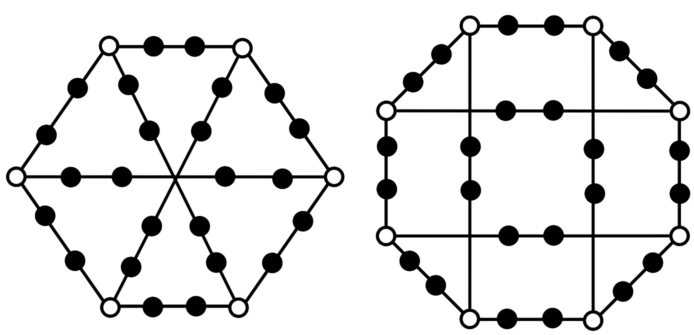


Figure 5.2.11: Two additional members of the infinite family of equitable partitions with quotient matrix A_M.

5.2.2 A graphical algorithm for solving $R(A)\boldsymbol{x} = \mathbf{0}$

In this section, we present a graph-based algorithm for finding positive integer solutions to $R(A)\boldsymbol{x} = \mathbf{0}$. As before, let A be a $k \times k$ matrix with non-negative integer entries and symmetric zeros. Associate with A a simple graph H with k vertices, $v_1, \ldots, v_k$, and let $\{v_i, v_j\} \in E(H)$ if and only if matrix entry $a_{i,j}$ (and thus also $a_{j,i}$) is nonzero. For any path $P_0 = v_i - v_j \cdots v_k - v_l$, let $\omega(P_0)$ be the product

$$\omega(P_0) = a_{i,j} \cdot a_{j,k} \cdots \cdot a_{k,l}.$$

Now let $C = v_i - v_j - v_k \cdots v_l - v_i$ be any cycle of H, and let $\tilde{C} = v_i - v_l \cdots v_k - v_j - v_i$ be the same cycle traversed in the opposite direction. We say that C is A-*invariant* if

$$\omega(C) = \omega(\tilde{C}).$$

That is, C is A-invariant if and only if

$$a_{i,j} \cdot a_{j,k} \cdots a_{l,i} = a_{i,l} \cdots a_{k,j} \cdot a_{j,i} \quad \text{or equivalently,} \quad \frac{a_{i,j}}{a_{j,i}} \cdot \frac{a_{j,k}}{a_{k,j}} \cdots \frac{a_{l,i}}{a_{i,l}} = 1.$$

Example 5.8. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}. \tag{5.12}$$

The associated graph H is shown in Figure 5.2.12. Let C and $\tilde{C}$ be the cycles $C = v_1 - v_4 - v_3 - v_1$ and $\tilde{C} = v_1 - v_3 - v_4 - v_1$. Then

$$\frac{\omega(C)}{\omega(\tilde{C})} = \frac{a_{1,4} \cdot a_{4,3} \cdot a_{3,1}}{a_{1,3} \cdot a_{3,4} \cdot a_{4,1}} = \frac{2 \cdot 1 \cdot 1}{1 \cdot 2 \cdot 1} = 1.$$

Therefore cycle C is A-invariant.

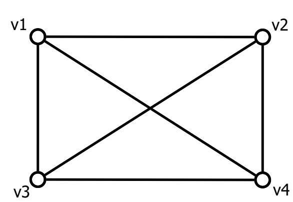


Figure 5.2.12: The simple graph H corresponding to A in (5.12). All cylces of H are A-invariant.

Theorem 5.3 will demonstrate that finding a positive integer solution to $R(A)\boldsymbol{x} = \mathbf{0}$ is equivalent to checking that all cycles of H are A-invariant. One direction of this proof is simple, while the other is constructive. In order to simplify the proof of Theorem 5.3, we first define a condition on H that is equivalent to all cycles being A-invariant. Suppose that H is connected, and let T be a spanning tree of H rooted at vertex v_1. For example, T could be constructed using depth-first-search. Note that if H is not connected, the following may be carried out independently on each connected component of H.

Recall that any two vertices in a tree are connected by a unique path. Let v_iTv_j denote the unique path from vertex v_i to vertex v_j in T. In particular, each vertex has a unique path to vertex v_1. This induces a partial ordering on $V(H)$. Namely for any $i \neq j$, we say that $v_i \preceq_T v_j$ if vertex v_i lies on the path v_1Tv_j. Under partial ordering $\preceq_T$, vertex v_1 is the unique minimal element, and all leaves of T are maximal elements. Moreover, any pair of vertices that are adjacent in T are necessarily comparable under $\preceq_T$. We will use partial ordering $\preceq_T$ to assign weights to all edges of T. For any edge $e = \{v_i, v_j\} \in E(T)$, suppose without loss of generality that $v_i \preceq_T v_j$. Define

$$w(e) = \frac{a_{i,j}}{a_{j,i}}.$$

Otherwise, for all edges $e = \{v_i, v_j\} \in E(H)\backslash E(T)$, suppose without loss of generality that $i < j$. Define

$$w(e) = \frac{a_{i,j}}{a_{j,i}}.$$

As such, every edge of H has now been assigned a weight. For any path $P_0 = uTv$ in spanning tree T, define the weight of the path to be the product of the weight of its edges. That is,

$$w(P_0) = \prod_{e \in E(P_0)} w(e).$$

Next consider the fundamental cycles of H with respect to T. T is a tree on k vertices and $k-1$ edges. Therefore there are $|E(H)| - (k-1)$ edges of H not contained in T. For each edge $e \in E(H)\backslash E(T)$, the graph $T + e$ contains *exactly* one cycle. Recall this cycle is called the *fundamental cycle* of e with respect to T, which we will denote by $C(e)$. Let $e = \{v_i, v_j\}$ be any edge of $E(H)\backslash E(T)$ and suppose, without loss of generality, that $i < j$.

Proposition 5.5. Fundamental cycle $C(e)$ contains a unique minimal vertex with respect to $\preceq_T$.

Proof. If $v_i = v_1$, then the result follows as v_1 is the unique minimal element with respect to $\preceq_T$. Otherwise, both v_1Tv_i and v_1Tv_j are paths on at least two vertices. Beginning at v_i and v_j, move down the paths v_iTv_1 and v_jTv_1. These two paths must eventually meet at a vertex (at the very least, at vertex v_1). Let v_e denote the first vertex at which these paths meet. Then fundamental cycle $C(e)$ is of the form

$$C(e) = (v_e T v_i) \cup (\{v_i, v_j\}) \cup (v_j T v_e).$$

As by definition of $\preceq_T$, $v_e \preceq_T v$ for all $v \in V(v_e T v_i)$ and $v \in V(v_e T v_i)$, the result follows. $\square$

We say that $C(e)$ is *A-resolvable* if

$$w(v_e T v_i) \cdot w(e) = w(v_e T v_j),$$

or equivalently,

$$w(e) = \frac{w(v_e T v_j)}{w(v_e T v_i)}.$$

Recalling that an empty product is defined to be 1, note that if $v_i = v_1$, then $w(v_e T v_i) = 1$ and this simply says $w(e) = w(v_1 T v_j)$.

Definition 5.1. Spanning tree T is A-***resolvable*** if for all $e \in E(H) \backslash E(T)$ the fundamental cycle $C(e)$ is A-resolvable.

This notion is best illustrated with an example.

Example 5.9. Let A be the 4×4 matrix given in (5.12),

$$A = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix},$$

and let H be the simple graph presented in Figure 5.2.12. Let T be the spanning tree, rooted at v_1, as shown in Figure 5.2.13

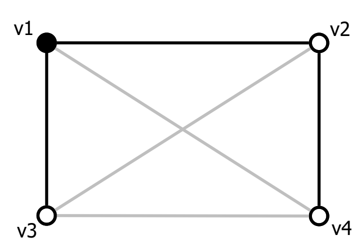


Figure 5.2.13: A spanning tree T, for graph H, rooted at vertex v_1.

Spanning tree T induces a partial ordering on $V(H)$, where:

$$v_1 \preceq_T v_2 \preceq_T v_4 \quad \text{and} \quad v_1 \preceq_T v_3.$$

We assign weights to the edges of T according to this partial ordering. Namely,

$$w(\{v_1, v_2\}) = \frac{a_{1,2}}{a_{2,1}} = 2 \quad w(\{v_2, v_4\}) = \frac{a_{2,4}}{a_{4,2}} = 1 \quad w(\{v_1, v_3\}) = \frac{a_{1,3}}{a_{3,1}} = 1.$$

Weights are assigned to the edges of $E(H)\backslash E(T)$ by

$$w(\{v_3, v_4\}) = \frac{a_{3,4}}{a_{4,3}} = 2 \quad w(\{v_2, v_3\}) = \frac{a_{2,3}}{a_{3,2}} = \frac{1}{2} \quad w(\{v_1, v_4\}) = \frac{a_{1,4}}{a_{4,1}} = 2.$$

These weights are summarized in Figure 5.2.14. In order to demonstrate that T is A-resolvable, we check that each fundamental cycle is A-resolvable.

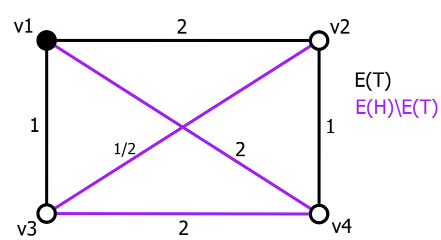


Figure 5.2.14: The edge weights of $E(T)$ and $E(H)\backslash E(T)$.

Beginning with $e = \{v_3, v_4\}$, observe that $v_e = v_1$. Then

$$v_eTv_3 = v_1 - v_3 \quad \text{and} \quad v_eTv_4 = v_1 - v_2 - v_4,$$

meaning

$$w(v_eTv_3) = 1 \quad \text{and} \quad w(v_eTv_4) = 2 \cdot 1 = 2.$$

Therefore,

$$w(v_eTv_3)w(e) = 2 = w(v_eTv_4)$$

so $C(e)$ is A-resolvable. This is illustrated in Figure 5.2.15.

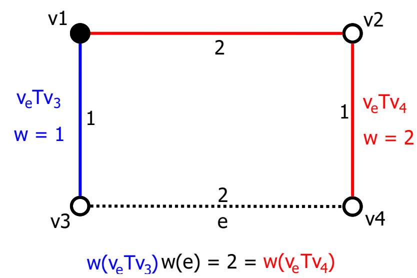


Figure 5.2.15: Fundamental cycle $C(\{v_3, v_4\})$ is A-resolvable.

Next consider $e = \{v_2, v_3\}$, where $v_e = v_1$. Then

$$v_eTv_2 = v_1 - v_2 \quad \text{and} \quad v_eTv_3 = v_1 - v_3,$$

meaning

$$w(v_eTv_2) = 2 \quad \text{and} \quad w(v_eTv_3) = 1.$$

Therefore

$$w(v_eTv_2)w(e) = 2 \cdot \frac{1}{2} = 1 = w(v_eTv_3)$$

so $C(e)$ is A-resolvable. This is illustrated in Figure 5.2.16.

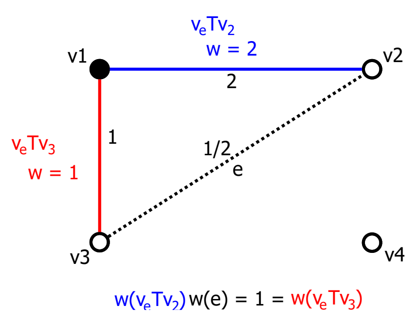


Figure 5.2.16: Fundamental cycle $C(\{v_2, v_3\})$ is A-resolvable.

Finally, consider $e = \{v_1, v_4\}$, where $v_e = v_1$. As one endpoint of e is v_1, we need only verify that $w(e) = w(v_1Tv_4)$. Observe that

$$w(v_1Tv_4) = 2 \cdot 1 = 2 = w(e)$$

and so $C(e)$ is A-resolvable. As as all fundamental cycles $C(e)$ are A-resolvable, it follows that spanning tree T is A-resolvable.

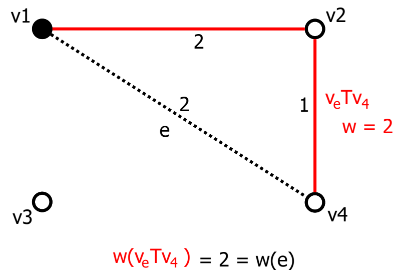


Figure 5.2.17: Fundamental cycle $C(\{v_1, v_4\})$ is A-resolvable.

A-resolvability of a spanning tree is closely related to A-invariance of cycles. Let A, H, and T be as before.

Lemma 5.4. *Spanning tree T is A-resolvable if and only if all cycles of H are A-invariant.*

Proof. First, suppose that T is A-resolvable. To show that all cycles of H are A-invariant, it suffices to check that a set of *fundamental* cycles are A-invariant. In particular, we will show the fundamental cycles of H with respect to T are all A-invariant. Let $e = \{v_i, v_j\}$ be any edge of $E(H)\backslash E(T)$, and suppose, without loss of generality, that $i < j$. Moreover, suppose that $v_e = v_N$ for some index N, and

$$\begin{aligned} v_N T v_i &= v_N - v_{i_1} - v_{i_2} - \cdots - v_{i_q} - v_i \\ v_N T v_j &= v_N - v_{j_1} - v_{j_2} - \cdots - v_{j_r} - v_j. \end{aligned} \tag{5.13}$$

Then to show that $C(e)$ is A-invariant, we must show that

$$\left(\frac{a_{N,i_1}}{a_{i_1,N}} \cdot \frac{a_{i_1,i_2}}{a_{i_2,i_1}} \cdots \frac{a_{i_q,i}}{a_{i_q,i}}\right) \left(\frac{a_{i,j}}{a_{j,i}}\right) \left(\frac{a_{j,j_r}}{a_{j_r,j}} \cdots \frac{a_{j_2,j_1}}{a_{j_1,j_2}} \cdot \frac{a_{j_1,N}}{a_{N,j_1}}\right) = 1. \tag{5.14}$$

However, because T is A-resolvable,

$$w(v_N T v_i) \cdot w(e) = w(v_N T v_j) \quad \text{that is,} \quad w(v_N T v_i) \cdot w(e) \frac{1}{w(v_N T v_j)} = 1. \tag{5.15}$$

But

$$w(v_N T v_i) = \frac{a_{N,i_1}}{a_{i_1,N}} \cdot \frac{a_{i_1,i_2}}{a_{i_2,i_1}} \cdots \frac{a_{i_q,i}}{a_{i_q,i}} \quad \text{and} \quad w(e) = \frac{a_{i,j}}{a_{j,i}}.$$

Moreover,

$$w(v_N T v_j) = \frac{a_{N,j_1}}{a_{j_1,N}} \cdot \frac{a_{j_1,j_2}}{a_{j_2,j_1}} \cdots \frac{a_{j_r,j}}{a_{j,j_r}} \quad \text{so} \quad \frac{1}{w(v_n T v_j)} = \frac{a_{j,j_r}}{a_{j_r,j}} \cdots \frac{a_{j_2,j_1}}{a_{j_1,j_2}} \cdot \frac{a_{j_1,N}}{a_{N,j_1}}.$$

Therefore by (5.15), equation (5.14) holds and so $C(e)$ is A-invariant. That is, every fundamental cycle of H with respect to T is A-invariant and, as a result, all cycles of H are

A-invariant.

Conversely, suppose every cycle of H is A-invariant. Let $e = \{v_i, v_j\}$ be any edge of $E(H)\backslash E(T)$ with $i < j$. Suppose $v_e = v_N$ for some index N, and let $v_N T v_i$ and $v_N T v_j$ be as in (5.13). As $C(e)$ is A-invariant, Equation (5.14) holds. That is,

$$\underbrace{\left(\frac{a_{N,i_1}}{a_{i_1,N}} \cdot \frac{a_{i_1,i_2}}{a_{i_2,i_1}} \cdots \frac{a_{i_q,i}}{a_{i_q,i}}\right)}_{w(v_N T v_i)} \underbrace{\left(\frac{a_{i,j}}{a_{j,i}}\right)}_{w(e)} \underbrace{\left(\frac{a_{j,j_r}}{a_{j_r,j}} \cdots \frac{a_{j_2,j_1}}{a_{j_1,j_2}} \cdot \frac{a_{j_1,N}}{a_{N,j_1}}\right)}_{\frac{1}{w(v_N T v_j)}} = 1.$$

Therefore $w(v_N T v_i) \cdot w(e) = w(v_N T v_j)$, and so $C(e)$ is A-resolvable. Thus, (5.15) holds for all fundamental cycles $C(e)$, and so T is A-resolvable. $\square$

Corollary 5.2. *If a spanning tree T of H is A-resolvable, then every spanning tree of H is A-resolvable.*

Finally, we may present an algorithm to find a positive integer solution to $R(A)\boldsymbol{x} = \mathbf{0}$. Let A be a $k \times k$ matrix with non-negative integer entries and symmetric zeros, and let graph H be as before.

Theorem 5.3. *The equation $R(A)\boldsymbol{x} = \mathbf{0}$ has a positive integer solution $\boldsymbol{x}$ if and only if every cycle of H is A-invariant.*

Proof. First, suppose that there exists a positive $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_k \end{bmatrix}^t$ such that $R(A)\boldsymbol{x} = \mathbf{0}$. In particular, in light of (5.7), for any $i \neq j$,

$$x_i(a_{i,j}) = x_j(a_{j,i}).$$

That is, for any $i \neq j$,

$$\frac{a_{i,j}}{a_{j,i}} = \frac{x_j}{x_i}.$$

Now let $C = v_i - v_{i_1} - v_{i_2} \cdots v_{i_{l-1}} - v_{i_l} - v_i$ be any cycle of H. Then

$$\frac{a_{i,i_1}}{a_{i_1,i}} \cdot \frac{a_{i_1,i_2}}{a_{i_2,i_1}} \cdots \frac{a_{i_{l-1},i_l}}{a_{i_l,i_{l-1}}} \cdot \frac{a_{i_l,i}}{a_{i,i_l}} = \frac{x_{i_1}}{x_i} \cdot \frac{x_{i_2}}{x_{i_1}} \cdot \frac{\cdot}{x_{i_2}} \cdots \frac{x_{i_{l-1}}}{\cdot} \cdot \frac{x_{i_l}}{x_{i_{l-1}}} \cdot \frac{x_i}{x_{i_l}} = 1.$$

Therefore C is A-invariant. As C was an arbitrary cycle in H, all cycles of H are A-invariant.

Conversely, suppose that all cycles of C are A-invariant. Let T be a spanning tree of H rooted at vertex v_1, and assign weights to each edge of H as before. In light of Lemma 5.4, T is A-resolvable. We will use T to find a solution $\boldsymbol{x}$ to the matrix equation $R(A)\boldsymbol{x} = \mathbf{0}$. In particular, it is sufficient to find positive integers $x_1, \ldots, x_k$ such that for all $i \neq j$,

$$x_i(a_{i,j}) = x_j(a_{j,i}).$$

For any two vertices v_i, v_j of T, let $d_T(v_i, v_j)$ be the *distance* in T between v_i and v_j. That is, $d_T(v_i, v_j)$ is the length of the (unique) $v_i - v_j$ path in T. Now let $\mathscr{D}_i$ be the set of vertices at distance i from vertex v_1,

$$\mathscr{D}_i = \{v \in V(H) : d_T(v_1, v) = i\}.$$

Moreover, for any vertex v (other than v_1), let $\alpha_T(v)$ be the *ancestor* of v in T. That is, $\alpha_T(v)$ is the neighbor of vertex v on the path $v_1 T v$. Therefore $\{\alpha_T(v), v\} \in E(T)$, and if $v \in \mathscr{D}_i$ then $\alpha_T(v) \in \mathscr{D}_{i-1}$.

We will now use an iterative process to assign a positive weight W to each vertex of H. Recall that each edge e of H was assigned a rational-valued weight, $w(e)$. For any set $\{e_1, \dots, e_p\}$ of edges, let $\delta(\{e_1, \dots, e_p\})$ be the least common multiple of the denominators of $w(e_1), \dots, w(e_p)$. In particular, for each $e_i \in \{e_1, \dots, e_p\}$,

$$\delta(\{e_1, \dots, e_p\}) \cdot w(e_i) \quad \text{is an integer.}$$

We now iteratively assign weights to the vertices of H.

Step 0. Let $W(v_1) = 1$

Step 1a. If for any $v \in \mathscr{D}_1$, $W(v_1) \cdot w(\{\alpha_T(v), v\})$ is not an integer, scale $W(v_1)$ by

$$\delta\Big(\big\{\{\alpha_T(v), v\} : v \in \mathscr{D}_1\big\}\Big).$$

Step 1b. For each $v \in \mathscr{D}_1$, let $W(v) = W(v_1) \cdot w(\{\alpha_T(v), v\})$.

Step 2a. If for any $v \in \mathscr{D}_2$, $W(\alpha_T(v)) \cdot w(\{\alpha_T(v), v\})$ is not an integer, for all $u \in \mathscr{D}_0 \cup \mathscr{D}_1$ scale $W(u)$ by

$$\delta\Big(\big\{\{\alpha_T(v), v\} : v \in \mathscr{D}_2\big\}\Big).$$

Step 2b. For each $v \in \mathscr{D}_2$, let $W(v) = W(\alpha_T(v)) \cdot w(\{\alpha_T(v), v\})$.

Having assigned $W(v)$ for all $v \in \mathscr{D}_0, \dots, \mathscr{D}_{i-1}$

Step ia. If for any $v \in \mathscr{D}_i$, $W(\alpha_T(v)) \cdot w(\{\alpha_T(v), v\})$ is not an integer, for all $u \in \mathscr{D}_0 \cup \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_{i-1}$ scale $W(u)$ by

$$\delta\Big(\big\{\{\alpha_T(v), v\} : v \in \mathscr{D}_i\big\}\Big).$$

Step ib. For each $v \in \mathscr{D}_i$, let $W(v) = W(\alpha_T(v)) \cdot w(\{\alpha_T(v), v\})$.

By construction, $W(v)$ is a positive integer for every vertex V. Finally, we will show that for all $i \neq j$

$$a_{i,j}W(v_i) = a_{j,i}W(v_j). \tag{5.16}$$

First suppose $e = \{v_i, v_j\} \in E(T)$, and $v_i \preceq_T v_j$. Then by the iterative process above, $W(v_i)w(e) = W(v_j)$. However $w(e) = a_{i,j}/a_{j,i}$ and therefore (5.16) holds. Otherwise, suppose $\{v_i, v_j\} \in E(H)\backslash E(T)$. Then the iterative construction and A-resolvability of T guarantees that $W(v_i)w(e) = W(v_j)$ and, as before, (5.16) holds. Finally, if $\{v_i, v_j\} \notin E(H)$, then by construction of H, $a_{i,j} = a_{j,i} = 0$, and so (5.16) trivially holds. That is, $W(v_1), \ldots, W(V_k)$ is a set of positive integers such that for all $i \neq j$,

$$a_{i,j}W(v_1) = a_{j,i}W(v_j).$$

Therefore taking $\boldsymbol{x}^t = \begin{bmatrix} W(v_1) & W(v_2) & \cdots & W(v_k) \end{bmatrix}$, $\boldsymbol{x}$ is a positive integer solution of $R(A)\boldsymbol{x} = \boldsymbol{0}$. □

Example 5.10. Let A be the 6×6 matrix as in Example 5.4,

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 & 2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 3 \\ 0 & 0 & 3 & 2 & 0 & 3 \\ 1 & 0 & 0 & 4 & 2 & 0 \end{bmatrix}$$

In that example, we exhibited a positive integer solution to $R(A)\boldsymbol{x} = \boldsymbol{0}$, where $\boldsymbol{x}^t = \begin{bmatrix} 3 & 2 & 3 & 4 & 2 & 3 \end{bmatrix}$. We now use the algorithm outlined above to show how we arrived at this solution. As A is 6×6, first construct a simple graph H on 6 vertices, $v_1, \ldots, v_6$, such that $\{v_i, v_j\} \in E(H)$ if and only if $a_{i,j} \neq 0 \neq a_{j,i}$. H is illustrated in Figure 5.2.18.

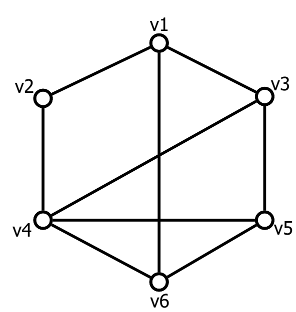


Figure 5.2.18: The simple graph H given by matrix A.

Next choose a spanning tree T of H rooted at v_1, such as that illustrated in Figure 5.2.19.

For any edge $e = \{v_i, v_j\} \in E(H)$, let

$$w(e) = \frac{a_{i,j}}{a_{j,i}} \text{ if } e \in E(T) \text{ and } v_i \preceq_T v_j, \text{ or when } e \in E(H)\backslash E(T) \text{ and } i < j.$$

These edge weights are illustrated in Figure 5.2.19.

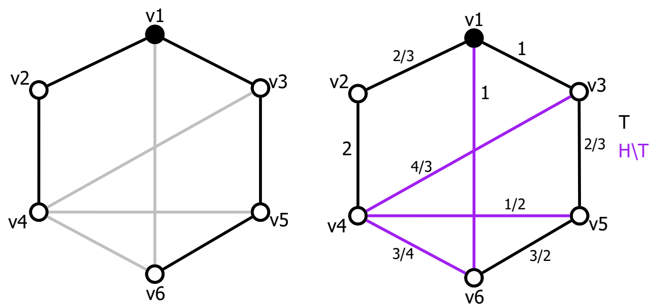


Figure 5.2.19: A spanning tree T of H and the resulting set of weights assigned to $E(H)$. One may verify that T is A-resolvable.

One may check that the fundamental cycles of H with respect to T are all A-resolvable. Therefore T is A-resolvable, and by Lemma 5.4 and Theorem 5.3, $R(A)\boldsymbol{x} = \mathbf{0}$ has a positive integer solution $\boldsymbol{x}$. To find such a solution, we assign positive integer weights to the vertices of H using the iterative process outlined in the proof of Theorem 5.3. This procedure is illustrated in Figure 5.2.20, where the distance sets $\mathscr{D}_i$ are distinguished using colors.

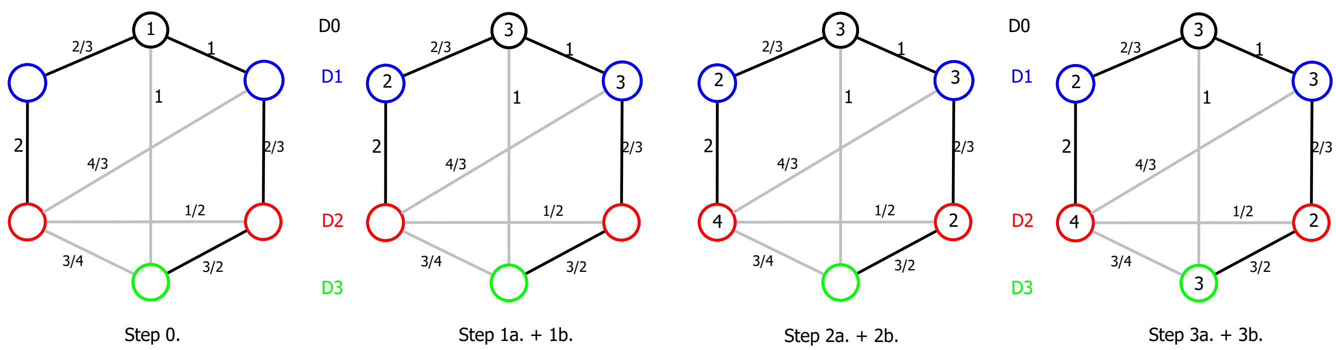


Figure 5.2.20: Iteratively assigning weights to the vertices of H as outlined in the proof of Theorem 5.3.

Observe that in Step 1a. $W(v_1)$ is scaled by 3 in order to guarantee that $W(v_2)$ and $W(v_3)$ are integers. No further scaling is required in Steps 2a. or 3a. The final weights are:

$$W(v_1) = W(v_3) = W(v_6) = 3 \quad W(v_2) = W(v_5) = 2 \quad W(v_4) = 4$$

The result is the vector $\boldsymbol{x}^t = \begin{bmatrix} 3 & 2 & 3 & 4 & 2 & 3 \end{bmatrix}$, the solution to $R(A)\boldsymbol{x} = \boldsymbol{0}$ originally presented in Example 5.4.

Example 5.11. Now consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

This matrix was originally used in Example 5.9. Letting the graph H be as in that example, it was shown that the weighted spanning tree in Figure 5.2.21 is A-resolvable.

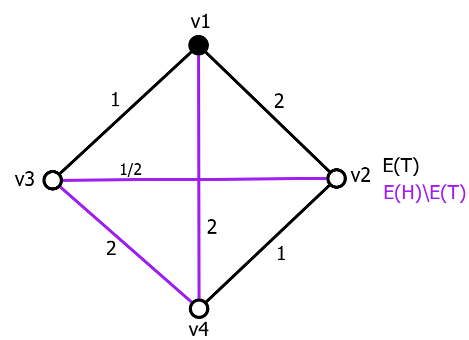


Figure 5.2.21: A spanning tree T of H and the resulting set of weights assigned to $E(H)$.

Iteratively assign weights to the vertices of H as outlined in the proof of Theorem 5.3. This process is illustrated in Figure 5.2.22. Once again, distance sets $\mathscr{D}_i$ are differentiated using colors, and observe that no rescaling is needed.

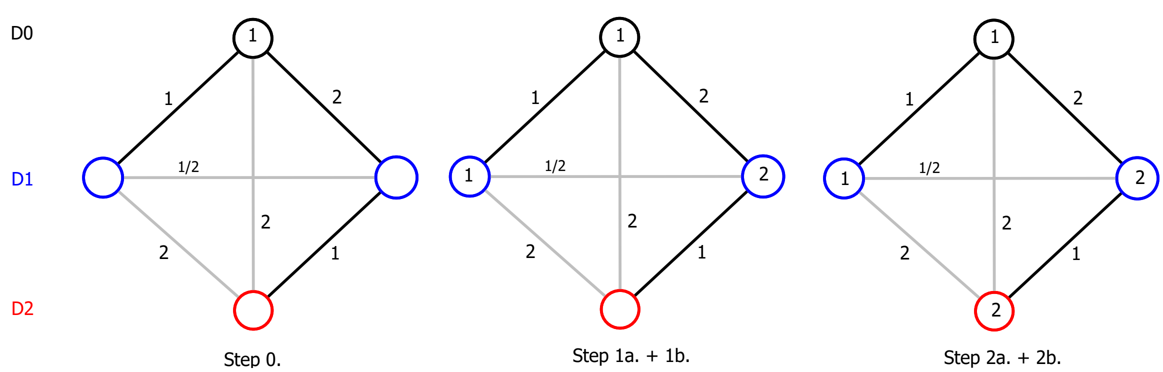


Figure 5.2.22: Iteratively assigning weights to the vertices of H as outlined in the proof of Theorem 5.3.

These weights give a positive integer solution $R(A)\boldsymbol{x} = \mathbf{0}$, where

$$\boldsymbol{x}^t = \left[\begin{array}{cccc} 1 & 2 & 1 & 2 \end{array}\right].$$

Therefore there exists a graph G and an equitable partition π with quotient matrix A. Given $\boldsymbol{x}$, G can be constructed with $1 + 2 + 1 + 2 = 6$ vertices. Partition π must have 4 classes, $\pi = (V_1, V_2, V_3, V_4)$ where

$$|V_1| = 1 \quad |V_2| = 2 \quad |V_3| = 1 \quad |V_4| = 2.$$

One such G and π are illustrated in Figure 5.2.23.

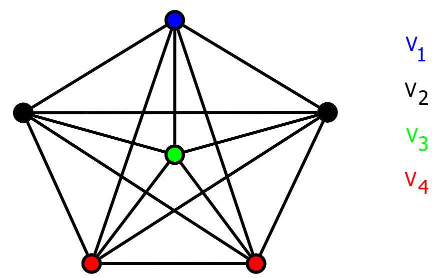


Figure 5.2.23: An equitable partition having quotient matrix A, constructed using solution vector $\boldsymbol{x}$.

This completes our discussion of equitable (vertex) partitions. In Chapter 6, we extend the notion of equitable partitioning to the edges of digraphs. After showing that these edge-partitions satisfy many of the same properties as their vertex counterparts, we demonstrate deep connections between equitable edge-partitions and Kirchhoff graphs.

Chapter 6

Edge-Partitions and Kirchhoff Graphs

In this chapter we extend the study of equitable partitions to the edges of multi-digraphs. In recent years, graph edge partitions have become an important research area , as they present useful methods in data decomposition and distributed computing. These methods typically involve partitioning the edges of a (large) graph into connected subgraphs, with a goal of minimizing some objective. For example [11] studies so-called balanced edge partitions, building upon the work of [5], [60], and [65]. Other authors have studied the computational complexity of these problems [61], or specific applications [44]. A number of additional references can be found in [67].

We take a more strictly combinatorial approach, partitioning the edges of a digraph into subsets, and studying the resulting structure. Section 6.1 begins by introducing a notion of signed adjacency between directed edges, which leads to a signed *edge* adjacency matrix of a digraph. Using this matrix, Section 6.1.1 defines equitable edge-partitions, and demonstrates that all of the classical results of Chapter 5 continue to hold, now with the edge-adjacency matrix A_E in place of adjacency matrix A. Section 6.1.2 examines orbit partitions of digraphs, and Theorem 6.1 demonstrates that every orbit edge-partition is equitable. To the author's knowledge, none of the material in Section 6.1 and its subsections, including signed edge adjacency matrix A_E, has been studied in the literature.

Building upon the results of Chapter 3, Section 6.2 presents a natural definition of *Kirchhoff* edge partitions. Notably, Theorem 6.2 shows that if the quotient matrix of an equitable edge-partition π is symmetric, then π is Kirchhoff. Sections 6.2.1 and 6.2.2 study the exactness of the correspondence given in Theorem 6.2. Specifically, Section 6.2.1 exhibits both

Kirchhoff partitions that are not equitable, and equitable partitions that are not Kirchhoff. Section 6.2.2 presents a partial converse to Theorem 6.2 by studying uniform edge-partitions. Finally, Section 6.3 makes the connection between Kirchhoff *partitions* and Kirchhoff *graphs* be defining vector edge-partitions, and demonstrating that every equitable edge-partition is a vector edge-partition.

6.1 Edge-Partitions of Directed Graphs

Let D be a directed graph, which may have multiple edges, with vertices $V(D) = \{v_i\}$ and edge set $E(D) = \{e_j\}$. Define two functions $\iota : E(D) \mapsto V(D)$ and $\tau : E(D) \mapsto V(D)$, where

$$\iota(e) = v \in V(D): \; v \text{ is the initial vertex of edge } e$$
$$\tau(e) = v \in V(D): \; v \text{ is the terminal vertex of edge } e.$$

That is, every edge e is of the form $e = (\iota(e), \tau(e))$. Now for each $i \neq j \in \{1, \ldots m\}$, define

$$s_{i,j} = \begin{cases} 1 & \text{if either } \iota(e_i) = \iota(e_j) \text{ or } \tau(e_i) = \tau(e_j) \\ 2 & \text{if } \iota(e_i) = \iota(e_j) \text{ and } \tau(e_i) = \tau(e_j) \\ -1 & \text{if either } \iota(e_i) = \tau(e_j) \text{ or } \tau(e_i) = \iota(e_j) \\ -2 & \text{if } \iota(e_i) = \tau(e_j) \text{ and } \tau(e_i) = \iota(e_j) \\ 0 & \text{Otherwise.} \end{cases} \tag{6.1}$$

That is, $|s_{i,j}| = 2$ if e_i and e_j share both endpoints, and is negative when they have opposite orientations. $|s_{i,j}| = 1$ if e_i and e_j share one endpoint, and is negative when that vertex is the initial vertex of one edge and the terminal vertex of the other. For each i, let $s_{i,i} = 0$. This is summarized in Figure 6.1.1.

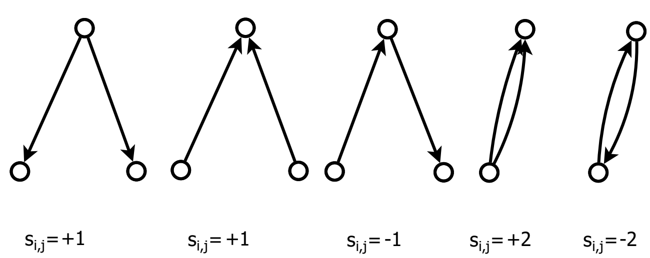


Figure 6.1.1: The 5 edge configurations for which $s_{i,j} \neq 0$.

Definition 6.1. The ***signed edge adjacency matrix*** of D, $A_E(D)$, is the $|E(D)|\times|E(D)|$ matrix with (i,j)-entry $s_{i,j}$.

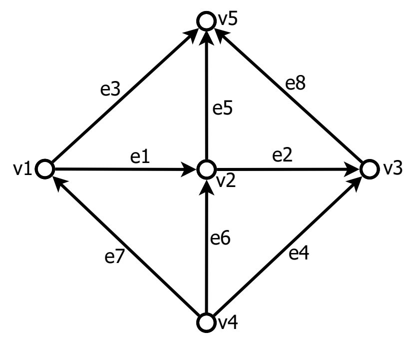


Figure 6.1.2: A digraph D with 5 vertices and 8 edges.

For example, the digraph D with 5 vertices and 8 edges in Figure 6.1.2 has signed edge adjacency matrix

$$A_E(D) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

For any digraph D, $A_E(D)$ is a symmetric matrix. Moreover, $A_E(D)$ can be computed from a number of other graph matrices.

Recall that $Q = Q(D)$ is the *incidence matrix* of digraph D. That is, Q is a $|V(D)|\times|E(D)|$ matrix with (i,j)-entry 1 if $v_i = \iota(e_j)$, and -1 if $v_i = \tau(e_j)$. Otherwise, $Q_{i,j}$ is 0 when vertex v_i is not an endpoint of edge e_j. For example, the digraph in Figure 6.1.2 has incidence matrix

$$Q(D) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array}.$$

Proposition 6.1. $A_E(D) = Q^tQ - 2I$

Proof. Let $\boldsymbol{q}_i$ denote the i^{th} column of matrix Q. Then,

$$(Q^tQ)_{i,j} = \boldsymbol{q}_i \cdot \boldsymbol{q}_j = \begin{cases} 1 & \text{if either } \iota(e_i) = \iota(e_j) \text{ or } \tau(e_i) = \tau(e_j) \\ 2 & \text{if } \iota(e_i) = \iota(e_j) \text{ and } \tau(e_i) = \tau(e_j) \\ -1 & \text{if either } \iota(e_i) = \tau(e_j) \text{ or } \tau(e_i) = \iota(e_j) \\ -2 & \text{if } \iota(e_i) = \tau(e_j) \text{ and } \tau(e_i) = \iota(e_j) \\ 0 & \text{Otherwise.} \end{cases}$$

Therefore by (6.1), $[Q^tQ - 2I]_{i,j} = s_{i,j}$. □

Remark 6.1. The equation $A_E(D) = Q^tQ - 2I$ motivates naming $A_E(D)$ the signed edge adjacency matrix of D. Let A be the adjacency matrix of any simple graph G. Arbitrarily orient the edges of G, and let Q be the incidence matrix of the resulting digraph. One may verify that[1]

$$A = \operatorname{diag}(QQ^t) - QQ^t.$$

That is, the adjacency matrix A matches–up to sign and the diagonal entries–the $|V(G)| \times |V(G)|$ matrix QQ^t. On the other hand, Q^tQ is the $|E(D)| \times |E(D)|$ analogue of QQ^t. Moreover, observe that $2I = \operatorname{diag}(Q^tQ)$, and so Proposition 6.1 can be rewritten as

$$A_E(D) = Q^tQ - \operatorname{diag}(Q^tQ).$$

That is, the matrix $A_E(D)$ is the same as the $|E(D)| \times |E(D)|$ matrix Q^tQ, up to the diagonal entries. Thus as A is the $|V(G)| \times |V(G)|$ adjacency matrix, we have chosen to call the $|E(D)| \times |E(D)|$ matrix $A_E(D)$ the (signed) *edge* adjacency matrix.

Recall that the incidence vector of a vertex v, $\lambda(v)$ is a vector with $|E(D)|$ entries, and j^{th} entry 1 if $v = \iota(e_j)$, -1 if $v = \tau(e_j)$, and 0 otherwise. In particular, $\lambda(v_j)$ is the j^{th} row of the incidence matrix $Q(D)$. In light of Proposition 6.1, the constants $s_{i,j}$, and therefore the rows of $A_E(D)$, can be written in terms of vertex incidence vectors.

Proposition 6.2. For each $i \neq j$,

$$s_{i,j} = \lambda(\iota(e_i))_j - \lambda(\tau(e_i))_j. \tag{6.2}$$

[1]Note the matrix QQ^t is also the classical combinatorial Laplacian of graph G. We will return to the Laplacian matrix in Chapter 7.

Proof. Observe that for $i \neq j$,

$$\lambda(\iota(e_i))_j = \begin{cases} 1 & \text{if } \iota(e_i) = \iota(e_j) \\ -1 & \text{if } \iota(e_i) = \tau(e_j) \\ 0 & \text{Otherwise} \end{cases} \quad \text{and} \quad -\lambda(\tau(e_i))_j = \begin{cases} 1 & \text{if } \tau(e_i) = \tau(e_j) \\ -1 & \text{if } v_{\tau(e_i)} = v_{\iota(e_j)} \\ 0 & \text{Otherwise} \end{cases} \tag{6.3}$$

Therefore by definition of $s_{i,j}$ (6.1), for $i \neq j$ equation (6.2) holds. □

On the other hand, for any i, $\lambda(\iota(e_i))_i = -\lambda(\tau(e_i))_i = 1$, meaning

$$\lambda(\iota(e_i))_i - \lambda(\tau(e_i))_i - 2 = 0 = s_{i,i}. \tag{6.4}$$

Let $V_{\mathcal{I}}$ and $V_{\mathcal{T}}$ be $|E(D)| \times |E(D)|$ matrices with j^{th} row $\lambda(\iota(e_j))$ and $\lambda(\tau(e_j))$ respectively. That is, $V_{\mathcal{I}}$ is a square matrix whose j^{th} row is the incidence vector of the *initial* vertex of edge e_j (hence the subscript $\mathcal{I}$ for "initial"). Similarly, the j^{th} row of matrix $V_{\mathcal{T}}$ is the incidence vector of the *terminal* vertex of edge e_j ($\mathcal{T}$ for "terminal"). Proposition 6.2 and (6.4) combine to prove,

Corollary 6.1. $A_E(D) = V_{\mathcal{I}} - V_{\mathcal{T}} - 2I$.

Remark 6.2. While the signed edge adjacency matrix A_E successfully captures interactions between the edges of a digraph, observe that reconstructing a graph from $A_E(D)$ is non-trivial. For example, in light of Figure 6.1.1, there are *two* edge configurations for which $s_{i,j} = 1$. Therefore given that $A_E(D)_{i,j} = 1$, we know that edges e_i and e_j meet at a vertex with matching orientations. However, we do not know if they share an initial vertex or a terminal vertex. This raises a number of interesting questions in its own right.

Question: Given the signed edge adjacency matrix A'_E of a digraph, can we reconstruct a digraph D such that $A_E(D) = A'_E$? Is the graph that is constructed unique?

6.1.1 Equitable Edge-Partitions

Let D be any digraph with signed edge adjacency matrix $A_E(D)$, and let $\pi = (E_1, \ldots, E_k)$ be any partition of $E(D)$.

Definition 6.2. π is an ***equitable edge-partition*** if for any $i, j \in \{1, \ldots, k\}$ and for any edge $e_p \in E_i$, the number

$$d_{i,j} = \sum_{q : e_q \in E_j} A_E(D)_{p,q} \tag{6.5}$$

depends only on i and j, and not on the choice of edge $e_p \in E_i$.

Comparing (5.1) to (6.5), observe that this definition of an equitable edge-partition is analogous to that of an equitable partition, with the signed edge adjacency matrix in (6.5) taking the place of the adjacency matrix in (5.1).

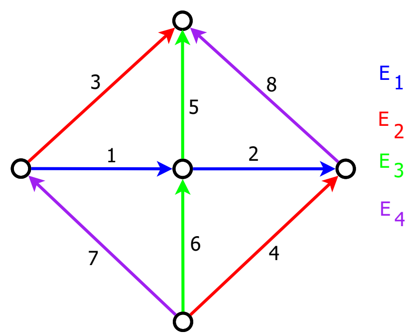


Figure 6.1.3: A digraph D with an equitable edge-partition π with 4 classes.

For example, the digraph D in Figure 6.1.2 has an equitable edge-partition $\pi = (E_1, E_2, E_3, E_4)$, where

$$E_1 = \{e_1, e_2\} \quad E_2 = \{e_3, e_4\} \quad E_3 = \{e_5, e_6\} \quad E_4 = \{e_7, e_8\}.$$

This edge-partition is illustrated with colored edges in Figure 6.1.3.

Remark 6.3. Every digraph has an edge-partition that is trivially equitable, namely the partition for which every cell contains exactly one edge. We will let π_I denote this trivially equitable edge-partition.

Let D be a digraph with equitable edge-partition $\pi = (E_1, \ldots, E_k)$. We can define a ***quotient*** of D with respect to π, denoted D/π. D/π is a labeled digraph with one vertex for each partition cell $E_1, \ldots, E_k$. For all $i, j \in \{1, \ldots, k\}$, D/π has $|d_{i.j}|$ edges directed from E_i to E_j, each having label $\operatorname{sign}(d_{i,j})$. In general, D/π may have loops and multiple edges. Given D and π as in Figure 6.1.2, the quotient D/π is illustrated in Figure 6.1.4.

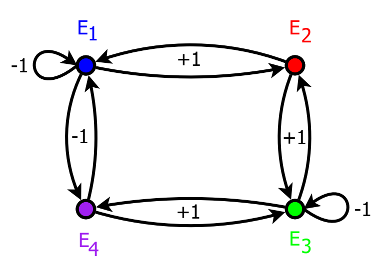


Figure 6.1.4: The quotient of D with respect to edge-partition π.

Now define $A_E(D/\pi)$ to be the $k \times k$ matrix with (i, j)-entry $d_{i,j}$. $A_E(D/\pi)$ is a signed adjacency matrix for quotient D/π in the sense that the magnitude of the (i, j)-entry of $A_E(D/\pi)$ indicates how many edges are directed from the i^{th} vertex to the j^{th} vertex of D/π, and the sign of the entry indicates the label on those edges. In this example,

$$A_E(D/\pi) = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Remark 6.4. Much as the matrix $A_E(D)$ encodes how some edge interacts with the other edges of a digraph, matrix $A_E(D/\pi)$ encodes how any edge of a particular *partition class* interacts with the other *partition classes*. For example under the coloring in Figure 6.1.4, $A_E(D/\pi)_{1,2} = 1$ indicates that every blue edge meets one red edge, and the orientations of those edges match at the incident vertex.

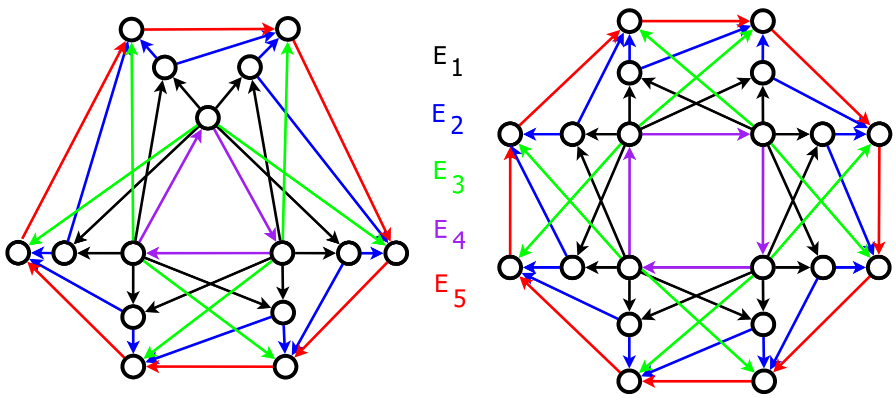


Figure 6.1.5: Two digraphs, one with 39 edges and one with 52, each having an equitable edge-partition with the same quotient matrix.

Once again, we call matrix $A_E(D/\pi)$ the ***quotient matrix*** of D with respect to π. Any such matrix must be square and have integer entries, and may be the quotient matrix of any number of digraph/equitable edge-partition pairs. For example, Figure 6.1.5 shows two equitable edge-partitions, one on a 39-edge digraph and one on a 52-edge digraph, both having quotient matrix

$$\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \left[\begin{array}{ccccc} 4 & -2 & 2 & 0 & 0 \\ -2 & 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right] \end{array}.$$

Quotient matrix $A_E(D/\pi)$ is closely related to the signed edge adjacency matrix $A_E(D)$. Let T be the characteristic matrix of edge-partition π. That is T is an $|E(D)| \times k$ matrix, and $T_{i,j} = 1$ when $e_i \in E_j$ and is otherwise 0. T provides the relationship between $A_E(D/\pi)$ and $A_E(D)$.

Lemma 6.1. *Let π be a partition of $E(D)$ with characteristic matrix T. If π is an equitable edge-partition then*

$$A_E(D)T = TA_E(D/\pi). \tag{6.6}$$

Proof. For any $p \in \{1, \ldots, k\}$, suppose that $e_p \in E_i$, and consider $[A_E(D)T]_{p,j}$. Noting that for all q, $T_{q,j} = 1$ if and only if $e_q \in E_j$,

$$[A_E(D)T]_{p,j} = \sum_{q=1}^{|E(D)|} A_E(D)_{p,q}T_{q,j} = \sum_{q:e_q \in E_j} A_E(D)_{p,q} = d_{i,j}$$

by (6.5). On the other hand, as $e_p \in E_i$, row p of T is zero except for a 1 in column i. Therefore row p of $TA_E(D/\pi)$ is row i of $A_E(D/\pi)$, and $[TA_E(D/\pi)]_{p,j} = A_E(D/\pi)_{i,j} = d_{i,j}$. Therefore $A_E(D)T = TA_E(D/\pi)$. □

The reverse implications of Lemma 6.1 is true as well.

Proposition 6.3. Let π be an edge-partition with characteristic matrix T. If

$$A_E(D)T = TR$$

for some $k \times k$ matrix R, then π is an equitable edge-partition.

Proof. As $A_E(D)T = TR$, the columns of $A_E(D)T$ are linear combinations of the columns of T, and therefore constant on the cells of π. That is, if e_p and e_q belong to class E_j of edge-partition π, then

$$[A_E(D)T]_{p,j} = [A_E(D)T]_{q,j}. \tag{6.7}$$

However for any i and j,

$$[A_E(D)T]_{i,j} = \sum_{q=1}^{|E(D)|} A_E(D)_{i,q}T_{q,j} = \sum_{q:e_q \in E_j} A_E(D)_{i,q}.$$

But by (6.7), this last sum depends only on the partition class containing edge e_i and not on the choice of edge. Therefore by definition, π is an equitable edge partition. $\square$

The quotient matrix $A_E(D/\pi)$ can also be written in terms of vertex incidence vectors and the characteristic matrix T. For each partition class $E_1, \ldots, E_k$, choose some representative edge $\varepsilon_i \in E_i$. Let $\tilde{V_{\mathcal{I}}}$ be a $k \times |E(D)|$ matrix whose i^{th} row is $\lambda(\iota(\varepsilon_i))$; that is, the incidence vector of the *initial* vertex of ε_i. Similarly, let $\tilde{V_{\mathcal{T}}}$ be a $k \times |E(D)|$ matrix whose i^{th} row is $\lambda(\tau(\varepsilon_i))$, the incidence vector of the *terminal* vertex of ε_i.

Proposition 6.4.

$$A_E(D/\pi) = (\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T - 2I_k. \tag{6.8}$$

Proof. For any $i \neq j \in \{1, \ldots, k\}$ suppose, without loss of generality, that $e_p \in E_i$ is the edge chosen for row i of $\tilde{V_{\mathcal{I}}}$ and $\tilde{V_{\mathcal{T}}}$. Then

$$\begin{aligned}
\left[(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T\right]_{i,j} &= \sum_{q=1}^{|E(D)|} \left[\lambda(\iota(e_p))_q - \lambda(\tau(e_p))_q\right] T_{q,j} \\
&= \sum_{q=1}^{|E(D)|} s_{p,q} T_{q,j} \quad \text{by Proposition 6.2} \\
&= \sum_{q:e_q \in E_j} s_{p,q} = \sum_{q:e_q \in E_j} A_E(D)_{p,q} \\
&= d_{i,j} = A_E(D/\pi)_{i,j}.
\end{aligned}$$

As $I_{i,j} = 0$ for $i \neq j$, it follows that (6.8) holds for all (i,j)-entries when $i \neq j$. A similar argument shows that $[(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{I}}})T]_{j,j} = d_{j,j} + 2$ and thus (6.8) holds. $\square$

Corollary 6.2. *If π is an equitable edge-partition, then*

$$(V_{\mathcal{I}} - V_{\mathcal{T}})\, T = T\left(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}}\right) T \tag{6.9}$$

Proof. Following previous results,

$$\begin{aligned}
(V_{\mathcal{I}} - V_{\mathcal{T}})T &= A_E(D)T + 2T && \text{by Corollary 6.1} \\
&= T A_E(D/\pi) + 2T && \text{by Lemma 6.1} \\
&= T\left((\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T - 2I_k\right) + 2T && \text{by Proposition 6.4} \\
&= T\left(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}}\right) T.
\end{aligned}$$

$\square$

Corollary 6.3. *If π is an equitable edge-partition of D, then for any two edges e_p and e_q in the same partition cell of π,*

$$\big[\lambda(\iota(e_p)) - \lambda(\tau(e_p))\big]T = \big[\lambda(\iota(e_q)) - \lambda(\tau(e_q))\big]T. \tag{6.10}$$

Proof. Suppose, without loss of generality, that $e_p, e_q \in E_i$. As rows p and q of matrix T are both zero except for a 1 in column i, it follows that rows p and q of $T(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T$ are the same. Thus by (6.9) rows p and q of $(V_{\mathcal{I}} - V_{\mathcal{T}})T$ are the same and so (6.11) holds. □

Proposition 6.5. The reverse implication of Corollary 6.3 is true as well. Let $\pi = (E_1, \ldots, E_k)$ be an edge-partition of a digraph D. If for any two edges e_p and e_q in the same partition cell of π,

$$\big[\lambda(\iota(e_p)) - \lambda(\tau(e_p))\big]T = \big[\lambda(\iota(e_q)) - \lambda(\tau(e_q))\big]T, \tag{6.11}$$

π is an equitable edge-partition.

Proof. For each partition class E_i, let e_{p_i} be some edge $e_{p_i} \in E_i$. Let R be the $k \times k$ matrix whose i^{th} row is

$$\big[\lambda(\iota(e_{p_i})) - \lambda(\tau(e_{p_i}))\big]T.$$

Then by (6.11),

$$\big(V_{\mathcal{I}} - V_{\mathcal{T}}\big)T = TR,$$

and as a result,

$$A_E(D)T = \big(V_{\mathcal{I}} - V_{\mathcal{T}} - 2I_m\big)T = TR - 2IT = T(R - 2I_k).$$

Thus by Proposition 6.3, π is an equitable edge-partition. □

Section 6.1.2, and later Section 7.3, will prove the existence of large classes of equitable edge-partitions.

6.1.2 Orbit Edge-Partitions

Let D be a digraph with vertices $V(D) = \{v_i\}$ and edges $E(D) = \{e_j\}$. An *automorphism* of D is a bijection $\phi : V(D) \to V(D)$ that preserves adjacencies, non-adjacencies, and edge orientations. That is, the directed edge $(v_i, v_j) \in E(D)$ if and only if $(\phi(v_i), \phi(v_j)) \in E(D)$. Any automorphism ϕ acts as a permutation of both $V(D)$ and $E(D)$. Once again, under function composition the automorphisms of D form a group, denoted $\mathrm{Aut}(D)$, called the *automorphism group* of D. Note that the identity map, which maps each vertex to itself, is an automorphism of every digraph, and is sometimes called the trivial automorphism.

For any subgroup Ω of $\mathrm{Aut}(D)$, the action of Ω on $E(D)$ naturally partitions the edges into equivalence classes, where two directed edges (v_i, v_j) and (v'_i, v'_j) are in the same class if and only if there is some $\phi \in \Omega$ such that $\phi(v_i) = v'_i$ and $\phi(v_j) = v'_j$. These equivalence classes are called the *orbits* of $E(D)$ under Ω. We will call any partition of $E(D)$ arising as orbits of some $\Omega \leq \mathrm{Aut}(D)$ an ***orbit edge-partition***.

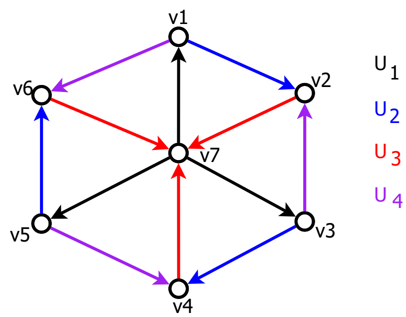


Figure 6.1.6: A digraph D with a 4-cell orbit edge-partition. This partition arises from the subgroup of $\mathrm{Aut}(D)$ generated by $\phi = (v_1v_3v_5)(v_2v_4v_6)(v_7)$.

Example 6.1. Consider the digraph D on 7 vertices presented in Figure 6.1.6. Let ϕ be a permutation of the vertices, with disjoint cycle form

$$\phi = (v_1v_3v_5)(v_2v_4v_6)(v_7).$$

One may verify that ϕ is a graph automorphism (given Figure 6.1.6, ϕ is the symmetry "rotation by 120°"). What is more, ϕ is a permutation of order 3 that generates a subgroup $\langle\phi\rangle \leq \mathrm{Aut}(G)$ of order 3. The edge orbits under $\langle\phi\rangle$ are U_1, U_2, U_3, U_4, where

$$U_1 = \big\{(v_7, v_1), (v_7, v_3), (v_7, v_5)\big\} \quad ; \quad U_2 = \big\{(v_1, v_2), (v_3, v_4), (v_5, v_6)\big\}$$
$$U_3 = \big\{(v_2, v_7), (v_4, v_7), (v_6, v_7)\big\} \quad ; \quad U_4 = \big\{(v_3, v_2), (v_5, v_4), (v_1, v_6)\big\}.$$

That is, $\pi = (U_1, U_2, U_3, U_4)$ is an orbit edge-partition of D. Moreover, π is an equitable edge-partition of D with quotient matrix

$$A_E(D/\pi) = \begin{array}{c} \\ {\scriptstyle U_1} \\ {\scriptstyle U_2} \\ {\scriptstyle U_3} \\ {\scriptstyle U_4} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle U_1} & {\scriptstyle U_2} & {\scriptstyle U_3} & {\scriptstyle U_4} \end{array} \\ \begin{bmatrix} 2 & -1 & -3 & -1 \\ -1 & 0 & -1 & 2 \\ -3 & -1 & 2 & -1 \\ -1 & 2 & -1 & 0 \end{bmatrix} \end{array}.$$

Choosing a different subgroup of automorphisms leads to a different equitable edge-partition of D. For example, let ψ be the permutation of $V(G)$ with disjoint cycle form

$$\psi = (v_2v_6)(v_3v_5)(v_1)(v_4)(v_7).$$

One may verify that ψ is a graph automorphism (given Figure 6.1.6, ψ is the symmetry "reflection across the line through v_1, v_7, v_4"). In particular, $\langle \phi, \psi \rangle \leq \mathrm{Aut}(D)$ is a subgroup of automorphisms of order 6. The edge orbits under $\langle \phi, \psi \rangle$ are U_1, U_2, U_3, where

$$U_1 = \big\{(v_7, v_1), (v_7, v_3), (v_7, v_5)\big\} \quad U_2 = \big\{(v_1, v_2), (v_3, v_4), (v_5, v_6), (v_3, v_2), (v_5, v_4), (v_1, v_6)\big\}$$

$$U_3 = \big\{(v_2, v_7), (v_4, v_7), (v_6, v_7)\big\}.$$

This orbit edge-partition is illustrated in Figure 6.1.7.

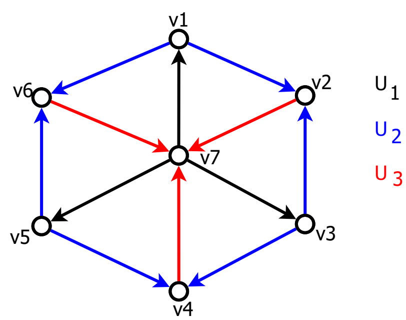


Figure 6.1.7: A 3-cell orbit edge-partition on digraph D. This partition arises from the subgroup $\langle \phi, \psi \rangle \leq \mathrm{Aut}(D)$, where ϕ is as in Figure 6.1.6 and $\psi = (v_2v_6)(v_3v_5)(v_1)(v_4)(v_7)$.

Once again, this edge-partition of D is equitable, with quotient matrix

$$A_E(D/\pi) = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \begin{array}{c} \begin{array}{ccc} U_1 & U_2 & U_3 \end{array} \\ \begin{bmatrix} 2 & -2 & -3 \\ -1 & 2 & -1 \\ -3 & -2 & 2 \end{bmatrix} \end{array}.$$

This is, in fact, a more general result. Theorem 6.1 is the edge-analogue of Proposition 5.3.

Theorem 6.1. *Every orbit edge-partition is an equitable edge-partition.*

Proof. Let $\Omega \leq \mathrm{Aut}(D)$ be a subgroup of automorphisms of a digraph D. Let $U_1, \ldots, U_r$ be the orbits of directed edges under the action of Ω on $E(D)$. For any index $i \in \{1, \ldots, r\}$, let e, f be two directed edges $e, f \in U_i$. To show that the orbit edge-partition $(U_1, \ldots, U_r)$ is equitable, it suffices to show that for any index $j \in \{1, \ldots, r\}$, the following 4 conditions hold.

(a.) The initial vertices of edges e and f are the initial vertices of the same number of edges in partition cell U_j,

$$\Big|\{\epsilon \in U_j : \iota(\epsilon) = \iota(e)\}\Big| = \Big|\{\epsilon \in U_j : \iota(\epsilon) = \iota(f)\}\Big|.$$

(b.) The terminal vertices of edges e and f are the initial vertices of the same number of edges in partition cell U_j,

$$\Big|\{\epsilon \in U_j : \iota(\epsilon) = \tau(e)\}\Big| = \Big|\{\epsilon \in U_j : \iota(\epsilon) = \tau(f)\}\Big|.$$

(c.) The terminal vertices of edges e and f are the terminal vertices of the same number of edges in partition cell U_j,

$$\Big|\{\epsilon \in U_j : \tau(\epsilon) = \tau(e)\}\Big| = \Big|\{\epsilon \in U_j : \tau(\epsilon) = \tau(f)\}\Big|.$$

(d.) The initial vertices of edges e and f are the terminal vertices of the same number of edges in partition cell U_j,

$$\Big|\{\epsilon \in U_j : \tau(\epsilon) = \iota(e)\}\Big| = \Big|\{\epsilon \in U_j : \tau(\epsilon) = \iota(f)\}\Big|.$$

We will prove (a.), the proofs of (b.) - (d.) being analogous. Let $U_j = \{\epsilon_1, \ldots, \epsilon_p\}$. As e, f are in the same cell of an orbit edge-partition, there exists an automorphism ψ such that

$$\psi(\iota(e)) = \iota(f) \quad \text{and} \quad \psi(\tau(e)) = \tau(f).$$

But now for any edge $\epsilon_l \in U_j$, $\iota(e)$ is the initial vertex of $\iota(\epsilon_l)$ if and only if $\psi(\iota(\epsilon_l))$ is the initial vertex of $\psi(\iota(e)) = \iota(f)$. However because U_j is an edge orbit, as a multi-set,

$$\Big\{\psi(\iota(\epsilon_1)), \psi(\iota(\epsilon_2)), \ldots, \psi(\iota(\epsilon_p))\Big\} = \Big\{\iota(\epsilon_1), \iota(\epsilon_2), \ldots, \iota(\epsilon_p)\Big\}.$$

Therefore $\iota(f)$ is the initial vertex of the same number of edges in U_j as $\iota(e)$, and thus (a.) holds. $\square$

While every orbit edge-partition is equitable, not all equitable edge-partitions are orbit edge-partitions. Consider, for example, the digraph D_6 on six vertices illustrated in Figure 6.1.8. This graph has a 6-cell equitable edge-partition $\pi = (E_1, E_2, E_3, E_4, E_5, E_6)$, with quotient matrix

$$A_E(D/\pi) = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 1 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

However, edge-partition π is not an orbit edge-partition. For example, there is no automorphism of D mapping vertex v_1 to v_2, but both are the terminal vertices of an edge in partition cell E_2.

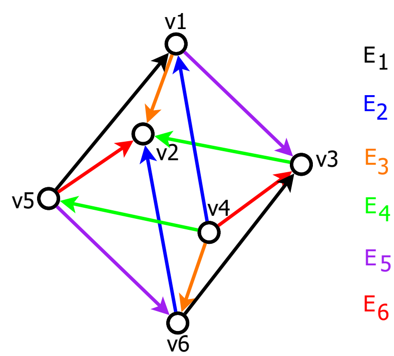


Figure 6.1.8: A digraph D_6 with a 6-cell equitable edge-partition. While equitable, this partition is *not* an orbit edge partition.

6.2 Kirchhoff Edge-Partitions

Let D be any digraph, which may have multiple edges. Corollary 3.2 suggests a natural definition of a Kirchhoff edge-partition of D.

Definition 6.3. An edge-partition $\pi = \{E_1, \ldots, E_k\}$ of $E(D)$ with characteristic matrix T is ***Kirchhoff*** if for all vertices v and cycles C,

$$\lambda(v)T \cdot \chi(C)T = 0. \tag{6.12}$$

Take a moment to consider the vectors $\lambda(v)T$ and $\chi(C)T$. Each has k entries, indexed by $E_1, \ldots, E_k$. The i^{th} entry of the vector $\lambda(v)T$ is the net number of edges of class E_i that exit vertex v. Similarly, the i^{th} entry of vector $\chi(C)T$ is the net number of edges of class E_i that cycle C traverses with the correct orientation.

Every digraph has an edge-partition that is Kirchhoff, namely the trivial edge-partition π_I. In this case, the characteristic matrix T is simply the $|E(D)| \times |E(D)|$ identity matrix. Therefore (10.1) becomes

$$\lambda(v) \cdot \chi(C) = 0,$$

which is the classical orthogonality of the cut and cycle spaces of digraphs, Corollary 1.1. We will be interested in nontrivial Kirchhoff edge-partitions.

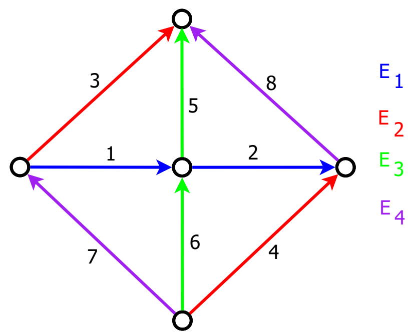


Figure 6.2.1: A digraph D with a 4-class Kirchhoff edge-partition.

Example 6.2. The equitable edge-partition presented in Figure 6.2.1 is Kirchhoff. This edge partition has characteristic matrix

$$T = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \end{array}.$$

One may verify that for any vertex v of D,

$$\lambda(v)T \in \Big\{ \pm \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \end{array}\right], \left[\begin{array}{cccc} 1 & 1 & 0 & -1 \end{array}\right], \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \end{array}\right] \Big\}. \tag{6.13}$$

Similarly for any cycle C,

$$\chi(C)T \in \mathrm{Span}\Big\{ \begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & 1 \end{bmatrix} \Big\}. \tag{6.14}$$

As every vector in (6.13) is orthogonal to every vector in (6.14), edge-partition π is Kirchhoff.

Theorem 6.2 will demonstrate that finding Kirchhoff partitions is closely related to finding equitable edge-partitions. Let D be any digraph with incidence matrix Q, and let $\pi = (E_1, \ldots, E_k)$ be an edge-partition of D with characteristic matrix T.

Proposition 6.6. Edge-partition $\pi = (E_1, \ldots, E_k)$ of $E(D)$ is Kirchhoff if and only if the row space of Q is invariant under right multiplication by TT^t.

Proof. Let N be a matrix with $|E(D)|$ rows whose columns are a basis for $\mathrm{Null}(Q)$. As every cycle vector of D lies in the null space of Q–and thus the row space of N^t–π is Kirchhoff if and only if $QT(N^tT)^t = 0$. That is, if and only if

$$Q(TT^t)N = 0. \tag{6.15}$$

But (6.15) holds if and only if the rows of $Q(TT^t)$ are orthogonal to the columns of N, i.e. if and only if the row space of Q is invariant under right multiplication by TT^t. □

Theorem 6.2. *Let D be a digraph with equitable edge-partition $\pi = (E_1, \ldots, E_k)$. If quotient matrix $A_E(D/\pi)$ is symmetric, then π is Kirchhoff.*

Proof. Let the matrices $V_{\mathcal{I}}, V_{\mathcal{T}}, \tilde{V_{\mathcal{I}}}, \tilde{V_{\mathcal{T}}}$ be defined as before. For any D, the matrix $A_E(D)$ is symmetric, and thus by Corollary 6.1, the matrix $(V_{\mathcal{I}} - V_{\mathcal{T}})$ is symmetric as well. By assumption $A_E(D/\pi)$ is symmetric, so by Proposition 6.4, $(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T$ is also symmetric. As π is equitable, by Corollary 6.2,

$$(V_{\mathcal{I}} - V_{\mathcal{T}})\, T = T \left(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}}\right) T. \tag{6.16}$$

Taking the transpose of (6.16),

$$\begin{aligned} T^t \left(V_{\mathcal{I}} - V_{\mathcal{T}}\right)^t &= T^t \left(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}}\right)^t T^t \\ &= \left[(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T\right]^t T^t. \end{aligned}$$

Thus as both $V_{\mathcal{I}} - V_{\mathcal{T}}$ and $(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T$ are symmetric,

$$T^t \left(V_{\mathcal{I}} - V_{\mathcal{T}}\right) = (\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})TT^t.$$

Therefore,

$$TT^t \left(V_{\mathcal{I}} - V_{\mathcal{T}}\right) = T(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})TT^t,$$

and by (6.16),

$$TT^t(V_{\mathcal{I}} - V_{\mathcal{T}}) = (V_{\mathcal{I}} - V_{\mathcal{T}})TT^t. \tag{6.17}$$

However, the rows of $(V_{\mathcal{I}} - V_{\mathcal{T}})$ span the row space of Q. Therefore the rows of $TT^t(V_{\mathcal{I}} - V_{\mathcal{T}})$ all lie in the row space of Q, and so by (6.17), each row of $(V_{\mathcal{I}} - V_{\mathcal{T}})TT^t$ lies in Row(Q) as well. Thus the row space of Q is invariant under right multiplication by TT^t, and by Proposition 6.6, π is Kirchhoff. □

Example 6.3. Consider the digraph D presented in Figure 6.2.2. D has equitable edge-partition $\pi = (E_1, E_2, E_3, E_4)$ where

$$E_1 = \{e_1, e_2\}; E_2 = \{e_3, e_4\}; E_3 = \{e_5, e_6\}; E_4 = \{e_7, e_8\}.$$

Edge-partition π has quotient matrix

$$A_E(D/\pi) = \begin{array}{c c} & \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} & \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \\ 2 & -2 & 2 & 0 \\ 2 & 2 & 0 & 2 \end{bmatrix} \end{array}.$$

As $A_E(D/\pi)$ is symmetric, Theorem 6.2 guarantees that this partition is Kirchhoff. This can also be checked directly. Letting T be the characteristic matrix of π, every vertex of D satisfies

$$\lambda(v)T \in \Big\{ \pm \begin{bmatrix} 1 & -1 & 2 & 0 \end{bmatrix}, \pm \begin{bmatrix} 1 & 1 & 0 & 2 \end{bmatrix} \Big\},$$

and every cycle C of D satisfies

$$\chi(C)T \in \text{Span}\Big\{ \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 & 0 \end{bmatrix} \Big\}. \tag{6.18}$$

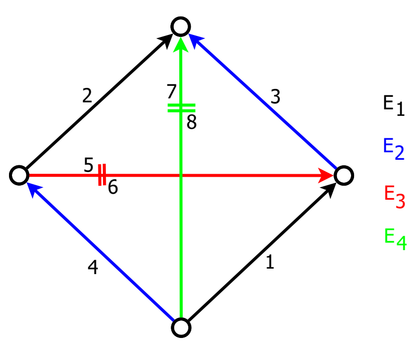


Figure 6.2.2: A digraph D with an equitable edge-partition π. $A_E(D/\pi)$ is symmetric, therefore π is Kirchhoff.

Next, we present an alternate proof of Theorem 6.2, that takes a different approach. First, we make two observations regarding cycle vectors of D and eigenvalues of $A_E(D)$. While we typically write cycle vectors as row vectors, for ease of notation in the following three proofs let all cycle vectors be *column* vectors.

Proposition 6.7. Let D be a digraph with signed edge adjacency matrix $A_E(D)$. Then any cycle vector of D is an eigenvector of $A_E(D)$ with eigenvalue -2.

Proof. Let $\boldsymbol{x}$ be any cycle vector of D. By Corollary 6.1, $A_E(D) = V_{\mathcal{I}} - V_{\mathcal{T}} - 2I$. However the rows of $V_{\mathcal{I}}$ and $V_{\mathcal{T}}$ all lie in the row space of incidence matrix $Q(D)$. Therefore as $\boldsymbol{x} \in \mathrm{Null}(Q(D))$,

$$\begin{aligned} A_E(D)\boldsymbol{x} &= V_{\mathcal{I}}\boldsymbol{x} - V_{\mathcal{T}}\boldsymbol{x} - 2I\boldsymbol{x} \\ &= 0 - 0 - 2I\boldsymbol{x} \\ &= -2\boldsymbol{x}. \end{aligned}$$

□

Proposition 6.8. If π is an equitable edge-partition with characteristic matrix T, then for any cycle vector $\boldsymbol{x}$ of D, the row vector $\boldsymbol{x}^t T$ is a left-eigenvector of $A_E(D/\pi)$ with eigenvalue -2. That is,

$$(\boldsymbol{x}^t T)A_E(D/\pi) = -2(\boldsymbol{x}^t T).$$

Proof. By Proposition 6.7 and symmetry of $A_E(D)$,

$$\begin{aligned} -2(\boldsymbol{x}^t T) = (-2\boldsymbol{x}^t)T &= (\boldsymbol{x}^t A_E(D))T \\ &= \boldsymbol{x}^t (A_E(D)T) \\ &= \boldsymbol{x}^t (T A_E(D/\pi)) \text{ by 6.1} \\ &= (\boldsymbol{x}^t T)A_E(D/\pi). \end{aligned}$$

□

Now we present the alternate proof of Theorem 6.2

Proof. Let D be a digraph with equitable edge-partition π. We will demonstrate that if π is not a Kirchhoff edge-partition, then $A_E(D/\pi)$ is not a symmetric matrix. Observe that if a square matrix B is symmetric, then for any column vector $\boldsymbol{x}$,

$$\boldsymbol{x}^t B = \lambda \boldsymbol{x}^t \text{ for some constant } \lambda \text{ if and only if } B\boldsymbol{x} = \lambda \boldsymbol{x}. \tag{6.19}$$

In particular, given a square matrix B if there is a vector $\boldsymbol{z}$ such that

$$\boldsymbol{z}^t B = \lambda \boldsymbol{z}^t \text{ for some constant } \lambda, \text{ but } B\boldsymbol{z} \neq \lambda \boldsymbol{z}, \tag{6.20}$$

then B is not symmetric. If π is not Kirchhoff, then by Proposition 6.6 there exists some cycle vector $\boldsymbol{y}$ of D such that $(QT)(T^t\boldsymbol{y}) \neq 0$. In particular,

$$(\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T(T^t\boldsymbol{y}) \neq 0.$$

Therefore by Proposition 6.4,

$$A_E(D/\pi)(T^t\boldsymbol{y}) = (\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T(T^t\boldsymbol{y}) - 2I(T^t\boldsymbol{y}) \neq -2(T^t\boldsymbol{y}).$$

However by Proposition 6.8,

$$(\boldsymbol{y}^tT)A_E(D/\pi) = -2(\boldsymbol{y}^tT).$$

Thus noting (6.20), $A_E(D/\pi)$ is not symmetric. □

6.2.1 Equitable versus Kirchhoff Edge-Partitions

Theorem 6.2 proved that any equitable edge-partition with symmetric quotient matrix $A_E(D/\pi)$ is Kirchhoff. This section addresses the exactness of the relationship between equitable edge-partitions and Kirchhoff edge-partitions. In particular, Example 6.4 will show that not every Kirchhoff edge-partition is equitable. Conversely, Example 6.5 shows that not every equitable edge-partition is Kirchhoff.

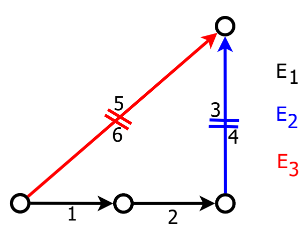


Figure 6.2.3: A digraph D with a Kirchhoff edge-partition that is not equitable.

Example 6.4. Consider the digraph D shown in Figure 6.2.3 and edge-partition $\pi = (E_1, E_2, E_3)$, where

$$E_1 = \{e_1, e_2\}; E_2 = \{e_3, e_4\}; E_3 = \{e_5, e_6\}. \tag{6.21}$$

Letting T be the characteristic matrix of partition π, all vertices v satisfy

$$\lambda(v)T \in \Big\{ \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \Big\},$$

and all cycles C satisfy

$$\chi(C)T \in \Big\{ \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc} 2 & 1 & -1 \end{array}\right] \Big\}.$$

Therefore edge-partition π is Kirchhoff. However, this partition **is not** equitable: for example, edges e_1 and e_2 belong to partition cell E_1, but e_1 is incident two two edges of cell E_3 while edge e_2 is incident to none.

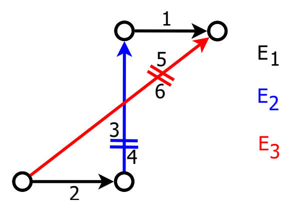


Figure 6.2.4: A digraph D', which can be viewed as a "rearrangement" of D, such that edge-partition π is both equitable and Kirchhoff.

Alternatively, consider the digraph D' shown in figure 6.2.4. D' can be derived from D by a slight rearrangement of vertices and edges. Let $\pi = (E_1, E_2, E_3)$ be the same edge-partition (6.21). Letting T be the characteristic matrix of partition π, all vertices v and cycles C satisfy

$$\lambda(v)T \in \Big\{ \pm\left[\begin{array}{ccc} 1 & 0 & 2 \end{array}\right], \pm\left[\begin{array}{ccc} -1 & 2 & 0 \end{array}\right] \Big\} \quad \chi(C)T \in \Big\{ \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc} 2 & 1 & -1 \end{array}\right] \Big\}. \tag{6.22}$$

Therefore edge-partition π is Kirchhoff. Moreover, in the case of digraph D', edge-partition π is now *equitable* as well, with quotient matrix

$$A_E(D'/\pi) = \begin{array}{c} \\ {\scriptstyle E_1} \\ {\scriptstyle E_2} \\ {\scriptstyle E_3} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} {\scriptstyle E_1} & {\scriptstyle E_2} & {\scriptstyle E_3} \end{array} \\ \left[\begin{array}{ccc} 0 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{array}\right] \end{array}.$$

This raises a number of questions about "rearranging" directed graphs.

Question: If an edge-partition is Kirchhoff, when can a "rearranged" digraph have the same Kirchhoff edge-partition?

Question: Can any Kirchhoff edge-partition become equitable by constructing a suitable "rearranged" digraph?

Remark 6.5. It should not go unnoticed that comparing (6.22) to the analogous sets in Example 6.4,

$$\Big\{\chi(C)T : C \text{ is a cycle of } D\Big\} = \Big\{\chi(C)T : C \text{ is a cycle of } D'\Big\}$$

$$\text{span}\Big\{\lambda(v)T : v \in V(D)\Big\} = \text{span}\Big\{\lambda(v)T : v \in V(D')\Big\}.$$

Therefore the digraph/edge-partition pairs (D, π) and (D', π) can be considered *equivalent* in the Kirchhoff sense.

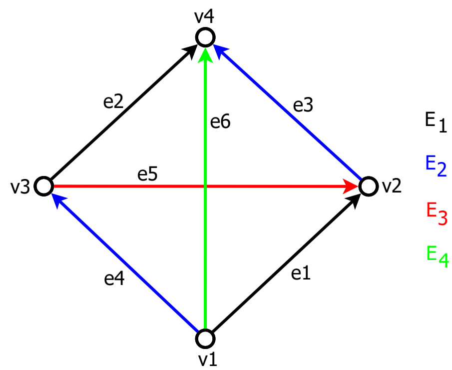


Figure 6.2.5: A digraph D'' with an equitable edge-partition that is not Kirchhoff.

Example 6.5. Now consider the digraph D'' in Figure 6.2.5, and edge-partition $\pi = (E_1, E_2, E_3, E_4)$, where

$$E_1 = \{e_1, e_2\}, E_2 = \{e_3, e_4\}, E_3 = \{e_5\}, E_4 = \{e_6\}.$$

Letting T be the characteristic matrix of π, and letting C be the cycle $v_1 - v_2 - v_4 - v_1$,

$$\lambda(v_1)T \cdot \chi(C)T = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix} = 1 \neq 0,$$

and so edge-partition π is not Kirchhoff. However, π *is* equitable, with quotient matrix (6.23).

$$A_E(D''/\pi) = \begin{array}{c} \\ {\scriptstyle E_1} \\ {\scriptstyle E_2} \\ {\scriptstyle E_3} \\ {\scriptstyle E_4} \end{array} \!\!\begin{array}{c} \begin{array}{cccc} {\scriptstyle E_1} & {\scriptstyle E_2} & {\scriptstyle E_3} & {\scriptstyle E_4} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix} \end{array}. \tag{6.23}$$

Figure 6.2.6 exhibits another edge-partition that is equitable, but not Kirchhoff. For example, letting C be the cycle $C = v_1 - v_2 - v_7 - v_1$,

$$\lambda(v_1)T \cdot \chi(C)T = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = 1 \neq 0.$$

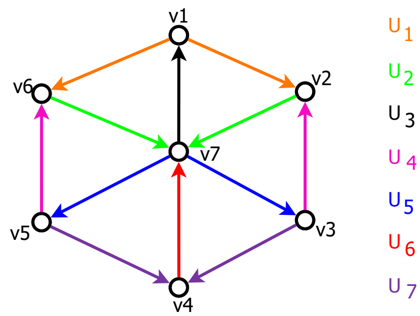


Figure 6.2.6: Another equitable edge-partition that is not Kirchhoff.

The preceding examples demonstrate a lack of exact correspondence between equitable and Kirchhoff edge-partitions. A natural next question is if the converse to Theorem 6.2 holds. That is, if an edge-partition π is *both* equitable and Kirchhoff, is the matrix $A_E(D/\pi)$ symmetric? This will be addressed in Section 6.2.2.

6.2.2 Uniform Edge-Partitions

Theorem 6.2 showed that if D is a digraph with equitable edge-partition π, and the quotient matrix $A_E(D/\pi)$ is symmetric, then π is Kirchhoff. However, the converse to Theorem 6.2 is false, as illustrated by Example 6.6.

Example 6.6. Consider the digraph D, with edge-partition π, presented in Figure 6.2.7. D has 9 directed edges, partitioned into 6 cells, $E_1, \ldots, E_6$.

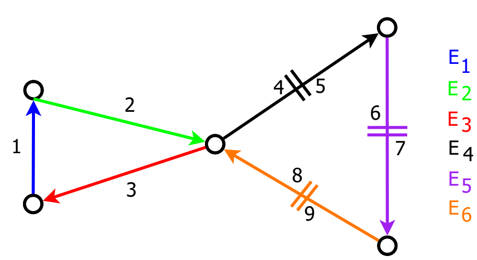


Figure 6.2.7: An equitable edge partition that is also Kirchhoff.

For any vertex v and cycle C, $\lambda(v)T \in \mathrm{Row}(R)$ and $\chi(C)T \in \mathrm{Row}(Z)$, where R and Z are the matrices

$$R = \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix} \end{array} \quad Z = \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

As the product RZ^t is the 5×2 zero matrix, (6.12) is satisfied for all vertices and all cycles. Therefore edge-partition π is Kirchhoff. Moreover, π is *equitable*, and has quotient matrix

$$A_E(D/\pi) = \begin{array}{c|c} & \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{array} & \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -2 & 0 & 2 \\ -1 & -1 & 0 & 2 & 0 & -2 \\ 0 & -1 & 1 & 2 & -2 & -2 \\ 0 & 0 & 0 & -2 & 2 & -2 \\ 0 & 1 & -1 & -2 & -2 & 2 \end{bmatrix} \end{array},$$

which is not symmetric. That is, edge-partition π is both equitable and Kirchhoff, but $A_E(D/\pi)$ is not symmetric.

While Example 6.6 shows that the converse of Theorem 6.2 does not hold in general, a partial converse (Corollary 6.5) can be obtained by considering the sizes of partition cells. Let π be any edge partition with k partition cells and characteristic matrix T. Let Λ be a $k \times k$ diagonal matrix with $\Lambda_{i,i} = 1/|E_i|$. Noting that T^tT is a $k \times k$ diagonal matrix with (i,i)-entry $|E_i|$, it follows that

$$\Lambda T^t T = I_k. \tag{6.24}$$

That is, the matrix ΛT^t is a left pseudoinverse of T. Moreover, if π is equitable, by (6.6),

$$A_E(D/\pi) = \Lambda T^t T A_E(D/\pi) = \Lambda T^t A_E(D)T. \tag{6.25}$$

We give special distinction to those edge-partitions for which $|E_i|$ is independent of i.

Definition 6.4. An edge-partition is ***uniform*** if all partition cells are the same size.

Theorem 6.3. *Let D be a connected digraph with equitable edge-partition π. Then $A_E(D/\pi)$ is symmetric if and only if π is uniform.*

Proof. First suppose π is uniform. In particular, $\Lambda = cI$ for some constant c. By (6.25),

$$A_E(D/\pi) = cT^t A_E(D)T$$

Therefore,

$$\begin{aligned} A_E(D/\pi)^t &= (cT^tA_E(D)T)^t \\ &= cT^tA_E(D)^tT \\ &= cT^tA_E(D)T = A_E(D/\pi) \end{aligned}$$

and $A_E(D/\pi)$ is symmetric. Conversely, suppose that $A_E(D/\pi)$ is symmetric. Then by (6.25), as $A_E(D)$ and Λ are both symmetric,

$$\Lambda(T^tA_E(D)T) = A_E(D/\pi) = A_E(D/\pi)^t = (\Lambda T^tA_E(D)T)^t = (T^tA_E(D)T)\Lambda.$$

Therefore for all $i \neq j$,

$$\Lambda_{i,i}(T^tA_E(D)T)_{i,j} = (\Lambda T^tA_E(D)T)_{i,j} = (T^tA_E(D)T\Lambda)_{i,j} = (T^tA_E(D)T)_{i,j}\Lambda_{j,j}. \tag{6.26}$$

As D is connected, it follows that no simultaneous permutation of rows and columns can transform $T^tA_E(D)T$ into a block-diagonal matrix of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Therefore as (6.26) is true for all $i \neq j$, it follows that all diagonal entries of Λ must be equal. Therefore $|E_i|$ is independent of i and, π is uniform. $\square$

Let D be a connected digraph with equitable edge-partition π. Corollary 6.4 is an immediate consequence of Theorem 6.3, and Corollary 6.5 is the partial converse of Theorem 6.2 that we desired.

Corollary 6.4. *Every uniform equitable edge-partition of D is Kirchhoff.*

Corollary 6.5. *If equitable edge-partition π is Kirchhoff and uniform, then quotient matrix $A_E(D/\pi)$ is symmetric.*

6.3 Vector Edge-Partitions

Thus far we have discussed Kirchhoff *edge-partitions*, though have not yet made any reference to Kirchhoff *graphs*. Generally speaking, Kirchhoff graphs can be thought of as vector graphs whose corresponding edge-partition is Kirchhoff. To begin we will make these ideas more precise.

Let D be any multi-digraph with vertices $V(D) = \{v_i\}$ and edges $E(D) = \{e_j\}$. Let

$S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k\}$ be a finite set of vectors in some vector space. Recall that a vector assignment φ (or a (D, S)-vector assignment) is any surjective function $\varphi : E(D) \to S$. Once again, let a cycle be any closed walk in D, with no repeated vertices, that may traverse edges regardless of orientation. Recall that a vector assignment is *consistent* if for any cycle C of D,

$$\sum_{e \in E(C)} \sigma_C(e)\varphi(e) = \mathbf{0},$$

where $\sigma_C(e) = 1$ if C traverses e in the direction of its orientation, and $\sigma_C(e) = -1$ if C traverses e against its orientation. That is, φ is consistent with D if the signed sum of vectors around any cycle is the zero vector.

Proposition 6.9. For any digraph D, and any matrix M with $|E(D)|$ rows, the vector assignment

$$\varphi(e) = \Big(\lambda(\tau(e)) - \lambda(\iota(e))\Big)M$$

is consistent.

Proof. Let $C = v_1 \cdot e_1 \cdot v_2 \cdot e_2 \cdot v_3 \cdots v_p \cdot e_p \cdot v_1$ be any cycle of D. Then for each i,

$$\sigma_C(e_i)\varphi(e_i) = \Big(\lambda(v_{i+1}) - \lambda(v_i)\Big)M.$$

Therefore,

$$\begin{aligned}
\sum_{e \in E(C)} \sigma_C \varphi(e) &= \sum_{e_i \in E(C)} \sigma_C(e_i)\big(\lambda(\tau(e_i)) - \lambda(\iota(e_i))\big)M \\
&= \Big((\lambda(v_2) - \lambda(v_1)) + (\lambda(v_3) - \lambda(v_2)) \cdots + (\lambda(v_1) - \lambda(v_p)) \Big)M \\
&= \Big(-\lambda(v_1) + \lambda(v_2) - \lambda(v_2) + \cdots + \lambda(v_p) - \lambda(v_p) + \lambda(v_1)\Big)M \\
&= \mathbf{0}M = \mathbf{0}.
\end{aligned}$$

As cycle C was arbitrary, vector assignment φ was consistent. $\square$

Corollary 6.6. *For any digraph D, the vector assignment*

$$\varphi(e) = \big(\lambda(\tau(e)) - \lambda(\iota(e))\big)$$

is consistent.

A vector graph is digraph together with a consistent (D, S)-vector assignment. Observe that every vector graph has a natural edge-partition, whose classes are the vectors to which the edges are assigned. Namely, for any consistent (D, S)-vector assignment φ, let π be the edge-partition $\pi = (E_1, \ldots, E_k)$ where

$$E_i = \{e \in E(D) : \varphi(e) = \boldsymbol{s}_i\}.$$

We call π the ***vector edge-partition***.

Corollary 6.7. *A Kirchhoff graph is a vector graph whose vector edge-partition is Kirchhoff.*

Moreover, observe that Corollary 6.3 and Proposition 6.5 combine to prove,

Proposition 6.10. Let $\pi = (E_1, \ldots, E_k)$ be an edge-partition of a digraph D. Then π is equitable if and only if for any two edges e and f in the same partition cell of π,

$$\big[\lambda(\tau(e)) - \lambda(\iota(e))\big]T = \big[\lambda(\tau(f)) - \lambda(\iota(f))\big]T, \tag{6.27}$$

where T is the characteristic matrix of edge-partition π.

Lemma 6.2. *Every equitable edge-partition is a vector edge-partition.*

Proof. Let D be a digraph with equitable edge-partition $\pi = (E_1, \ldots, E_k)$ and characteristic matrix T. Then by Proposition 6.9, the vector assignment

$$\varphi(e) = \big(\lambda(\tau(e)) - \lambda(\iota(e))\big)T$$

is consistent, so (D, φ) is a vector graph. Moreover by (6.27), for each partition class E_i, every edge of E_i is assigned the same vector under φ. Therefore π is the vector edge-partition of (D, φ). □

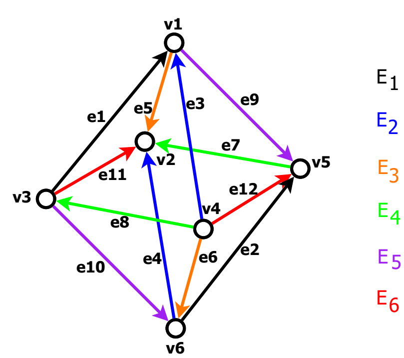


Figure 6.3.1: A digraph D with a 6-cell equitable edge-partition, π.

Example 6.7. Let D be the digraph illustrated in Figure 6.3.1. Let $\pi = (E_1, \dots E_6)$ be the edge-partition of D, where

$$\begin{array}{lll} E_1 = \{e_1, e_2\} & E_2 = \{e_3, e_4\} & E_3 = \{e_5, e_6\} \\ E_4 = \{e_7, e_8\} & E_5 = \{e_9, e_{10}\} & E_6 = \{e_{11}, e_{12}\}. \end{array}$$

Edge-partition π is equitable. Therefore by Lemma 6.2, π is also a vector edge-partition. Let T be the characteristic matrix of π, and let $\varphi : E(D) \to \mathbb{R}^6$ where

$$\varphi(e_i) = [\lambda(\tau(e_i)) - \lambda(\iota(e_i))]T.$$

Vector assignment φ is consistent, and any edges in the same partition class of π are assigned the same vector under φ. For example,

$$\begin{aligned} \varphi(e_1) = [\lambda(v_1) - \lambda(v_3)]T &= \begin{bmatrix} -2 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{bmatrix} T \\ &= \begin{bmatrix} -2 & -1 & 1 & 1 & 0 & -1 \end{bmatrix} \\ \varphi(e_2) = [\lambda(v_4) - \lambda(v_6)]T &= \begin{bmatrix} 0 & -2 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 \end{bmatrix} T \\ &= \begin{bmatrix} -2 & -1 & 1 & 1 & 0 & -1 \end{bmatrix} \end{aligned}$$

More completely,

$$\begin{array}{ccccc} \varphi(e_1) & = & \varphi(e_2) & = & [\ -2\ \ -1\ \ \ 1\ \ \ 1\ \ \ 0\ \ -1\ ] \\ \varphi(e_3) & = & \varphi(e_4) & = & [\ -1\ \ -2\ \ \ 0\ \ -1\ \ \ 1\ \ -1\ ] \\ \varphi(e_5) & = & \varphi(e_6) & = & [\ 1\ \ \ 0\ \ -2\ \ -1\ \ -1\ \ -1\ ] \\ \varphi(e_7) & = & \varphi(e_8) & = & [\ 1\ \ -1\ \ -1\ \ -2\ \ \ 1\ \ \ 0\ ] \\ \varphi(e_9) & = & \varphi(e_{10}) & = & [\ 0\ \ \ 1\ \ -1\ \ \ 1\ \ -2\ \ -1\ ] \\ \varphi(e_{11}) & = & \varphi(e_{12}) & = & [\ -1\ \ -1\ \ -1\ \ \ 0\ \ -1\ \ -2\ ]. \end{array} \tag{6.28}$$

Moreover, as φ is consistent, digraph D can be embedded in $\mathbb{R}^6$ in such a way that each edge is drawn as the vector it is assigned under φ. Figure 6.3.1 shows the projection in $\mathbb{R}^2$ of one such embedding. Observe that any two edges belonging to the same vector edge-partition class are drawn as the same $\mathbb{R}^2$ vector.

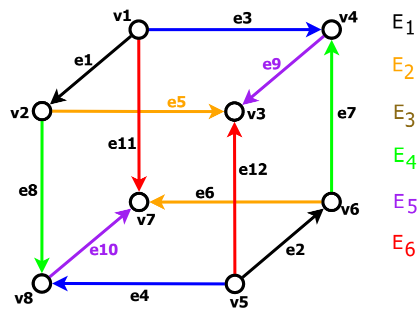


Figure 6.3.2: An equitable edge-partition π of D_3

Example 6.8. Let D_3 be the oriented cube illustrated in Figure 6.3.2. Let $\pi = (E_1, \ldots, E_6)$ be the edge partition given by

$$\begin{array}{lll} E_1 = \{e_1, e_2\} & E_2 = \{e_3, e_4\} & E_3 = \{e_5, e_6\} \\ E_4 = \{e_7, e_8\} & E_5 = \{e_9, e_{10}\} & E_6 = \{e_{11}, e_{12}\}. \end{array}$$

Edge-partition π is equitable. Therefore by Lemma 6.2, π is also a vector edge-partition. Let T be the characteristic matrix of π, and let $\varphi : E(D) \to \mathbb{R}^6$ where

$$\varphi(e_i) = [\lambda(\tau(e_i)) - \lambda(\iota(e_i))]T$$

Vector assignment φ is consistent, and any edges in the same partition class of π are assigned the same vector under φ. For example,

$$\begin{aligned} \varphi(e_1) = [\lambda(v_2) - \lambda(v_1)]T &= \begin{bmatrix} -2 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} T \\ &= \begin{bmatrix} -2 & -1 & 1 & 1 & 0 & -1 \end{bmatrix} \\ \varphi(e_2) = [\lambda(v_6) - \lambda(v_5)]T &= \begin{bmatrix} 0 & -2 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} T \\ &= \begin{bmatrix} -2 & -1 & 1 & 1 & 0 & -1 \end{bmatrix} \end{aligned}$$

More completely,

$$\begin{array}{ccccc} \varphi(e_1) & = & \varphi(e_2) & = & [\ -2 \ \ -1 \ \ 1 \ \ 1 \ \ 0 \ \ -1\ ] \\ \varphi(e_3) & = & \varphi(e_4) & = & [\ -1 \ \ -2 \ \ 0 \ \ -1 \ \ 1 \ \ -1\ ] \\ \varphi(e_5) & = & \varphi(e_6) & = & [\ 1 \ \ 0 \ \ -2 \ \ -1 \ \ -1 \ \ -1\ ] \\ \varphi(e_7) & = & \varphi(e_8) & = & [\ 1 \ \ -1 \ \ -1 \ \ -2 \ \ 1 \ \ 0\ ] \\ \varphi(e_9) & = & \varphi(e_{10}) & = & [\ 0 \ \ 1 \ \ -1 \ \ 1 \ \ -2 \ \ -1\ ] \\ \varphi(e_{11}) & = & \varphi(e_{12}) & = & [\ -1 \ \ -1 \ \ -1 \ \ 0 \ \ -1 \ \ -2\ ]. \end{array} \tag{6.29}$$

Note that these vectors assigned by φ are identical to those of (6.28). In this case, vector assignment φ is consistent with D_3. However, embedding this graph in $\mathbb{R}^6$ is less straightforward than the digraph in Example 6.7. Observe, for example, that

$$v_1 \cdot (e_1) \cdot (e_5) \cdot (-e_{12}) \cdot v_5$$

is a path in D_3. However,

$$\begin{aligned}\varphi(e_1) + \varphi(e_5) - \varphi(e_{12}) &= \left[\begin{array}{cccccc} -2 & -1 & 1 & 1 & 0 & -1 \end{array}\right] + \left[\begin{array}{cccccc} 1 & 0 & -2 & -1 & -1 & -1 \end{array}\right] \\ &\qquad - \left[\begin{array}{cccccc} -1 & -1 & -1 & 0 & -1 & -2 \end{array}\right] \\ &= \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].\end{aligned}$$

Therefore in any embedding of D_3 in $\mathbb{R}^6$, if each edge is drawn as the vector it is assigned under φ, vertex v_1 and vertex v_5 must be embedded at the same point. In a similar manner, one finds that the pairs v_2 and v_6, v_3 and v_7, and v_4 and v_8 must each be embedded at the same points. The result of any such embedding is illustrated in Figure 6.3.3. This leads to a new digraph, which maintains an equitable edge-partition. More importantly, under this edge-partition the digraph can be embedded in $\mathbb{R}^6$ on vector edges.

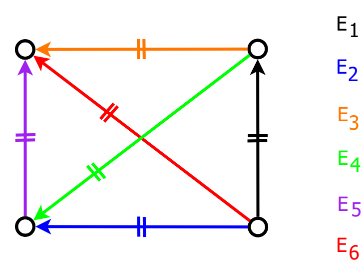


Figure 6.3.3: An equitable vector edge-partition arising from a vector embedding of D_3 in $\mathbb{R}^6$.

Remark 6.6. The three examples in Figures 6.3.1, 6.3.2, and 6.3.3 are intentionally presented together. Figure 6.3.1 is an equitable edge-partition that is not an orbit partition. Using the corresponding vector edge-partition, the digraph may be embedded in $\mathbb{R}^6$ on vector edges. Figure 6.3.2 is an *orbit* edge-partition, and therefore equitable. However, the digraph collapses under an embedding in $\mathbb{R}^6$ on the vector edges of the associated the vector edge-partition. The collapsed embedding is presented in Figure 6.3.3. This edge-partition is now an orbit partition, as was that of Figure 6.3.2, but also embeddable in $\mathbb{R}^6$ on vector edges, as was that of Figure 6.3.1.

Although each respective edge-partition has different properties, each is a Kirchhoff partition. Moreover, the three partitions may be considered *equivalent* in the Kirchhoff sense.

That is, if one considers cycles with respect to edge-partition classes, each digraph-partition pair has exactly the same set of cycles. More specifically, for any cycle C in any of these three graphs, letting T be the appropriate characteristic matrix,

$$\chi(C)T \in \operatorname{span}\left\{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}\right\}.$$

This example inspires a number of further questions. Specifically, understanding this "collapsing" phenomenon is an interesting problem. For example,

Question: Given an equitable edge-partition π of a digraph D, can we determine when the associated vector edge-partition requires that vertices be identified once edges are taken to be vectors?

Chapter 7

Additional Topics in Graph Partitions

This chapter presents a collection of additional results on both vertex and edge-partitions. With the exception of Sections 7.1 and 7.2, each section should be considered independently of the others. Although the sections of Chapter 7 are not directly interrelated, each presents interesting examples or natural extensions of the results in Chapters 5 and 6. These results do not necessarily fit easily into the framework of those chapters, and therefore have been collected here as additional topics.

Section 7.1 reviews so-called *almost* equitable partitions, introduced by Cardoso [13] as a generalization of equitable partitions. Almost equitable partitions share many classical properties with equitable partitions when the Laplacian matrix is used in place of the adjacency matrix. Section 7.2 then combines the ideas of Section 7.1 and Chapter 6 to define almost equitable edge-partitions. Theorem 7.1 demonstrates that an almost equitable edge-partition with a symmetric quotient matrix is Kirchhoff. Next, Section 7.3 shows that equitable edge-partitions can be constructed from certain equitable (vertex) partitions. Section 7.3.1 then uses the results of Section 7.3 to construct three infinite families of equitable edge-partitions by examining 3 types of equitable (vertex) partitions on prism graphs. Finally, Section 7.4 explores the interactions between *layers* of partitions, giving an example when a pair of quotients–taken in either order–leads to the same resultant quotient.

7.1 Almost Equitable Partitions

Let G be a graph with equitable partition $\pi = (V_1, \ldots, V_k)$ of $V(G)$. Recall that for any $v_p \in V_i$,

$$c_{i,j} = \sum_{q:v_q \in V_j} A(G)_{p,q}$$

is independent of p. Observe that for each i, the induced subgraph $G[V_i]$ must be regular of degree $c_{i,i}$. Relaxing this condition results in so-called almost equitable partitions. More precisely, an ***almost equitable partition*** is a partition $\pi = \{V_1, \ldots, V_k\}$ of $V(G)$ such that for all $i \neq j$ and any $v_p \in V_i$,

$$c_{i,j} = \sum_{q:v_q \in V_j} A(G)_{p,q} \tag{7.1}$$

is independent of p. Note that the definitions of equitable and almost equitable partitions are nearly identical, only $c_{i,i}$ need not be well-defined for a partition to be almost equitable. In particular, any equitable partition is almost equitable.

Example 7.1. Consider the partition $\pi = (V_1, V_2, V_3)$ of the graph in Figure 7.1.1, where $V_1 = \{v_1, v_2, v_3, v_4\}$ and $V_2 = \{v_5, v_6\}$, and $V_3 = \{v_7\}$. Every vertex in V_1 has 1 neighbor in V_2, and every vertex in V_2 has 2 neighbors in V_1. Similarly every vertex in V_1 or V_2 has 1 neighbor in V_3, and every vertex in V_3 has 4 neighbors in V_1 and 2 neighbors in V_2. Therefore π is an almost equitable partition with

$$c_{1,2} = 1, c_{2,1} = 2, c_{1,3} = c_{2,3} = 1, c_{3,1} = 4, c_{3,2} = 2.$$

However, $V_1 = \{v_1, v_2, v_3, v_4\}$, and v_1 and v_4 each have 1 neighbor in V_1 while v_2 and v_3 each have 2, meaning π is not equitable.

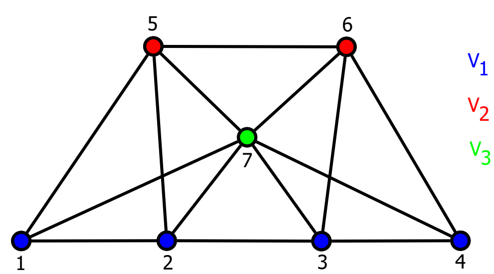


Figure 7.1.1: An almost equitable vertex partition.

Remark 7.1. Much as the definitions of equitable and almost equitable are similar, the examples one constructs are closely related as well. By definition, every equitable partition is almost equitable. More than that, deleting all edges of an almost equitable partition with two endvertices in the same cell results in an equitable partition. Conversely, beginning with any equitable partition, one may freely add and delete edges between vertices of the same partition cell, and the result is an almost equitable partition.

Let G be a graph with vertices $V(G) = \{v_i\}$ and almost equitable partition $\pi = (V_1, \ldots, V_k)$. Many of the results in Section 5.1 continue to hold, with the Laplacian matrix $L(G)$ now in place of $A(G)$. Let $\Delta(G)$ be a $|V(G)| \times |V(G)|$ diagonal matrix with (i,i)-entry $\sum_{k \neq i} A(G)_{i,k}$. Then the ***Laplacian matrix*** of G, $L(G)$, is given by

$$L(G) = \Delta(G) - A(G). \tag{7.2}$$

Equivalently,

$$L(G)_{i,j} = \begin{cases} -A(G)_{i,j} & \text{if } i \neq j \\ \sum_{k \neq i} A(G)_{i,k} & \text{if } i = j \end{cases}. \tag{7.3}$$

We define the Laplacian matrix of G/π analogously to (7.3).

$$L(G/\pi)_{i,j} = \begin{cases} -c_{i,j} & \text{if } i \neq j \\ \sum_{k \neq i} c_{i,k} & \text{if } i = j \end{cases}. \tag{7.4}$$

For example, given G and π as in Figure 7.1.1,

$$L(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad L(G/\pi) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \end{array} \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 \\ -2 & 3 & -1 \\ -4 & -2 & 6 \end{bmatrix} \end{array}.$$

These Laplacian matrices now satisfy the almost equitable analogue of Lemma 5.1.

Lemma 7.1. *Let π be a partition of $V(G)$ with characteristic matrix P. If π is almost equitable then*

$$L(G)P = PL(G/\pi).$$

Proof. Let v_p be any vertex of G, and suppose that $v_p \in V_i$. For any $j \neq i$, by (7.3),

$$[L(G)P]_{p,j} = \sum_{q : v_q \in V_j} L(G)_{p,q} = \sum_{q : v_q \in V_j} -A(G)_{p,q} = -c_{i,j}.$$

Similarly by (7.3),

$$[L(G)P]_{p,i} = \sum_{k \neq i} A(G)_{i,k} - \sum_{q: v_q \in V_i} A(G)_{p,q} = \sum_{k \neq i} c_{i,k}.$$

On the other hand, as $v_p \in V_i$, row p of P is zero except for a 1 in column i. Therefore row p of $PL(G/\pi)$ is row i of $L(G/\pi)$. Thus by (7.4),

$$[PA(G/\pi)]_{p,j} = \begin{cases} -c_{i,j} & \text{when } j \neq i \\ \sum_{k \neq i} c_{i,k} & \text{when } j = i. \end{cases}$$

Therefore $L(G)P = PL(G/\pi)$. □

7.2 Almost Equitable Edge-Partitions

Combining the ideas presented in Sections 6.1 and 7.1, we now consider almost equitable edge-partitions. Let D be a digraph with signed edge-adjacency matrix $A_E(D)$.

Definition 7.1. An edge-partition $\pi = (E_1, \ldots, E_k)$ is ***almost equitable*** if for all $i \neq j$ and any $e_p \in E_i$,

$$d_{i,j} = \sum_{q: e_q \in E_j} A_E(D)_{p,q} \tag{7.5}$$

depends only on i and j, and not on the choice of edge $e_p \in E_i$.

Comparing (7.5) to (7.1), the definition of an almost equitable edge-partition is analogous to that of an almost equitable partition, with the signed edge-adjacency matrix $A_E(D)$ in (7.5) taking the place of adjacency matrix $A(G)$ in (7.1). Moreover, observe that the definition of an equitable edge-partition (Definition 6.2) and an almost equitable edge-partition (Definition 7.1) are nearly identical, only $d_{i,i}$ need not be defined for an edge-partition to be almost equitable. If $d_{i,i}$ *is* defined for all i, then the partition is equitable and, in particular, every equitable edge-partition is almost equitable.

For example, the digraph in Figure 7.2.1 has an almost equitable edge-partition $\pi = (E_1, E_2, E_3, E_4)$ with

$$E_1 = \{e_1, e_2, e_3, e_4\} \quad E_2 = \{e_5, e_6\} \quad E_3 = \{e_7, e_8\} \quad E_4 = \{e_9, e_{10}\}.$$

For all $i \neq j$, the constant $d_{i,j}$ is given by

$$\begin{array}{cccc} & d_{1,2}=0 & d_{1,3}=1 & d_{1,4}=-1 \\ d_{2,1}=0 & & d_{2,3}=1 & d_{2,4}=1 \\ d_{3,1}=2 & d_{3,2}=1 & & d_{3,4}=-1 \\ d_{4,1}=-2 & d_{4,2}=1 & d_{4,3}=-1 & \end{array}$$

Note that while π is almost equitable, it is *not* equitable. In particular, $E_1 = \{e_1, e_2, e_3, e_4\}$, and edges e_1 and e_2 are incident to each other at a single vertex, while e_3 and e_4 are not incident to any other edges of E_1.

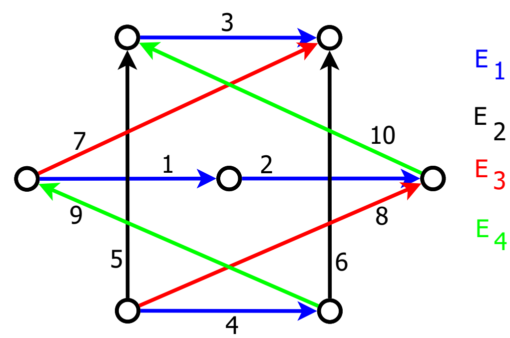


Figure 7.2.1: An almost equitable edge-partition.

Let D be a digraph with edges $\{e_j\}$ and almost-equitable partition $\pi = (E_1, \ldots E_k)$. Many of the results of Section 6.1 continue to hold, now with a Laplacian matrix in place of $A_E(D)$. Let $\Delta_E(D)$ be an $|E(D)| \times |E(D)|$ diagonal matrix with (i,i)-entry

$$\Delta_E(D)_{i,i} = \sum_{k \neq i} A_E(D)_{i,k}.$$

That is, $\Delta_E(D)$ is the edge-incidence analogue of degree matrix $\Delta(G)$. Define the ***signed edge Laplacian matrix*** of D, $L_E(D)$, by

$$L_E(D) = \Delta_E(D) - A_E(D). \tag{7.6}$$

Equivalently,

$$L_E(D)_{i,j} = \begin{cases} -A_E(D)_{i,j} & \text{if } i \neq j \\ \sum_{k \neq i} A_E(D)_{i,k} & \text{if } i = j \end{cases}. \tag{7.7}$$

Or, in light of Proposition 6.1,

$$L_E(D) = (\Delta_E(D) + 2I) - Q^t Q. \tag{7.8}$$

Matrix $L_E(D)$ is always symmetric. Comparing (7.2) and (7.3) to (7.6) and (7.7), the definition of the signed edge Laplacian is analogous to that of the Laplacian, with signed edge-adjacency matrix $A_E(D)$ in the place of $A(G)$. Combining (7.4) with (7.6), we naturally define $L_E(D/\pi)$ as the $k \times k$ matrix such that

$$L_E(D/\pi)_{i,j} = \begin{cases} -d_{i,j} & \text{if } i \neq j \\ \sum_{k \neq i} d_{i,k} & \text{if } i = j \end{cases}. \tag{7.9}$$

For example, given D and π as in Figure 7.2.1,

$$L_E(D) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 2 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -2 \end{bmatrix} \end{array},$$

and

$$L_E(D/\pi) = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 2 & -1 & -1 \\ -2 & -1 & 2 & 1 \\ 2 & -1 & 1 & -2 \end{bmatrix} \end{array}.$$

Once again, matrices $L_E(D)$ and $L_E(D/\pi)$ are related by the characteristic matrix of π.

Lemma 7.2. *Let π be a partition of $E(D)$ with characteristic matrix T. If π is almost equitable then*

$$L_E(D)T = TL_E(D/\pi).$$

Proof. For any edge e_p of D, suppose that $e_p \in E_i$. For any $j \neq i$, by (7.7),

$$[L_E(D)T]_{p,j} = \sum_{q:e_q \in E_j} L_E(D)_{p,q} = \sum_{q:e_q \in E_j} -A_E(D)_{p,q} = -d_{i,j}.$$

Similarly by (7.7),

$$[L_E(D)T]_{p,i} = \sum_{k \neq i} A_E(D)_{i,k} - \sum_{q:e_q \in E_i} A_E(D)_{p,q} = \sum_{k \neq i} d_{i,k}.$$

On the other hand, as $e_p \in E_i$, row p of T is zero except for a 1 in column i. Therefore row p of $TL_E(D/\pi)$ is row i of $L_E(D/\pi)$. Thus by (7.9),

$$[TL_E(D/\pi)]_{p,j} = \begin{cases} -d_{i,j} & \text{when } j \neq i \\ \sum_{k\neq i} d_{i,k} & \text{when } j = i. \end{cases}$$

Therefore $L_E(D)T = TL_E(D/\pi)$. □

Similar to Section 6.1, Laplacian matrices $L_E(D)$ and $L_E(D/\pi)$ can be written in terms of vertex incidence vectors. Recall by (7.6),

$$L_E(D) = \Delta_E(D) - A_E(D),$$

where $\Delta_E(D)$ was the $|E(D)| \times |E(D)|$ diagonal matrix with (i,i)-entry $\sum_{k\neq i} A_E(D)_{i,k}$. Letting $V_{\mathcal{I}}$ and $V_{\mathcal{T}}$ be defined as previously, by Corollary 6.1,

$$A_E(D) = V_{\mathcal{I}} - V_{\mathcal{T}} - 2I.$$

Therefore,

$$L_E(D) = (V_{\mathcal{T}} - V_{\mathcal{I}}) + 2I_m + \Delta_E(D). \tag{7.10}$$

On the other hand, letting $\Delta_E(D/\pi)$ be the $k \times k$ diagonal matrix with (i,i)-entry $\sum_{k\neq i} d_{i,k}$,

$$L_E(D/\pi) = \Delta_E(D/\pi) - A_E(D/\pi).$$

Moreover, taking $\tilde{V_{\mathcal{I}}}$ and $\tilde{V_{\mathcal{T}}}$ as defined previously, by Proposition 6.4,

$$A_E(D/\pi) = (\tilde{V_{\mathcal{I}}} - \tilde{V_{\mathcal{T}}})T - 2I_k.$$

Therefore,

$$L_E(D/\pi) = (\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}})T + 2I_k + \Delta_E(D/\pi). \tag{7.11}$$

Lemma 7.3. *If π is an almost equitable-edge partition, then*

$$(V_{\mathcal{T}} - V_{\mathcal{I}})\, T = T\left(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}}\right) T.$$

Proof. First, observe that by definition of $\Delta_E(D)$ and $\Delta_E(D/\pi)$,

$$\Delta_E(D)T = T\Delta_E(D/\pi). \tag{7.12}$$

Therefore,

$$\begin{aligned}
(V_{\mathcal{T}} - V_{\mathcal{I}})\,T &= L_E(D)T - 2T - \Delta_E(D)T && \text{by (7.10)}\\
&= TL_E(D/\pi) - 2T - \Delta_E(D)T && \text{by Lemma 7.2}\\
&= T\left((\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}})T + 2I_k + \Delta_E(D/\pi)\right) - 2T - \Delta_E(D)T && \text{by (7.11)}\\
&= T\left(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}}\right)T - 2T + 2T + T\Delta_E(D/\pi) - \Delta_E(D)T\\
&= T\left(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}}\right)T. && \text{by (7.12)}
\end{aligned}$$

□

At long last, Lemma 7.3 allows us to understand the relationship between almost equitable edge-partitions and Kirchhoff graphs.

Theorem 7.1. *Let D be a digraph with almost equitable edge-partition $\pi = (E_1, \ldots, E_k)$. If the matrix $L_E(D/\pi)$ is symmetric, then π is Kirchhoff.*

Proof. For any D, the matrix $L_E(D)$ is symmetric and thus by (7.10), $V_{\mathcal{T}} - V_{\mathcal{I}}$ is symmetric as well. By assumption $L_E(D/\pi)$ is symmetric, so by (7.11), $(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}})T$ is also symmetric. As π is almost equitable,

$$(V_{\mathcal{T}} - V_{\mathcal{I}})\,T. = T\left(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}}\right)T. \tag{7.13}$$

Taking the transpose of (7.13),

$$\begin{aligned}
T^t\,(V_{\mathcal{T}} - V_{\mathcal{I}})^t &= T^t\left(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}}\right)^t T^t\\
&= \left[(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}})T\right]^t T^t.
\end{aligned}$$

Thus as both $V_{\mathcal{T}} - V_{\mathcal{I}}$ and $(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}})T$ are symmetric,

$$T^t\,(V_{\mathcal{T}} - V_{\mathcal{I}}) = (\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}})TT^t.$$

Therefore,

$$TT^t\,(V_{\mathcal{T}} - V_{\mathcal{I}}) = T(\tilde{V_{\mathcal{T}}} - \tilde{V_{\mathcal{I}}})TT^t,$$

and by (7.13),

$$TT^t(V_{\mathcal{T}} - V_{\mathcal{I}}) = (V_{\mathcal{T}} - V_{\mathcal{I}})TT^t. \tag{7.14}$$

However, the rows of $V_{\mathcal{T}} - V_{\mathcal{I}}$ span the row space of incidence matrix $Q = Q(D)$. Therefore the rows of $TT^t(V_{\mathcal{T}} - V_{\mathcal{I}})$ all lie in the row space of Q, and so by (7.14), each row of $(V_{\mathcal{T}} - V_{\mathcal{I}})TT^t$ lies in Row(Q) as well. Thus the row space of Q is invariant under right multiplication by TT^t, and by Proposition 6.6, π is Kirchhoff. □

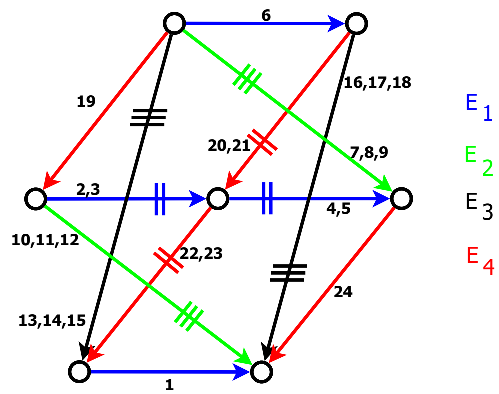


Figure 7.2.2: An almost equitable edge-partition with symmetric quotient.

Example 7.2. Consider the digraph D with 24 edges presented in Figure 7.2.2. D has almost equitable edge-partition $\pi = (E_1, E_2, E_3, E_4)$, where

$$\begin{aligned} E_1 &= \{e_1, e_2, e_3, e_4, e_5, e_6\} \\ E_2 &= \{e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\} \\ E_3 &= \{e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}\} \\ E_4 &= \{e_{19}, e_{20}, e_{21}, e_{22}, e_{23}, e_{24}\}. \end{aligned}$$

Under edge-partition π,

$$L_E(D/\pi) = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 1 & 0 & -3 & 2 \end{bmatrix}.$$

Matrix $L_E(D/\pi)$ is symmetric, and therefore by Theorem 7.1, π is Kirchhoff. One may verify that for any vertex v of D,

$$\lambda(v)T \in \left\{ \pm \begin{bmatrix} 1 & 3 & 3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 3 & 2 \end{bmatrix}, \pm \begin{bmatrix} 2 & 3 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \right\}. \tag{7.15}$$

Similarly for any cycle C,

$$\chi(C)T \in \operatorname{Span}\left\{ \begin{bmatrix} 2 & -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix} \right\}. \tag{7.16}$$

As every vector in (7.15) is orthogonal to those in (7.16), π is Kirchhoff. For completeness, the 24×24 matrix $L_E(D)$ is given in (7.17). One may verify that, as guaranteed by Lemma 7.2, $L_E(D)T = TL_E(D/\pi)$.

$$L_E(D) = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 \\
0 & 2 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 \\
0 & -2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 \\
0 & 1 & 1 & 2 & -2 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 \\
0 & 1 & 1 & -2 & 2 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 10 & -2 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 & -1 & -2 & 10 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 & -1 & -2 & -2 & 10 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & -2 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 10 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 10 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 10 & -2 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -2 & 10 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -2 & -2 & 10 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 10 & -2 & -2 & 0 & -1 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -2 & 10 & -2 & 0 & -1 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -2 & -2 & 10 & 0 & -1 & -1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 2 & -2 & 1 & 1 & 0 \\
0 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -2 & 2 & 1 & 1 & 0 \\
1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & -2 & 0 \\
1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 2 & 0 \\
-1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix} \tag{7.17}$$

Finally, the almost equitable analogue of Theorem 6.3 holds. Once again, for any edge-partition $\pi = (E_1, \ldots, E_k)$ with characteristic matrix T, let Λ be the $k \times k$ diagonal matrix with $\Lambda_{i,i} = 1/|E_i|$. In particular,

$$\Lambda T^t T = I_k. \tag{7.18}$$

Moreover, if π is almost equitable, by Lemma 7.2,

$$L_E(D/\pi) = \Lambda T^t T L_E(D/\pi) = \Lambda T^t L_E(D) T. \tag{7.19}$$

Theorem 7.2. *Let D be a connected digraph with almost equitable edge-partition π. Then $L_E(D/\pi)$ is symmetric if and only if π is uniform.*

Proof. First suppose that π is uniform. Then in particular, $\Lambda = cI$ for some constant c. By (7.19),

$$L_E(D/\pi) = cT^t L_E(D) T.$$

Therefore,

$$\begin{aligned} L_E(D/\pi)^t &= (cT^t L_E(D) T)^t \\ &= cT^t L_E(D)^t T \\ &= cT^t L_E(D) T = L_E(D/\pi) \end{aligned}$$

and $L_E(D/\pi)$ is symmetric. Conversely, suppose that $L_E(D/\pi)$ is symmetric. Then by (7.19), as $L_E(D)$ and Λ are both symmetric,

$$\Lambda(T^t L_E(D)T) = L_E(D/\pi) = L_E(D/\pi)^t = (\Lambda T^t L_E(D)T)^t = (T^t L_E(D)T)\Lambda.$$

Therefore for all $i \neq j$,

$$\Lambda_{i,i}(T^t L_E(D)T)_{i,j} = (\Lambda T^t L_E(D)T)_{i,j} = (T^t L_E(D)T\Lambda)_{i,j} = (T^t L_E(D)T)_{i,j}\Lambda_{j,j}. \tag{7.20}$$

As D is connected, it follows that no simultaneous permutation of rows and columns can transform $T^t L_E(D)T$ into a block-diagonal matrix of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Therefore as (7.20) is true for all $i \neq j$, it follows that all diagonal entries of Λ must be equal. Therefore $|E_i|$ is independent of i and, π is uniform. □

7.2.1 A Word on Spectral Graph Theory

The Laplacian matrix of a graph was introduced in section 7.1. The study of graph Laplacian eigenvalues–spectral graph theory–has developed into an extraordinarily impactful research area. From its algebraic beginnings, when eigenvalues were used to study problems such as connectivity and graph coloring, the applications of spectral graph theory are now broad and far-reaching. These include graph drawing, machine learning, optimization, random walks, Riemannian geometry, randomized algorithms, communication networks, stability of chemical molecules, rapidly mixing Markov chains, and quasi-randomness, among countless others. It is beyond the scope of this text to present a literature review of even the most important spectral graph theory results, but we would be remiss if we did not acknowledge the significance of the field. We refer the reader to the texts of Biggs [8], and Cvetković, Doob, and Sachs [21] for an introduction to the algebraic aspects of spectral graph theory. A thorough and more geometric treatment can be found in Chung [16]. Each of these texts also contain a wealth of additional references. More modern approaches, including many applications, are addressed in the texts of Brouwer and Haemmers [12], and Spielman [101]. The surveys of Mohar [80] and Merris [79], and their references, also identify a number of interesting applications.

The Laplacian matrix L of a graph G satisfies 3 main properties.

(1.) L is a square $|V(G)| \times |V(G)|$ matrix

(2.) L is symmetric

(3.) All rows and columns of L sum to 0.

Section 7.2 presented a natural definition of an *edge* Laplacian matrix, $L_E(D)$, of a digraph D. As defined, $L_E(D)$ satisfies the same properties as its vertex counterpart,

(1.) $L_E(D)$ is a square $|E(D)| \times |E(D)|$ matrix

(2.) $L_E(D)$ is symmetric

(3.) All rows and columns of $L_E(D)$ sum to zero.

Moreover, via the incidence matrix of a graph, the edge Laplacian matrix is related to the to the combinatorial Laplacian. Specifically, let G be a graph, and let D be a digraph obtained from G via an arbitrary orientation of the edges. Then if $Q = Q(D)$ is the incidence matrix of the resulting digraph,

$$L = L(G) = QQ^t.$$

On the other hand, up to rescaling diagonal entries, $L_E(D)$ can be written in terms of Q^tQ (see (7.8)). To the author's knowledge edge Laplacian $L_E(D)$ is not well-studied, and exploring the spectral properties of $L_E(D)$ presents an intriguing research topic in its own right. For example,

Question: Can the eigenvalues (or eigenvectors) of $L_E(D)$ be used to determine if D has a Kirchhoff edge-partition?

Question: Can the eigenvalues (or eigenvectors) of $L_E(D)$ be used to determine if D has an equitable edge-partition?

The answer to these questions may well lie in (7.8). Letting D be any multi-digraph, then the edge Laplacian matrix was defined to be

$$L_E(D) = \Delta_E(D) - A_E(D).$$

Alternatively,

$$L_E(D) = (\Delta_E(D) + 2I) - Q^tQ.$$

As any vector $\boldsymbol{x}$ is an eigenvector of the identity matrix, one way to study eigenvectors (and eigenvalues) of $L_E(D)$ is to study the eigenvectors of $\Delta_E(D)$ in comparison to those of Q^tQ. For example, let $\boldsymbol{x}$ be any vector in the cycle space of D. That is,

$$\boldsymbol{x}^t = \chi(C_1) + \chi(C_2) + \cdots + \chi(C_p)$$

for some cycles C_i of D. Then as the cycle vectors of D span the null space of Q,

$$Q^t Q\boldsymbol{x} = \mathbf{0}.$$

Therefore $\boldsymbol{x}$ is an eigenvector of $L_E(D)$ if and only if it is an eigenvector of $\Delta_E(D)$, and letting σ_L and σ_Δ denote the associated eigenvalues,

$$\sigma_L = \sigma_\Delta - 2.$$

Even moving beyond the topics discussed in previous chapters,

Question: Can $L_E(D)$ be used to study connectivity or flow in digraphs?

Answers to this question would likely involve deriving eigenvalue bounds for $L_E(D)$ based on the structure of digraph D and, more importantly, understanding why the eigenvalues of $L_E(D)$ are significant. That is, an *interpretation* of what the eigenvalues of $L_E(D)$ mean in terms of the properties of a digraph would be incredibly beneficial. For example, the second largest eigenvalue of the Laplacian matrix of an undirected graph is often referred to as the *spectral gap*. It is well-known that this eigenvalue is closely related to the expansion properties of a graph [16]. A similar interpretation of an eigenvalue of $L_E(D)$ would further motivate the study of edge Laplacian $L_E(D)$. Many interesting results pertaining to the graph Laplacian–including eigenvalue bounds and associated graph properties–are accomplished via quadratic forms derived from the Laplacian matrix. A similar course of action, now using the edge Laplacian $L_E(D)$, seems a natural place to start. Although spectral properties of $L_E(D)$ will not be discussed further in this text, this would be an excellent topic to begin and inspire future research.

In addition to edge Laplacian $L_E(D)$, another closely related matrix may prove illuminating. Specifically, observe that

$$L = QQ^t, \tag{7.21}$$

where Q is the incidence matrix of a digraph. That is, Q is a $|V(D)| \times |E(D)|$ matrix whose i^{th} row is $\lambda(v_i)$, the incidence vector of vertex v_i. Now let π be any edge-partition $\pi = (E_1, \ldots, E_k)$ of $E(D)$ with characteristic matrix T. Then the π-incidence vector of vertex v_i is given by

$$\lambda_\pi(v_i) = \lambda(v_i)T.$$

The j^{th} entry of the vector $\lambda_\pi(v_i)$ represents the net number of edges of partition cell E_j incident with vertex v_i. Moreover, this vector is precisely the i^{th} row of the matrix product QT. As such, we will call QT the *π-incidence matrix* of D (with respect to edge-partition

π). Building upon (7.21), consider the matrix

$$L_\pi = (QT)(QT)^t = Q(TT^t)Q^t.$$

Observe that L_π is also $|V| \times |V|$ and symmetric, and we call L_π the π-*Laplacian matrix.*

Without loss of generality, we may reorder the columns of Q (and thus the rows of T) so that T is of the form in (3.19). That is, the first $|E_1|$ rows of T contain a 1 in column 1, the next $|E_2|$ rows contain a 1 in column 2, and so on. Then the product TT^t is a block-diagonal matrix of the form

$$TT^t = \begin{bmatrix} J_{|E_1|} & 0 & 0 & 0 \\ 0 & J_{|E_2|} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{|E_k|} \end{bmatrix},$$

where J_n is the $n \times n$ all-ones matrix. Alternatively, letting

$$H_n = J_n - I_n$$

be the matrix of all-ones except for a zero-diagonal, define H to be the matrix

$$H = \begin{bmatrix} H_{|E_1|} & 0 & 0 & 0 \\ 0 & H_{|E_2|} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{|E_k|} \end{bmatrix}.$$

Then H is the adjacency matrix of a graph G, where G the disjoint union of complete graphs

$$G = K_{|E_1|} \cup K_{|E_2|} \cup \cdots \cup K_{|E_k|},$$

the eigenvalues and eigenvectors of which are well-understood. Moreover,

$$TT^t = H + I_{|V(D)|}.$$

Therefore,

$$\begin{aligned} L_\pi &= (QT)(QT)^t \\ &= Q(TT^t)Q^t \\ &= Q(H + I)Q^t \\ &= QHQ^t + QQ^t \\ &= QHQ^t + L. \end{aligned}$$

That is, given edge-partition π, the π-Laplacian matrix L_π can be written as the sum of the classical Laplacian matrix and QHQ^t, a term which presumably accounts for the added structure of the edge partition. Once again, the eigenvectors (and eigenvalues) of L_π can be understood by studying those of L compared to those of QHQ^t. While this will not be explored further here, this presents another excellent starting point for research into the spectra of edge-partitioned graphs.

7.3 Equitable Partitions Generate Equitable Edge-Partitions

In order to rationalize the study of equitable edge-partitions, one must verify that reasonable classes of them exist. In fact, many graphs with equitable (vertex) partitions lead to digraphs with equitable edge-partitions.

Lemma 7.4. *Let G be a graph with equitable partition $\pi = (V_1, \ldots, V_k)$. If $A(G/\pi)_{i,i} = 0$ for all $i \in \{1, \ldots, k\}$, then G and π induce a digraph with an equitable edge-partition.*

Proof. Derive a digraph D from G by adding an orientation to each edge such that the initial vertex is that belonging to the partition cell with the lower index. By assumption $A(G/\pi)_{i,i} = 0$ for all i, so there are no edges with both vertices in the same partition cell. Let Π be a (doubly-indexed) edge-partition $\Pi = (E_{(i,j)} : i < j \in \{1, \ldots k\})$ such that for all $i < j$,

$$E_{(i,j)} = \{e : \iota(e) \in V_i \text{ and } \tau(e) \in V_j\}.$$

Note that for ease of notation we use double indexing, and allow for empty partition cells. To show that Π is an equitable edge-partition, we must show that for any pair of partition cells $E_{(i,j)}$ and $E_{(i',j')}$, and any edge $e_p \in E_{(i,j)}$, the sum

$$\sum_{q : e_q \in E_{(i',j')}} A_E(D)_{p,q} \tag{7.22}$$

is independent of p. Observe that for any $e_p \in E_{(i,j)}$ and any $e_q \in E_{(i',j')}$,

$$\iota(e_p) \in V_i \text{ and } \tau(e_p) \in V_j \quad ; \quad \iota(e_q) \in V_{i'} \text{ and } \tau(e_q) \in V_{j'}. \tag{7.23}$$

The proof is now reduced to checking 5 simple cases. Note that by construction, in all cases $i < j$ and $i' < j'$.

Case 1: $\{i, j, i', j'\}$ are distinct. By (7.23) $A_E(D)_{p,q} = 0$ for all $e_p \in E_{(i,j)}$ and $e_q \in E_{(i',j')}$ therefore the sum (7.22) is 0, and independent of p.

Case 2: $i = i'$ **and** $j \neq j'$ For any $e_p \in E_{(i,j)}$, $\iota(e_p) \in V_{i'}$ and $\iota(e_p)$ has $A(G/\pi)_{i',j'}$ neighbors in $V_{j'}$. As $i = i' < j'$, there are $A(G/\pi)_{i',j'}$ edges in $E_{(i',j')}$ with initial vertex $\iota(e_p)$. As $j \neq j'$, $\tau(e_p)$ is not the endvertex of any edge in $E_{(i',j')}$. Therefore (7.22) is equal to $A(G/\pi)_{i',j'}$, and independent of p.

Case 3: $i \neq i'$ **and** $j = j'$**.** For any $e_p \in E_{(i,j)}$, $\tau(e_p) \in V_{j'}$ and $\tau(e_p)$ has $A(G/\pi)_{i',j'}$ neighbors in $V_{i'}$. As $i' < j' = j$, there are $A(G/\pi)_{i',j'}$ edges in $E_{(i',j')}$ with terminal vertex $\tau(e_p)$. As $i \neq i'$, $\iota(e_p)$ is not the endvertex of any edge in $E_{(i',j')}$. Therefore (7.22) is equal to $A(G/\pi)_{i',j'}$, which is independent of p.

Case 4: $i = i'$ **and** $j = j'$**.** For any e_p in $E_{(i,j)}$, $\iota(e_p)$ has $A(G/\pi)_{i,j}$ neighbors in V_j. As $i < j$, there are $A(G/\pi)_{i,j} - 1$ edges other than e_p in $E_{i,j}$ with initial vertex $\iota(e_p)$. Similarly, $\tau(e_p)$ has $A(G/\pi)_{j,i}$ neighbors in V_i. Thus there are $A(G/\pi)_{j,i} - 1$ edges other than e_p in $E_{(i,j)}$ with terminal vertex $\tau(e_p)$. Therefore (7.22) is equal to $A(G/\pi)_{i,j} + A(G/\pi)_{j,i} - 2$, and independent of p.

Case 5: $i' = j$**.** Then for any $e_p \in E_{(i,j)}$, $\tau(e_p) \in V_{i'}$ and $\tau(e_p)$ has $A(G/\pi)_{i',j'}$ neighbors in $V_{j'}$. As $i < j = i' < j'$, there are $A(G/\pi)_{i',j'}$ edges in $E_{(i',j')}$ with *initial* vertex $\tau(e_p)$. As $i < i' < j'$, $\iota(e_p)$ is not the endvertex of any edges of $E_{(i',j')}$. Therefore (7.22) is equal to $-A(G/\pi)_{i',j'}$, and independent of p.

□

Example 7.3. Lemma 7.4 can be illustrated with the following example. Figure 7.3.1 shows an orbit–and therefore equitable–partition

$$\pi = \Big(\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v6\}, \{v_7, v_8\}\Big)$$

of the classical 3-cube, $\mathcal{Q}_3$.

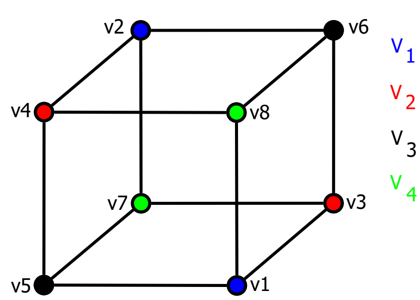


Figure 7.3.1: An equitable partition π of $\mathcal{Q}_3$ with no edges between vertices of the same cell.

Observe that under partition π,

$$A(\mathcal{Q}_3/\pi) = \begin{array}{c c} & \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array},$$

and in particular, $A(\mathcal{Q}_3/\pi)_{i,i} = 0$ for all $i \in \{1, \ldots, 4\}$. Now let D_3 be the digraph obtained by orienting each edge in $\mathcal{Q}_3$, where the initial vertex of each edge is that belonging to the lower-indexed cell of partition π. D_3 is illustrated in Figure 7.3.2.

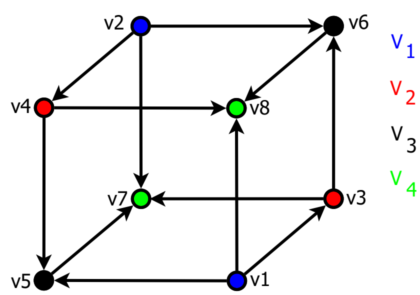


Figure 7.3.2: A digraph D_3, obtained by orienting the edges of $\mathcal{Q}_3$ according to equitable partition π, where the initial vertex of each edge is that in the lower-indexed cell of π.

Equitable partition π of $\mathcal{Q}_3$ induces an equitable edge-partition Π on D_3,

$$\Pi = \big(E_{(1,2)}, E_{(1,3)}, E_{(1,4)}, E_{(2,3)}, E_{(2,4)}, E_{(3,4)}\big),$$

where for each $i < j$, $E_{(i,j)}$ consists of the pair of edges with initial vertex in V_i and terminal vertex in V_j. Equitable edge-partition Π is illustrated in Figure 7.3.3.

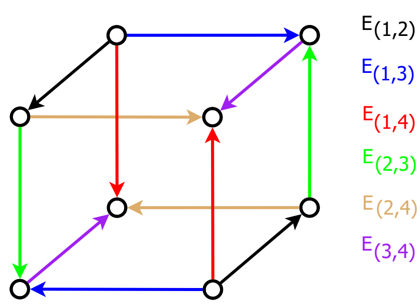


Figure 7.3.3: An equitable edge-partition Π of D_3.

Under edge-partition Π,

$$A_E(D_3/\Pi) = \begin{array}{c} E_{(1,2)} \\ E_{(1,3)} \\ E_{(1,4)} \\ E_{(2,3)} \\ E_{(2,4)} \\ E_{(3,4)} \end{array} \overset{\begin{array}{cccccc} E_{(1,2)} & E_{(1,3)} & E_{(1,4)} & E_{(2,3)} & E_{(2,4)} & E_{(3,4)} \end{array}}{\begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 1 & 0 \end{bmatrix}}.$$

Two other examples of equitable edge-partitions arising in this manner are depicted in Figure 7.3.4. Before moving on, observe that equitable partitions of this type are plentiful.

Proposition 7.1. Let G be a graph with equitable (vertex) partition $\pi = (V_1, \ldots, V_k)$. Let G' be the graph obtained from G by deleting all edges with endvertices in the same class of π. Then π is an equitable partition of G', and $A(G'/\pi)$ has a zero diagonal. Alternatively, let G'' be the graph on vertex set $V(G)$, whose edges connect all pairs of vertices from distinct cells of π that are nonadjacent in G. Then π is an equitable partition of G'' and $A(G''/\pi)$ has a zero diagonal.

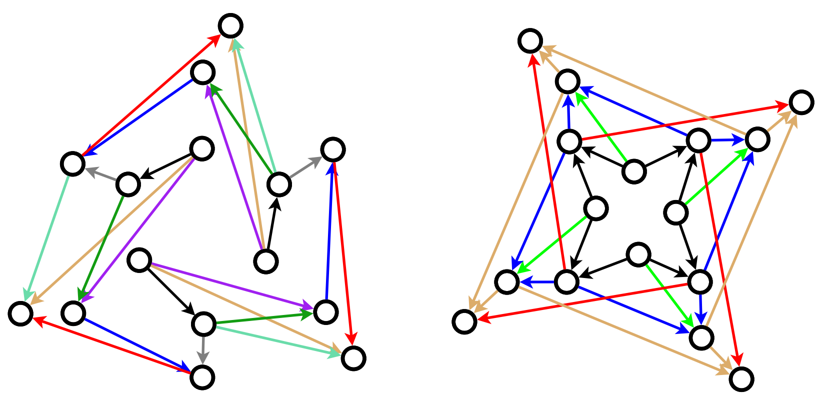


Figure 7.3.4: Two equitable edge-partitions arising from equitable vertex partitions.

7.3.1 Infinite Families of Equitable Edge-Partitions

Section 7.3 demonstrated that equitable edge-partitions can be derived from equitable (vertex) partitions in which no two vertices in the same partition cell are joined by an edge. In this section, we illustrate three infinite families of equitable partitions of this type. Each of these examples will consider equitable partitions on prism graphs.

The prism graph, denoted Y_n, is a graph corresponding to the skeleton of the regular n-prism. Specifically, Y_n has $2n$ vertices and $3n$ edges. One can enumerate the vertices of Y_n by considering Y_n as the Cartesian product of two common graphs. Let C_n be the cycle graph on n vertices. That is, C_n has n vertices $V(C_n) = \{v_1, \ldots, v_n\}$, and n edges,

$$E(C_n) = \Big\{\{v_i, v_{[i+1]_n} : 1 \le i \le n\}\Big\}.$$

We use the notation $[j]_n$ to denote

$$[j]_n = \begin{cases} n & \text{if } j \equiv 0 \pmod{n} \\ j \pmod{n} & \text{Otherwise.} \end{cases}.$$

Let K_2 be the complete graph on 2 vertices u_1 and u_2, so $E(K_2) = \{u_1, u_2\}$. C_5 and K_2 are illustrated in Figure 7.3.5.

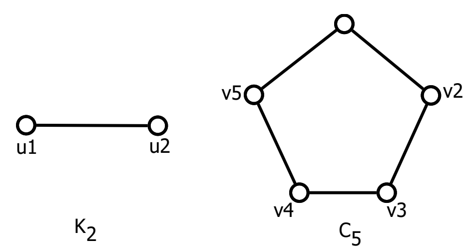


Figure 7.3.5: The complete graph K_2 and cycle graph C_5.

The prism graph Y_n is defined to be the graph Cartesian product

$$Y_n = K_2 \otimes C_n.$$

That is, Y_n has $2n$ vertices,

$$V(Y_n) = \Big\{(u_i, v_j) : u_i \in V(K_2) \text{ and } v_j \in V(C_n)\Big\},$$

and $3n$ edges. Specifically,

$$\begin{aligned} E(Y_n) = &\Big\{\{(u_1, v_i), (u_1, v_{[i+1]_n})\} : 1 \le i \le n\Big\} \cup \\ &\Big\{\{(u_2, v_i), (u_2, v_{[i+1]_n})\} : 1 \le i \le n\Big\} \cup \\ &\Big\{\{(u_1, v_i), (u_2, v_i)\} : 1 \le i \le n\Big\}. \end{aligned}$$

Remark 7.2. Throughout this paper, we have used (u, v) to denote an edge directed from vertex u to vertex v. The three examples of this section, however, will deal with undirected graph products. Parentheses are the standard notation for labeling vertices in graph products, and we will stay consistent with this convention.

The prism graph Y_5 is illustrated in Figure 7.3.6.

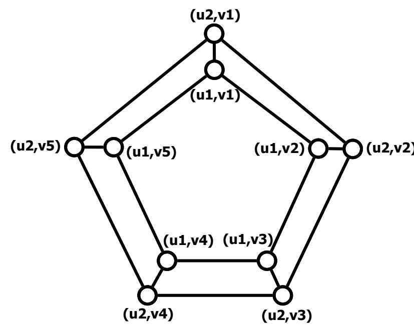


Figure 7.3.6: The prism graph Y_5.

The prism graphs are excellent candidates for constructing equitable partitions as they are *vertex transitive*. That is, any vertex of Y_n can be mapped to any other vertex by an automorphism of Y_n. As a result, the prism graphs have large automorphism groups, and many orbit partitions. What is more, while these graphs are vertex transitive they are *not* edge-transitive. The full automorphism group of Y_n partitions the edges into exactly 2 edge orbits. If orientations are added to the edges, the directed edges are partitioned into at least 3 orbits. Thus although there are many orbit vertex partitions, the edge partitions arising from these are more interesting.

Example 7.4. For even n, let Y_n be the prism graph on $2n$ vertices enumerated as above. For any vertex $v \in V(Y_n)$, there is a unique vertex of Y_n with distance $n/2+1$ from V. Let σ be the permutation on $V(Y_n)$ that interchanges all such pairs of vertices (σ is more commonly known as the *antipodal map*). Given this enumeration, for each $1 \leq j \leq n$, permutation σ interchanges the vertices (u_1, v_j) and $(u_2, v_{[j+n/2]_n})$. That is, σ can be written as a product of n disjoint transpositions,

$$\sigma = \prod_{j=1}^{n} \Big((u_1, v_j) \ \ (u_2, v_{[j+n/2]_n}) \Big).$$

One may verify that σ is a graph automorphism of order 2. Therefore σ has n vertex orbits, each of size 2. These induce an equitable partition $\pi = (V_1, \ldots, V_n)$ on Y_n, where

$$V_j = \Big\{ (u_1, v_j), (u_2, v_{[j+n/2]_n}) \Big\},$$

and there are no edges between vertices of the same partition class. The case $n = 6$ is illustrated in Figure 7.3.7.

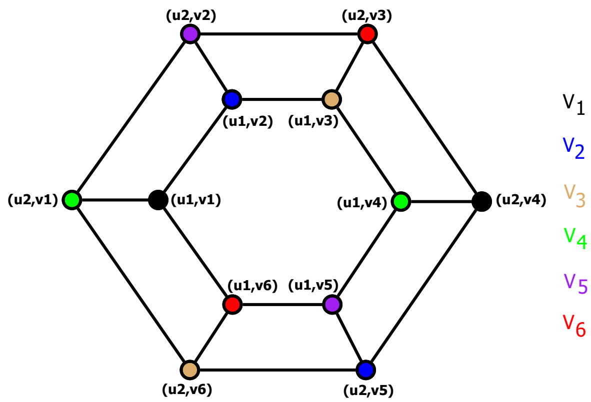


Figure 7.3.7: A 6-cell orbit partition of the prism graph Y_6.

Example 7.5. For even n, let Y_n be the prism graph with vertices enumerated as above. Let $\pi = (V_1, \ldots, V_4)$ be a 4-cell partition of $V(Y_n)$ defined as follows:

$$V_1 = \Big\{(u_1, v_j) : j \equiv 1 (\text{mod } 2)\Big\} \qquad V_2 = \Big\{(u_1, v_j) : j \equiv 0 (\text{mod } 2)\Big\}$$
$$V_3 = \Big\{(u_2, v_j) : j \equiv 1 (\text{mod } 2)\Big\} \qquad V_4 = \Big\{(u_2, v_j) : j \equiv 0 (\text{mod } 2)\Big\}$$

One may verify that π is an equitable partition of Y_n with quotient matrix

$$A(Y_n/\pi) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}.$$

Note that the diagonal entries of $A(Y_n/\pi)$ are all zero: there are no edges between vertices of the same partition class. In fact, π is also an orbit partition. It arises from the automorphism group generated by a single permutation ψ, that is the product of 4 disjoint $n/2$-cycles,

$$\psi = \Big((u_1, v_1)\ \ (u_1, v_3)\ \ \ldots (u_1, v_{n-1})\Big)\Big((u_1, v_2)\ \ (u_1, v_4)\ \ \ldots (u_1, v_n)\Big)$$
$$\Big((u_2, v_1)\ \ (u_2, v_3)\ \ \ldots (u_2, v_{n-1})\Big)\Big((u_2, v_2)\ \ (u_2, v_4)\ \ \ldots (u_2, v_n)\Big)$$

The case of $n = 6$ is illustrated in Figure 7.3.8.

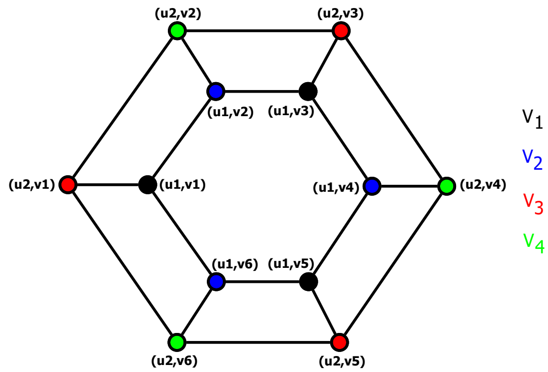


Figure 7.3.8: A 4-cell orbit partition of the prism graph Y_6.

Example 7.6. For even n, let Y_n be the prism graph with vertices enumerated as above. Let ρ be the permutation of order 2 on $V(Y_n)$,

$$\rho = \prod_{j=1}^{n/2-1} \Big((u_1, v_{[1+j]_n}) \ \ (u_1, v_{[1-j]_n}) \Big) \Big((u_2, v_{[1+j]_n}) \ \ (u_2, v_{[1-j]_n}) \Big).$$

Permutation ρ is a product of $n-2$ disjoint transpositions, and one may verify that ρ is an automorphism of Y_n. Letting σ be the antipodal map automorphism as in Example 7.4, the subgroup generated by σ and ρ, $\langle \sigma, \rho \rangle$ is an automorphism group of Y_n. The vertex orbits of Y_n form an equitable partition. If $n \equiv 0 \mod 4$ this partition consists of 4 cells each of size 2, and $(n-4)/2$ cells of size 4. Otherwise when $n \equiv 2 \mod 4$, this partition consists of 2 cells of size 2, and $(n-2)/2$ cells of size 4.

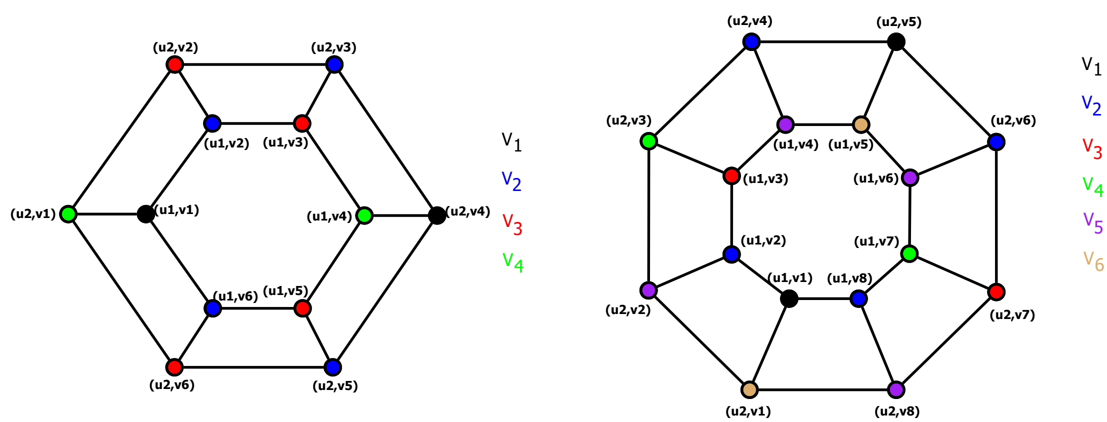


Figure 7.3.9: The orbit partitions of Y_6 and Y_8 from the automorphism group $\langle \sigma, \rho \rangle$.

7.4 An Interesting Example

Example 6.1 presented two orbit partitions of the same digraph. This section takes a more in-depth look at these partitions. Specifically, consider the 3-cell equitable edge-partition π depicted in Figure 7.4.1.

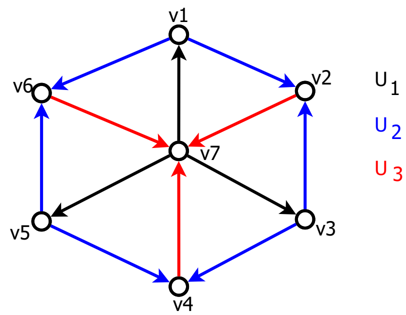
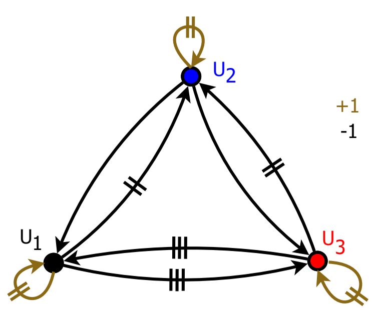


Figure 7.4.1: A 3-cell orbit edge-partition on digraph D, arising from the subgroup $\langle\phi, \psi\rangle \leq \mathrm{Aut}(D)$.

This partition is, in fact, an orbit partition arising from the automorphism group $\langle\phi, \psi\rangle$ generated by the two automorphisms

$$\phi = (v_1v_3v_5)(v_2v_4v_6)(v_7) \quad \text{and} \quad \psi = (v_2v_6)(v_3v_5)(v_1)(v_4)(v_7).$$

The quotient D/π is depicted in Figure 7.4.2. The -1 edges are drawn in black, and the +1 edges in brown. Cross-hatches are used to denote multiple edges.



$$A_E(D/\pi) = \begin{array}{c c} & \begin{array}{ccc} \mathbf{U_1} & \mathbf{U_2} & \mathbf{U_3} \end{array} \\ \begin{array}{c} \mathbf{U_1} \\ \mathbf{U_2} \\ \mathbf{U_3} \end{array} & \begin{bmatrix} 2 & -2 & -3 \\ -1 & 2 & -1 \\ -3 & -2 & 2 \end{bmatrix} \end{array}$$

Figure 7.4.2: The quotient D/π. Cross-hatches are used to indicate multiple edges with the same endvertices.

The automorphism group $\langle\phi, \psi\rangle$ generates the quotient shown in Figure 7.4.2. In this section we demonstrate that this quotient can be constructed by taking 2 layers of equitable partitions. First we will consider the orbit edge-partitions generated by the subgroups $\langle\phi\rangle \leq \langle\phi, \psi\rangle$

and $\langle\psi\rangle \leq \langle\phi, \psi\rangle$ individually. After taking quotients under the respective equitable edge-partitions, we then study *vertex* partitions of those quotients to arrive at Figure 7.4.2.

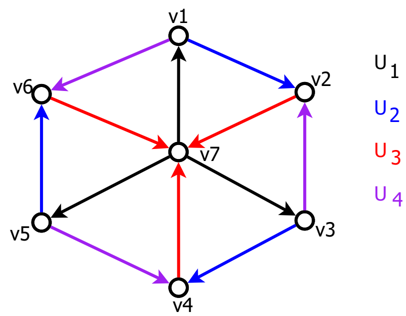
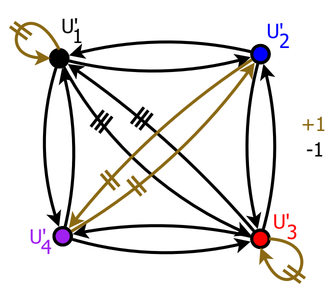


Figure 7.4.3: An orbit edge-partition π' of D generated by $\langle\phi\rangle$.

First, consider the orbit edge-partition generated by the automorphism group $\langle\phi\rangle$. This edge-partition π', illustrated in Figure 7.4.3, has 4 cells of size 3

$$U_1' = \Big\{(v_7, v_1), (v_7, v_3), (v_7, v_5)\Big\} \quad U_2' = \Big\{(v_1, v_2), (v_3, v_5), (v_5, v_6)\Big\}$$
$$U_3' = \Big\{(v_2, v_7), (v_4, v_7), (v_6, v_7)\Big\} \quad U_4' = \Big\{(v_3, v_2), (v_5, v_4), (v_1, v_6)\Big\}.$$

The quotient D/π', and quotient matrix $A_E(D/\pi')$ are given in Figure 7.4.4. Note that the *vertices* of the quotient D/π' correspond to the edges of D. Therefore in the next step we will consider *vertex* partitions of quotient D/π'.



$$A_E(D/\pi') = \begin{array}{c} \\ \mathbf{U_1'} \\ \mathbf{U_2'} \\ \mathbf{U_3'} \\ \mathbf{U_4'} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{U_1'} & \mathbf{U_2'} & \mathbf{U_3'} & \mathbf{U_4'} \end{array} \\ \begin{bmatrix} 2 & -1 & -3 & -1 \\ -1 & 0 & -1 & 2 \\ -3 & -1 & 2 & -1 \\ -1 & 2 & -1 & 0 \end{bmatrix} \end{array}$$

Figure 7.4.4: The quotient D/π', and quotient matrix $A_E(D/\pi')$.

One can view the quotient D/π as the union of two graphs, which we will call the +1 graph and the -1 graph, denoted $(D/\pi')_{+1}$ and $(D/\pi')_{-1}$. Each has vertex set $V(D/\pi) = \{U_1', U_2', U_3', U_4'\}$, and $(D/\pi')_{+1}$ (respectively $(D/\pi')_{-1}$) contains all +1 (respectively -1) edges of D/π'. The +1 and -1 graphs are illustrated in Figure 7.4.5.

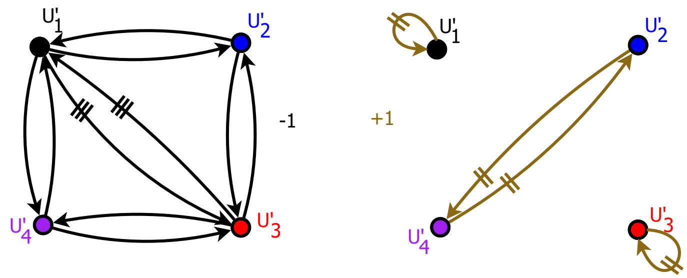


Figure 7.4.5: The -1 and +1 graphs of D/π', $(D/\pi')_{-1}$ and $(D/\pi')_{+1}$

The definition of an equitable (vertex) partition naturally extends from graphs to multi-digraphs. Namely, the number of edges directed from any vertex to the set of vertices in any partition class must be independent of the choice of vertex. We will say a vertex partition of the quotient D/π' is equitable if it is an equitable partition of both $(D/\pi')_{+1}$ and $(D/\pi')_{+1}$.

Observing the +1 and -1 graphs shown in Figure 7.4.5, one sees that these graphs share a common automorphism. Specifically, the permutation of order 2

$$\omega = (U_2'U_4')(U_1')(U_3')$$

is an automorphism of both $(D/\pi')_{+1}$ and $(D/\pi')_{-1}$. The vertex orbits under $\langle\omega\rangle$ are

$$U_1 = \left\{U_1'\right\} \quad U_2 = \left\{U_2', U_4'\right\} \quad U_3 = \left\{U_3'\right\}$$

and so (U_1, U_2, U_3) is an orbit partition–and therefore an equitable partition–of both $(D/\pi')_{+1}$ and $(D/\pi')_{-1}$. That is, (U_1, U_2, U_3) is an equitable partition of quotient D/π. The quotient of each digraph (+1 and -1) under partition (U_1, U_2, U_3) is shown in Figure 7.4.6. It is easy to see that the union of these two quotients is precisely the original quotient D/π.

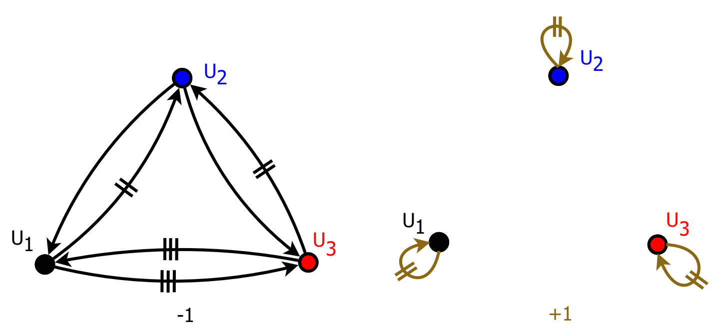


Figure 7.4.6: The quotients of $(D/\pi')_{-1}$ and $(D/\pi')_{+1}$ under orbit partition (U_1, U_2, U_3).

Now consider the orbit edge-partition generated by the automorphism group $\langle\psi\rangle$. This edge-partition π'', illustrated in Figure 7.4.7, has 7 cells, 5 of size 2 and 2 of size 1

$$U_1'' = \Big\{(v_1, v_2), (v_1, v_6)\Big\} \quad U_2'' = \Big\{(v_2, v_7), (v_6, v_7)\Big\} \quad U_3'' = \Big\{(v_7, v_1)\Big\}$$
$$U_4'' = \Big\{(v_3, v_2), (v_5, v_6)\Big\} \quad U_5'' = \Big\{(v_7, v_3), (v_7, v_5)\Big\} \quad U_6'' = \Big\{(v_4, v_7)\Big\} \quad U_7'' = \Big\{(v_3, v_4), (v_5, v_4)\Big\}.$$

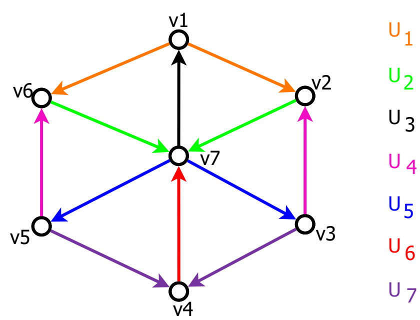


Figure 7.4.7: An orbit edge-partition π'' of D generated by $\langle\psi\rangle$.

Partition π'' is an orbit edge-partition, and therefore equitable with quotient matrix

$$A_E(D/\pi) = \begin{array}{c} \\ \mathbf{U_1''} \\ \mathbf{U_2''} \\ \mathbf{U_3''} \\ \mathbf{U_4''} \\ \mathbf{U_5''} \\ \mathbf{U_6''} \\ \mathbf{U_7''} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{U_1''} & \mathbf{U_2''} & \mathbf{U_3''} & \mathbf{U_4''} & \mathbf{U_5''} & \mathbf{U_6''} & \mathbf{U_7''} \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & -1 & 1 & 0 \\ -2 & -2 & 0 & 0 & 2 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -2 & 1 & -1 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \end{array}.$$

The quotient D/π'', is given in Figure 7.4.8. Once again, we will consider *vertex* partitions of D/π''. Let $(D/\pi'')_{+1}$ and $(D/\pi'')_{-1}$ be the +1 and -1 graphs of the quotient D/π''. In the previous case, we found $(D/\pi')_{+1}$ and $(D/\pi')_{-1}$ shared a common automorphism, and therefore a common orbit partition. Inspection shows that $(D/\pi'')_{+1}$ and $(D/\pi'')_{-1}$ have very few automorphisms, and none in common. However, consider the vertex partition (U_1, U_2, U_3) of both $(D/\pi'')_{+1}$ and $(D/\pi'')_{-1}$, where

$$U_1 = \big\{U_3'', U_5''\big\} \quad U_2 = \big\{U_1'', U_4'', U_7''\big\} \quad U_3 = \big\{U_2'', U_6''\big\}$$

This partition is illustrated in Figure 7.4.9.

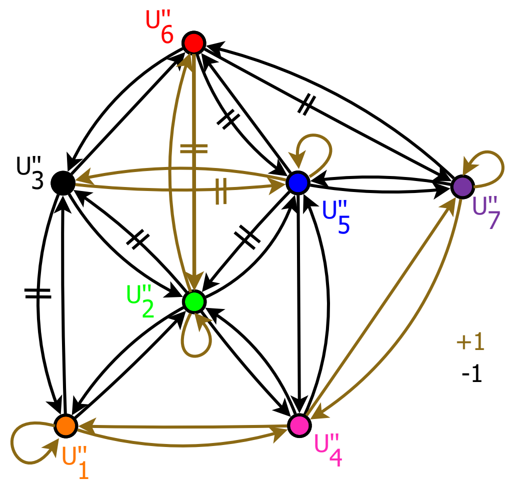


Figure 7.4.8: The quotient D/π'', and quotient matrix $A_E(D/\pi'')$.

This partition is an *equitable* partition of both $(D/\pi'')_{+1}$ and $(D/\pi'')_{-1}$. In $(D/\pi'')_{+1}$, for each $i \in \{1, 2, 3\}$ every vertex of U_i has 2 edges directed to the vertices of U_i, and no edges to any vertex of U_j for $j \neq i$. On the other hand, in $(D/\pi'')_{-1}$ every vertex of U_1 has no neighbors in U_1, 2 edges directed to the vertices of U_2, and 1 edge directed to the vertices of U_3, for example. As a result, (U_1, U_2, U_3) is an *equitable* partition of quotient D/π''.

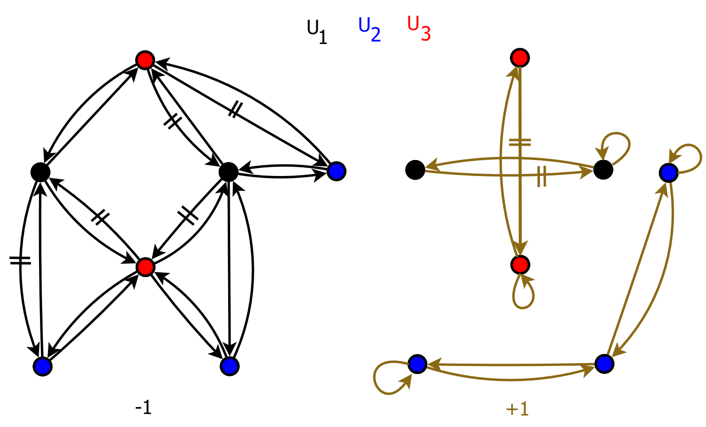


Figure 7.4.9: An equitable partition (U_1, U_2, U_3) of both $(D/\pi'')_{+1}$ and $(D/\pi'')_{-1}$.

The quotient of each digraph (+1 and -1) under partition (U_1, U_2, U_3) is shown in Figure

7.4.10. It is easy to see that the union of these two quotients is precisely the original quotient D/π.

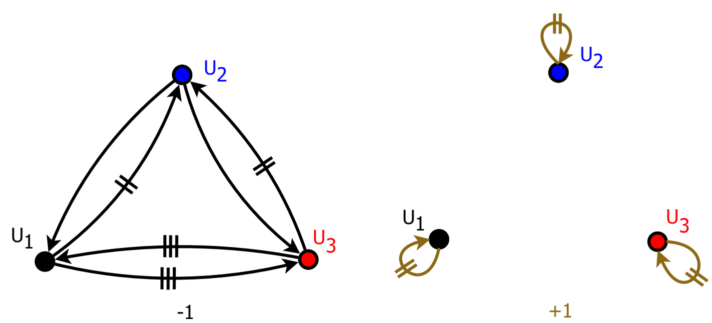


Figure 7.4.10: The quotients of $(D/\pi'')_{-1}$ and $(D/\pi'')_{+1}$ under orbit partition (U_1, U_2, U_3).

On this note we conclude our discussion of equitable partitions, and thus the second part of this text. We will return to the notion of edge-partitioning in Chapter 10, where we discuss $\mathbb{Z}_2$-Kirchhoff edge partitions of *undirected* graphs. We will use these edge-partitions to explore the relationship between $\mathbb{Z}_2$-valued matrices and graphs. This will be accomplished through a study of matroids, which are the topic of the final part of this text (Chapters 8 through 10). To begin, Chapter 8 will review the fundamentals of matroid theory. After Chapter 9 discusses classical relationships between binary matroids and graphs, Chapter 10 concludes by studying binary matroids in the context of $\mathbb{Z}_2$-Kirchhoff edge partitions.

Chapter 8

Matroids

In our study of Kirchhoff graphs, the preceding chapters have covered a broad range of topics, from Cayley color graphs to equitable partitions. A common theme throughout has been understanding relationships between graphs and matrices. At the heart of studying graphs and matrices is the theory of matroids. Matroids were introduced by Whitney in 1935 [110], aimed at understanding the fundamental properties of dependence as it relates to graphs and matrices. This was later extended in a series of papers by Tutte, [104], [105], and [106], whose work we will review in Chapter 9. Since then, the study of matroids has generated interest in a variety of disciplines. Harary [56] considered a graph-theoretic approach to matroids, and while matroids have been used to understand graphs, graph methods have also been used to better understand matroids; see, for example, [14], [24], [72], [73], and [100]. In [109], Whitely reviews a number of applications of matroids in discrete applied geometry, and Murota [81] illustrates how matroids can be used to model and solve a variety of engineering problems.

Tutte [105] specifically studied the so-called cycle matroid of a graph, and that will be the matroid we consider in this chapter. However, graphs give rise to a number of matroids that have generated interest in recent years. In particular, the rigidity matroid is used to study undirected graphs embedded in Euclidean space [46]. Also known as *frameworks*, these embedded graphs have edges of fixed lengths. The rigidity matroid can model stresses within a framework, and has been used to study the *unique realization problem* of d-dimensional frameworks [64]. A framework has a unique realization in $\mathbb{R}^d$ if every equivalent framework (i.e. one with the same edge lengths) is congruent to it. This has applications, for example, in studying molecule configurations [59]. Hendrickson [58] showed that every generic framework–one whose vertex coordinates are algebraically independent–that has a unique realization in $\mathbb{R}^d$ is $(d+1)$-connected and is redundantly rigid (i.e. is rigid after removal of any one of its edges). It was shown that this is a complete characterization for $d = 1$ or 2 [64], but Connely showed it is not when $d \geq 3$ [18].

This chapter introduces the fundamentals of matroid theory. For a complete introduction, the reader is referred to the standard texts of Oxley [84] or Welsh [108], or the more recent survey by Pitsoulis [91]. Section 8.1 begins by defining matroids in terms of both independent sets and circuits. Section 8.1.2 introduces the cycle matroid of a graph, and after 8.1.3 defines matroid duality, Section 8.1.4 introduces representable matroids. Section 8.2 reviews substructure of matroids. Specifically, 8.2.1 and 8.2.2 define deletion and contraction in matroids, which leads to a discussion of matroid minors in Section 8.2.4. This allows us to present the classic excluded-minor theorem of Tutte, which classifies all representable matroids that are graphic. Finally, the chapter concludes with Section 8.3, that presents an observation of the author regarding when a binary matroid is graphic. In particular, Theorem 8.9 is used to verify that the classical excluded minors are not graphic. This chapter all builds towards Chapter 9, which presents an algorithm due to Tutte that determines when a binary matroid is graphic.

8.1 Basic Properties of Matroids

We begin with an introduction to the theory of matroids. At its core, matroids aim to capture the abstract notion of mathematical dependence. Matroids can be defined in many different but equivalent ways. In fact, for most standard matroid notions (such as circuits, bases, and rank functions) is an associated set of axioms that can be used as a definition. We will present each of these axiom systems in turn, but will not explicitly demonstrate their equivalence. For proofs of equivalence the reader is referred to [84], [91], or [108].

8.1.1 Independent Sets and Circuits

Perhaps the most common definition of matroids, given in terms of *independent sets*, is Definition 8.1, also known as the *independence axioms*.

Definition 8.1. A ***matroid*** $M(S, \mathscr{I})$ consists of a finite set of elements S together with a family of subsets $\mathscr{I}$ of S, called ***independent sets***, such that:

(I1) $\emptyset \in \mathscr{I}$.

(I2) If $X \in \mathscr{I}$ and $Y \subseteq X$ then $Y \in \mathscr{I}$.

(I3) If $X, Y \in \mathscr{I}$ and $|X| > |Y|$ then there exists $x \in X - Y$ such that $Y \cup x \in \mathscr{I}$.

We often denote a matroid simply by M when the set S, called the ground set, and the family $\mathscr{I}$ are clear by context. Properties (I2) and (I3) are often referred to as the *hereditary* and *augmentation* properties of matroids, respectively.

Example 8.1. The *uniform matroid* $U_{k,n}$ has ground set S with $|S| = n$ and independence family

$$\mathscr{I} = \{X \subseteq S : |X| \leq k\}.$$

Naturally, any subset X of S not in $\mathscr{I}$ is called ***dependent***. A minimal (with respect to inclusion) dependent set is called a ***circuit***. A 1-element circuit is called a *loop*. The family of circuits will be denoted by $\mathscr{C}(M)$. Given any subset $X \subseteq S$, the collection of circuits of X is defined as:

$$\mathscr{C}(X) = \{Y \subseteq X : Y \notin \mathscr{I} \text{ and } Y - \{x\} \in \mathscr{I} \text{ for all } x \in Y\}.$$

An equivalent definition of matroids may be given in terms of circuits.

Definition 8.1a. A ***matroid*** $M(S, \mathscr{C})$ consists of a finite set S of elements together with a family $\mathscr{C} = \{C_1, C_2, \dots\}$ of nonempty subsets of S, called *circuits*, satisfying the axioms:

(C1) $\emptyset \notin \mathscr{C}$.

(C2) If $C_1, C_2 \in \mathscr{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathscr{C}$, $C_1 \neq C_2$ and $x \in C_1 \cap C_2$ then there exists $C_3 \in \mathscr{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - \{x\}$.

Property (C3) is often called the *circuit elimination* axiom.

Example 8.1a. The uniform matroid $U_{k,n}$ has ground set S with $|S| = n$ and circuits

$$\mathscr{C} = \{X \subseteq X : |X| = k + 1\}.$$

Equivalence of (I1)-(I3) and (C1)-(C3) is demonstrated by the following theorem, the proof of which can be found in [84].

Theorem 8.1. *Let S be a finite set and $\mathscr{C}$ a family of subsets of S satisfying (C1)-(C3). Let $\mathscr{I}$ be the family of subsets of S containing no member of $\mathscr{C}$. Then $(S, \mathscr{I})$ is a matroid having $\mathscr{C}$ as its family of circuits.* $\square$

Two matroids M_1 and M_2 are *isomorphic*, denoted $M_1 \cong M_2$, if there is a bijection $\phi : S(M_1) \to S(M_2)$ such that $X \in \mathscr{I}(M_1)$ if and only if $\phi(X) \in \mathscr{I}(M_2)$ (or equivalently, $X \in \mathscr{C}(M_1)$ if and only if $\phi(X) \in \mathscr{C}(M_2)$). Before proceeding with further definitions, we will illustrate these notions via a large class of matroids arising from graphs.

8.1.2 Graphic Matroids: The Cycle Matroid of a Graph

Let G be any undirected, though not necessarily simple, graph. We may associate a matroid with G by taking the set of edges $E(G)$ as the ground set S, and the cycles of G as circuits. That is,

$$\mathscr{C} = \{X \subseteq E(G) : G[X] \text{ is a cycle } \},$$

where $G[X]$ denotes the induced subgraph of G on edge set X. It is easy to check that both axioms (C1) and (C2) are satisfied. This matroid, denoted by $M(G)$, is called the ***cycle matroid*** of graph G. Alternatively, one may define the cycle matroid of a graph by taking $E(G)$ to be the ground set S, and the family of all acyclic subgraphs of G to be the independent sets. That is,

$$\mathscr{I} = \{X \subseteq E(G) : G[X] \text{ is a forest } \}.$$

Example 8.2. Consider the graph G given in Figure 8.1.1. It is clear that G has cycle matroid $M(G)$ with ground set and circuits

$$S = \{e_1, e_2, e_3, e_4, e_5, e_6\} \qquad \mathscr{C} = \{\{e_1, e_2, e_3\}, \{e_4, e_5, e_6\}\}.$$

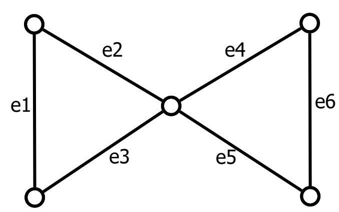


Figure 8.1.1: Bowtie graph, G.

Definition 8.2. A matroid M is ***graphic*** if there exists some graph G such that M is isomorphic to the cycle matroid $M(G)$.

Example 8.3. Consider the matroid M having ground set and circuits:

$$S = \{x_1, x_2, x_3, x_4, x_5, x_6\} \qquad \mathcal{C} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}.$$

It is easy to check that M is isomorphic to the cycle matroid of the graph G' given in Figure 8.1.2.

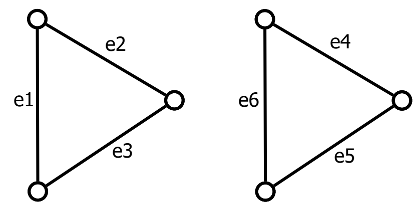


Figure 8.1.2: A distinct graph G' such that M is isomorphic to $M(G')$

Therefore matroid M is graphic.

Remark 8.1. Observe that matroid M in Example 8.3 is isomorphic to the cycle matroid of the graph G in Example 8.2. Graph G can be obtained from G' by identifying a vertex in one component of G' with a vertex in the other component, and joining the two components at that vertex. In particular, this process of "joining at a vertex" of two connected components *does not change* the set of cycles in a graph. Therefore one may observe that any graphic matroid is the cycle matroid of some *connected* graph. Thus in what follows, it is sufficient to consider connected graphs.

Remark 8.2. A graphic matroid may be isomorphic to the cycle matroid of *many* non-isomorphic graphs. For example, consider the matroid $\mathcal{I}_n$ on n elements in which all subsets are independent sets (also known as the *independent matroid*). As $\mathcal{I}_n$ does not have any circuits, $\mathcal{I}_n$ is isomorphic to the cycle matroid of *any* forest on n edges.

8.1.3 Bases and Duality

In light of axioms (I1) - (I3), in order to identify all independent sets of a matroid M, it suffices to specify the maximal independent sets. A maximal independent set of a matroid M is called a ***base*** or a ***basis*** of M. The family of bases is denoted by $\mathscr{B}$. For any $X \subseteq S$ the collection of bases for X, $\mathscr{B}(X)$, is defined as:

$$\mathscr{B}(X) = \{Y \subseteq X : Y \in \mathscr{I},\ \text{and } Y \cup \{x\} \notin \mathscr{I} \text{ for all } x \in X - Y\}.$$

Proposition 8.1. Given a matroid $M(S, \mathscr{I})$ and any $X \subseteq S$, all bases of X have the same cardinality. In particular, all bases in $\mathscr{B}$ have the same cardinality.

Proof. Let X be any subset of S. Assume, for contradiction, there exist bases $\mathcal{B}_1, \mathcal{B}_2 \in \mathscr{B}(X)$ with $|\mathcal{B}_1| > |\mathcal{B}_2|$. Since by definition $\mathcal{B}_1, \mathcal{B}_2 \in \mathscr{I}$, by axiom (I3) there exists some $x \in \mathcal{B}_1 - \mathcal{B}_2$ such that $\mathcal{B}_2 \cup \{x\} \in \mathscr{I}$. Then since $x \in X$, $\mathcal{B}_2$ is not a maximally independent subset of X, a contradiction. $\square$

The cardinality of any base $\mathcal{B} \in \mathscr{B}$ is the ***rank*** of M. We will return to a more thorough discussion of rank in Section 8.1.5. Much as matroids can be defined in terms of independent sets or circuits, a matroid can also be defined in terms of its bases. The equivalence of Definition 8.2a, known as the *basis axioms*, with (I1) - (I3) can be found in [91].

Definition 8.2a. A *matroid* M consists of a finite set S together with a family $\mathscr{B}$ of bases satisfying:

(B1) $\mathscr{B} \neq \emptyset$.

(B2) If $\mathcal{B}_1, \mathcal{B}_2 \in \mathscr{B}$ and $x \in \mathcal{B}_1 - \mathcal{B}_2$, then there exists $y \in \mathcal{B}_2 - \mathcal{B}_1$ such that $(\mathcal{B}_1 - \{x\}) \cup \{y\} \in \mathscr{B}$.

Let $\mathcal{B}$ be a basis of matroid M. Then any $x \in S(M) - \mathcal{B}$ must be dependent upon the elements of $\mathcal{B}$. More specifically,

Proposition 8.2. Let $\mathcal{B}$ be a basis of matroid M. If $x \in S(M) - \mathcal{B}$, then x is contained in a *unique* circuit of $M \cap (\mathcal{B} \cup x)$, denoted $C(x, \mathcal{B})$. In particular, $x \in C(x, \mathcal{B})$, and we call $C(x, \mathcal{B})$ the *fundamental circuit* of x with respect to $\mathcal{B}$.

Proof. By maximal independence of $\mathcal{B}$, $\mathcal{B} \cup x$ must contain a circuit, and all such circuits must contain x. Now suppose C_1 and C_2 are two distinct circuits in $\mathcal{B} \cup x$ containing x. Then by (C3), $(C_1 \cup C_2) - x$ contains a circuit. However,

$$(C_1 \cup C_2) - x = (C_1 - x) \cup (C_2 - x) \subseteq \mathcal{B} \subset \mathscr{I}(M)$$

which is a contradiction. Therefore the circuit of $\mathcal{B} \cup x$ containing x must be unique. □

For any matroid M with bases $\mathscr{B}$, we can show the existence of another matroid on the same ground set.

Theorem 8.2. *Given a matroid M with finite ground set S, the family*

$$\mathscr{B}^* = \{X \subseteq S : \text{ there exists a base } \mathcal{B} \in \mathscr{B} \text{ such that } X = S - \mathcal{B}\}$$

is the family of bases of a matroid on ground set S.

Proof. It can be shown that the elements of $\mathscr{B}^*$ satisfy basis axioms (B1) and (B2). Full details may be found in [91]. □

Definition 8.3. Let $M = (S, \mathscr{B})$ be a matroid, and let $\mathscr{B}^*$ be as in Theorem 8.2. Then let M^* denote the matroid $(S, \mathscr{B}^*)$, known as the ***dual matroid*** of M.

Because of the complementarity of bases of M and of M^*, the elements of $\mathscr{B}^*$ are often referred to as *cobases* of M. Similarly, the prefix *co* is used whenever referring to an element of the dual matroid. So for example, circuits of M^* are *cocircuits* of M, and independent sets of M^* are *coindependent* sets of M. A loop of M^* is called a *coloop* of M.

Example 8.4. The uniform matroid $U_{2,4}$ has bases

$$\mathscr{B}(U_{2,4}) = \Big\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\Big\}.$$

Now consider the family

$$\mathscr{B}^*(U_{2,4}) = \Big\{\{1,2,3,4\} - X : X \subseteq \mathscr{B}(U_{2,4})\Big\}.$$

These sets form the bases of the dual matroid $U^*_{2,4}$ where,

$$\mathscr{B}(U^*_{2,4}) = \mathscr{B}^*(U_{2,4}) = \Big\{\{3,4\},\{2,4\},\{2,3\},\{1,4\},\{1,3\},\{1,2\}\Big\}.$$

Observation reveals that $S(U_{2,4}) = S(U^*_{2,4})$ and $\mathscr{B}(U_{2,4}) = \mathscr{B}(U^*_{2,4})$. Therefore $U_{2,4} \cong U^*_{2,4}$ and we say that $U_{2,4}$ is *self-dual*.

Remark 8.3. Observe that by 2 applications of Theorem 8.2,

$$(M^*)^* = M.$$

Before proceeding with further definitions, we will illustrate these notions through another large class of matroids, arising from matrices.

8.1.4 Representable Matroids: The Column Matroid of a Matrix

Let A be any $m \times n$ matrix with entries in a field $\mathbb{F}$. A gives rise to a matroid by letting S be the set of columns $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ of A. Let a set $X = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k\} \in \mathscr{I}$ if and only if the vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k$ are linearly independent in $\mathbb{F}^m$. It is easy to check that $\mathscr{I}$ satisfies (I1)-(I3), and thus $\mathscr{I}$ is the collection of independent sets of a matroid M. We call this matroid the ***column matroid*** of matrix A, denoted by $M[A]$.

Example 8.5. Let A be the matrix given in (8.1), with entries taken in $\mathbb{Z}_2$.

$$A = \begin{array}{c} \begin{matrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \tag{8.1}$$

Letting $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ denote the columns of A, $c_1 = [1,0]^t, c_2 = [0,1]^t$, and $c_3 = [1,1]^t$. Then $M[A]$ has ground set $S = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ and independent sets

$$\mathscr{I}(M[A]) = \Big\{\emptyset, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \{\boldsymbol{c}_1, \boldsymbol{c}_2\}, \{\boldsymbol{c}_1, \boldsymbol{c}_3\}, \{\boldsymbol{c}_2, \boldsymbol{c}_3\}\Big\}.$$

Definition 8.4. A matroid M with ground set S of size n is ***representable*** over a field $\mathbb{F}$ if and only if there exists an $m \times n$ matrix A with entries belonging to $\mathbb{F}$, such that

$$M \cong M[A].$$

Such a matroid may also be called $\mathbb{F}$-representable, and a ***binary matroid*** is a representable matroid with $\mathbb{F} = \mathbb{Z}_2$

Example 8.6. Let M be a matroid with ground set $S = \{c_1, c_2, c_3, c_4, c_5\}$, and independent sets

$$\mathscr{I} = \emptyset \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3,$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ consist of all 1- and 2- subsets of S respectively, and

$$\mathcal{C}_3 = \Big\{\{c_1, c_2, c_4\}, \{c_1, c_2, c_5\}, \{c_1, c_3, c_4\}, \{c_1, c_3, c_5\}, \{c_1, c_4, c_5\}, \{c_2, c_3, c_4\}, \{c_2, c_3, c_5\}, \{c_2, c_4, c_5\}\Big\}.$$

One may check that this matroid is isomorphic to the column matroid of

$$A = \begin{array}{c} \begin{array}{ccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Therefore M is a binary matroid.

Let M be any representable matroid with ground set of size $S(M) = \{x_1, \ldots, x_n\}$ and rank m. Choose some base $\mathcal{B} \in \mathscr{B}$ of M. We may re-label the elements of M so that $\mathcal{B} = \{x_1, \ldots, x_m\}$, and $S(M) - \mathcal{B} = \{x_{m+1}, \ldots, x_n\}$. Then for each $i \geq m+1$, element x_i is dependent upon $\{x_1, \ldots x_m\}$. Therefore in any matrix representation A for M, the first m columns are linearly independent and the remaining $n - m$ columns can be written as a linear combination of those m columns.

In particular, let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ denote the columns of A. Then we may choose $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m$ to be the m standard basis vectors of $\mathbb{F}^m$. Then the remaining columns $\boldsymbol{c}_{m+1}, \ldots, \boldsymbol{c}_n$ can be determined by considering the fundamental circuits $C(x_i, \mathcal{B})$ for each element $x_i \in \{x_{m+1}, \ldots, x_n\}$. That is, for each $j \in \{1, \ldots, m\}$ and each $i \in \{m+1, \ldots, n\}$, define the entries:

$$\boldsymbol{c}_i(j) = \begin{cases} 1 & \text{If element } x_j \text{ is contained in fundamental circuit } C(x_i, \mathcal{B}) \\ 0 & \text{Otherwise} \end{cases}$$

The resulting matrix has the form

$$
\begin{array}{c}
\begin{array}{ccccccc} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m & & \boldsymbol{c}_{m+1} & \cdots & \boldsymbol{c}_n \end{array} \\
A = \left[\begin{array}{c|c} \quad I_m \quad & \quad H \quad \end{array} \right],
\end{array}
$$

where H is a $m \times (n-m)$ nonzero matrix, and $M \cong M[A]$. This is called a ***standard representation matrix*** for M, and the matrix H is called a ***compact representation matrix***. (Note for duality reasons that will become clear shortly, we will also refer to a matrix of the form $\left[\begin{array}{c|c} H & I \end{array} \right]$ as a standard representation matrix). Every pair of standard and compact representation matrices of a matroid M are generated by a choice of base of M.

Remark 8.4. Observe that standard representation of matroids is closely tied to row reduction. Given a matrix A over field $\mathbb{F}$, the sets of linearly independent columns of A are the independent sets of the matroid $M[A]$. Any matrix row-equivalent to A has the same dependence and independence of columns, and thus the same column matroid. Therefore if A does not have full row-rank, it contains more information than necessary to define the matroid $M[A]$: we may delete any row that is linearly dependent on the others without changing the column matroid. In particular, given any $\mathbb{F}$-representable matroid with matrix A, we can obtain a representation in standard form $\left[\begin{array}{c|c} I & H \end{array} \right]$ via a sequence of elementary row operations, deleting zero rows, and column interchanges.

The standard representation matrix of a matroid M easily provides a standard representation of the dual matroid, M^*.

Theorem 8.3. *If $A = \left[\begin{array}{c|c} I & H \end{array} \right]$ is a standard representation matrix for a representable matroid M, then M^* is representable over the same field as A, and*

$$A_* = \left[\begin{array}{c|c} -H^t & I \end{array} \right]$$

is a standard representation matrix of the dual matroid M^*.

Proof. [1] It is easy to verify that the columns of A_*^t constitute a basis for the null space of matrix A, Null(A). Now consider any base $\mathcal{B} = \{x_1, \ldots, x_m\}$ of $M[A]$. Let $\boldsymbol{c}_i$ denote the i^{th} column of matrix A. Then for each $k \geq m+1$ there exist scalars $\alpha_{k,i}$ such that:

$$\boldsymbol{c}_k + \alpha_{k,1}\boldsymbol{c}_1 + \cdots + \alpha_{k,m}\boldsymbol{c}_m = \mathbf{0}.$$

Let A' be the $(n-m) \times m$ matrix with $A'(i,j) = \alpha_{m+i,j}$, and let A'' be the $(n-m) \times n$ matrix:

$$A'' = \left[\begin{array}{c|c} A' & I_{(n-m)} \end{array} \right].$$

[1] A version of this proof can be found in [91].

Then the columns corresponding to $x_{m+1}, \ldots, x_n$ form a basis for the column space of A''. Moreover, since the rows of A'' are in Null(A) it follows that $M[A''] = M[A_*]$ and $S - \mathcal{B}$ is a basis of $M[A_*]$. By reversing the roles of A and A_* in the above argument, it can be shown that if $S - \mathcal{B}$ is a base for $M[A_*]$ then $\mathcal{B}$ is a base of $M[A]$. □

Corollary 8.1. *If M is a representable matroid with compact representation matrix H, then $-H^t$ is a compact representation matrix of the dual matroid M^*.*

Example 8.7. The Fano matroid F_7, related to the Fano plane in finite geometry, is a binary representable matroid with standard representation matrix A and compact representation matrix H,

$$A = \begin{array}{c} \begin{array}{ccccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \end{array} \qquad H = \begin{array}{c} \begin{array}{cccc} \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

By Theorem 8.3 the dual of the Fano matroid, F_7^* has standard representation A_* and compact representation H^t,

$$A_* = \begin{array}{c} \begin{array}{ccccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \qquad H^t = \begin{array}{c} \begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \end{array}$$

In the special case of binary matroids, there is a close connection between the dependent sets of M and the rows in a representation of M^*. Let M be a matroid on ground set $S = \{s_1, \ldots, s_n\}$. For any $X \subseteq S$ there is a naturally corresponding ***characteristic vector***, the binary n-tuple whose j^{th} entry is 1 if and only if $s_j \in X$.

Theorem 8.4. *Let M and M^* be a pair of dual binary matroids with ground set $S = \{x_1, x_2, \ldots, x_n\}$ and standard matrix representations A and A_* respectively, as in Theorem 8.3.*

(i) The rows of A_* correspond to dependent sets in M. That is, each row of A_* is the characteristic vector of some dependent set in M.

(ii) If $X \subseteq S$ is dependent in M, the characteristic vector lies in the row space of A_*.

(iii) In fact, as A_* is a standard representation matrix, each row corresponds to a circuit in M.

Proof. Let $\boldsymbol{a}_i$ denote the i^{th} row of A_*. As discussed in the proof of Theorem 8.3, each $\boldsymbol{a}_i^t$ lies in Null(A). Therefore the columns of A corresponding to the unit entries of $\boldsymbol{a}_i$ sum to zero, are linearly dependent, and thus (i) follows. As the columns of A_*^t are in fact a *basis* for Null(A), (ii) follows. To prove (iii) assume, for contradiction, that some row $\boldsymbol{a}_i$ does not correspond to a circuit. Let X be the subset of S corresponding to $\boldsymbol{a}_i$. By (i) X is dependent in M, but (by assumption) must contain some proper subset Y that is also dependent. Then by (ii), the characteristic vector $\boldsymbol{v}_Y$ of Y must lie in the row space of A. Therefore $\boldsymbol{v}_{X-Y}$, the characteristic vector of $X - Y$ must lie in Row(A) as well, and $\boldsymbol{v}_Y + \boldsymbol{v}_{X-Y} = \boldsymbol{a}_i$. But then since A_* is a standard representation matrix of M^*, one of Y or $X - Y$ includes no elements of M^* associated with the identity block of A_*. This contradicts that both $\boldsymbol{v}_Y$ and $\boldsymbol{v}_{X-Y}$ lie in the row space of A_*, completing the proof of (iii). □

Corollary 8.2. *Let M be a binary matrix with standard representation matrix A. The rows of A correspond to cocircuits of M.*

We have now seen that a matroid can be defined on a collection of (column) vectors. There are two linear algebraic concepts naturally associated with sets of vectors, namely dimension and span. These two notions both have matroid analogues, which we discuss in Section 8.1.5.

8.1.5 Matroid Rank and Closure

As introduced in Section 8.1.3, the rank of a matroid M is the cardinality of any base $\mathcal{B} \in \mathscr{B}(M)$. We may extend this notion to any subset $X \subseteq S(M)$. Let

$$\mathscr{I}|X = \{I \subseteq X : I \in \mathscr{I}\}.$$

One may verify that $\mathscr{I}|X$ satisfy $(I1) - (I3)$ and thus $(X, \mathscr{I}|X)$ is a matroid. Denoted by $M|X$, we call this matroid the ***restriction of*** M ***to*** X. We will return to restrictions when we discuss deletion in the next section, but this now allows us to define the rank of a subset X.

Definition 8.5. For any subset $X \subset S(M)$, the *rank* of X, denoted $r(X)$, is the cardinality of any basis $\mathcal{B}_X$ of $M|X$. In this case we call $\mathcal{B}_X$ a ***basis*** of X.

Observe that under this definition, $r : 2^S \to \mathbb{Z}$ is a function mapping all subsets of $S(M)$ into the nonnegative integers. We call r the ***rank function of*** M. Moreover, r satisfies the following three properties.

(R1) If $X \subseteq S$, then $0 \leq r(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq S$ then $r(X) \leq r(Y)$.

(R3) If $X, Y \subseteq S$, then

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

Properties (R1) and (R2) are clear. Property (R3) is often called *submodularity*, and verifying that it is satisfied by rank function r requires proof.

Proposition 8.3. Given $r : 2^S \to \mathbb{Z}$ as defined in Definition 8.5, r satisfies (R3).

Proof. [84] Let $\mathcal{B}_{X\cap Y}$ be a basis of $X \cap Y$. Then since $X \cap Y \subseteq X \cup Y$, $\mathcal{B}_{X\cap Y}$ is independent in $M|(X \cup Y)$. In particular, there exists a basis $\mathcal{B}_{X\cup Y}$ for $M|(X \cup Y)$ which contains $\mathcal{B}_{X\cap Y}$. Therefore $\mathcal{B}_{X\cup Y} \cap (X \cup Y) = \mathcal{B}_{X\cup Y}$, and $\mathcal{B}_{X\cup Y} \cap (X \cap Y) = \mathcal{B}_{X\cap Y}$. Moreover, by definition of r it follows that:

$$\begin{aligned}
|\mathcal{B}_{X\cup Y} \cap (X \cup Y)| &= |\mathcal{B}_{X\cup Y}| = r(X \cup Y) \\
|\mathcal{B}_{X\cup Y} \cap (X \cap Y)| &= |\mathcal{B}_{X\cap Y}| = r(X \cap Y).
\end{aligned}$$

Observe that $\mathcal{B}_{X\cup Y} \cap X$ is independent in $M|X$. Thus by $(R1)$ and $(R2)$, $|\mathcal{B}_{X\cup Y} \cap X| \leq r(X)$. Similarly, $|\mathcal{B}_{X\cup Y} \cap Y| \leq r(Y)$. Therefore:

$$\begin{aligned}
r(X \cup Y) + r(X \cap Y) &= |\mathcal{B}_{X\cup Y} \cap (X \cup Y)| + |\mathcal{B}_{X\cup Y} \cap (X \cap Y)| \\
&= |(\mathcal{B}_{X\cup Y} \cap X) \cup (\mathcal{B}_{X\cup Y} \cap Y)| + |(\mathcal{B}_{X\cup Y} \cap X) \cap (\mathcal{B}_{X\cup Y} \cap Y)| \\
&= |\mathcal{B}_{X\cup Y} \cap X| + |\mathcal{B}_{X\cup Y} \cap Y| \\
&\leq r(X) + r(Y).
\end{aligned}$$

□

As suggested by the notation, (R1)-(R3) are known as the *rank axioms* of a matroid, and may be used as an equivalent definition. The equivalence of (R1)-(R3) and (I1)-(I3) is deduced by the following theorem, the proof of which can be found in [84].

Theorem 8.5. *Let S be a set and r a function mapping 2^S to the set of non-negative integers that satisfies (R1)-(R3). Let $\mathscr{I}$ be the collection of subsets of $X \subseteq S$ for which $r(X) = |X|$. Then $(S, \mathscr{I})$ is a matroid having rank function r.*

Independent sets, bases, and circuits can be characterized in terms of the rank function.

Proposition 8.4. Let M be a matroid with rank function r and let $X \subseteq S(M)$. Then:

(i) X is independent if and only if $|X| = r(X)$.

(ii) X is a basis if and only if $|X| = r(X) = r(M)$.

(iii) X is a circuit if and only if X is non-empty and for all $x \in X$, $r(X - x) = |X| - 1 = r(X)$.

Proof.

$$\begin{aligned}(i) X \in \mathscr{I}(M) &\iff X \text{ is a basis of } M|X \\ &\iff r(X) = |X|.\end{aligned}$$

$$\begin{aligned}(ii) X \in \mathscr{B}(M) &\iff X \in \mathscr{I}(M) \text{ and } |X| = r(M) \\ &\iff r(X) = |X| = r(M) \text{ by (i).}\end{aligned}$$

$$\begin{aligned}(iii) X \in \mathscr{C}(M) &\iff X \notin \mathscr{I}(M) \text{ and } X - x \in \mathscr{I}(M) \quad \forall x \in X \\ &\iff r(X) < |X| \text{ and } r(X-x) = |X-x| = |X| - 1 \quad \forall x \in X \\ &\iff r(X-x) = |X| - 1 = r(X) \quad \forall x \in X.\end{aligned}$$

□

Rank is the matroid analogue of dimension. Recall that in a vector space, a vector v is in the span of $\{v_1, \ldots, v_n\}$ if the subspaces spanned by $\{v_1, \ldots, v_n\}$ and $\{v_1, \ldots, v_n, v\}$ have the same dimension. This idea allows the notion of *span* to be extended to matroids.

Definition 8.6. Let M be a matroid with ground set S and rank function r. Let cl be the function $cl : 2^S \to 2^S$ defined for all $X \subset S$ by:

$$cl(X) = \{x \in S : r(X \cup x) = r(X)\}.$$

The function cl is known as the ***closure operator*** and we call $cl(X)$ the ***closure*** of X in M.

As one might expect, a subset and its closure have the same rank.

Lemma 8.1. *For every $X \subseteq S(M)$, $r(X) = r(cl(X))$.*

Proof. Let $\mathcal{B}_X$ be a basis for X, and take $x \in cl(X) - X$. Then since $\mathcal{B}_X \subseteq \mathcal{B}_X \cup x \subseteq X \cup x$,

$$r(\mathcal{B}_x \cup x) \leq r(X \cup x) = r(X) = |\mathcal{B}_X| = r(\mathcal{B}_x) \leq r(\mathcal{B}_x \cup X)$$

Therefore all relations in this equation are in fact equalities, and $r(\mathcal{B}_X \cup x) = |\mathcal{B}_X| < |\mathcal{B}_X \cup x|$. Thus $\mathcal{B}_X \cup x$ is dependent, $\mathcal{B}_X$ is a basis of $cl(X)$, and so $r(X) = r(cl(X))$. □

The closure operator satisfies the following four properties, known as the *closure axioms* of a matroid. A proof that the closure operator cl satisfies (CL1) - (CL4) can be found in [84].

(CL1) If $X \subseteq S$ then $X \subseteq cl(X)$.

(CL2) If $X \subseteq Y \subseteq S$, then $cl(X) \subseteq cl(Y)$.

(CL3) If $X \subseteq S$ then $cl(cl(X)) = cl(X)$.

(CL4) If $X \subseteq S$ and $x \in S$, and $y \in cl(X \cup x) - cl(X)$, then $x \in cl(X \cup y)$.

Once again, these closure axioms may be used as an equivalent matroid definition. The equivalence of (CL1)-(CL4) with (I1)-(I3) can be deduced from the following theorem, the proof of which can be found in [84].

Theorem 8.6. *Let S be a set and cl be a function $cl : 2^S \to 2^S$ satisfying (CL1)-(CL4). Let $\mathscr{I} = \{X \subset S : x \notin cl(X - x)$ for all $x \in X\}$. Then $(S, \mathscr{I})$ is a matroid having closure operator cl.*

The closure operator on a matroid allows us to extend a number of more familiar concepts. A *closed set* (sometimes called a *flat*) is any subset $X \subseteq S(M)$ for which $cl(X) = X$. A *hyperplane* is a closed set of rank $r(M) - 1$. For any $X, Y \subset S(M)$, we say that X *spans* Y if $Y \subset cl(X)$. X is *spanning* if $cl(X) = S(M)$. Bases, hyperplanes, circuits, and spanning sets are all closely related, as demonstrated by the following proposition of [84].

Proposition 8.5. Let M be a matroid and $X \subset S(M)$,

(i) X is a spanning set if and only if $r(X) = r(M)$.

(ii) X is a basis if and only if it is both spanning and independent.

(iii) X is a basis if and only if it is a minimal spanning set.

(iv) X is a hyperplane if and only if it is a maximal non-spanning set.

(v) X is a circuit if and only if X is a minimal nonempty set such that $x \in cl(X - x)$ for all $x \in X$.

(vi) $cl(X) = X \cup \{x : \exists C \in \mathscr{C}(M)$ such that $x \in C \subseteq X \cup x\}$.

Much as subsets of dual matroid M^* are given the prefix co- in relation to M, the spanning sets and hyperplanes of M^* are called *cospanning* and *cohyperplanes*. These are closely tied to subsets of M, as described by the following proposition of [84].

Proposition 8.6. Let M be a matroid and let $X \subseteq S(M)$,

(i) X is independent if and only if $S - X$ is cospanning.

(ii) X is spanning if and only if $S - X$ is coindependent.

(iii) X is a hyperplane if and only if $S - X$ is a cocircuit.

(iv) X is a circuit if and only if $E - X$ is a cohyperplane.

The ***corank*** of $X \subset S(M)$, denoted by $r^*(X)$, is the rank of X in dual matroid M^*. The function r^*, often called the *corank function* of M, can be defined using rank function r.

Proposition 8.7. For all $X \subseteq S(M)$,

$$r^*(X) = r(S - X) + |X| - r(M) \tag{8.2}$$

The proof of this proposition is deduced from the following lemma. The proof of this lemma, which follow from Proposition 8.6, as well as the proof of Proposition 8.7, can be found in [84].

Lemma 8.2. *Let $\mathcal{I}$ and $\mathcal{I}^*$ be disjoint subsets of $S(M)$ that are independent and coindependent respectively. Then M has a basis $\mathcal{B}$ and a cobasis $\mathcal{B}^*$ such that $I \subseteq \mathcal{B}$, $I^* \subseteq \mathcal{B}^*$, and $\mathcal{B}$ and $\mathcal{B}^*$ are disjoint.*

8.2 Matroid Sub-Structure

Given a matroid M and a subset $X \subseteq S(M)$, we have already seen that a new matroid $M|X$ can be defined by letting $\mathscr{I}(M|X)$ be the elements of $\mathscr{I}(M)$ contained in X. That is, a matroid can be used to define new matroids on subsets of its ground set. This section examines various methods of producing such sub-matroids.

8.2.1 Deletion and Restriction

Let M be a matroid on ground set S, and let X be any subset $X \subseteq S$. Matroid M can be used to define a new matroid on ground set $S - X$, denoted by $M \backslash X$.

Proposition 8.8. For a matroid $M(S, \mathscr{C})$ and $X \subseteq S$, the set

$$\mathscr{C}(M \backslash X) = \{C \subseteq S - X : C \in \mathscr{C}(M)\}$$

is the family of circuits of a matroid on $S - X$.

Proof. It is easy to verify that the circuit axioms (C1) and (C2) are satisfied, as $\mathscr{C}(M \backslash X) \subseteq \mathscr{C}(M)$. □

The matroid having circuits $\mathscr{C}(M \backslash X)$ is called the ***deletion*** of X from M. The matroid that deletes all elements *other* than X, $M \backslash (S(M) - X)$, is called the *deletion of M to X*

or ***restriction of*** M ***to*** X and will be denoted by $M|X$. It is easy to verify that the independent sets of $M\backslash X$ are:

$$\mathscr{I}(M\backslash X) = \{I \in \mathscr{I}(M) : I \subseteq X\}.$$

Let r_M denote the rank function on M. Then one may verify that the rank function of $M\backslash X$, $r_{M\backslash X}$, is the restriction of r_M to the subsets of $S - X$. That is, for all $Y \subseteq S - X$,

$$r_{M\backslash X}(Y) = r_M(Y). \tag{8.3}$$

Example 8.8. We return to the example of $U_{2,4}$ to illustrate matroid deletion. The circuits of $U_{2,4}$ consist of all 3-element subsets of $\{1, 2, 3, 4\}$:

$$\mathscr{C}(U_{2,4}) = \Big\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\Big\}.$$

Now consider the one-element subset $X = \{1\}$. The matroid $U_{2,4}\backslash X$ has ground set $S(U_{2,4}\backslash X) = \{2, 3, 4\}$ and circuits:

$$\mathscr{C}(U_{2,4}\backslash X) = \Big\{C \subseteq \{2, 3, 4\} : C \in \mathscr{C}(U_{2,4})\Big\} = \Big\{\{2, 3, 4\}\Big\}.$$

As $\{2, 3, 4\}$ is a minimal dependent set in ground set $S(U_{2,4}\backslash X) = \{2, 3, 4\}$ it follows that

$$\mathscr{I}(U_{2,4}\backslash X) = \Big\{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\Big\}.$$

8.2.2 Contraction

Given the definition of matroid deletion, we may introduce a second matroid operation, contraction. Recall that in a planar graph G (with dual graph G^*), the operations of contraction and deletion of edges are dual operations. That is, for any $e \in E(G)$, $G/e = (G^*\backslash e)^*$ [23]. This suggests how to extend the notion of contraction to matroids.

Definition 8.7. For a matroid M with ground set S, the ***contraction of*** X ***in*** M is the matroid M/X on $S - X$ defined as:

$$M/X = (M^*\backslash X)^*. \tag{8.4}$$

The matroid that contracts all elements *other* than X in M, $M/(S(M) - X)$, is called the *contraction of* M *to* X and is denoted by $M.X$.

Example 8.9. Continuing Example 8.8 we now observe the effect of contraction on $U_{2,4}$.

Recall that $U_{2,4}$ was self-dual, $U_{2,4} \cong U^*_{2,4}$, so for any $X \subseteq S(U_{2,4})$.

$$U^*_{2,4} \backslash X = U_{2,4} \backslash X.$$

Letting $X = \{1\}$ as in Example 8.8, $U^*_{2,4} \backslash X$ has ground set $\{2, 3, 4\}$ and bases

$$\mathscr{B}(U^*_{2,4} \backslash X) = \Big\{\{2, 3\}, \{2, 4\}, \{3, 4\}\Big\}.$$

Therefore the dual matroid $(U^*_{2,4} \backslash X)^*$ has bases

$$\mathscr{B}((U^*_{2,4} \backslash X)^*) = \mathscr{B}^*(U^*_{2,4} \backslash X) = \Big\{\{2, 3, 4\} - \mathcal{B} : \mathcal{B} \subseteq \mathscr{B}(U^*_{2,4} \backslash X)\Big\} = \Big\{\{2\}, \{3\}, \{4\}\Big\}.$$

That is, the matroid contraction $U_{2,4}/X$ has ground set $S(U_{2,4}/X) = \{2, 3, 4\}$ and:

$$\mathscr{I}(U_{2,4}/X) = \Big\{\emptyset, \{2\}, \{3\}, \{4\}\Big\} \quad ; \quad \mathscr{C}(U_{2,4}/X) = \Big\{\{2, 3\}, \{2, 4\}, \{3, 4\}\Big\}$$

Observe there is a distinct difference between deletion and contraction–the 2-subsets of $\{2, 3, 4\}$ are maximal independent sets of $U_{2,4} \backslash X$, but minimal dependent sets of $U_{2,4}/X$.

Contraction and deletion may be considered as dual operations, in the following sense.

Proposition 8.9. For a matroid M and $X \subseteq S(M)$,

(i) $(M \backslash X)^* = M^*/X$.

(ii) $(M/X)^* = M^* \backslash X$.

Proof. Recall from Remark 8.3 that $(M^*)^* = M$. Therefore taking the dual of expression 8.4,

$$(M/X)^* = ((M^* \backslash X)^*)^* = M^* \backslash X.$$

giving (i). By considering a contraction on the dual M^*, the roles of M and M^* in expression 8.4 reverse, giving (ii):

$$M^*/X = (M \backslash X)^*.$$

□

Let M be a matroid and $X \subseteq S(M)$. We can identify the rank function, independent sets, and circuits of M/X from those of M. Let $r_{M/X}$ denote the rank function on M/X.

Proposition 8.10. For all $Y \subseteq S(M) - X$,

$$r_{M/X}(Y) = r_M(Y \cup X) - r_M(X)$$

Proof. The proof relies on the equations (8.2) and (8.3) and the fact that $M/X = (M^*\backslash X)^*$. Full details can be found in [84]. $\square$

In the following three propositions, let M be a matroid on ground set S and X any subset $X \subseteq S$. Full proofs of each proposition can be found in [84]

Proposition 8.11. Let $\mathcal{B}_X$ be a basis of $M|X$. Then

$$\begin{aligned}\mathscr{I}(M/X) &= \{I \subseteq S - X : I \cup \mathcal{B}_X \in \mathscr{I}(M)\}\\ &= \{\mathcal{I} \subseteq S - X : M|X \text{ has a basis } \mathcal{B} \text{ such that } \mathcal{B} \cup \mathcal{I} \in \mathscr{I}(M)\ \}\end{aligned} \quad \square$$

Proposition 8.12. Let $\mathcal{B}_X$ be a basis of $M|X$. Then

$$\begin{aligned}\mathscr{B}(M/X) &= \{\mathcal{B}' \subseteq S - X : \mathcal{B}' \cup \mathcal{B}_X \in \mathscr{B}(M)\}\\ &= \{\mathcal{B}' \subseteq S - X : M|X \text{ has a basis } \mathcal{B} \text{ such that } \mathcal{B}' \cup \mathcal{B} \in \mathscr{B}(M)\ \}\end{aligned} \quad \square$$

Proposition 8.13. The circuits of M/X consist of the minimal non-empty members of $\{C - X : C \in \mathscr{C}(M)\}$ $\square$

Take a moment to consider what this last proposition implies. Let Y be any subset of $S - X$. Then if Y is dependent in M, Y remains dependent in M/X. In particular, for any circuit $C \in \mathscr{C}(M)$ such that $C \subseteq S - X$, C is dependent in M/X. In order to determine $\mathscr{C}(M/X)$, what remains is to check minimality.

8.2.3 Connectivity

We have already seen that familiar graph theoretic notions—such as contraction and deletion—can be extended to matroids. In this section we extend the concept of graph connectivity. As discussed previously, any graphic matroid is isomorphic to the cycle matroid of a connected graph: if G_1 is disconnected there exists a connected graph G_2 such that $M(G_1) \cong M(G_2)$. Thus there is no matroid notion analogous to 1-connectedness of graphs. As we will illustrate in this section, 2-connectedness of graphs has a natural matroid analogue.

Recall that a graph G is 2-connected if the removal of any vertex $v \in V(G)$ does not separate the graph. Notice, however, that matroids do not have a natural counterpart for the notion of *vertices*, so this this definition of 2-connectedness does not easily extend to matroids. Instead, we may consider the following characterization of 2-connected graphs [23].

Proposition 8.14. Let G be a simple graph with $|V(G)| \geq 3$. Then G is 2-connected if and only if any pair of distinct edges e and f are contained in a common cycle.

Proof. Let $e = \{x, y\}$ and $f = \{u, v\}$ (where $|\{x, y\} \cap \{u, v\}| \leq 1$). Since G is 2-connected, by Menger's theorem [23] there exist 2 totally disjoint paths in G joining $\{x, y\}$ to $\{u, v\}$ (where one of these paths may be the trivial path of one point in the case that $|\{x, y\} \cap \{u, v\}| = 1$). These two paths, in combination with edges e and f form a cycle containing both e and f. □

This notion of 2-connectedness can be adapted to matroids. Define a relation R on ground set $S(M)$ by xRy if either $x = y$ or M has a circuit containing both x and y.

Proposition 8.15. For every matroid M, the relation R is an equivalence relation on $S(M)$.

Proof. Clearly R is reflexive and symmetric. Then it suffices to show that if $x, y, z \in S(M)$ are distinct, and both xRy and yRz, then xRz. By distinctness of x, y, z there exist circuits of M containing $\{x, y\}$ and $\{y, z\}$. Choose circuits $C_1 \ni x$ and $C_2 \ni z$ such that $C_1 \cap C_2 \neq \emptyset$ and among all such circuits, $|C_1 \cup C_2|$ is minimal. The goal is to show that there exists $C \in \mathscr{C}(M)$ containing both x and z. The proof, which proceeds by contradiction, repeatedly appeals to the circuit elimination axiom. Full details can be found in [84]. □

Definition 8.8. The R-equivalence classes of M are called the ***(connected) components*** of matroid M.

Observe that every loop of M is a component, as is every coloop. Moreover, if X is a component of M, observe that every circuit of M is either contained in X or in $S(M) - X$.

Definition 8.9. If $S(M)$ is a component of M (or if $S(M)$ is empty), we say that matroid M is ***connected***. If M is not connected, then it is *disconnected*.

There are many alternative characterizations of matroid connectivity. A ***separator*** of matroid M is a union of components of M. Alternatively, a separator of matroid M is a subset $X \subseteq S(M)$ such that any circuit $C \in \mathscr{C}(M)$ is contained in either X or $S(M) - X$. Some authors (and specifically Tutte in [105]), call a non-null separator which contains no other non-null separator ***elementary***.

Remark 8.5. Clearly the elementary separators of M are just the connected components. In future sections, we will review an algorithm of Tutte, who prefers elementary separator terminology. We will follow this convention, but the reader should be aware that the elementary separators are precisely the connected matroid components.

We can also characterize separators via the rank function. This will allow us to show that the class of connected matroids is closed under duality.

Proposition 8.16. Let M be a matroid and $X \subseteq S(M)$. Then X is a separator of M if and only if

$$r(X) + r(S - X) = r(M).$$

Proof. Let X be any separator of M and $\mathcal{B}_X$ and $\mathcal{B}_{S-X}$ bases of $M|X$ and $M\backslash X$. Since all circuits of M are contained in either X or $S-X$, no circuit of M is contained in $\mathcal{B} = \mathcal{B}_X \cup \mathcal{B}_{S-X}$. Therefore $\mathcal{B}$ is independent in M. By maximality of $\mathcal{B}_X$ and $\mathcal{B}_{S-X}$ in X and $S-X$, $\mathcal{B}$ is a maximal independent subset of M. Therefore:

$$r(X) + r(S-X) = |\mathcal{B}_X| + |\mathcal{B}_{S-X}| = |\mathcal{B}| = r(M)$$

The proof of the converse is similar. □

Let X be any subset of matroid M. By extending formula (8.2), one can verify that:

$$r(X) + r(S-X) - r(M) = r(X) + r^*(X) - |X| \tag{8.5}$$

This allows us to prove the following,

Proposition 8.17. A matroid M is disconnected if and only if for some proper non-empty subset $X \subseteq S(M)$,

$$r(X) + r^*(X) - |X| = 0 \tag{8.6}$$

Proof. M is disconnected if and only if $S(M)$ is not a component of M. $S(M)$ is not a component of M if and only if some proper subset $X \subseteq S(M)$ is a separator of M. By Proposition 8.16 this occurs if and only if a proper subset $X \subseteq S(M)$ satisfies $r(X) + r(S-X) = r(M)$, or equivalently, $r(X) + r(S-X) - r(M) = 0$. Then by equation (8.5), this is true if and only if for some proper subset $X \subseteq S(M)$, $r(X) + r^*(X) - |X| = 0$. □

Corollary 8.3. *A matroid M is connected if and only if M^* is connected.*

Proof. Observe that equation (8.6) is self-dual. □

8.2.4 Minors and Excluded Minors

The basic properties one desires of contraction and deletion in a matroid hold, as summarized in Proposition 8.18.

Proposition 8.18. For a matroid M and disjoint subsets $X, Y \subseteq S(M)$,

(i) $(M\backslash X)\backslash Y = M\backslash(X \cup Y)$.

(ii) $(M/X)/Y = M/(X \cup Y)$.

(iii) $(M/X)\backslash Y = (M\backslash Y)/X$.

This demonstrates that the order in which elements are contracted or deleted from a matroid is irrelevant. Let $M_0, M_1, M_2, \ldots, M_n$ be a sequence of matroids such that M_{i+1} is obtained from M_i by deletion or contraction of a single element. Let X be the set of elements contracted on, and Y the set of elements deleted in this sequence. Then Proposition 8.18 says that:

$$M_n = (M_0/X)\backslash Y = (M_0\backslash Y)/X.$$

Definition 8.10. For a matroid M and disjoint subsets $X, Y \subseteq S(M)$, the matroid $M\backslash X/Y$ is called a ***minor*** of M.

A class of matroids is *minor-closed* if every minor of each matroid in that class also belongs in the class. The two classes of matroids we have introduced, graphic matroids and representable matroids, are both minor-closed as we will now demonstrate. There are classes of matroids that are not minor-closed, such as transversal matroids and strict gammoids, the interested reader can find details in [84].

Proposition 8.19. The class of graphic matroids if minor-closed.

Proof. Let G be any graph with graphic matroid $M(G)$, and X any subset of edges $X \subseteq E(G)$. Recall that $G\backslash X$ is the graph obtained from G by removing the edges in X. Similarly, G/X is obtained from G by contracting all edges in X. Then to prove that the graphic matroids are minor closed it suffices to show that

$$M(G)\backslash X = M(G\backslash X) \quad \text{and} \quad M(G)/X = M(G/X).$$

The fact that $M(G)\backslash X = M(G\backslash X)$ is straightforward. To show that $M(G)/X = M(G/X)$, it suffices to show that for any edge $e \in E(G)$, $M(G)/e = M(G/e)$. The result then follows by induction on $|X|$. If e is a loop of G then $G/e = G\backslash e$, and the result follows by the first case. Otherwise, let $\mathcal{J} \subseteq E(G)-e$. Then $\mathcal{J}\cup e$ contains no cycle of G if and only if $\mathcal{J}$ contains no cycle of $G\backslash e$. Therefore $\mathscr{I}(M(G)/e) = \mathscr{I}(M(G/e))$ and thus $M(G)/e = M(G/e)$. □

Proposition 8.20. For any field $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids is minor-closed.

Proof. We have already seen that for any $\mathbb{F}$-representable matroid M, the dual matroid M^* is also $\mathbb{F}$-representable. Now for any $X \subseteq S(M)$, by definition of contraction $M/X = (M^*\backslash X)^*$. Therefore since $\mathbb{F}$-representable matroids are closed under duality, to show that the $\mathbb{F}$-representable matroids are minor-closed it suffices to show that $M\backslash X$ is $\mathbb{F}$-representable. But now let A be any matrix over $\mathbb{F}$ such that $M[A] \cong M$. Then X represents a subset of the column labels of A, and let $A\backslash X$ be the matrix obtained from A by deleting all columns whose labels are in X. Then it is easy to check that

$$M\backslash X \cong M[A]\backslash X = M[A\backslash X].$$

Therefore $M\backslash X$ is $\mathbb{F}$-representable, and the result follows. □

One advantage of studying matroid minors is that large classes of matroids may be characterized using *excluded minor* theorems. The following gives a characterization of all graphic matroids, in terms of a (small) finite number of forbidden minors. Here K_5 denotes the complete graph on 5 vertices, and $K_{3,3}$ the complete bipartite graph. F_7 is the Fano matroid, introduced in Section 8.1.4.

Theorem 8.7. *A matroid M is graphic if and only if it has no minor isomorphic to $U_{2,4}, F_7, F_7^*, M^*(K_5)$ and $M^*(K_{3,3})$.*

The result of this theorem, due to Tutte, is powerful–any matroid that is *not* graphic must have a minor isomorphic to one matroid in this finite (and very small) family. The proof relies, in part, on the relationship between planar graphs and dual matroids. The proof, which also uses Kuratowski's theorem on forbidden minors in planar graphs, will not be presented here (see, for example, [108]).

Another excluded minor theorem, presented by Tutte, classifies binary representable matroids.

Theorem 8.8. *A matroid M is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

8.3 An Observation on Graphic Binary Matroids

In this section we present a way of recognizing when a binary matroid is graphic. This method may be more efficient than checking all possible minors and relying on the standard excluded minor result in Theorem 8.7.

Before proceeding to the main results, we first introduce some terminology. Let $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ be any vector in $\mathbb{Z}_2^n$ and let G be any graph on n labeled edges, $\{e_1, \ldots, e_n\}$.

Definition 8.11. $\boldsymbol{x}$ is a ***path vector for*** G if there exists a path P in G such that:

$$e_i \in E(P) \text{ if and only if } x_i = 1.$$

For example, given G in Figure 8.1.1, the vector $\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$ is a path vector for G.

Lemma 8.3. *Let M be any $\mathbb{Z}_2$-representable matroid with $|S(M)| = n$ and rank r. M is a graphic matroid if and only if there exists a compact representation matrix H for dual matroid M^*, and a labeled tree T with r edges such that each row of H is a path vector for T.*

Proof. ($\Rightarrow$) Suppose M is graphic with $|S| = n$, and let A' be any matrix representation of M. Let G be a connected graph for which M is isomorphic to $M(G)$. Let T be a spanning tree of G. Without loss of generality, by relabeling the edges of G (and the elements of M) we may assume that $E(T) = \{e_1, \dots, e_r\}$ and $E(G) - E(T) = \{e_{r+1}, \dots, e_n\}$. A standard graph-theoretic result[2] shows that for each $e_i \in E(G) - E(T)$, the graph $T \cup e_i$ contains exactly one cycle, C_i. The cycle-vectors $\boldsymbol{c}_i$ corresponding to these fundamental cycles form a basis for the cycle space of G. Since M is isomorphic to $M(G)$, there is a one-to-one correspondence between cycles in G and circuits of M. By definition of a column matroid, the vectors corresponding to the circuits of M generate $\text{Null}(A)$. Therefore $\{\boldsymbol{c}_{r+1}^t, \dots, \boldsymbol{c}_n^t\}$ is a basis for $\text{Null}(A)$, and the matrix N with rows $\boldsymbol{c}_{r+1}, \dots, \boldsymbol{c}_n$ is a representation matrix of M^*. By choice of labeling, N is a standard representation matrix of M^* with compact representation matrix H, $N = \left[\begin{array}{c|c} H & I \end{array} \right]$. As each row of N corresponds to a cycle in G, and each row has a single nonzero entry corresponding to a non-tree edge (located in the last $n - r$ columns), each row of H must be a path vector for T, and as a result, G.

($\Leftarrow$) Conversely suppose there exists a compact $(n - r) \times r$ representation matrix H for dual matroid M^* and a labeled tree T on r edges. For each column j of H, label the corresponding edge as e_j in T. For each row $\boldsymbol{h}_i$ of matrix H, let P_i be the corresponding path in T. For each i ($1 \le i \le n-r$) add an edge labeled e_{r+i} between the two endpoints of P_i in T (if $\boldsymbol{h}_i$ is the zero vector, and a loop edge at an arbitrary vertex of T). Let G be the graph obtained from T by adding $n-r$ edges $e_{r+1}, \dots, e_n$ in this manner, and let $N = \left[\begin{array}{c|c} H & I_{(n-r)} \end{array} \right]$. By construction, the rows of N are the cycle vectors of the fundamental cycles of G with respect to T, so N is a standard representation matrix of $M^*(G)$. Therefore since H is a compact representation matrix of M^*, $M^* \cong M^*(G)$, so $M \cong M(G)$ and so M is graphic. $\square$

This result gives a method of identifying those binary matroids that are graphic. However, it is an impractical means of determining whether or not a given binary matroid is *not* graphic. In particular, to use Lemma 8.3 to show that a binary matroid is non-graphic, one would need to check all possible standard representation matrices. However, we can prove a strictly stronger result that eases identification of non-graphic matroids. In particular, Theorem 8.9 demonstrates that we need only check *one* compact representation matrix of the dual matroid.

Theorem 8.9. *Let M be any $\mathbb{Z}_2$-representable matroid as in Lemma 8.3. M is a graphic matroid if and only if for any compact representation matrix H of dual M^*, there exists a labeled tree T with r edges such that each row of H is a path vector for T.*

Proof. ($\Leftarrow$) The reverse implication follows from the proof of Lemma 8.3.

($\Rightarrow$) Let M be a graphic matroid, and suppose there exists a compact representation matrix

[2]See, for example, [55]

H for M^* for which there exist *no* labeled tree T on r edges having the rows of H as path vectors. Let G be any connected graph for which M is isomorphic to $M(G)$, and let the edges $e_1, \ldots e_r$ be those corresponding to the columns of H. Then the edge set $\{e_1, \ldots e_r\}$ does not induce a forest in G. Therefore G contains a cycle on $\{e_1, \ldots, e_r\}$. The cycle vector for such a cycle is clearly linearly independent of the rows of the matrix $N = \begin{bmatrix} H & | & I \end{bmatrix}$, contradicting that H is a compact representation matrix for M^*. Therefore such a labeled tree must exist. □

8.3.1 Non-Graphic Matroids

Theorem 8.9 provides a method of determining when a binary matroid is not graphic. Using this method, we will demonstrate that the excluded matroids of Theorem 8.7 are not graphic.

Recall from Example 10.3 that the Fano matroid, F_7, and its dual, F_7^*, have compact representation matrices H and $H_* = H^t$ respectively,

$$H = \begin{array}{c} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_3 \end{array} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad H_* = \begin{array}{c} \boldsymbol{h}_{1*} \\ \boldsymbol{h}_{2*} \\ \boldsymbol{h}_{3*} \\ \boldsymbol{h}_{4*} \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Proposition 8.21. The Fano matroid F_7 is not graphic.

Proof. As H^* is a compact representation matrix of F_7^*, it suffices to show there is no labeled tree on 3 edges having the rows of H^* as path vectors. Suppose there is a tree T on 3 edges having the rows of H^* as path vectors. Letting $\boldsymbol{h}_i*$ denote the i^{th} row of H_*, the only tree T having $\boldsymbol{h}_1*, \boldsymbol{h}_2*$ and $\boldsymbol{h}_3*$ as path vectors is the star on four vertices, shown in Figure 8.3.1.

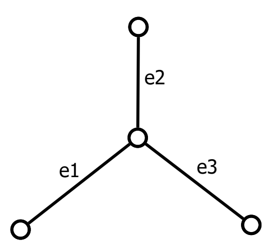


Figure 8.3.1: The only T having $\boldsymbol{h}_1*, \boldsymbol{h}_2*$ and $\boldsymbol{h}_3*$ as path vectors.

Clearly this labeled tree T *does not* have $\boldsymbol{h}_4* = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ as a path vector, and so F_7 is not graphic by Theorem 8.9. □

Proposition 8.22. The dual of the Fano matroid, F_7^* is not graphic.

Proof. As H is a compact representation matrix of $F_7 = (F_7^*)^*$ it suffices to show there is no labeled tree on 4 edges having the rows of H as path vectors. Let $\boldsymbol{h}_i$ denote the i^{th} row of H. There are three trees on 3 edges having $\boldsymbol{h}_3$ as a path vector, given as T_1, T_2, and T_3 in Figure 8.3.2.

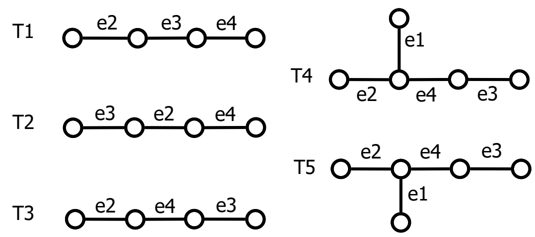


Figure 8.3.2: T_1, T_2, T_3: Three trees having $\boldsymbol{h}_3$ as a path vector. T_4, T_5: The only two trees with $\boldsymbol{h}_3$ and $\boldsymbol{h}_2$ as path vectors and no T_1 or T_2 subtree.

It is clear that tree T_1 does not have $\boldsymbol{h}_1$ as a path vector, and T_2 does not have $\boldsymbol{h}_2$ as a path vector. Therefore neither T_1 nor T_2 can be a sub-tree of the desired labeled tree. There are only two trees having $\boldsymbol{h}_3$ and $\boldsymbol{h}_2$ as path vectors, and no T_1 or T_2 subtree. These are T_4 and T_5 in Figure 8.3.2. It is clear that neither has $\boldsymbol{h}_1$ as a path vector, and thus no labeled tree on 4 edges has the rows of H as path vectors. Therefore F_7^* is not graphic by Theorem 8.9. □

Proposition 8.23. $M^*(K_{3,3})$ is not graphic.

Proof. By choosing a suitable spanning tree of $K_{3,3}$, $M(K_{3,3})$ has compact representation matrix

$$H = \begin{array}{c} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_3 \\ \boldsymbol{h}_4 \\ \boldsymbol{h}_5 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore H is a compact representation matrix of $(M^*(K_{3,3}))^*$, and thus it suffices to show there is no labeled tree on 4 edges having the rows of H as path vectors. As before, let $\boldsymbol{h}_i$ represent i^{th} row of H. There are exactly 4 trees having $\boldsymbol{h}_1$, $\boldsymbol{h}_2$ and $\boldsymbol{h}_5$ as path vectors. These are shown in Figure 8.3.3. It is clear that T_1, T_3, and T_4 do not have $\boldsymbol{h}_4$ as a path vector, and T_2 does not have $\boldsymbol{h}_3$ as a path vector. Thus no labeled tree on 4 edges has the rows of H as path vectors, and $M^*(K_{3,3})$ is not graphic by Theorem 8.9.

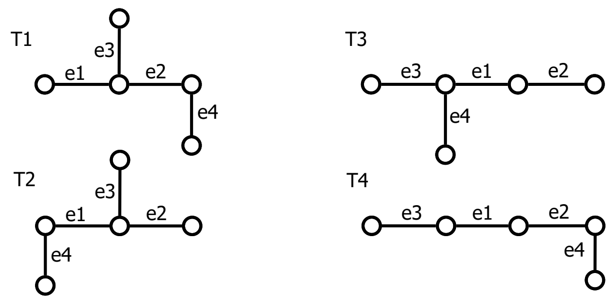


Figure 8.3.3: The 4 trees having $\boldsymbol{h}_1, \boldsymbol{h}_2$ and $\boldsymbol{h}_5$ as path vectors

□

Proposition 8.24. $M^*(K_5)$ is not graphic.

Proof. By choosing a suitable spanning tree of K_5, $M(K_5)$ has compact representation

$$H = \begin{array}{c} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_3 \\ \boldsymbol{h}_4 \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

As H is a compact representation matrix of $(M^*(K_5))^*$, it suffices to show there is no labeled tree on 6 edges having the rows $\boldsymbol{h}_i$ of H as path vectors. Any tree having the rows of H as path vectors must have one of the following three paths in Figure 8.3.4

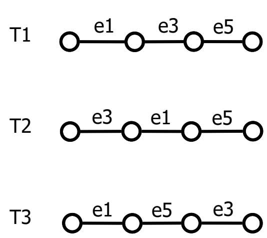


Figure 8.3.4: Three trees paths having $\boldsymbol{h}_1$ as a path vector.

We will show that there is no such tree having T_1 as a sub-tree. There are exactly 4 trees on five edges having $\boldsymbol{h}_2$ as a path vector and T_1 as a sub-tree. These are shown in Figure 8.3.5.

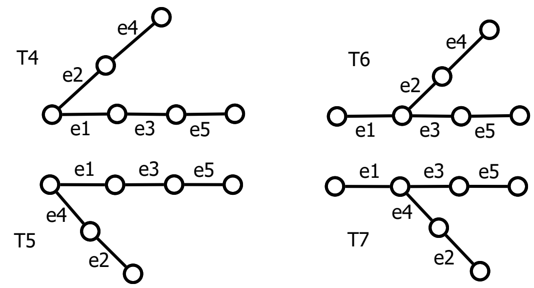


Figure 8.3.5: 4 five-edge trees having $\boldsymbol{h}_2$ as a path vector and a T_1 sub-tree.

It is clear that in adding e_6 to T_4, T_5 or T_7, if $\boldsymbol{h}_3$ is a path vector then we must introduce a cycle. On the other hand, there are only two trees having T_6 as a subgraph and $\boldsymbol{h}_3$ as a path vector. These are shown in Figure 8.3.6.

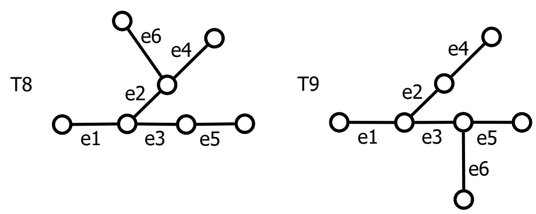


Figure 8.3.6: 2 six-edge trees having $\boldsymbol{h}_3$ as a path vector and a T_6 sub-tree.

It is clear that neither T_8 nor T_9 have $\boldsymbol{h}_4$ as a path vector. Therefore there is no labeled tree on 6 edges having the rows of H as path vectors and T_1 as a subtree. A similar case-by-case argument shows that no such labeled tree has T_2 or T_3 as a subtree either. Thus no labeled tree on 6 edges has the rows of H as path vectors, and so $M^*(K_5)$ is not graphic by Theorem 8.9. $\square$

Theorem 8.9 presents one method of recognizing when a binary matroid is graphic. This section then used this result to verify that the standard excluded minors of Theorem 8.7 are not graphic. The relationship between binary matroids and graphic matroids was studied extensively by Tutte in the 1960's. His methods are the topic of Chapter 9.

Chapter 9

Graphic Binary Matroids: Tutte's Algorithm

As alluded to in the introduction of Chapter 8, our interest in matroids was motivated by understanding relationships between graphs and matrices. A classical question in this vein is the following.

Which binary matrices naturally correspond to undirected graphs?

In terms of matroids, this question can be rephrased by asking when a binary matroid is graphic. Corollary 9.1, which follows directly from Theorem 8.7 in Chapter 8, provides the standard characterization of graphic binary matroids.

Corollary 9.1. *A binary matroid is graphic if and only if it has no minor isomorphic to* $F_7, F_7^*, M^*(K_5)$ *and* $M^*(K_3,3)$.

While Corollary 9.1 completely characterizes those binary matroids that are graphic, if one wants to determine if some *specific* binary matroid is graphic, it would require checking all possible minors. In 1960, Tutte presented a deterministic algorithm for deciding whether a binary matroid is graphic [105]. The notation and terminology preferred by Tutte were not adopted in the years since that publication, so we dedicate this chapter to presenting Tutte's method using current mathematical conventions. A version of this process is presented as a decomposition result in [91]. We will also work-through (in detail) the example originally presented by Tutte in order to illustrate his process. Beyond notational updates, there is one important difference in Tutte's description of representable matroids that leads to changes in our presentation. When Tutte refers to the "standard representation matrix" of a matroid

M, the matrix he describes, by current conventions, is actually a representation of the *dual matroid* M^*.

Before presenting Tutte's method, Section 9.1 will first introduce a number of definitions and results. In particular, Theorems 9.1, 9.2, 9.3, and 9.4 present a set of necessary conditions of graphic matroids. Tutte's method, presented in Section 9.2.1, begins with a binary matroid and sequentially checks each of these necessary conditions. If any are not met, we conclude that the matroid is not graphic. Otherwise the process is repeated on a specified set of matroid minors. Section 9.2.2 illustrates a worked example of this process. If after iteration none of the necessary conditions prove false, Tutte renders these conditions *sufficient* by constructing the desired graph. Section 9.2.3 introduces the machinery used in Tutte's graph construction method, which is presented in Section 9.2.4. Section 9.2.5 continues the example of Section 9.2.2, and constructs a graph.

9.1 Preliminaries to Tutte's Algorithm

For a graph G with $V(G) = \{v_i\}$ and $E(G) = \{e_j\}$, the ***binary incidence matrix*** R_G is the $|V(G)| \times |E(G)|$ matrix with a 1 in entry (i, j) if and only if edge e_j is incident to vertex v_i.

Proposition 9.1. Let G be a graph with binary incidence matrix R_G. R_G is a matrix representation of $M(G)$. That is, $M(G) \cong M[R_G]$.

Proof. [91] For any $X \subseteq E(G)$, it suffices to show X is a linearly dependent set of columns in R_G if and only if the induced subgraph $G[X]$ contains a cycle. First assume that $G[X]$ contains a cycle, C. The columns of R_G corresponding to C form a submatrix R_C, the incidence matrix of C. Then since each row of R_C has exactly 2 ones, it follows that the sum of the columns of R_G corresponding to C must be zero. Conversely, suppose that X is a linearly dependent set of columns of R_G, and let $C \subseteq X$ be a minimally dependent subset of X. Then the columns of R_G corresponding to C must sum to zero. In particular, $G[C]$ has no vertex of odd degree. Therefore $G[C]$ has no vertex of degree one, and thus is not a forest [23]. Therefore $G[C]$ and hence $G[X]$ must contain a cycle. □

Corollary 9.2. *Every graphic matroid is a binary matroid.*

By a similar argument, one can show that graphic matroids are representable over *any* field $\mathbb{F}$. Such matroids are given special distinction.

Definition 9.1. A matroid is ***regular*** if it is representable over every field $\mathbb{F}$.

In particular, all graphic matroids are regular. In Section 8.1.2 we introduced the cycle matroid of a graph. In fact, there is another matroid naturally associated with a graph, known as the ***bond matroid***. Recall that a *bond* of a graph G is a minimal (with respect to inclusion) nonempty set of cut edges [23]. The bond matroid of G, $M'(G)$, is defined by taking the set of edges $E(G)$ as the ground set S, and the bonds of G as circuits. One may check that, as defined, this matroid satisfies circuit axioms $(C1)$ and $(C2)$.

Remark 9.1. The orthogonality of the cycle space and cut space of graphs leads to a natural connection between the cycle and bond matroids of a graph. Specifically for any graph G,

$$(M(G))^* = M'(G).$$

This can be seen by considering binary matrix representations of $M(G)$ and $M(G)^*$. Let A and A_* be standard matrix representations of $M(G)$ and $M(G)^*$ respectively. While A is a matrix representation of the cycle matroid of G, the *rows* of A correspond to bonds of G. On the other hand, while A_* is a matrix representation of the bond matroid of G, the rows of A_* correspond to cycles of G.

In view of these remarks, this leads to another description binary graphic matroids.

Proposition 9.2. A binary matroid M is graphic if and only if dual matroid M^* is the bond matroid of some graph G.

9.1.1 Bridges and Y-components

Recall that a separator of a matroid M is a subset $X \subseteq S(M)$ such that any circuit $C \in \mathscr{C}(M)$ is contained in either X or $S(M) - X$. It is clear that $S(M)$ and $\emptyset$ are always separators of M. Moreover, recall that a matroid that has no separators other than $S(M)$ and $\emptyset$ is called *connected*.

The elementary separators (i.e. connected components) of a matroid M are disjoint and their union is M. When M is binary, the elementary separators can be recognized by partitioning the columns of a standard matrix representation or, in view of Theorem 8.4, by considering the standard matrix representation of M^*.

Example 9.1. Consider the binary matroid $M[R]$, where

$$R = \begin{array}{c} \begin{array}{cccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 \end{array} \\ \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \end{array}.$$

Letting $s_1, \ldots, s_6$ denote the columns of R (and thus the ground set of $M[R]$), partition the columns as $S_1 = \{s_1, s_2, s_5\}$ and $S_2 = \{s_3, s_4, s_6\}$. Reordering the elements, we find the following block form

$$\begin{array}{c} \begin{array}{cccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 5 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

It is clear that any minimal dependent set of columns (i.e. any circuit of $M[R]$) must lie either completely in S_1 or S_2, and no further partitioning of the columns maintains this property. Therefore S_1 and S_2 are elementary separators of $M[R]$. Alternatively, the matrix R_* (9.1) is a standard representation of dual matroid $M[R]^*$.

$$R_* = \begin{array}{c} \begin{array}{cccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 \end{array} \\ \left[\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \end{array} \tag{9.1}$$

By Theorem 8.4, the rows of R_* correspond to circuits of $M[R]$, and any circuit must lie in the row space of R_*. Therefore it is clear that any circuit of $M[R]$ must lie in either $S_1 = \{s_1, s_2, s_5\}$ or $S_2 = \{s_3, s_4, s_6\}$.

The following rule, which we present without proof, is often useful in calculations involving matroids.

Proposition 9.3. A subset $X \subseteq S(M)$ is a separator of matroid M if and only if

$$M|S = M.S.$$

Example 9.2. We can check the "only if" direction of this statement in the context of $M[R]$ as in Example 9.1. Taking elementary separator $S_1 = \{s_1, s_2, s_5\}$, a representation of $M[R]|S_1$ is obtained by deleting from R all columns other than those of S_1. The resulting matrix is given in (9.2).

$$\begin{array}{c} \begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 5 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \tag{9.2}$$

On the other hand, $M[R].S_1 = (M[R]^*|S_1)^*$. Beginning with representation R_* for $M[R]^*$, observe $M[R]^*|S_1$ has representation R_0 and standard representation R_0', where

$$R_0 = \begin{array}{c} \begin{matrix} 1 & 2 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad R_0' = \begin{array}{c} \begin{matrix} 1 & 2 & & 5 \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 1 & 1 \end{array}\right] \end{array}.$$

Matrix R_0', in turn, is used to obtain a standard representation matrix of $(M[R]^*|S_1)^* = M[R].S_1$, given in (9.3). Clearly the representation of $M[R].S_1$ in (9.3) is row-equivalent to the representation of $M[R]|S_1$ in (9.2). Therefore $M[R]|S_1 = M[R].S_1$.

$$\begin{array}{c} \begin{matrix} 1 & 2 & & 5 \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \end{array} \tag{9.3}$$

Now let M be any binary matroid and Y any cocircuit of M.

Definition 9.2. The ***bridges*** of Y in M are the elementary separators of the matroid $M\backslash Y$.

A detailed example of finding representations of bridges in the binary case will be given in Section 9.2.2. The number of bridges of a cocircuit are of particular interest in studying graphic matroids, as illustrated by the following theorem.

Theorem 9.1. *Let M be the cycle matroid of a graph G, and Y a cocircuit of M having at most one bridge in M. Then there is a vertex of G such that Y is the set of edges incident to that vertex (i.e. Y is the star of a vertex in G).*

Proof. The edges of Y are a bond (minimal separating set of edges) of graph G. Thus $G - Y$ has two connected components, G_1 and G_2, and $Y = E(G_1, G_2)$. That is, Y is the set of all edges in G with one end in G_1 and one end in G_2. Recall that $E(G_1)$ is always a separator of $M(G_1)$, so if $E(G_1)$ is nonempty, $M[G_1]$ must contain a minimal nonempty separator– i.e. an elementary separator. But now observe that the elementary separators of $M(G_1)$ and $M(G_2)$ are also elementary separators of $M\backslash Y$. Therefore if $E(G_1)$ and $E(G_2)$ are both nonempty, $M\backslash Y$ must have at least two elementary separators, a contradicting that Y had at most one bridge. Therefore without loss of generality, G_1 is connected and contains no edges. That is, G_1 is a single vertex v. As $Y = E(G_1, G_2)$, Y is the star of vertex v. $\square$

Definition 9.3. For each bridge B of cocircuit Y, the matroid $M.(B \cup Y)$ is known as a Y***-component*** of M.

The following result is well-known [91][105].

Lemma 9.1. *If Y is a cocircuit of a connected matroid M, then each Y-component of M is connected.*

9.1.2 Bridge Partitioning and Even Cocircuits

Let M be a matroid and Y any cocircuit of M. For any bridge B of cocircuit Y, let $\pi(M, B, Y)$ be the collection of all non-null subsets of Y that can be written as an intersection of cocircuits $M.(B \cup Y)$. Tutte showed that in certain cases, the members of $\pi(M, B, Y)$ partition Y[104].

Lemma 9.2. *If M is a regular matroid and Y is a cocircuit of M, then the members of $\pi(M, B, Y)$ are disjoint and their union is Y.*

On the other hand, in the case of binary matroids the members of $\pi(M, B, Y)$ are related to a certain minor of Y-component $M.(B \cup Y)$ [91]:

Lemma 9.3. *If Y is a cocircuit of a binary matroid M, then*

$$\pi(M, B, Y) = \mathscr{C}^*(M.B \cup Y)|Y)$$

Since every regular matroid is also binary, if M is regular then the cocircuits of $M.(B \cup Y)|Y$ form disjoint subsets $S_1, S_2, \ldots, S_k$ of Y whose union is Y. In this case we say B *partitions* Y and that $\{S_1, \ldots, S_k\}$ is the *partition* of Y determined by B. A binary standard representation matrix of $M.(B \cup Y)|Y$ will have exactly one nonzero element in each column. As every graphic matroid is regular, we have the following necessary condition for graphic matroids.

Theorem 9.2. *If M is a graphic matroid, every bridge partitions its respective cocircuit.*

Given a cocircuit Y of matroid M, let B and B' be bridges of Y that partition Y into $\{S_1, \ldots, S_k\}$ and $\{T_1, \ldots, T_m\}$ respectively.

Definition 9.4. If there exist S_i and T_j such that $S_i \cup T_j = Y$, we say that bridges B and B' are *non-overlapping*, otherwise they are said to *overlap*. Bridges which overlap are sometimes said to *avoid*. We call Y an ***even*** cocircuit of M if each bridge of Y partitions Y, and the bridges of Y can be separated into two disjoint classes such that no members of the same class overlap.

An example of recognizing non-overlapping bridges and even cocircuits in binary matroids will be given in Section 9.2.2. Once again, even cocircuits are a necessary condition of graphic matroids.

Theorem 9.3. *In a graphic matroid, every cocircuit is even [104].*

Proof. [91] Let M be a graphic matroid and $Y \in \mathscr{C}^*(M)$. Each bridge of Y partitions Y by Theorem 9.2, so it suffices to show the bridges can be partitioned into non-overlapping classes. Let G be a graph such that $M = M(G)$, and let G_1 and G_2 be the two connected components of $G \backslash Y$. Partition the bridges of Y by their membership in components G_1 and

G_2 (note that by minimality, each bridge must be contained entirely in G_1 or G_2). That is, for $i = 1, 2$, define:

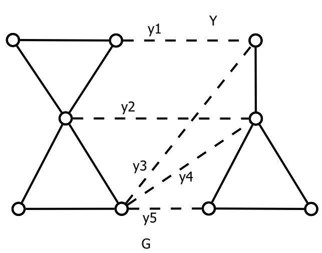
$$\mathscr{L}_i = \{B \subseteq E(M) : B \text{ is a bridge of } Y \text{ and } G[B] \text{ is a subgraph of component } G_i\}$$

Assume for contradiction that Y is not even: then any two-class partition of its bridges must have some class containing an overlapping pair. Without loss of generality, suppose $\mathscr{L}_1$ contains bridges B_1 and B_2 that overlap. For any vertex $v \in V(G_1 \backslash B_1)$, let $C(B_1, v)$ denote the component of $G_1 \backslash B_1$ containing v, and $Y(B_1, v)$ the set of edges in Y with one end in $C(B_1, v)$. One may show, using results of Section 9 in [106] for example, that there exists a vertex $v_1 \in V(G[B_1])$ such that $G[B_2]$ is a subgraph of $C(B_1, v_1)$. Similarly, there exists $v_2 \in V(G[B_2])$ such that $G[B_1]$ is a subgraph of $C(B_2, v_2)$ and $V(G_1) \subseteq V(C(B_1, v_1)) \cup V(C(B_2, v_2))$. Moreover, in [106] it is then shown that $Y(B_1, v_1) \cup Y(B_2, v_2) = Y$, contradicting that B_1 and B_2 overlap. □

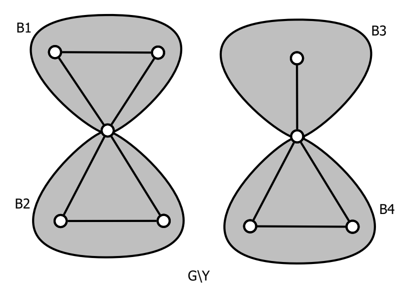
9.1.3 A Worked Example of Necessary Conditions

The following example demonstrates some of the necessary conditions of graphic matroids discussed in the previous sections.

Example 9.3. Consider the graph G and cocircuit Y as given in Figure 9.1.1a.



(a) Graph G with cocircuit Y



(b) Elementary separators of $M(G) \backslash Y$

Figure 9.1.1: Bridges of a cocircuit in a graph

Figure 9.1.1b shows the elementary separators of $M(G)\backslash Y$–that is, the bridges of Y. Note that B_i is the set of *edges* in each circled subgraph. Every cycle of $G\backslash Y$ is either contained in $G[B_i]$ or $G[(E(G\backslash Y) - B_i)]$, so each B_i is a separator. As each edge set B_i is minimal with respect to this property, these are the elementary separators of $M(G)\backslash Y$. Figure 9.1.2 illustrates the Y-component $G.(B_i \cup Y)$ for each bridge B_i. Observe that for each i, the Y-component $G.(B_i \cup Y)$ is obtained by contracting all edges $e \notin B_i \cup Y$.

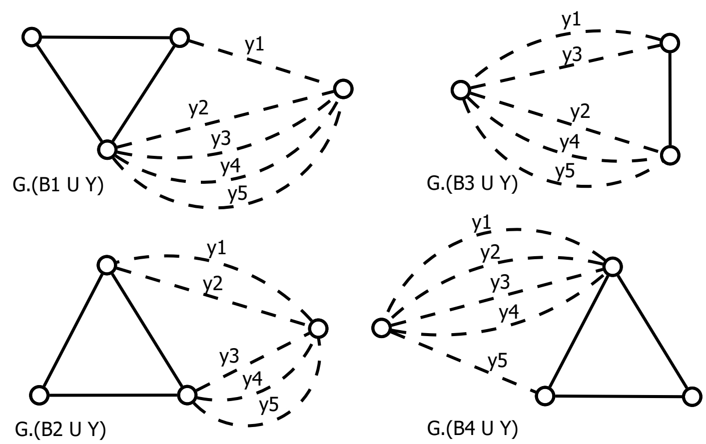


Figure 9.1.2: Y-components $G.(B_i \cup Y)$.

Finally, we construct the graphs $G.(B_i \cup Y)|Y$. These are obtained from the Y-components in Figure 9.1.2 by deleting the edges of B_i for each i. This is illustrated in Figure 9.1.3.

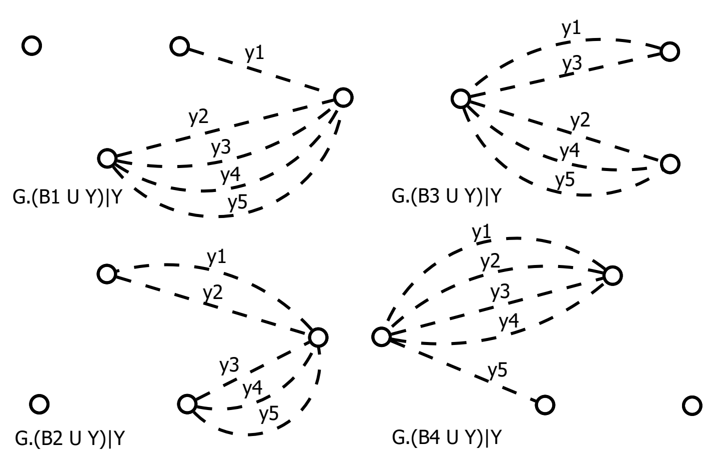


Figure 9.1.3: Graphs of $G.(B_i \cup Y)|Y$.

By considering the four graphs in Figure 9.1.3, we see that the bonds of each graph $G.(B_i \cup Y)|Y$ partition the edges of Y. Therefore each bridge B_i partitions Y, as necessitated by

Theorem 9.2. In particular,

$$\pi(M(G), B_1, Y) = \Big\{\{y_1\}, \{y_2, y_3, y_4, y_5\}\Big\} \qquad \pi(M(G), B_3, Y) = \Big\{\{y_1, y_3\}, \{y_2, y_4, y_5\}\Big\}$$
$$\pi(M(G), B_2, Y) = \Big\{\{y_1, y_2\}, \{y_3, y_4, y_5\}\Big\} \qquad \pi(M(G), B_4, Y) = \Big\{\{y_1, y_2, y_3, y_4\}, \{y_5\}\Big\}$$

Moreover, observe that $\{y_2, y_3, y_4, y_5\} \cup \{y_1, y_2\} = Y$, so B_1 and B_2 are non-overlapping. Similarly, $\{y_2, y_4, y_5\} \cup \{y_1, y_2, y_3, y_4\} = Y$, meaning B_3 and B_4 are non-overlapping as well. Therefore if we partition the bridges as $\{B_1, B_2\}$ and $\{B_3, B_4\}$, no two bridges of the same class overlap and Y is an even cocircuit of $M(G)$, as necessitated by Theorem 9.3. Notice that B_1 and B_2 lie in the same connected component of $G \backslash Y$, which is different from that component that contains B_3 and B_4. Thus the partition $\{B_1, B_2\}$ and $\{B_3, B_4\}$ is exactly that considered in the proof of Theorem 9.3.

Before moving to the presentation of Tutte's algorithm, we need a few results relating graphic matroids and minors.

9.1.4 Graphic Matroids and Minors

Lemma 9.4. *If M is graphic and $X \subseteq S(M)$, then $M.X$ is also graphic.*

The proof of this theorem is straightforward, arising because the notions of matroid contraction and deletion align with the well-defined notions of edge deletion and contraction in graphs. A partial converse to this statement, given in terms of Y-components (recall Definition 9.3), is a key step in Tutte's procedure.

Theorem 9.4. *Let Y be an even cocircuit of a connected binary matroid M such that every Y-component of M is graphic. Then M is graphic.*

Proof. [91] Assume there exists a binary matroid M and a cocircuit $Y \in \mathscr{C}^*(M)$ such that the result fails. Among such M and Y, choose one for which $|S(M)|$ is minimal. We will reach a contradiction by finding a graph G such that $M = M(G)$. If M has only one bridge, $M = M.(B \cup Y)$ which is graphic by assumption. Thus we may assume that Y has at least two bridges. Since Y is even, its bridges can be partitioned into classes $\mathscr{U}_1$ and $\mathscr{U}_2$ such that any pair of bridges in the same class avoid each other. For $i = 1, 2$, let E_i be the set of edges contained in all bridges of class i. That is,

$$E_i = \bigcup_{B \in \mathscr{U}_i} B.$$

From here the proof can be divided into three parts. First, we demonstrate that the matroids $M.(E_1 \cup Y)$ and $M.(E_2 \cup Y)$ are graphic. Next we construct two graphs, G_1 and G_2 such that $M(G_i) = M.(E_i \cup Y)$, and combine these graphs into a third graph G. Finally, we show that $M = M(G)$. These steps can be found in detail in [91]. In addition to the theory that guarantees existence, an example of constructing the graphs G_1, G_2 and G will be given in Sections 9.2.4 and 9.2.5. $\square$

9.2 Tutte's Algorithm

Section 9.1 presented a number of necessary conditions for a matroid to be graphic. If a cocircuit has only one bridge, it must be the star of a vertex in the corresponding graph (Theorem 9.1). Any bridge must partition its respective cocircuit (Theorem 9.2), and every cocircuit must be even (Theorem 9.3). Tutte's algorithm begins with a binary matroid, and sequentially checks these necessary conditions. If at any point one such condition fails, the algorithm terminates and concludes that the matroid is not graphic. Otherwise if each condition is satisfied, it then appeals to Theorem 9.4 and repeats to successively determine if each Y component is graphic. A flow-chart of this algorithm is presented in Figure A.1 of Appendix A. The reader will likely find it helpful to follow this reference in parallel to the following description.

9.2.1 Tutte's Recognition Algorithm

Let M be any connected binary matroid. First, construct a standard matrix representation of M, R. If no column of R has more than 2 nonzero entries, adjoin to R (as a new row) the mod 2 sum of the rows of R. The result is the binary incidence matrix of a graph whose cycle matroid, by Proposition 9.1, is M. Otherwise some column of R contains at least three nonzero elements. Without loss of generality, assume the last column of R has nonzero elements in the first, second, and third rows.

By Theorem 8.4, the first row of R is the characteristic vector of a cocircuit Y_1 of M. We wish to consider the Y_1 components of M, so first we construct a representation of the matroid $M \backslash Y_1$. This matrix R' is obtained by removing the first row from R, as well as all columns that have a unit entry in the first row. From R', we identify the elementary separators of Y_1 (for example, by either method described in Example 9.1), giving the bridges $B_1, \ldots, B_k$ of Y_1.

It may be the case that Y_1 has only one bridge in M. If so, repeat this process with Y_2, the cocircuit corresponding to the second row of R. If Y_2 has only one bridge, continue with cocircuit Y_3 corresponding to row 3. If this cocircuit also has only one bridge, we may conclude that the matroid is not graphic. If M were graphic, then by Theorem 9.1, the last column of R corresponds to an edge that is incident with three distinct vertices. This is clearly impossible.

Otherwise, assume without loss of generality that Y_1 has at least two bridges in M. For each bridge B_i, construct a standard representation matrix of the Y_1-component $M.(B_i \cup Y_1)$. This is accomplished with a few straightforward steps:

(1) Identify the rows of R' (the representation matrix of $M \backslash Y_1$) representing bridge B_i.

(2) Extract the corresponding rows of matrix R.

(3) Adjoin the first row (i.e. the row corresponding to Y_1) and delete any zero columns.

Now for each i, use this matrix representation of $M.(B_i \cup Y_1)$ to find a standard representation matrix of $M.(B_i \cup Y_1)|Y_1$. This can be accomplished by first deleting all columns without a 1 in the first row, and then performing row operations. If each column of the resulting matrix has exactly one unit entry, then B_i partitions Y_1. Otherwise bridge B_i does not partition Y_1, and by Theorem 9.2 M is not graphic.

If each bridge B_i partitions Y_1, we examine to see which bridges overlap. If Y_1 is not even, then by Theorem 9.3, M is not graphic. Otherwise if Y_1 is even, we have now simplified the problem. Specifically, M is graphic if and only if its Y_1 components are graphic by Lemma 9.4 and Theorem 9.4. Notice we have already constructed standard representation matrices for the Y_1-components of M. Moreover, by Lemma 9.1, each of these Y_1-components is connected and (because Y_1 has more than one bridge) has lower rank than M. Therefore we repeat the above procedure for each Y_1-component of M, and continue in this manner until the process eventually terminates.

If there exist graphs for each Y-component of a graphic matroid M, then these graphs can be used to construct a graph for matroid M. These constructions are described in [104] and will be presented in Section 9.2.4.

9.2.2 A Worked Example of Tutte's Algorithm

The following is the example originally used by Tutte to illustrate his method. We will work through this example in detail. Consider the binary matroid $M = M[R]$, where R is the standard binary representation given in (9.4). Throughout this example, we use the integers $\{1, \ldots, 15\}$ to denote the elements of matroid M. To help illustrate the various matrix operations performed, the j^{th} row of matrix R will be labeled j^*. These conventions will be used through the end of this chapter, as well as in Chapter 10.

$$R = \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \\ 5^* \\ 6^* \\ 7^* \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \end{array} \tag{9.4}$$

There are several columns with at least 3 nonzero entries, let us consider column 13. The first row having a unit entry in column 13 is row 5–let Y denote the cocircuit of M corresponding to row 5. That is, $Y = \{5, 8, 9, 10, 11, 12, 13\}$. Removing the 5th row of R and every column with a 1 in this row, we obtain the standard representation matrix R' of $M \backslash Y$,

$$R' = \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \\ 6^* \\ 7^* \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 6 & 7 & & 14 & 15 \end{array} \\ \left[\begin{array}{cccccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \end{array}.$$

Next we determine the bridges of Y in M, that is, the elementary separators of $M \backslash Y$. Given the standard representation matrix R', the rule for identifying an elementary separator is as follows.

(1) Choose an arbitrary row of R'.

(2) Add to this row vector every row of R' having a 1 in the same column as a 1 in the row from (1).

(3) Add to the result every row having a 1 in the same column as a 1 in the row chosen in (2).

(4) Continue in this manner until no such further rows exist. The elementary separator corresponds to the columns with a nonzero entry in the resulting row vector.

(5) To find another elementary separator, choose a row which was not previously chosen in (1)-(4), and begin again at (1).

Using this process, we see that the bridges of Y in M are $B_1 = \{6\}$, $B_2 = \{1, 7, 15\}$ and $B_3 = \{2, 3, 4, 14\}$. For each bridge B_i, we construct a standard representation matrix for

$M.(B_i \cup Y)|Y$ via the three-step procedure described in Section 9.2 (notice in step 3, we adjoin the row corresponding to Y, in this case that is the 5th row of R). The following matrices, R_1, R_2, R_3 are the standard representation matrices of the Y-components $M.(B_i \cup Y)$. In each case the last row represents cocircuit Y.

$$R_1 = \begin{array}{c} \\ 6^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccc|cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

$$R_2 = \begin{array}{c} \\ 1^* \\ 7^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccc|cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

$$R_3 = \begin{array}{c} \\ 2^* \\ 3^* \\ 4^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccc|cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

If zero columns are removed, by permuting the columns each R_i can be seen to be a standard binary representation matrix. For each $1 \leq i \leq 3$, let $M_i = M[R_i]$. We will return to the matroids M_1, M_2 and M_3 shortly.

The next step is to examine if each of B_1, B_2, and B_3 partition Y. To do so, we construct a matrix representation of $M.(B_i \cup Y)|Y$. Given that R_i is a matrix representation of $M.(B_i \cup Y)$, this is accomplished by removing all columns of R_i having a zero in the last row (i.e. the row corresponding to Y). This gives R'_1, R'_2, and R'_3 as in (9.2.2), the matrix representations of $M.(B_i \cup Y)|Y$.

$$R'_1 = \begin{array}{c} \\ 6^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 5 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} \qquad R'_2 = \begin{array}{c} \\ 1^* \\ 7^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 5 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

$$R'_3 = \begin{array}{c} \\ 2^* \\ 3^* \\ 4^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 5 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{9.5}$$

Via row operations, each of these matrices is transformed to a matrix R''_i that is, up to a permutation of columns, a standard representation matrix of $M.(B_i \cup Y)|Y$. These are given in (9.2.2).

$$R_1'' = \begin{array}{c} \\ 6^* \\ (5^*{+}6^*) \end{array} \begin{array}{c} \begin{array}{ccccccc} 5 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad R_2'' = \begin{array}{c} \\ 1^* \\ (7^*{+}1^*) \\ (5^*{+}7^*) \end{array} \begin{array}{c} \begin{array}{ccccccc} 5 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$R_3'' = \begin{array}{c} \\ 2^* \\ (3^*{+}2^*) \\ (4^*{+}3^*) \\ (5^*{+}4^*) \end{array} \begin{array}{c} \begin{array}{ccccccc} 5 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{9.6}$$

Clearly each standard representation matrix has a single 1 in each column, and thus each of B_1, B_2, and B_3 partitions Y. Letting P_i denote the partition of Y determined by B_i, we have:

$$\begin{aligned} P_1 &= \Big\{\{5,11\},\{8,9,10,12,13\}\Big\} \\ P_2 &= \Big\{\{5,10,11\},\{8,9\},\{12,13\}\Big\} \\ P_3 &= \Big\{\{5,13\},\{8,10\},\{9,12\},\{11\}\Big\}. \end{aligned}$$

If the bridges B_i had not all partitioned Y–that is, if one of the matrices R_i'' had two nonzero entries in some column–then the algorithm would terminate and conclude that M was not graphic. However since each B_i partitions Y we must check if Y is even. Notice that $\{8,9,10,12,13\}$ is a member of partition P_1, $\{5,10,11\}$ is a member of partition P_2, and $\{8,9,10,12,13\} \cup \{5,10,11\} = Y$. Then bridges B_1 and B_2 are non-overlapping, and the bridges can be divided into disjoint classes $\{B_1,B_2\}$ and $\{B_3\}$ such that no two members of the same class overlap. Therefore Y is even. If we had found the bridges could not be arranged in this manner, then Y is not even, and the algorithm would terminate concluding that M is not graphic.

Thus we have now reduced the problem, and we can assert that M is graphic if and only if matroids M_1, M_2 and M_3 are graphic. We repeat this algorithm, now applied to standard representation matrices R_1, R_2 and R_3 respectively.

M_1 is graphic because R_1 has at most two 1's in each column. By applying the algorithm to R_2 and R_3, we find that both M_2 and M_3 are graphic as well. Therefore the algorithm terminates and concludes that matroid M is graphic.

9.2.3 Graph Operations: Twisting and Star Composition

While Tutte's algorithm determines whether a binary matroid is graphic, constructing the associated graph is also desirable. The matroid minors considered by Tutte's algorithm can be utilized to construct such a graph. Before describing this construction process, some theory is in order.

A ***2-separation*** of a graph G is a partition $\{X, Y\}$ of the edges such that

$$\min\{|X|, |Y|\} \geq 2 \text{ and } |V(G[X]) \cap V(G[Y])| = 2$$

Let $\{X, Y\}$ be a 2-separation in a graph G such that $V(G[X]) \cap V(G[Y]) = \{u, v\}$. The ***twisting*** of G about u and v is obtained by interchanging u and v in every edge of X. Of particular importance is that if G' is obtained from G by a twisting, then *the cycles and bonds of G' are exactly the cycles and bonds of G*. Figure 9.2.1 gives an example of twisting, where one may observe that the cycles and bonds are unchanged. As described in [91], we can think of twisting as separating $G[X]$ and $G[Y]$ at the vertices u and v, twisting either subgraph about an axis perpendicular to the line that passes through u and v, and reconnecting.

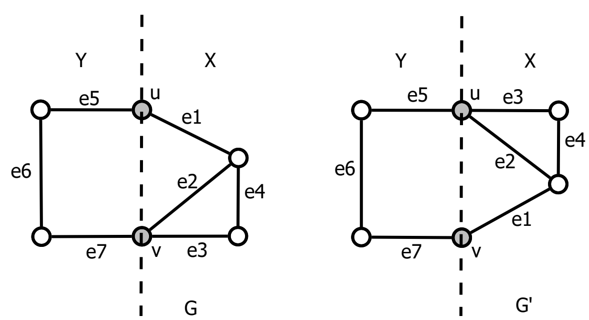


Figure 9.2.1: An example of Twisting.

Theorem 9.5. *Let Y be a cocircuit of a connected graphic matroid such that any two bridges of Y avoid each other. Then there exists a 2-connected graph G where Y is a star of a vertex in G, and $M = M(G)$.*

Proof. Since M is graphic, there exists a 2-connected graph G such that $M(G) = M$ and Y is a bond in G. Let G_1, G_2 be the two components of $G \backslash Y$. The proof proceeds by finding a sequence of 2-separations which, by a series of twistings on the associated intersection vertices, reduce the size of G_1 successively until we are left with a single vertex whose star is Y. As each successive graph is obtained by a twisting, the bonds and cycles of each graph in the sequence are identical to those of G. Thus we arrive at a 2-connected graph G' such that $M(G') \cong M(G) \cong M$ and Y is the star of a vertex in G'. Full details of this proof can be found in Theorem 5.5 of [91]. $\square$

Example 9.4. The sequence of graphs G, G', and G'' in Figure 9.2.2 that illustrates the process sketched above. Each is obtained from the previous by a twisting, and we see that $Y = \{y_1, y_2, y_3\}$ is a cocircuit in each. In particular, Y is the star of vertex w in G''.

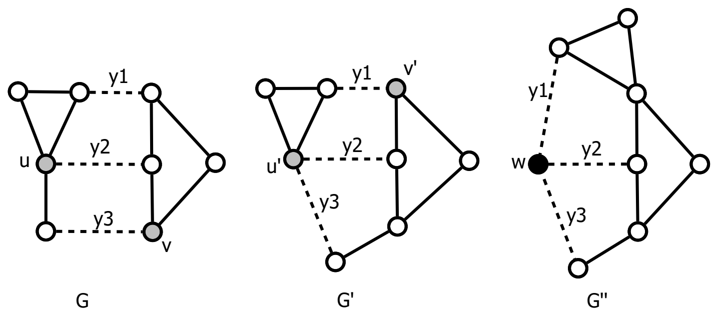


Figure 9.2.2: A sequence of twistings.

Let G_1 and G_2 be two labeled graphs, and Y set of edge labels such that $Y \in E(G_1) \cap E(G_2)$. Moreover, suppose there are vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that Y is the star of vertices v_1 and v_2. We construct a graph G with $E(G) = E(G_1) \cup E(G_2)$ by adjoining $G_1 \backslash v_1$ and $G_2 \backslash v_2$ as follows. For any edge $e \in Y$, add an edge between the endvertex of e (other than v_1) in G_1, and the endvertex of e (other than v_2) in G_2. Label this new edge e. By construction, Y is a bond of the resulting graph, G. G is known as the ***star composition*** of G_1 and G_2 in Y, denoted $G = G_1 \star_Y G_2$.

Now consider the incidence matrices of G_1 and G_2, and examine the matrix operation associated with the graph operation of star composition. By re-ordering the columns and vertices, each incidence matrix may be written in the form

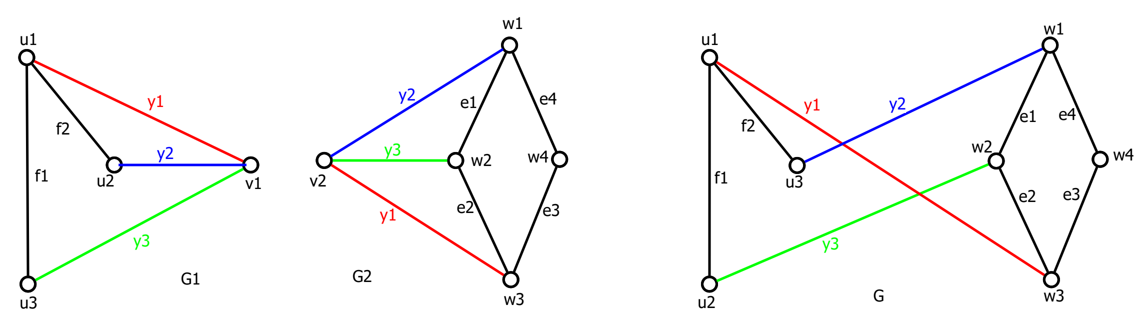
$$R_{G_1} = \begin{array}{c} \\ V(G_1)\backslash v_1 \\ \hline v_1 \end{array} \begin{array}{c} \begin{array}{cc} E(G_1)\backslash Y & \qquad Y \end{array} \\ \left[\begin{array}{ccc|ccc} & R_{1,1} & & & R_{1,2} & \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 \end{array}\right] \end{array} \qquad R_{G_2} = \begin{array}{c} \\ V(G_2)\backslash v_2 \\ \hline v_2 \end{array} \begin{array}{c} \begin{array}{cc} E(G_2)\backslash Y & \qquad Y \end{array} \\ \left[\begin{array}{ccc|ccc} & R_{2,1} & & & R_{2,2} & \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 \end{array}\right] \end{array}$$

The incidence matrix of the star composition $G = G_1 \star_Y G_2$ of G_1 and G_2 is given by the star composition of the corresponding matrices, defined as

$$R_G = R_{G_1 \star_Y G_2} = R_{G_1} \star R_{G_2} = \begin{array}{c} \\ V(G_1)\backslash v_1 \\ \\ V(G_2)\backslash v_2 \end{array} \begin{array}{c} \begin{array}{ccc} E(G_1)\backslash Y & E(G_2)\backslash Y & Y \end{array} \\ \left[\begin{array}{c|c|c} R_{1,1} & \mathbf{0} & R_{1,2} \\ \hline \mathbf{0} & R_{2,1} & R_{2,2} \end{array}\right] \end{array},$$

where $\mathbf{0}$ represents the all-zeros matrix of the correct size.

Example 9.5. Consider the two labeled graphs, G_1 and G_2 in Figure 9.2.3a. In each graph, $Y = \{y_1, y_2, y_3\}$ is the star of a vertex labeled v_i, as described above.



(a) Two labeled graphs G_1 and G_2 **(b)** Star composition $G = G_1 \star_Y G_2$

Figure 9.2.3: Star composition of graphs.

The binary incidence matrices of G_1 and G_2 are

$$R_{G_1} = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ v_1 \end{array} \begin{array}{c} \begin{array}{ccccc} f_1 & f_2 & y_1 & y_2 & y_3 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array} \qquad R_{G_2} = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ v_2 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_1 & e_3 & e_4 & y_1 & y_2 & y_3 \end{array} \\ \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}.$$

Figure 9.2.3b is the graph $G = G_1 \star_Y G_2$, the star composition of G_1 and G_2 in Y. The incidence matrix of G, R_G, is given by the star composition of the incidence matrices, R_{G_1} and R_{G_2},

$$
R_G = R_{G_1} \star R_{G_2} = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{array}
\begin{array}{c}
\begin{array}{cc|cccc|ccc} f_1 & f_2 & e_2 & e_2 & e_3 & e_4 & y_1 & y_2 & y_3 \end{array} \\
\left[\begin{array}{cc|cccc|ccc}
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{array}\right]
\end{array}.
$$

One may check that this matrix R_G is, in fact, the binary incidence matrix of the star composition $G = G_1 \star_Y G_2$ in Figure 9.2.3b.

9.2.4 Tutte's Graph Construction Method

The graph and matrix operations introduced in Section 9.2.3 can be harnessed to construct a graph when Tutte's algorithm determines that a binary matroid is graphic. Begin with a binary matroid M and a cocircuit Y, having bridges $\{B_1, B_2, \ldots, B_n\}$. If Tutte's algorithm determines M is graphic, a number of necessary conditions must be true:

- Each bridge B_i partitions Y.
- The bridges can be partitioned into classes $\mathscr{U}_1$ and $\mathscr{U}_2$ such that in each class, any pair of bridges is non-overlapping. By re-labeling bridges if necessary, we may assume

$$\mathscr{U}_1 = \{B_1, B_2, \ldots, B_m\} \text{ and } \mathscr{U}_2 = \{B_{m+1}, \ldots, B_n\}.$$

- Each Y-component $M.(B_i \cup Y)$ is graphic.

As each $M.(B_i \cup Y)$ is graphic, let R_i be the incidence matrix of a graph G_i such that $M(G_i) \cong M.(B_i \cup Y)$. Such a matrix may be obtained in the course of the algorithm, or by beginning with the matroid $M.(B_i \cup Y)$ and applying the following procedure (here described for M).

Observe that each Y-component $M.(B_i \cup Y)$ is connected, has Y as a cocircuit, and B_i is the only bridge of Y in $M.(B_i \cup Y)$. Therefore by Theorem 9.5 there exists a 2-connected graph G_i' such that $M(G_i') \cong M.(B_i \cup Y)$ and Y is the star of a vertex in G_i'. This graph can be obtained from the graph G_i by applying a sequence of twistings, as described in the proof of Theorem 9.5, or by applying a sequence of elementary row operations to matrix R_i. Let R_i' be the resulting binary incidence matrix of graph G_i'.

Next consider the bridge-class $\mathscr{U}_1 = \{B_1, B_2, \ldots, B_M\}$. The goal is to use a sequence of successive star compositions to obtain a graph $\mathscr{G}_1$ for the matroid

$$M_1 \cong M.\Big(\Big(\bigcup_{B\in\mathscr{U}_1} B\Big) \cup Y\Big).$$

A flowchart of this process is presented in Figure A.2 of Appendix A. The basic ideas of this construction are as follows. Suppose we can construct a graph H_i' such that Y is the star of a vertex in H_i', and

$$M(H_i') \cong M.\Big((B_1 \cup \cdots \cup B_i) \cup Y\Big).$$

Then taking the star composition $G_{i+1}' \star_Y H_i'$ gives a graph H_{i+1} satisfying

$$M(H_{i+1}) \cong M.\Big((B_1 \cup \cdots \cup B_i \cup B_{i+1}) \cup Y\Big).$$

Via a sequence of twistings we can obtain a graph H_{i+1}' having Y as a star, and continue in this manner.

Begin by taking the star composition $H_2 = G_2' \star_Y G_1'$. The graph H_2 satisfies

$$M(H_2) \cong M.(B_1 \cup B_2 \cup Y)$$

(this is proven, for example, in the course of Theorem 5.6 in [91]). Therefore $M.(B_1 \cup B_2 \cup Y)$ is a graphic matroid with cocircuit Y and bridges B_1 and B_2. As $B_1, B_2 \in \mathscr{U}_1$, they are non-overlapping, and by Theorem 9.5, there exists a 2-connected graph H_2' such that $M(H_2') \cong M.(B_1 \cup B_2 \cup Y)$ and Y is the star of a vertex in H_2'. H_2' may be obtained from H_2 by a sequence of twistings.

Next take the star composition of *this* graph with G_3'. That is, we construct $H_3 = G_3' \star_Y H_2'$, satisfying

$$M(H_3) \cong M.(B_1 \cup B_2 \cup B_3 \cup Y).$$

Once again, Y is a cocircuit of $M.(B_1 \cup B_2 \cup B_3 \cup Y)$ having all-avoiding bridges B_1, B_2 and B_3. Thus by Theorem 9.5, a sequence of twistings on H_3 results in a 2-connected graph H_3', with $M(H_3') \cong M(H_3)$, having Y as the star of a vertex.

Continuing in this manner, by successive star compositions we obtain a graph $\mathscr{G}_1$ such that

$$M(\mathscr{G}_1) \cong M.\Big(\Big(\bigcup_{B\in\mathscr{U}_1} B\Big) \cup Y\Big)$$

and $\mathscr{G}_1$ has Y as the star of a vertex. On the other hand, starting with bridge class $\mathscr{U}_2 = \{B_{m+1}, \ldots, B_n\}$, the same procedure leads to a graph $\mathscr{G}_2$ such that Y is the star of a vertex and

$$M(\mathscr{G}_2) \cong M.\Big(\Big(\bigcup_{B \in \mathscr{U}_2} B\Big) \cup Y\Big).$$

Finally, we take the star composition $G = \mathscr{G}_1 \star_Y \mathscr{G}_2$. As proven, for example, in Theorem 5.6 of [91], the resulting graph G satisfies $M(G) \cong M$. Therefore, not only has Tutte's method concluded that binary matroid M was graphic, but we have constructed a graph, G, such that $M(G) \cong M$.

9.2.5 A Worked Example of Graph Construction

Section 9.2.2 examined Tutte's algorithm on a specific example, the binary matroid M with standard representation matrix

$$R = \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \\ 5^* \\ 6^* \\ 7^* \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1
\end{array}\right] \end{array}$$

We chose cocircuit $Y = \{5, 8, 9, 10, 11, 12, 13\}$, with bridges $B_1 = \{6\}$, $B_2 = \{1, 7, 15\}$ and $B_3 = \{2, 3, 4, 14\}$. Each bridge partitioned Y, and Y was even with non-overlapping bridge classes $\mathscr{U}_1 = \{B_1, B_2\}$ and $\mathscr{U}_2 = \{B_3\}$. Proceeding as described in Section 9.2.4, we will construct a graph for M.

Begin by considering bridge class $\mathscr{U}_1$. Both $M.(B_1 \cup Y)$ and $M.(B_2 \cup Y)$ are graphic, with standard representations

$$R_1 = \begin{array}{c} \\ 6^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{array}\right] \end{array}$$

$$R_2 = \begin{array}{c} \\ 1^* \\ 7^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{array}\right] \end{array}$$

Via elementary row operations (and adding new rows within the row space, if necessary), we can transform these matrices into binary incidence matrices. That is, for $i = 1, 2$ we construct matrices R_i' such that $M[R_i'] \cong M[R_i] \cong M.(B_i \cup Y)$, and R_i' is the binary incidence matrix of a graph G_i' having Y as the star of a vertex.

$$R_1' = \begin{array}{c} \begin{array}{ccccccc|cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$R_2' = \begin{array}{c} \begin{array}{ccccccc|cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

In particular, R_1' obtained from R_1 by adding a third row that is the sum of rows 1 and 2 of R_1. R_2' arises by adding row 1 of R_2 to row 2, and adding a fourth row that is the column sums of the resulting matrix. The graphs G_1' and G_2', having R_1' and R_2' as binary incidence matrices, are shown in Figure 9.2.4. For clarity, the edges have been labeled only with subscript numbers (i.e. 5 instead of e_5).

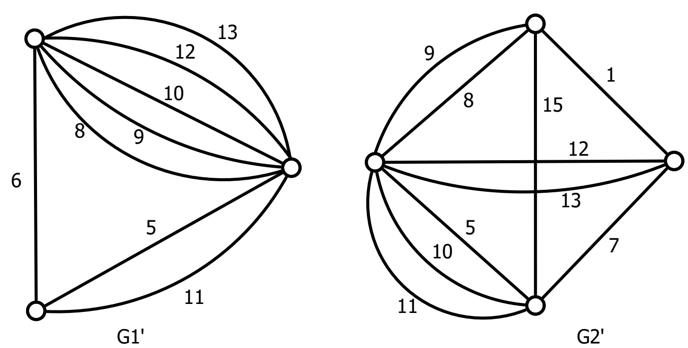
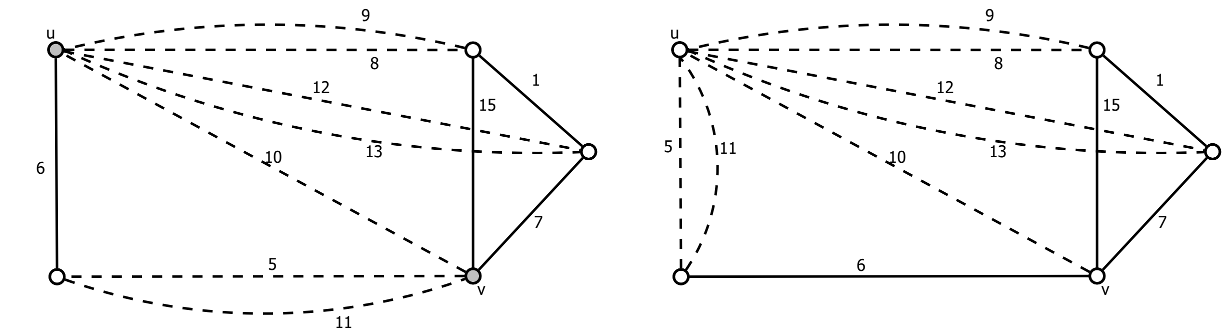


Figure 9.2.4: Graphs G_1' and G_2'.

Next we take the star composition $H_2 = G_1' \star_Y G_2'$ of G_1' and G_2' along Y. The resulting graph H_2 satisfies $M(H_2) \cong M.(B_1 \cup B_2 \cup Y)$, and is shown in Figure 9.2.5a.

Via Theorem 9.5, construct a graph H_2' such that $M(H_2') \cong M(H_2)$ and Y is the star of a vertex in H_2'. This can be accomplished either by a sequence of twistings on the graph H_2 or a sequence of elementary row operations on the incidence matrix of H_2. Observe that a single twisting on $\{u, v\}$ in H_2 produces the desired graph H_2', shown in Figure 9.2.5b.



(a) Star composition $H_2 = G'_1 \star_Y G'_2$. **(b)** The graph H'_2 obtained by twisting on $\{u, v\}$.

Figure 9.2.5: Constructing the graph H'_2.

That is, $M(H'_2) \cong M.(B_1 \cup B_2 \cup Y)$, and Y is the star of a vertex in H'_2. As $\mathscr{U}_1 = \{B_1, B_2\}$, let $\mathscr{G}_1 = H'_2$. Therefore, we have constructed a graph $\mathscr{G}_1$ for

$$M.\Big(\Big(\bigcup_{B \in \mathscr{U}_1} B\Big) \cup Y\Big),$$

and now turn our attention to $\mathscr{U}_2 = \{B_3\}$. Recall that $M.(B_3 \cup Y)$ had standard representation matrix:

$$R_3 = \begin{array}{c} \\ 2^* \\ 3^* \\ 4^* \\ 5^* \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

Once again, by elementary row operations we may obtain from R_3 a matrix R'_3 that is the binary incidence matrix for a graph with Y as the star of a vertex. Specifically, add rows 1 and 4 of R_3 to rows 2 and 3, respectively. Then adjoin a fifth row that is the sum of the resulting 4 rows. This gives

$$R'_3 = \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \left[\begin{array}{ccccccc|cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

The graph with binary incidence matrix R_3', G_3', is shown in Figure 9.2.6. As $\mathscr{U}_2 = \{B_3\}$, it follows that G_3' is the graph $\mathscr{G}_2$ for

$$M.\Big(\Big(\bigcup_{B\in\mathscr{U}_2} B\Big) \cup Y\Big).$$

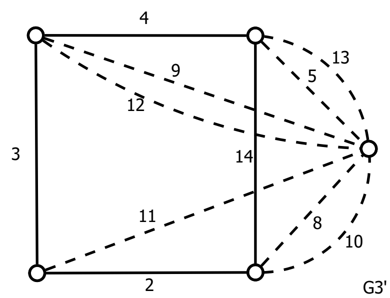


Figure 9.2.6: The graph G_3'.

The final step is to take the star composition $G = \mathscr{G}_1 \star_Y \mathscr{G}_2$. This is shown in Figures 9.2.7 and 9.2.8.

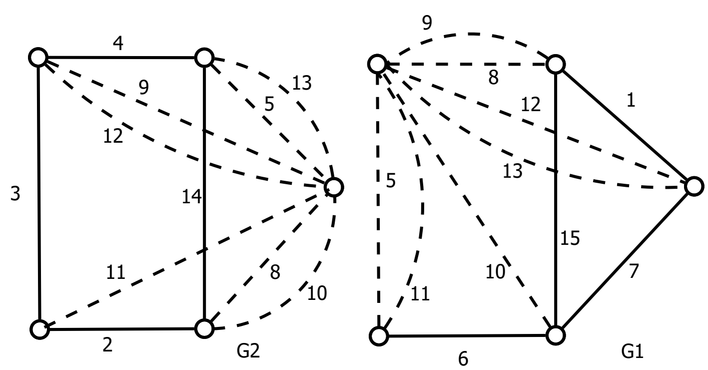


Figure 9.2.7: The graphs $\mathscr{G}_1$ and $\mathscr{G}_2$.

The graph G, illustrated in Figure 9.2.8, satisfies $M(G) \cong M = M[R]$. To verify this, one may take the incidence matrix of G (equivalently, the star composition of the incidence matrices of $\mathscr{G}_1$ and $\mathscr{G}_2$), and check that it is row-equivalent to R.

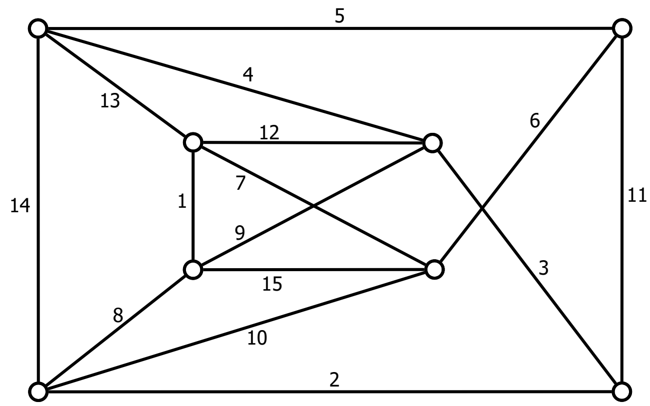


Figure 9.2.8: A graph G such that $M(G) = M$.

This concludes our exploration of Tutte's algorithm for determining when a binary matroid is graphic. The idea behind Tutte's method is beautifully simple. After determining a set of necessary conditions of a graphic matroid, we sequentially check each of these conditions on a given binary matroid. If any condition fails, we may conclude that the binary matroid is not graphic. On the other hand if all conditions are met, these necessary conditions are rendered sufficient by constructing the desired graph.

On the other hand, there are many binary matroids that are not graphic. In Chapter 2 we showed that every binary matrix has a $\mathbb{Z}_2$-Kirchhoff graph. We will reconcile these notions in Chapter 10 by introducing $\mathbb{Z}_2$-Kirchhoff *partitions*. Moreover, we will illustrate a number of examples where a slight modification of Tutte's algorithm permits us to construct $\mathbb{Z}_2$-Kirchhoff partitions for binary matroids that are not graphic.

Chapter 10

Binary Matroids and $\mathbb{Z}_2$-Kirchhoff Partitions

This chapter synthesizes the discussions of Chapters 6, 8, and 9. Specifically, although every graphic matroid is binary, not every binary matroid is graphic. This chapter seeks a full converse of Corollary 9.2, in the form of a graphical structure that can capture the dependencies of *any* binary matroid. Before proceeding, we first determine the properties of graphic matroids that we aim to preserve in this structure, outlined by Proposition 10.1.[1]

Proposition 10.1. Suppose M is a graphic matroid, $M \cong M(G)$ for some graph G. Then

(1) Each circuit of M corresponds to a cycle in G.

(2) Each cycle of G corresponds to a dependent set in M.

(3) Each vertex cut of G corresponds to a disjoint union of cocircuits, i.e. circuits of dual matroid M^*.

Keeping properties (1)-(3) in mind, the main motivation for this chapter is the following. In some sense, graphs are limited by the fact that each edge, and thus each matroid element, appears *only once*. By allowing cells of an edge partition–rather than the edges themselves–to correspond to matroid elements, we can thereby achieve *repetition* of matroid elements in the graph. The added flexibility of repeated elements will permit these structures to capture additional dependencies of binary matroids.

[1]Note that these are not the strongest-possible conditions: (2) could be strengthened by replacing “dependent” with “a circuit.”

Section 10.1 begins by introducing $\mathbb{Z}_2$-Kirchhoff edge-partitions on undirected graphs. Section 10.2 then reinterprets Proposition 10.1 in order to define a $\mathbb{Z}_2$-Kirchhoff matroid. Specifically, a matroid is $\mathbb{Z}_2$-Kirchhoff if the cycles and vertex cuts of some graph–with respect to a $\mathbb{Z}_2$-Kirchhoff edge-partition–satisfy Proposition 10.1. Examples 10.3 and 10.4 demonstrate that two classically non-graphic binary matroids, F_7 and $M^*(K_{3,3})$, are both $\mathbb{Z}_2$-Kirchhoff. These are special cases of Theorem 10.1, the main result of this chapter, which states that every binary matroid is $\mathbb{Z}_2$-Kirchhoff. Section 10.2.1 presents the proof of this result. Finally, Section 10.3 presents a sequence of examples that construct $\mathbb{Z}_2$-Kirchhoff edge-partitions. Specifically, we execute Tutte's algorithm on non-graphic binary matroids. At the point at which Tutte's algorithm terminates (and concludes "not graphic") we introduce edge-partitions and follow an analogue of Tutte's method to construct the desired $\mathbb{Z}_2$-Kirchhoff edge-partitions. This process is carried out on the each of the standard excluded minors, $M^*(K_{3,3})$ (Section 10.3.1), $M^*(K_5)$ (Section 10.3.2), F_7 (Section 10.3.3), and F_7^* (Section 10.3.4).

10.1 $\mathbb{Z}_2$-Kirchhoff Edge-Partitions

Let $G = (V, E)$ be a finite undirected graph with vertex set $V(G) = \{v_i\}$ and edge set $E(G) = \{e_j\}$. All graphs in this chapter will be undirected, and may have multiple edges. For any vertex v, let $\lambda(v)$ be the *binary incidence vector* of v. That is, $\lambda(v)$ is a vector with $|E(D)|$ entries, and j^{th} entry 1 if v is an endpoint of edge e_j. As before, let a *cycle* in G be any closed walk with non-repeating vertices. For any cycle C, let the *binary cycle vector* of C, denoted $\chi(C)$, be a vector with $|E(D)|$ entries, and j^{th} entry 1 if C traverses edge e_j. It is well-known that for all vertices v and all cycles C,

$$\lambda(v) \cdot \chi(C) \equiv 0 \pmod 2. \tag{10.1}$$

Now let $\pi = (E_1, \ldots, E_k)$ be an edge-partition of G. The following definition is analogous to that of Section 6.2.

Definition 10.1. An edge-partition $\pi = \{E_1, \ldots, E_k\}$ of $E(G)$ with characteristic matrix T is $\mathbb{Z}_2$***-Kirchhoff*** if for all vertices v and cycles C,

$$\lambda(v)T \cdot \chi(C)T \equiv 0 \pmod 2. \tag{10.2}$$

We say that a $\mathbb{Z}_2$-Kirchhoff edge partition is ***nontrivial*** if for some vertex v, the vector $\lambda(v)T$ is nonzero (mod 2).

Take a moment to consider the vectors $\lambda(v)T$ and $\chi(C)T$. Each has k entries, indexed by

$E_1, \ldots, E_k$. The i^{th} entry of the vector $\lambda(v)T$ is the net number of edges of class E_i that have vertex v as an endpoint. That is, $\lambda(v)T$ can be thought of as a vertex incidence vector with respect to edge-partition π. We will call this vector the ***π-incidence vector***, and denote it $\lambda_\pi(v)$. Similarly, the i^{th} entry of vector $\chi(C)T$ is the net number of edges of class E_i that cycle C traverses. Thus $\chi(C)T$ can be thought of as a cycle vector with respect to edge-partition π. We will call this vector the ***π-cycle vector***, and denote it $\chi_\pi(C)$. Therefore an edge-partition is $\mathbb{Z}_2$-Kirchhoff when the classical orthogonality of vertex and cycle vectors is maintained with respect to the structure of the edge-partition,

$$\lambda_\pi(v) \cdot \chi_\pi(C) \equiv 0 \pmod 2$$

for all vertices v and cycles C.

Example 10.1. In this chapter, an edge-partition of a graph G will typically be presented via a drawing with edge labels that indicate the subscript of the partition cell containing each edge. For example, Figure 10.1.1 presents a graph with a 5 cell edge-partition $\pi = (E_1, \ldots, E_5)$. Let C be the cycle around the perimeter of G_1, and v the central vertex as drawn. Then

$$\chi_\pi(C) = \begin{array}{c} \begin{array}{ccccc} \scriptstyle E_1 & \scriptstyle E_2 & \scriptstyle E_3 & \scriptstyle E_4 & \scriptstyle E_5 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad \lambda_\pi(v) = \begin{array}{c} \begin{array}{ccccc} \scriptstyle E_1 & \scriptstyle E_2 & \scriptstyle E_3 & \scriptstyle E_4 & \scriptstyle E_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 3 & 2 & 1 \end{bmatrix} \end{array}.$$

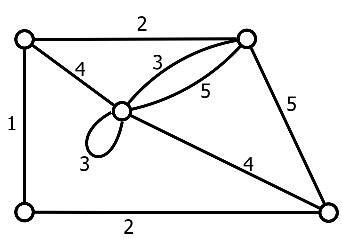


Figure 10.1.1: An undirected multigraph G_1 with edge partition π.

Once again, every graph has an edge-partition that is trivially $\mathbb{Z}_2$-Kirchhoff, namely the partition for which every cell contains exactly one edge. In this case the characteristic matrix T is simply the $|E(G)| \times |E(G)|$ identity matrix, and equation (10.2) becomes (10.1).

Example 10.2. Consider graph G_2 with edge-partition π as presented in Figure 10.1.2. Observe that any vertex v satisfies

$$\lambda_\pi(v) \in U = \left\{ \begin{array}{c} \begin{array}{cccc} \scriptstyle E_1 & \scriptstyle E_2 & \scriptstyle E_3 & \scriptstyle E_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \begin{array}{cccc} \scriptstyle E_1 & \scriptstyle E_2 & \scriptstyle E_3 & \scriptstyle E_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \end{array}, \begin{array}{c} \begin{array}{cccc} \scriptstyle E_1 & \scriptstyle E_2 & \scriptstyle E_3 & \scriptstyle E_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 2 \end{bmatrix} \end{array} \right\},$$

and any cycle C satisfies

$$\chi_\pi(C) \in Z = \Bigg\{ \begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 & E_4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \end{array}, \begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 & E_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} \end{array}, \begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 & E_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 2 & 0 \end{bmatrix} \end{array},$$
$$\begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 & E_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 2 \end{bmatrix} \end{array}, \begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 & E_4 \end{matrix} \\ \begin{bmatrix} 2 & 2 & 0 & 0 \end{bmatrix} \end{array} \Bigg\}.$$

Moreover, each element of U is orthogonal (mod 2) to each element of Z. Therefore, edge-partition π of graph G_2 is $\mathbb{Z}_2$-Kirchhoff.

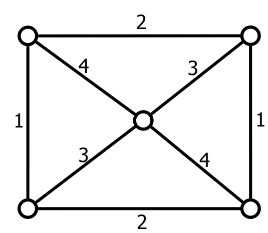


Figure 10.1.2: An undirected graph G_2 with a $\mathbb{Z}_2$-Kirchhoff edge-partition.

10.2 $\mathbb{Z}_2$-Kirchhoff Matroids

The cycle matroids of graphs are classically constructed by considering edge dependencies, specifically cycles, with respect to edge enumeration. That is, given a graph G, enumerate the edges and let each be an element of the ground set $S(M(G))$. Edge enumeration, however, is an edge-partition in which each edge is assigned to a distinct partition class. In this section we explore the relationship between binary matroids and edge-partitions in general. By studying graph structure with respect to the partition, edge-partitions offer the added flexibility of multiple-element partition classes. This equates to allowing a matroid element to appear more than once in the graph.

Let M be a binary matroid with ground set S.

Definition 10.2. For any $X \subseteq S$ the ***indicator of*** X, denoted Υ_X, is a binary vector with entries labeled by S, such that $\Upsilon_X(j) = 1$ if and only if $j \in X$.

Given this notation, Proposition 10.1 may be re-formulated as follows.

Corollary 10.1. *Let M be a binary matroid that is graphic, say $M \cong M(G)$.*

(1) For any circuit X of M, $\Upsilon_X \equiv \chi(C)$ (mod 2) for some cycle C of G.

(2) If C is a cycle of G, then $\chi(C) \equiv \Upsilon_X$ (mod 2) for some dependent set X of M.

(3) For any vertex v, $\lambda(v)$ is the indicator of a disjoint union of circuits of M^*.

For any $\mathbb{Z}_2$-Kirchhoff graph, the cycle vectors $\chi_\pi(C)$ (mod 2) are well-defined and have all entries 0 or 1. This suggests a natural way to extend Corollary 10.1.

Definition 10.3. Let M be a matroid with ground set S of cardinality k. M is ***$\mathbb{Z}_2$-Kirchhoff*** if there exists a graph G with nontrivial $\mathbb{Z}_2$-Kirchhoff edge-partition π (on k classes) such that

(i) For any circuit X of M, $\Upsilon_X \equiv \chi_\pi(C)$ (mod 2) for some cycle C of G.

(ii) For any cycle C of G, if $\chi_\pi(C) \neq \mathbf{0}$ (mod 2) then there exists some dependent set X in M such that $\chi_\pi(C) \equiv \Upsilon_X$ (mod 2).

We also say that edge-partition π of G is ***$\mathbb{Z}_2$-Kirchhoff with respect to M.***

Remark 10.1. Observe that Definition 10.3 does not specifically address property (3), as it is guaranteed by the orthogonality maintained by $\mathbb{Z}_2$-Kirchhoff edge-partitions.

Remark 10.2. The restriction to nontrivial $\mathbb{Z}_2$-Kirchhoff graphs in Definition 10.3 is included rule-out undesirable trivial cases introduced by the field $\mathbb{Z}_2$. For example, one could construct a sequence of disjoint cycles (corresponding to circuits of M), and double all labeled edges. This forces all binary π-incidence vectors to be the zero vector (mod 2). The result is a graph that is trivially $\mathbb{Z}_2$-Kirchhoff, and satisfies (i) and (ii). It does not, however, illuminate any properties of the underlying matroid.

Corollary 10.2. *Every graphic matroid is $\mathbb{Z}_2$-Kirchhoff.*

Proof. Let M be a graphic matroid and G any graph such that $M(G) \cong M$. Then if $E(G) = \{e_1, \ldots, e_n\}$, the trivial edge-partition $\pi = (\{e_1\}, \ldots, \{e_n\})$ is $\mathbb{Z}_2$-Kirchhoff. Moreover, as $M(G) \cong M$, the conditions of Definition 10.3 are satisfied. $\square$

$\mathbb{Z}_2$-Kirchhoff matroids may be viewed as an extension of graphic matroids. In this chapter, we will demonstrate that all binary matroids are $\mathbb{Z}_2$-Kirchhoff. As Examples 10.3 and 10.4 will show, the most fundamental non-graphic binary matroids (namely, those listed in Theorem 8.7) are $\mathbb{Z}_2$-Kirchhoff.

Example 10.3. The Fano Matroid is $\mathbb{Z}_2$-Kirchhoff. The Fano matroid, F_7, is a binary matroid on 7 elements with the following standard representation matrix

$$A_{F_7} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}. \tag{10.3}$$

The Fano matroid has 14 circuits: 7 each of cardinalities 3 and 4. Under this representation,

$$\mathscr{C}(F_7) = \left\{ \begin{array}{l} \{1,2,4\}, \{1,3,5\}, \{1,6,7\}, \{2,3,6\}, \{2,5,7\}, \{3,4,7\}, \{4,5,6\} \\ \{1,4,5,7\}, \{1,2,3,7\}, \{1,3,4,6\}, \{1,2,5,6\}, \{2,3,4,5\}, \{2,4,6,7\}, \{3,5,6,7\} \end{array} \right\}. \tag{10.4}$$

Now consider the graph G_{F_7} with edge-partition π as depicted in Figure 10.2.1. One may verify that edge-partition π is, in fact, $\mathbb{Z}_2$-Kirchhoff. As before, cross-hatches are used to denote edges (with the same endpoints) contained in the same partition class.

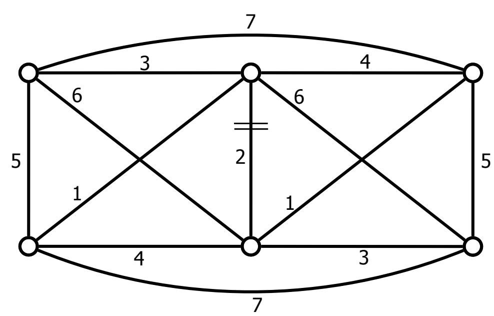


Figure 10.2.1: A graph G_{F_7} with $\mathbb{Z}_2$-Kirchhoff edge-partition with respect for the Fano matroid F_7.

One may check that for any circuit X of F_7 listed in (10.4), there is a cycle C of G_{F_7} such that

$$\Upsilon_X \equiv \chi_\pi(C) \pmod 2. \tag{10.5}$$

Conversely, for any cycle C of G_{F_7} such that $\chi_\pi(C) \not\equiv 0 \pmod 2$, there exists a dependent set X of F_7 such that (10.5) is satisfied. This can easily be checked by showing that for any cycle C,

$$[A_{F_7}]\,[\chi_\pi(C)]^t \equiv \mathbf{0} \pmod 2.$$

Therefore *the Fano matroid is $\mathbb{Z}_2$-Kirchhoff.* This demonstrates the flexibility afforded by $\mathbb{Z}_2$-Kirchhoff partitions: although F_7 is not graphic, an edge-partition can capture the structure of F_7.

Example 10.4. $M^*(K_{3,3})$ **is $\mathbb{Z}_2$-Kirchhoff.** The following is a binary standard representation of $M^*(K_{3,3})$,

$$A_{K_{3,3}} = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}. \tag{10.6}$$

Consider the graph $G_{K_{3,3}}$ with edge-partition π as depicted in Figure 10.2.2. Edge-partition π is $\mathbb{Z}_2$ Kirchhoff.

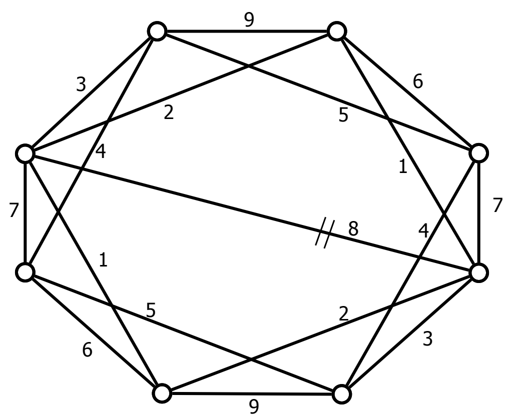


Figure 10.2.2: A graph $G_{K_{3,3}}$, with edge-partition π, a $\mathbb{Z}_2$-Kirchhoff edge-partition for $M^*(K_{3,3})$.

One may check that the for any minimally dependent set of columns X of matrix (10.6), there exists a cycle C in graph $G_{K_{3,3}}$ such that

$$\Upsilon_X \equiv \chi_\pi(C) \pmod 2.$$

Conversely, for any cycle C of $G_{K_{3,3}}$,

$$\left[A_{K_{3,3}}\right] \left[\chi_\pi(C)\right]^t \equiv \mathbf{0} \pmod 2.$$

Therefore matroid $M^*(K_{3,3})$ is $\mathbb{Z}_2$-Kirchhoff. Observe that although $K_{3,3}$ has no dual graph under any classical notions, it has a dual $\mathbb{Z}_2$-Kirchhoff edge-partition.

It can also be demonstrated that the remaining two excluded minors of Theorem 8.7, F_7^* and $M^*(K_5)$, are both $\mathbb{Z}_2$-Kirchhoff. This can be concluded from the following overarching theorem.

Theorem 10.1. *All binary matroids are $\mathbb{Z}_2$-Kirchhoff.*

The remainder of this section will present the full proof of Theorem 10.1. First, Theorem 10.2, reduces the problem to studying a single binary matroid of each rank. Next, Theorem 10.3 shows that those selected matroids satisfy the desired properties to prove Theorem 10.1.

10.2.1 Proof of Theorem 10.1

Recall that for a matroid M, any $X \subseteq S(M)$ not containing a circuit is *independent* and a maximal independent set is a *base*. As shown in [84] (or any standard matroid text), all bases of M have the same cardinality, called the *rank* of M and often denoted $r(M)$. If M is representable, any standard representation matrix for M has exactly $r(M)$ rows.

Lemma 10.1. *Every binary matroid of rank 1 or 2 is graphic.*

Proof. Let M be any binary matroid of rank 1 or 2 with standard representation matrix A. Form a new matrix A' by adjoining to A an additional row that is the (mod 2) sum of the row(s) of A. As this additional row lies in the row space of A, $M[A] \cong M[A']$. Moreover, every column of A' has exactly two unit entries. Therefore A' is the binary incidence matrix of some graph G, and $M[A'] \cong M(G)$. Therefore $M = M[A] \cong M[A'] \cong M(G)$ and so M is graphic. □

Corollary 10.3. *Every binary matroid of rank 1 or 2 is $\mathbb{Z}_2$-Kirchhoff.*

Next consider binary matroids of larger rank. A similar argument to Corollary 10.3 and Lemma 10.1 cannot be used: the Fano matroid F_7 is a binary matroid of rank 3 that is not graphic. For any matroid of rank n, Theorem 10.2 will reduce the problem to studying exactly one matroid of rank n.

Let B be any binary matrix with $k-1$ columns. Let A be any k-column binary matrix obtained from B by adjoining a k^{th} column, that duplicates an existing column of B. Without loss of generality, assume that this k^{th} column is identical to column $k-1$. Now suppose $M[B]$ is $\mathbb{Z}_2$-Kirchhoff. That is, there exists a graph G_B with edge-partition π_B that is $\mathbb{Z}_2$-Kirchhoff with respect to B. Necessarily, edge-partition π_B has $k-1$ classes, $E_1, \ldots, E_{k-1}$. Construct a new graph, G_A, with edge-partition π_A, as follows. For every edge of G_B in partition class E_{k-1}, add a new edge. Let all such additional edges belong to a new cell, E_k, and let $\pi_A = \pi_B \cup E_k$.

Lemma 10.2. *Edge-partition π_A of G_A is $\mathbb{Z}_2$-Kirchhoff with respect to $M[A]$.*

Proof. First, we show that edge-partition π_A of graph G_A is $\mathbb{Z}_2$-Kirchhoff. Let v be any vertex in G_A, and let x_i denote the i^{th} entry of incidence vector $\lambda_{\pi_A}(v)$. By construction, $x_{k-1} = x_k$, so

$$\lambda_{\pi_A}(v) = \left[\begin{array}{ccccc} x_1 & x_2 & \cdots & x_{k-1} & x_{k-1} \end{array} \right].$$

Moreover,

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_{k-1} \end{bmatrix} = \lambda_{\pi_B}(\widetilde{v})$$

for some vertex $\widetilde{v}$ in graph G_B. Now let C be any cycle of G_A with

$$\chi_{\pi_A}(C) = \begin{bmatrix} y_1 & y_2 & \cdots & y_{k-1} & y_k \end{bmatrix}.$$

We will show that $\lambda_{\pi_A}(v) \cdot \chi_{\pi_A}(C) \equiv 0 \pmod 2$. By construction, the cycles C of G_A with nonzero cycle vectors break into three cases.

Case 1: C has cycle vector $\begin{array}{c} \begin{array}{cccccc} & & & E_{k-1} & E_k & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \end{array}$. Then

$$\lambda_{\pi_A}(v) \cdot \chi_{\pi_A}(C) \equiv \sum_{i=1}^{k} x_i y_i = x_{k-1} + x_{k-1} \equiv 0 \pmod 2.$$

Case 2: C does not traverse any edges of cell E_k. Then $y_k = 0$, and $\begin{bmatrix} y_1 & y_2 & \cdots & y_{k-1} \end{bmatrix} = \chi(\widetilde{C})$ for some cycle $\widetilde{C}$ of graph G_B. Then since edge-partition π_B of G_B is $\mathbb{Z}_2$-Kirchhoff,

$$\lambda_{\pi_A}(v) \cdot \chi_{\pi_A}(C) \equiv \sum_{i=1}^{k} x_i y_i \equiv \sum_{i=1}^{k-1} x_i y_i + 0 \equiv \sum_{i=1}^{k-1} x_i y_i \equiv \lambda_{\pi_B}(\widetilde{v}) \cdot \chi_{\pi_B}(\widetilde{C}) \equiv 0 \pmod 2.$$

Case 3: C traverses some edges of cell E_k. Let $\widetilde{C}$ be the cycle in G_B obtained from C by replacing each occurrence of an edge of cell k with its matching edge of cell $k-1$. Then

$$\chi_{\pi_B}(\widetilde{C}) = \begin{bmatrix} y_1 & y_2 & \cdots & y_{k-2} & (y_{k-1} + y_k) \end{bmatrix}.$$

Thus as edge-partition π_B is $\mathbb{Z}_2$-Kirchhoff,

$$\begin{aligned} \lambda_{\pi_A}(v) \cdot \chi_{\pi_A}(C) &\equiv \sum_{i=1}^{k} x_i y_i \\ &\equiv \sum_{i=1}^{k-2} x_i y_i + x_{k-1} y_{k-1} + x_{k-1} y_k \\ &\equiv \sum_{i=1}^{k-2} x_i y_i + x_{k-1}(y_{k-1} + y_k) \\ &\equiv \lambda_{\pi_B}(\widetilde{v}) \cdot \chi_{\pi_B}(\widetilde{C}) \equiv 0 \pmod 2. \end{aligned}$$

This completes the proof that edge-partition π_A of graph G_A is $\mathbb{Z}_2$-Kirchhoff.

Now consider the circuits of $M[A]$ compared to those of $M[B]$. As matrix A is obtained from B by adding an additional column equal to the final column of B, the minimal column dependencies of A are

$$\mathscr{C}(M[A]) = \big\{k{-}1, k\big\} \cup \big\{X : X \in \mathscr{C}(M[B])\big\} \cup \big\{X \backslash (k{-}1) \cup k : X \in \mathscr{C}(M[B]) \text{ and } k{-}1 \in X\big\}.$$

Observe that each of these three types of circuits fall exactly into the three cases presented above. Clearly $\Upsilon_{\{k-1,k\}}$ is a π_A-cycle vector of G_A. For any circuit $X \in \mathscr{C}(M[B])$, since edge-partition π_B is $\mathbb{Z}_2$-Kirchhoff, it follows by Case 2 that there is a cycle in G_A with π_A-cycle vector Υ_X. Finally consider any cycle of the form $X \backslash (k-1) \cup k$ for $X \in \mathscr{C}(M[B])$. Let C_B be the cycle in G_B with cycle vector Υ_X, and let C_A be the cycle in G_A obtained replacing all occurrences of an edge of cell $k-1$ in C_B with its matching edge of cell k. It follows from Case 3 that $\chi_{\pi_A}(C_A) = \Upsilon_{X \backslash (k-1) \cup k}$. Therefore, For any $X \in \mathscr{C}(M) = \mathscr{C}(M[A])$ there is a cycle C in G_A such that $\chi_{\pi_A}(C) \equiv \Upsilon_X \pmod 2$.

Conversely given Cases 1- 3, it is clear that any nonzero π_A-cycle vector of G_A is the indicator of a dependent set in M. As G_B is nonzero, so is G_A. Therefore $M[A]$ and G_A satisfy all conditions of Definition 10.3. That is, partition π_A of graph G_A is $\mathbb{Z}_2$-Kirchhoff graphic with respect to $M[A]$. □

Corollary 10.4. *Let B be any binary matrix, and suppose a binary matrix A can be obtained from B by appending any number of additional columns, each of which is a column of B. If $M[B]$ is $\mathbb{Z}_2$-Kirchhoff, then $M[A]$ is also $\mathbb{Z}_2$-Kirchhoff.*

Thus given a binary matrix B, if $M[B]$ is $\mathbb{Z}_2$-Kirchhoff then we can freely replicate the columns of matrix B and construct a graph with a $\mathbb{Z}_2$-Kirchhoff edge-partition for the resulting matrix. As Lemma 10.3 will show, a similar result holds for deleting columns.

Remark 10.3. Corollary 10.4 demonstrates that it is sufficient to study binary matrices with distinct columns. In terms of matroids, this means it is sufficient to consider binary matroids with no 2-element circuits.

Let M be a binary matroid, and suppose that M is $\mathbb{Z}_2$-Kirchhoff with respect to some graph G and edge-partition π.

Definition 10.4. If for any circuit $X \in \mathscr{C}(M)$, there is a cycle C in G such that

$$\chi_\pi(C) = \Upsilon_X \tag{10.7}$$

we say that M is ***strongly $\mathbb{Z}_2$-Kirchhoff***. Moreover, we say that edge-partition π of G is strongly $\mathbb{Z}_2$-Kirchhoff with respect to M.

Observe that (10.7) is vector equality *over the rational numbers*, and thus is a stronger assumption than Υ_X being equivalent (mod 2) to a π-cycle vector of G. For a matroid to be

$\mathbb{Z}_2$-Kirchhoff, we only require that $\chi_\pi(C) \equiv \Upsilon_X \pmod 2$ for some cycle C. In a strongly $\mathbb{Z}_2$-Kirchhoff matroid, however, each circuit occurs *as precisely the set of edge labels* of a cycle. This will play a crucial role when considering element deletion.

Let A be any k-column binary matrix, and let B be obtained from A by deleting some (labeled) subset X of p columns. Without loss of generality, we may suppose these are the last p columns of A, $\{k-p+1, \ldots, k-1, k\}$. Suppose that there exists a graph G_A with edge-partition π_A that is *strongly* $\mathbb{Z}_2$-Kirchhoff with respect to $M[A]$. Moreover, suppose that for some $j \notin X$ and some vertex v, $\lambda_{\pi_A}(v)_j \not\equiv 0 \pmod 2$. Let G_B be a graph obtained from G_A by deleting all edges of partition classes $k-p+1, \ldots, k$ of π_A. Let π_B be the $(k-p)$-cell edge-partition of G_B induced by π_A.

Lemma 10.3. *Edge-partition π_B of G_B is $\mathbb{Z}_2$-Kirchhoff with respect to the matroid $M[B]$.*

Proof. First we prove that π_B is $\mathbb{Z}_2$-Kirchhoff. Let C be any cycle of G_B, and let y_i be the i^{th} entry of cycle vector $\chi_{\pi_B}(C)$. Clearly the edges of C are precisely the edges of a cycle $\widetilde{C}$ in G_A, with cycle vector

$$\chi_{\pi_A}(\widetilde{C}) = \begin{bmatrix} y_1 & \cdots & y_{k-p} & 0 & \cdots & 0 \end{bmatrix}.$$

Now let v be any vertex of G_B, and let x_i denote the i^{th} entry of incidence vector $\lambda_{\pi_B}(v)$. Let $\widetilde{v}$ be the corresponding vertex in G_A, so

$$\lambda_{\pi_A}(\widetilde{v}) = \begin{bmatrix} x_1 & \cdots & x_{k-p} & * & \cdots & * \end{bmatrix},$$

where a $*$ may be any integer including 0. Then as π_A is $\mathbb{Z}_2$-Kirchhoff,

$$0 \equiv \lambda_{\pi_A}(\widetilde{v}) \cdot \chi_{\pi_A}(\widetilde{C}) \equiv \sum_{i=1}^{k-p} x_i y_i + 0 \equiv \lambda_{\pi_B}(v) \cdot \chi_{\pi_B}(C) \pmod 2.$$

Therefore since C and v were arbitrary in G_B, π_B is $\mathbb{Z}_2$-Kirchhoff. Observe $M[B] = M[A]\backslash X$, so to show that $M[B]$ is $\mathbb{Z}_2$-Kirchhoff, it suffices to show that $M[A]\backslash X$ is $\mathbb{Z}_2$-Kirchhoff. Recall from Section 8.2.1 that

$$\mathscr{C}(M[A]\backslash X) = \{Y \subseteq S(M[A]) - X : Y \in \mathscr{C}(M[A])\}.$$

That is, the circuits of $M[A]\backslash X$ are precisely the circuits of $M[A]$ that do not contain any elements of X. For any circuit $Y \in \mathscr{C}(M[A])$, because π_A is strongly $\mathbb{Z}_2$-Kirchhoff, there is a cycle in G_A with edges whose partition cells are exactly those elements of C. Therefore in deleting all edges with labels from X, each such cycle (corresponding to a circuit of $M[A]\backslash X$) is maintained. Therefore for each circuit $Y \in \mathscr{C}(M[A]\backslash X)$ there exists a cycle in G_B whose π_B-cycle vector is Υ_Y. Conversely, since every π_A-cycle vector of G_A is the indicator of a dependent set in $M[A]$, it follows that every π_B-cycle vector of G_B is the indicator of a dependent set in $M[B]$. Finally, π_B is guaranteed to be nontrivial by the assumption that for some $j \notin X$ and some vertex v, the $\lambda_{\pi_A}(v)_j \neq 0 \pmod 2$. Therefore $M[B]$ and π_B satisfy the conditions of Definition 10.3, and π_B is $\mathbb{Z}_2$-Kirchhoff with respect to $M[B]$. □

Now let M_n be the $n \times 2^n$ matrix where column i, denoted $\boldsymbol{m}_i$, is the binary representation of the integer $(i-1)$. Corollary 10.4 and Lemma 10.3 lead to the following.

Theorem 10.2. *Suppose there exists a graph G_n, with edge-partition π_n, that is strongly $\mathbb{Z}_2$-Kirchhoff with respect to $M[M_n]$. Moreover, suppose that for some row $\boldsymbol{r}$ of M_n, and every vertex v of G_n, $\lambda_{\pi_n}(v) \equiv \boldsymbol{r} \pmod 2$. Then every binary matroid of rank n is $\mathbb{Z}_2$-Kirchhoff.*

Proof. Let M be any binary matroid of rank n with standard representation matrix A. By construction, the columns of M_n are unique, and are all possible vectors in $\mathbb{Z}_2^n$. Therefore the matrix A can be obtained, up to column relabeling, from M_n in two steps.

> **(1.)** First delete a (possibly empty) set of columns Y from M_n. Let M'_n denote the resulting matrix.
>
> **(2.)** Add a (possibly empty) set of new columns to M'_n, each of which is a column of M'_n. Let M''_n denote the resulting matrix.

By assumption, there exists a graph G_n, with edge-partition π_n, that is strongly $\mathbb{Z}_2$-Kirchhoff with respect to $M[M_n]$. We claim that there exists a label $p \notin Y$, and a vertex v, such that $\lambda_{\pi_n}(v)_p \neq 0 \pmod 2$. Suppose not. By assumption, for any vertex v, $\lambda_{\pi_n}(v) = \boldsymbol{r} \pmod 2$. Therefore, it must be the case that $\{j : r_j = 1\} \subseteq Y$. Then if $\boldsymbol{r}$ was the i^{th} row of M_n, it follows that the i^{th} row of the matrix M'_n must contain only zeros. However, this contradicts that matrix A (and thus matroid M) has rank n. Therefore by Lemma 10.3, $M[M'_n]$ is $\mathbb{Z}_2$-Kirchhoff. Furthermore, by Corollary 10.4, $M[M''_n]$ is also $\mathbb{Z}_2$-Kirchhoff. Since $M[M''_n] \cong M[A] \cong M$, M is $\mathbb{Z}_2$-Kirchhoff. As M was an arbitrary binary matroid of rank n, the result is proven. □

Theorem 10.2 is significant in that it greatly reduces the problem of proving that all binary matroids are $\mathbb{Z}_2$-Kirchhoff. In particular, to prove that any binary matroid of rank n is $\mathbb{Z}_2$-Kirchhoff, it suffices to show the following.

Theorem 10.3. *For any n there exists a graph G_n, with edge-partition π_n, that is strongly $\mathbb{Z}_2$-Kirchhoff with respect to $M[M_n]$. Moreover, for some row $\boldsymbol{r}$ of M_n, and every vertex v of G_n, $\lambda_{\pi_n}(v) \equiv \boldsymbol{r} \pmod 2$.*

Proof. The proof of this theorem is constructive. It utilizes the fact that the columns of M_n are the complete set of elements of the vector space $\mathbb{Z}_2^n$, and essentially builds the Cayley color graph. For completeness, an alternate (inductive) proof is given in Appendix B.

Let n be given, and let G_n be a graph on 2^n vertices, $\{v_1, \dots, v_{2^n}\}$, with an edge between all pairs of vertices, and a loop at each vertex. Let $\pi_n = (E_1 \dots E_{2^n})$ be a 2^n-cell edge-partition of G_n defined as follows. For each $1 \le i, j \le 2^n$, edge $\{v_i, v_j\} \in E_k$, where k is the index such that

$$\boldsymbol{m}_k \equiv \boldsymbol{m}_i + \boldsymbol{m}_j \pmod 2. \tag{10.8}$$

For any edge in a partition cell E_k with odd index $k > 1$, add a second edge, also in cell E_k, with the same end points. Thus G_n is a complete graph with a loop (contained in cell E_1) at each vertex, and all edges in odd-indexed cells of π_n occur in pairs.

Proposition 10.2. Let v_i be any vertex of G_n. For any index k, v_i has a unique neighbor v_j to which it is connected by an edge of partition class E_k.

Proof. Let j be the index such that $\boldsymbol{m}_j \equiv \boldsymbol{m}_k + \boldsymbol{m}_i \pmod 2$. Such a j must exist since the columns of M_n are all possible binary vectors of length n. Moreover,

$$\boldsymbol{m}_i + \boldsymbol{m}_j \equiv \boldsymbol{m}_i + (\boldsymbol{m}_k + \boldsymbol{m}_i) \equiv \boldsymbol{m}_k \pmod 2,$$

so by (10.8), $\{v_i, v_j\} \in E_k$. Now suppose there exist j_1, j_2 such that $\{v_i, v_{j_1}\}, \{v_i, v_{j_2}\} \in E_k$. Then by (10.8), $\boldsymbol{m}_i + \boldsymbol{m}_{j_1} \equiv \boldsymbol{m}_i + \boldsymbol{m}_{j_2}$. Therefore $\boldsymbol{m}_{j_1} \equiv \boldsymbol{m}_{j_2}$ and $j_1 = j_2$ since the columns of M_n are pairwise distinct. $\square$

Observe that by construction if $k = 1$, $i = j$ and $\{v_i, v_j\}$ is a loop. If k is even, edge $\{v_i, v_j\}$ is unique, and if $k > 1$ is odd, there are exactly two edges between v_i and v_j.

Corollary 10.5. *For any vertex v_i,*

$$\lambda_{\pi_n}(v_i) = \begin{bmatrix} 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix} \ (\textit{mod } 2).$$

Moreover, observe that $\boldsymbol{r}_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix}$ is the first row of matrix M_n, and so for all vertices v_i of G,

$$\lambda_{\pi_n}(v_i) \equiv \boldsymbol{r}_1 \pmod 2.$$

Proposition 10.3 illustrates the key property of this edge-partition. For any edge e, let $\varphi(e)$ be the index k such that $e \in E_k$. In particular, by (10.8),

$$\boldsymbol{m}_{\varphi(\{v_i, v_j\})} \equiv \boldsymbol{m}_i + \boldsymbol{m}_j \pmod 2.$$

Proposition 10.3. For any walk $W = v_{i_1} \cdot v_{i_2} \cdot v_{i_3} \cdots v_{i_N}$ in G_n with edges $E(W) = \big\{\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots \{v_{i_{N-1}}, v_{i_N}\}\big\}$,

$$\sum_{e \in E(W)} \boldsymbol{m}_{\varphi(e)} \equiv \boldsymbol{m}_{i_1} + \boldsymbol{m}_{i_N} \pmod 2.$$

Proof. By (10.8), if $e = \{v_{i_l}, v_{i_{l+1}}\}$, then $\boldsymbol{m}_{\varphi(e)} \equiv \boldsymbol{m}_{i_l} + \boldsymbol{m}_{i_{l+1}}$. As a result the sum is telescoping,

$$\begin{aligned}\sum_{e \in E(W)} \boldsymbol{m}_{\varphi(e)} &\equiv (\boldsymbol{m}_{i_1} + \boldsymbol{m}_{i_2}) + (\boldsymbol{m}_{i_2} + \boldsymbol{m}_{i_3}) + \cdots + (\boldsymbol{m}_{i_{N-1}} + \boldsymbol{m}_{i_N}) \\ &\equiv \boldsymbol{m}_{i_1} + (\boldsymbol{m}_{i_2} + \boldsymbol{m}_{i_2}) + (\boldsymbol{m}_{i_3} + \boldsymbol{m}_{i_3}) + \cdots + (\boldsymbol{m}_{i_{N-1}} + \boldsymbol{m}_{i_{N-1}}) + \boldsymbol{m}_{i_N} \\ &\equiv \boldsymbol{m}_{i_1} + \boldsymbol{m}_{i_N} \pmod 2.\end{aligned}$$

□

Lemma 10.4. *Let $C = v_{i_1} \cdot v_{i_2} \cdots v_{i_N} \cdot v_{i_1}$ be any cycle of G_n with edges $E(C)$. Then*

$$[M_n]\,[\chi_{\pi_n}(C)]^t \equiv \mathbf{0} \ (mod\ 2).$$

Proof. Let X be the indices of the unit entries of $\chi_{\pi_n}(C)$ (mod 2). Then it suffices to show that $\sum_{k \in X} \boldsymbol{m}_k \equiv \mathbf{0}$ (mod 2). Observe that for any index $k \in X$, partition cell E_k contains an odd number of edges of $E(C)$, whereas for any label $k \notin X$, cell E_k must contain an even number. Therefore,

$$\begin{aligned}\sum_{k \in X} \boldsymbol{m}_k &\equiv \sum_{e \in E(C)} \boldsymbol{m}_{\varphi(e)} \pmod 2 \\ &\equiv \boldsymbol{m}_{i_1} + \boldsymbol{m}_{i_1} \pmod 2 \text{ by Proposition 10.3} \\ &\equiv \mathbf{0} \pmod 2.\end{aligned}$$

□

Corollary 10.6. *Edge-partition π_n of G_n is $\mathbb{Z}_2$-Kirchhoff.*

Proof. Let v be any vertex, and let C be any cycle of G_n. By Proposition 10.2,

$$\lambda_{\pi_n}(v) = \begin{bmatrix} 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix} \pmod 2 \ \in \text{Row}\,(M_n),$$

and by Lemma 10.4,

$$[M_n]\,[\chi_{\pi_n}(C)]^t \equiv 0 \pmod 2.$$

Therefore,

$$\lambda_{\pi_n}(v) \cdot \chi_{\pi_n}(C) \equiv 0 \pmod 2.$$

□

Next we show that π_n is strongly $\mathbb{Z}_2$-Kirchhoff with respect to the matroid $M[M_n]$.

Corollary 10.7. *For any cycle C of G_n, if $\chi_{\pi_n}(C)$ is nonzero (mod 2), then*

$$\chi_{\pi_n}(C) \equiv \Upsilon_X \ (mod\ 2)$$

for some dependent set X of $M[M_n]$.

Proof. Let C be any cycle of G_n for which $\chi_{\pi_n}(C) \neq 0 \pmod 2$. Then by Lemma 10.4, $[M_n]\,[\chi_{\pi_n}(C)]^t \equiv 0 \pmod 2$. Therefore letting X be the indices of the unit entries of $\chi_{\pi_n}(C) \pmod 2$, X is a set of columns of M_n that sum to 0, and thus a dependent set of $M[M_n]$. □

Lemma 10.5. *Let X be any circuit of $M[M_n]$. Then there exists a cycle C in G_n such that*

$$\chi_{\pi_n}(C) = \Upsilon_X.$$

Proof. Suppose $X = \{k_1, k_2, \ldots, k_N\}$. Choose any initial vertex v_{i_0}. By Proposition 10.2 there exists a unique vertex v_{i_1} such that $\varphi(\{v_{i_0}, v_{i_1}\}) = k_1$. Similarly, v_{i_1} has a unique neighbor v_{i_2} such that $\varphi(\{v_{i_1}, v_{i_2}\}) = k_2$. Continuing in this manner, construct a walk $C = v_{i_0} \cdot v_{i_1} \cdot v_{i_2} \cdots v_{i_N}$ such that $\varphi(\{v_{i_{m-1}}, v_{i_m}\}) = k_m$. Since X is a circuit of $M[M_n]$, $\sum_{j=1}^{N} \boldsymbol{m}_{k_j} \equiv \boldsymbol{0} \pmod 2$. Thus by Proposition 10.3,

$$\boldsymbol{0} \equiv \sum_{j=1}^{N} \boldsymbol{m}_{k_j} \equiv \sum_{e \in E(C)} m_{\varphi(e)} \equiv \boldsymbol{m}_{i_0} + \boldsymbol{m}_{i_N} \pmod 2.$$

As the columns of M_n are unique, necessarily $i_0 = i_N$. Therefore $v_{i_0} = v_{i_N}$ and C is a closed walk in G_n. Suppose C is not a cycle of G_n. Then there exists $1 \leq s < t \leq N$ such that $t - s < N$ and $v_{i_{s-1}} \cdot v_{i_s} \cdots v_{i_t} \cdot v_{i_{s-1}}$ is a cycle in G_n. Then by Corollary 10.7, $\{k_s, \ldots, k_t\}$ is dependent in $M[M_n]$. However $\{k_s, \ldots, k_t\} \subsetneq X$, contradicting that X is a circuit. Therefore C is a cycle of G_n and clearly $\chi_{\pi_n}(C) = \Upsilon_X$. □

This completes the proof of Theorem 10.3 and, as a result, the proof of Theorem 10.1. □

10.3 Constructing $\mathbb{Z}_2$-Kirchhoff Edge-Partitions

The remaining sections of Chapter 10 will illustrate various constructions of $\mathbb{Z}_2$-Kirchhoff partitions. Specifically, the classical non-graphic binary matroids $M^*(K_{3,3})$, $M^*(K_5)$, F_7, and F_7^* will be considered. In each case, beginning with a binary representation matrix we proceed as in Tutte's algorithm, described in Chapter 9. At the point where Tutte's algorithm would classically terminate and conclude "not graphic," we will utilize the flexibility of repeated elements (afforded by edge-partitions) to construct a graph with the desired $\mathbb{Z}_2$-Kirchhoff edge-partition. The intermediate steps of each example will be presented as graphs with edge labels. The final product of each construction will be presented in color to aid visualization of the edge partition.

Remark 10.4. This Section will give 4 interesting examples of constructing $\mathbb{Z}_2$-Kirchhoff partitions based on extensions of Tutte's algorithm. However, determining a general construction method is *not* an open problem. A universal method of building a $\mathbb{Z}_2$-Kirchhoff edge partition for any binary matroid was used to prove Theorem 10.1.

10.3.1 The Dual of the Cycle Matroid of $K_{3,3}$, $M^*(K_{3,3})$

The matrix A_3 given in (10.9) is a standard binary representation of the matroid $M^*(K_{3,3})$.

$$A_3 = \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{10.9}$$

Remark 10.5. Note that we index the rows of A_3 with starred integers. This construction will include many matrix operations, and this row indexing will add transparency to many steps. The stars are included to distinguish row indices from the column labels, which correspond to matroid elements.

Letting $M_3 = M[A_3]$, it is well-known that $M^*(K_{3,3})$, and thus M_3, is not graphic. Therefore Tutte's algorithm performed on matrix A_3 will always terminate at a step where a graph cannot be constructed. Instead of constructing a *graph*, we will modify Tutte's algorithm to construct a graph with an edge partition that is $\mathbb{Z}_2$-Kirchhoff with respect to M_3.

Begin with the cocircuit Y of M_3 corresponding to the first row of A_3, $Y = \{1, 2, 3, 4, 5, 6\}$. First, find the bridges of Y (i.e. the elementary separators of $M_3 \backslash Y$). The binary matrix representation of $M_3 \backslash Y$ is given in (10.10).

$$M_3 \backslash Y : \quad \begin{array}{c} \begin{array}{ccc} 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array} \tag{10.10}$$

Clearly the elementary separators are

$$B_1 = \{7\} \qquad B_2 = \{8\} \qquad B_3 = \{9\}.$$

Consider these bridges one at a time.

To begin, take the bridge $B_1 = \{7\}$. The matrix representation of Y-component $M_3.(B_1 \cup Y)$ is

$$M_3.(B_1 \cup Y) : \begin{array}{c} \\ 1^* \\ 2^* \end{array} \begin{array}{c} \begin{array}{ccccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}.$$

Therefore, the restriction of $M_3.(B_1 \cup Y)$ to Y has matrix representation

$$M_3.(B_1 \cup Y)|Y : \begin{array}{c} \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}. \tag{10.11}$$

Adding the second row of (10.11) to the first,

$$\begin{array}{c} \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array},$$

row operations show that B_1 partitions Y, with partition

$$\{1, 2, 3\}, \{4, 5, 6\}.$$

Moreover, $M_3.(B_1 \cup Y)$ is a binary matroid of rank 2, and thus is graphic. The incidence matrix of a graph can be obtained by appending a third row to the matrix representation of $M_3.(B_1 \cup Y)$,

$$M_3.(B_1 \cup Y) : \begin{array}{c} \\ 1^* \\ 2^* \end{array} \begin{array}{c} \begin{array}{ccccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \\ 1^* \\ 2^* \\ (1^*{+}2^*) \end{array} \begin{array}{c} \begin{array}{ccccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}.$$

This graph, G_1, has $Y = \{1, 2, 3, 4, 5, 6\}$ as the star of a vertex, and is illustrated in Figure 10.3.1.

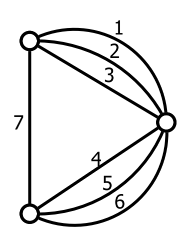


Figure 10.3.1: A graph G_1 for Y-component $M_3.(B_1 \cup Y)$.

Next take the bridge $B_2 = \{8\}$. The matrix representation of Y-component $M_3.(B_2 \cup Y)$ is

$$M_3.(B_2 \cup Y) : \begin{array}{c} \\ 1^* \\ 3^* \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

The restriction of $M_3.(B_2 \cup Y)$ to Y has matrix representation

$$M_3.(B_2 \cup Y)|Y : \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}. \tag{10.12}$$

Adding the second row of (10.12) to the first,

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

row operations show that B_2 partitions Y, with partition

$$\{1, 5, 6\}, \{2, 3, 4\}.$$

Moreover, $M_3.(B_2 \cup Y)$ is a binary matroid of rank 2, and thus is graphic. Once again, appending a third row gives the incidence matrix of a graph,

$$M_3.(B_2 \cup Y) : \begin{array}{c} \\ 1^* \\ 3^* \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \\ 1^* \\ 3^* \\ (1^*{+}3^*) \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

This graph, G_2, has $Y = \{1, 2, 3, 4, 5, 6\}$ as the star of a vertex, and is illustrated in Figure 10.3.2.

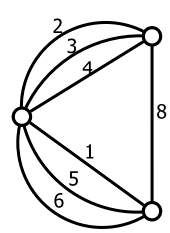


Figure 10.3.2: A graph G_2 for Y-component $M_3.(B_2 \cup Y)$.

Finally, take the bridge $B_3 = \{9\}$. The matrix representation of $M_3.(B_3 \cup Y)$ is

$$M_3.(B_3 \cup Y) : \begin{array}{c c} & \begin{array}{ccccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 1^* \\ 4^* \end{array} & \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

The restriction of $M_3.(B_3 \cup Y)$ to Y has matrix representation

$$M_3.(B_3 \cup Y)|Y : \begin{array}{c} \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}. \tag{10.13}$$

Adding the second row of (10.13) to the first,

$$\begin{array}{c} \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

row operations show that B_3 partitions Y, with partition

$$\{1, 2, 6\}, \{3, 4, 5\}.$$

Moreover, $M_3.(B_3 \cup Y)$ is a binary matroid of rank 2, and thus is graphic. Row operations lead to the incidence matrix of a graph,

$$M_3.(B_3 \cup Y) : \begin{array}{c c} & \begin{array}{ccccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 1^* \\ 4^* \end{array} & \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \longrightarrow \begin{array}{c c} & \begin{array}{ccccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 1^* \\ 4^* \\ (1^*{+}4^*) \end{array} & \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

This graph, G_3, has $Y = \{1, 2, 3, 4, 5, 6\}$ as the star of a vertex, and is illustrated in figure 10.3.3.

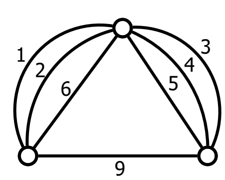


Figure 10.3.3: A graph G_3 for Y-component $M_3.(B_3 \cup Y)$.

Therefore cocircuit Y has 3 bridges, each of which partitions Y. Specifically, the partitions are:

$$B_1 : \{1,2,3\}, \{4,5,6\}$$
$$B_2 : \{1,5,6\}, \{2,3,4\}$$
$$B_3 : \{1,2,6\}, \{3,4,5\}.$$

Observe that B_1 and B_2 overlap, B_1 and B_3 overlap, and B_2 and B_3 overlap. Therefore we cannot find the non-overlapping families of bridges $\mathscr{U}_1$ and $\mathscr{U}_2$ required by Tutte's algorithm. At this point, Tutte's algorithm terminates and concludes that $M^*(K_{3,3})$ is not graphic.

However, we have already demonstrated that each Y-component is graphic. Moreover, Tutte's star-composition technique can be used to build a graph $G_{3,3}$, with edge-partition $\pi_{3,3}$, that is $\mathbb{Z}_2$-Kirchhoff with respect to M_3.

Remark 10.6. Throughout the following graph construction, lines were chosen to make each intermediate step as simply-drawn as possible. Line segments that appear congruent do not reflect edges contained in the same partition cell.

First, let H_2 be the star composition $H_2 = G_1 \star_Y G_2$, illustrated in Figure 10.3.4.

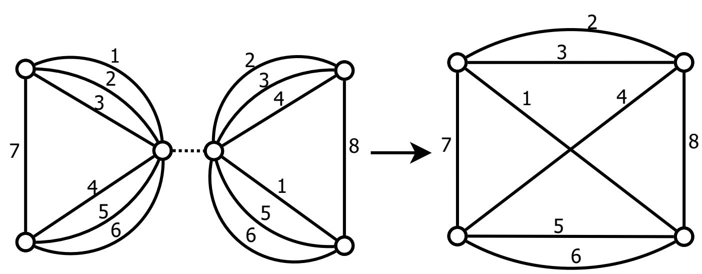
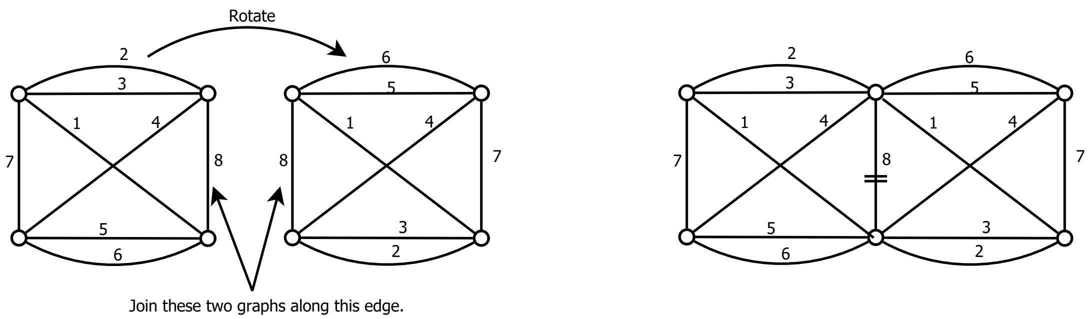


Figure 10.3.4: The star composition $H_2 = G_1 \star_Y G_2$.

Notice that H_2 does not have $Y = \{1,2,3,4,5,6\}$ as the star of a vertex. Beginning with H_2, we will construct a graph H_2', with edge-partition π_2', such that $Y = \{1,2,3,4,5,6\}$ *is* the star of a vertex with respect to partition π_2'. That is, there exists a vertex $v \in H_2'$ such that

$$\lambda_{\pi_2'}(v) \equiv \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \equiv \Upsilon_Y \pmod 2.$$

Moreover, H_2' can be constructed so that for any cycle C of H_2', the vector $\chi_{\pi_2'}(C) \pmod 2$ is a cycle vector of H. Therefore with respect to matroid elements, the cycles and vertices of H_2' are identical to those of H_2. Specifically, H_2' can be constructed by duplicating, rotating, and adjoining a second copy of H_2 to itself. This construction is illustrated in Figure 10.3.5a, and H_2' is given in Figure 10.3.5b.



(a) Constructing H_2' **(b)** The result, H_2'

Figure 10.3.5: A graph H_2', with an edge partition π_2', consisting of 8 cells each of cardinality 2.

Observe that H_2' has *two* vertices with $Y = \{1, 2, 3, 4, 5, 6\}$ as the star, that is,

$$\lambda_{\pi_2'}(v) \equiv \Upsilon_Y \pmod{2}. \tag{10.14}$$

Thus we may now take the star composition of H_2' with G_3. In fact, we take the star composition of H_2' with *two* copies of G_3, one at each vertex of H_2' satisfying (10.14), as illustrated in Figure 10.3.6.

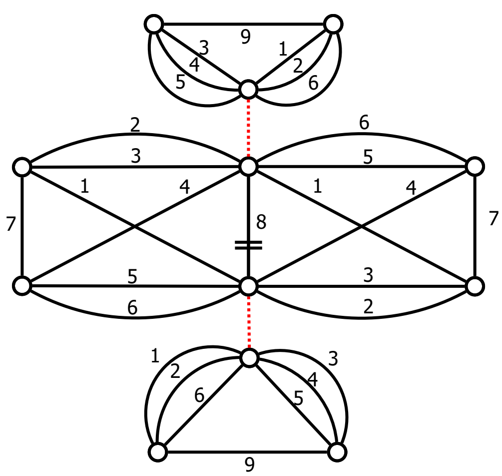


Figure 10.3.6: Double star composition of H_2' with G_3.

In the process of star composition, choose one vertex of G_3 (*other* than the one with star Y), and let the two copies of that vertex be the endpoints of all edges of partition cell 8. The result is the graph $G_{3,3}$, with edge-partition $\pi_{3,3}$, depicted in Figure 10.3.7.

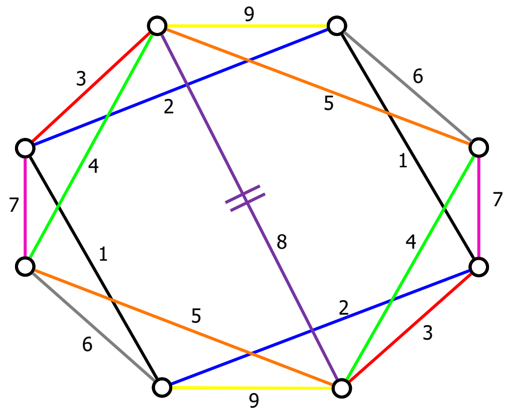


Figure 10.3.7: A graph $G_{3,3}$ with edge-partition $\pi_{3,3}$. $\pi_{3,3}$ is $\mathbb{Z}_2$-Kirchhoff with respect to M_3, and therefore $M^*(K_{3,3})$.

One may verify that for any vertex v and any cycle C of $G_{3,3}$,

$$\lambda_{\pi_{3,3}}(v) \cdot \chi_{\pi_{3,3}}(C) \equiv 0(\mod 2).$$

Moreover, $\pi_{3,3}$ is $\mathbb{Z}_2$-Kirchhoff with respect to M_3, and thus with respect to $M^*(K_{3,3})$. Perhaps more interestingly, this edge-partition was constructed by following the machinery of Tutte's algorithm, and utilizing the multiplicity afforded by edge-partitions when the classical algorithm terminated.

10.3.2 The Dual of the Cycle Matroid of K_5, $M^*(K_5)$

The matrix A_5 as in (10.15) is a standard binary representation of the matroid $M^*(K_5)$.

$$A_5 = \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \\ 5^* \\ 6^* \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{cccc|cccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \qquad (10.15)$$

Letting $M_5 = M[A_5]$, it is well-known that $M^*(K_5)$, and thus M_5, is not graphic. As in Section 10.3.1, Tutte's algorithm can be modified to produce a graph, G_5, with an edge-partition that is $\mathbb{Z}_2$-Kirchhoff with respect to M_5. Begin with the cocircuit Y of M_5 corresponding to the last row of A_5, $Y = \{2, 3, 4, 10\}$. The bridges of Y are the elementary separators of $M_5 \backslash Y$. The binary matrix representation of $M_5 \backslash Y$ is given in (10.16).

$$M_5\backslash Y: \quad \begin{array}{c} \begin{array}{cccccc} 1 & & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{c|ccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{10.16}$$

Therefore the elementary separators of $M_5 \backslash Y$ are

$$B_1 = \{1, 5, 7, 9\} \qquad B_2 = \{6\} \qquad B_3 = \{8\}.$$

Consider these bridges one at a time. First, take the bridge $B_1 = \{1, 5, 7, 9\}$. The matrix representation of Y-component $M_5.(B_1 \cup Y)$ is

$$M_5.(B_1 \cup Y): \quad \begin{array}{c|cccc|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1^* & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3^* & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 5^* & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6^* & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}.$$

Therefore, the restriction of $M_5.(B_1 \cup Y)$ to Y has matrix representation

$$M_5.(B_1 \cup Y)|Y: \quad \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 10 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \end{array}. \tag{10.17}$$

No manner of row operations can transform this matrix into one with exactly one nonzero entry in each column. Therefore, bridge B_1 *does not* partition Y, and Tutte's algorithm terminates at this step, concluding that $M^*(K_5)$ is not graphic. Nevertheless, we will continue to follow the outline of Tutte's algorithm, and ultimately construct a graph with a $\mathbb{Z}_2$-Kirchhoff edge-partition.

Although B_1 does not partition Y, $M_5.(B_1 \cup Y)$ *is* graphic. Row operations on (10.17) and appending a new row (in the row space of (10.17)) result in the incidence matrix of a graph,

$$\begin{array}{cc} & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{array}{c} 1^* \\ 3^* \\ 5^* \\ 6^* \end{array} & \left[\begin{array}{cccc|cccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \longrightarrow \begin{array}{cc} & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{array}{c} (1^*{+}6^*) \\ 3^* \\ (3^*{+}5^*) \\ 6^* \\ (1^*{+}5^*) \end{array} & \left[\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

This graph, G_1, has $Y = \{2, 3, 4, 10\}$ as the star of a vertex, and is given in Figure 10.3.8.

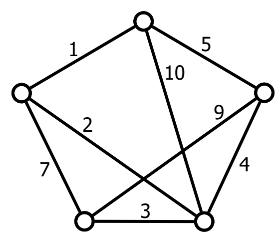


Figure 10.3.8: A graph G_1 for Y-component $M_5.(B_1 \cup Y)$.

Next take the bridge $B_2 = \{6\}$. The matrix representation of Y-component $M_5.(B_2 \cup Y)$ is

$$M_5.(B_2 \cup Y): \begin{array}{cc} & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{array}{c} 2^* \\ 6^* \end{array} & \left[\begin{array}{cccc|cccccc} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

The restriction of $M_5.(B_2 \cup Y)$ to Y has matrix representation

$$M_5.(B_2 \cup Y)|Y: \quad \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 10 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \end{array}. \tag{10.18}$$

Adding the first row of (10.18) to the second, row operations show that B_2 partitions Y, with partition $\{2, 3\}, \{4, 10\}$. Moreover, $M_5.(B_2 \cup Y)$ is a binary matroid of rank 2, and is thus graphic. The incidence matrix of a graph can be derived via row operations,

$$\begin{array}{cc} & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{array}{c} 2^* \\ 6^* \end{array} & \left[\begin{array}{cccc|cccccc} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \longrightarrow \begin{array}{cc} & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{array}{c} 2^* \\ 6^* \\ (2^*{+}6^*) \end{array} & \left[\begin{array}{cccc|cccccc} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

This graph, G_2, has $Y = \{2, 3, 4, 10\}$ as the star of a vertex, illustrated in Figure 10.3.9.

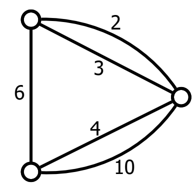


Figure 10.3.9: A graph G_2 for Y-component $M_5.(B_2 \cup Y)$.

Finally, take the bridge $B_3 = \{8\}$. The matrix representation of $M_5.(B_3 \cup Y)$ is

$$M_5.(B_3 \cup Y) : \begin{array}{c} \\ 4^* \\ 6^* \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{cccc|cccccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

The restriction of of $M_5.(B_3 \cup Y)$ to Y has matrix representation

$$M_5.(B_3 \cup Y)|Y : \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 10 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \end{array}. \tag{10.19}$$

Adding the first row of (10.19) to the second, row operations show that B_3 partitions Y with partition $\{2, 10\}, \{3, 4\}$. Moreover, $M_5.(B_3 \cup Y)$ is a binary matroid of rank 2 and is thus graphic. The incidence matrix of a graph can be derived via row operations,

$$\begin{array}{c} \\ 4^* \\ 6^* \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{cccc|cccccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \\ 4^* \\ 6^* \\ (4^*{+}6^*) \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{cccc|cccccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \end{array}.$$

This graph, G_3, has $Y = \{2, 3, 4, 10\}$ as the star of a vertex, illustrated in Figure 10.3.10.

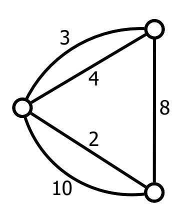


Figure 10.3.10: A graph G_3 for Y-component $M_5.(B_3 \cup Y)$.

Therefore, cocircuit Y of M_5 has 3 bridges, and each Y-component is graphic. As in Section 10.3.1, Tutte's star-composition technique can be used to construct a graph with a suitable $\mathbb{Z}_2$-Kirchhoff edge-partition. First, let the graph H_2 be the star composition $H_2 = G_2 \star_Y G_3$, illustrated in Figure 10.3.11.

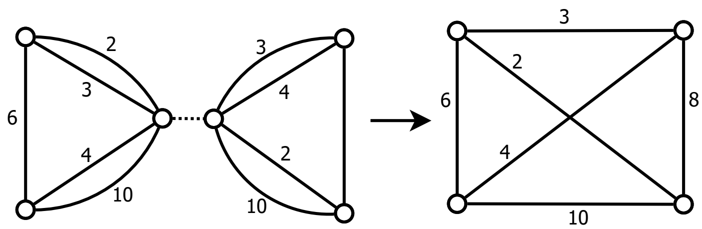
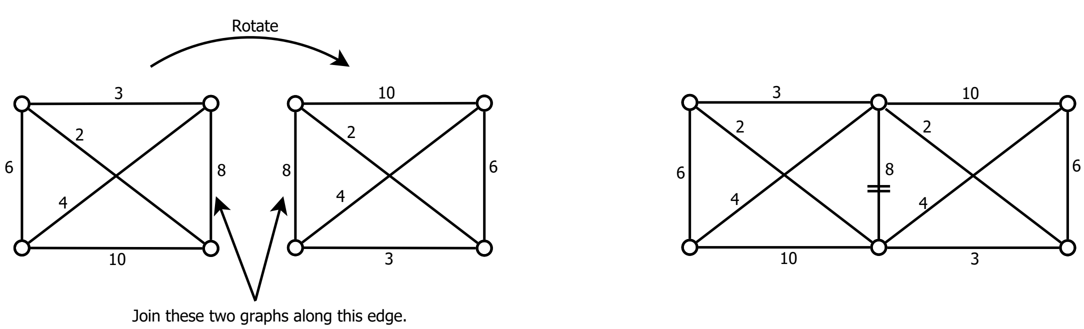


Figure 10.3.11: The star composition $H_2 = G_2 \star_Y G_3$.

Notice that H_2 does not have $Y = \{2, 3, 4, 10\}$ as the star of a vertex. Beginning with H_2, we will construct a graph H_2', with edge-partition π_2', such that $Y = \{2, 3, 4, 10\}$ *is* the star of a vertex with respect to partition π_2'. That is, there exists a vertex $v \in H_2'$ such that

$$\begin{array}{cc} & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \lambda_{\pi_2'}(v) \equiv & \left[\begin{array}{cccccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \pmod 2. \end{array}$$

Moreover, H_2' can be constructed so that for any cycle C of H_2', the vector $\chi_{\pi_2'}(C) \pmod 2$ is a cycle vector of H. Therefore with respect to matroid elements, the cycles and vertices of H_2' are identical to those of H_2. Specifically, H_2' can be constructed by duplicating, rotating, and adjoining a second copy of H_2 to itself. This construction is illustrated in Figure 10.3.12a, and H_2' is illustrated in Figure 10.3.12b.



(a) Constructing H_2'. **(b)** The result, H_2'

Figure 10.3.12: A graph H_2', with edge-partition π_2' consisting of 6 cells, each of cardinality 2.

Observe that H_2' has *two* vertices with $Y = \{2, 3, 4, 10\}$ as the star, that is,

$$\lambda_{\pi_2'}(v) \equiv \Upsilon_Y \pmod 2. \tag{10.20}$$

Thus we may now take the star composition of H_2' with G_1. In fact, we take the star composition of H_2' with *two* copies of G_1, illustrated in Figure 10.3.13.

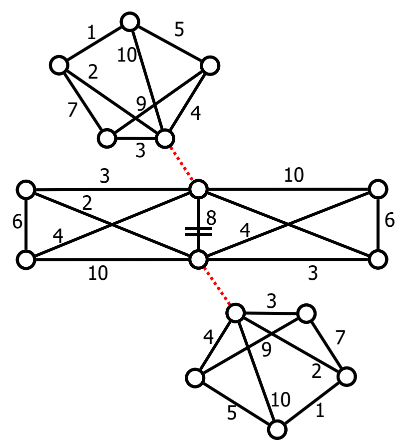


Figure 10.3.13: Double star composition of H_2' with G_1.

Once again, in the process of star composition, choose one vertex of G_3 (*other* than the one with star Y), and let the two copies of that vertex be the endpoints of all edges of partition cell 8. The result is the graph G_5, with edge-partition π_5, depicted in Figure 10.3.14.

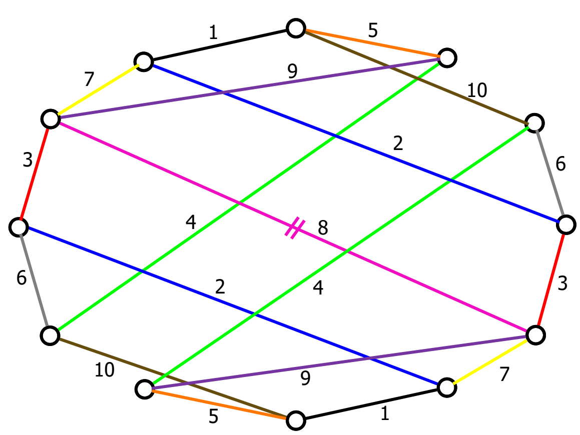


Figure 10.3.14: A graph G_5 with edge-partition π_5. π_5 is $\mathbb{Z}_2$-Kirchhoff with respect to M_5, and therefore $M^*(K_5)$.

One may verify that for any vertex, v, and any cycle, C, of G_5,

$$\lambda_{\pi_5}(v) \cdot \chi_{\pi_5}(C) \equiv 0 \pmod 2.$$

Moreover, π_5 is $\mathbb{Z}_2$-Kirchhoff with respect to M_5, and thus with respect to $M^*(K_5)$. Once again, this edge-partition was constructed by following the machinery of Tutte's algorithm, and utilizing edge-partitions when the classical algorithm terminated.

10.3.3 The Fano Matroid, F_7

The Fano matroid, F_7, is a binary matroid on 7 elements that is not graphic. The standard binary representation of F_7 is given in (10.21).

$$F_7 : \begin{array}{c} \\ 1^* \\ 2^* \\ 2^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{10.21}$$

Notice that the only element contained in more than 2 rows of this standard representation matrix is element 7. First consider the cocircuit $Y_1 = \{1, 4, 5, 7\}$ corresponding to the first row of (10.21). The binary representation of $F_7 \backslash Y_1$ is given in (10.22).

$$F_7 \backslash Y_1 : \begin{array}{c} \begin{array}{ccc} 2 & 3 & 6 \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \end{array} \quad F_7 \backslash Y_2 : \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \end{array} \quad F_7 \backslash Y_3 : \begin{array}{c} \begin{array}{ccc} 1 & 2 & 4 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \end{array} \tag{10.22}$$

This matrix demonstrates that $F_7 \backslash Y_1$ has only one elementary separator, meaning Y_1 has only one bridge. Next consider the cocircuit $Y_2 = \{2, 4, 6\}$, corresponding to the second row of (10.21). The binary representation of $F_7 \backslash Y_2$ is given in (10.22). Observe that $F_7 \backslash Y_2$ has only one elementary separator, so Y_2 has only one bridge as well. Lastly consider the cocircuit $Y_3 = \{1, 4, 5, 7\}$ corresponding to the third row of (10.21). The binary representation of $F_7 \backslash Y_3$ is given in (10.22), and notice that Y_3 has only one bridge as well. Therefore Y_1, Y_2 and Y_3 are three cocircuits of F_7, each having exactly one bridge, and each containing element 7. Therefore Tutte's algorithm terminates at this point, and concludes that F_7 is not graphic.

Once again, we would like to continue in a constructive manner to find a graph with an edge-partition that is $\mathbb{Z}_2$-Kirchhoff with respect to F_7. In the case of F_7, we cannot proceed via star composition as in Sections 10.3.1 and 10.3.2, as each cocircuit Y_i has only a single Y-component. Instead, consider the underlying theme of Tutte's construction. The algorithm considers a very specific set of matroid minors. In the case that these minors are graphic, Tutte joins the graphs of these minors in a specific way (star composition) to build a graph for the desired matroid.

A similar construction can be used to build an edge-partition that is $\mathbb{Z}_2$-Kirchhoff with respect to F_7. In particular, we know that every minor of F_7 is graphic (and thus also $\mathbb{Z}_2$-Kirchhoff). Let F_7' be the minor obtained from F_7 by deleting element 7. The standard representation matrix F_7' is given in (10.23).

$$F_7' : \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array} \tag{10.23}$$

Each row of this matrix has at most two nonzero entries—adjoining a fourth row gives the incidence matrix of a graph, given in (10.24). This graph, G_1, is illustrated in Figure 10.3.15.

$$\begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ (1^*+2^*+3^*) \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{10.24}$$

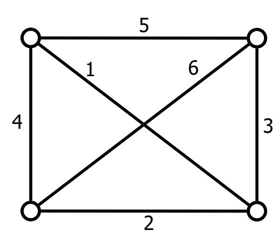


Figure 10.3.15: A graph G_1 for matroid $F_7' = F_7\backslash 7$.

Next follow a similar method as the constructions in Sections 10.3.1 and 10.3.2. By replicating, rotating, and joining a second copy of this graph G_1, we can construct a new graph G_2, with edge-partition π_2, as illustrated in Figure 10.3.16. Edge-partition π_2 has 6 partition cells and, moreover, is $\mathbb{Z}_2$-Kirchhoff with respect to F_7'

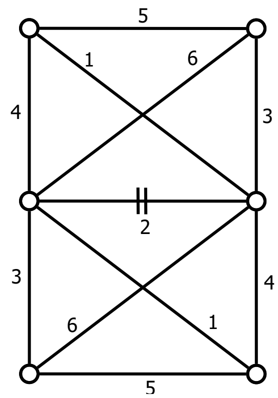


Figure 10.3.16: A graph G_2 with edge-partition π_2. π_2 is $\mathbb{Z}_2$-Kirchhoff with respect to the matroid $F_7' = F_7 \backslash 7$.

The matroid F_7' was obtained by deletion of element 7 from the Fano matroid F_7. Therefore rather than proceed via star composition, the goal is to re-introduce element 7 into our edge-partition. That is, beginning with G_2 and π_2, we construct a new graph with a 7-cell edge partition that is $\mathbb{Z}_2$-Kirchhoff with respect to F_7. For example, consider the graph G_7, with edge-partition π_7, as illustrated in Figure 10.3.17. G_7 and π_7 were obtained from G_2 and π_2 by adding two new edges, each contained in cell 7 of π_7.

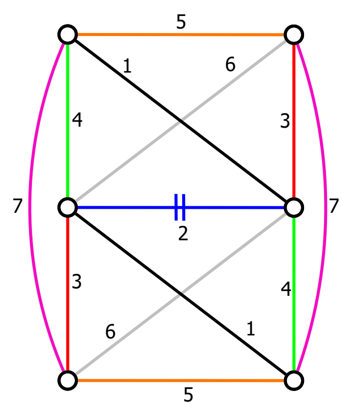


Figure 10.3.17: A graph G_7 with edge-partition π_7. π_7 is $\mathbb{Z}_2$-Kirchhoff with respect to F_7.

For any vertex, v, and any cycle, C, of G_7, one may check that $\lambda_{\pi_7}(v) \cdot \chi_{\pi_7}(C) \equiv 0 \pmod 2$. Therefore edge-partition π_7 is $\mathbb{Z}_2$-Kirchhoff. Moreover, by introducing 2 edges of cell 7 in this manner, one may verify that π_7 is $\mathbb{Z}_2$-Kirchhoff with respect to F_7.

10.3.4 The Dual of the Fano Matroid F_7^*

The dual of the Fano matroid, F_7^*, is the last of the excluded minors. The standard binary matrix representation of of F_7^* is given in (10.25).

$$F_7^* : \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{10.25}$$

Sections 10.3.1, 10.3.2, and 10.3.3 demonstrated two methods of constructing $\mathbb{Z}_2$-Kirchhoff partitions for non-graphic binary matroids. For $M^*(K_{3,3})$ and $M^*(K_5)$, Sections 10.3.1 and 10.3.2 followed Tutte's method of graph construction, now utilizing the repetition of matroid elements afforded by edge-partitions. In the case of F_7, Section 10.3.3 instead deleted a matroid element, constructed a $\mathbb{Z}_2$-Kirchhoff partition for the resulting matroid minor, and then re-introduced the deleted element. This section will show that a $\mathbb{Z}_2$-Kirchhoff partition F_7^* can be constructed via either method.

First we proceed by Tutte's method. Consider the cocircuit $Y = \{4, 5, 6, 7\}$ corresponding to the fourth row of (10.25). $F_7^* \backslash Y$ has matrix representation

$$F_7^* \backslash Y : \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \end{array}.$$

Thus Y has three bridges, $B_1 = \{1\}$, $B_2 = \{2\}$, and $B_3 = \{3\}$, and we consider the three Y-components, $F_7^*.(B_i \cup Y)$. Y-component $F_7^*.(B_1 \cup Y)$ has standard representation matrix

$$F_7^*.(B_1 \cup Y) : \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}.$$

Sections 10.3.1 and 10.3.2 presented—in full—six examples of Y-components of rank 2, so similar details will be omitted here. B_1 partitions Y, and $F_7^*.(B_1 \cup Y)$ is graphic. Figure 10.3.18 illustrates a graph G_1 such that $M(G_1) \cong F_7^*.(B_1 \cup Y)$.

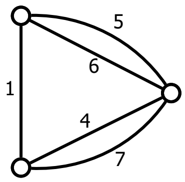


Figure 10.3.18: A graph G_1 for $F_7^*.(B_1 \cup Y)$.

Next, Y-component $F_7^*.(B_2 \cup Y)$ has standard representation matrix

$$F_7^*.(B_2 \cup Y): \begin{array}{c} \\ 1^* \\ 4^* \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc|ccc} 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}.$$

Once again, $F_7^*.(B_2 \cup Y)$ has rank 2, so B_2 partitions Y, and $F_7^*.(B_2 \cup Y)$ is graphic. Figure 10.3.19 presents a graph G_2 for which $M(G_2) \cong F_7^*.(B_2 \cup Y)$.

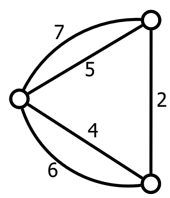


Figure 10.3.19: A graph G_2 for $F_7^*.(B_2 \cup Y)$.

Lastly, Y-component $F_7^*.(B_3 \cup Y)$ has standard representation matrix

$$F_7^*.(B_3 \cup Y): \begin{array}{c} \\ 1^* \\ 4^* \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc|ccc} 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}.$$

$F_7^*.(B_3 \cup Y)$ has rank 2, so B_3 partitions Y, and $F_7^*.(B_3 \cup Y)$ is graphic. Figure 10.3.20 provides a graph G_3 satisfying $M(G_3) \cong F_7^*.(B_3 \cup Y)$.

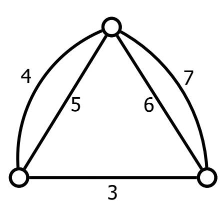


Figure 10.3.20: A graph G_3 for $F_7^*.(B_3 \cup Y)$.

Proceeding as before, graphs G_1, G_2, and G_3 will be used to construct the desired graph and edge-partition. First, let H_2 be the star composition $G_1 \star_Y G_2$ as illustrated in Figure 10.3.21.

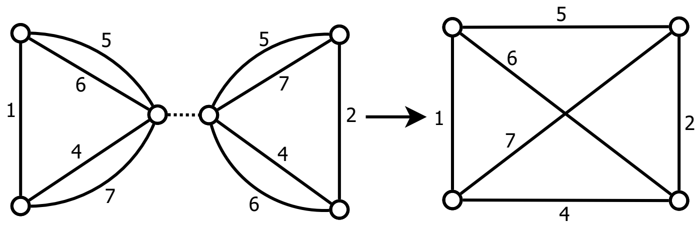


Figure 10.3.21: The star composition $H_2 = G_1 \star_Y G_2$.

Notice that H_2 does not have $Y = \{4, 5, 6, 7\}$ as the star of a vertex. As before, duplicate, rotate, and then join a second copy of H_2, resulting in a new graph H_2', with edge-partition π_2', as illustrated in Figure 10.3.22.

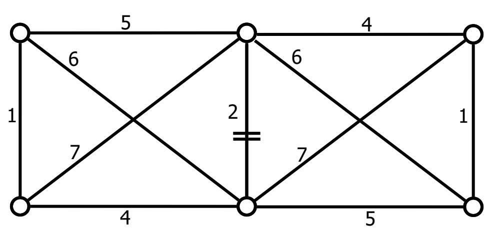


Figure 10.3.22: A graph H_2' with edge-partition π_2'.

Observe there are *two* vertices of H_2' such that

$$\lambda_{\pi_2'} \equiv \Upsilon_Y \equiv \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \pmod 2. \tag{10.26}$$

Therefore take the double star composition of H_2' with *two* copies of G_3, one at each vertex of H_2' satisfying (10.26), as illustrated in Figure 10.3.23.

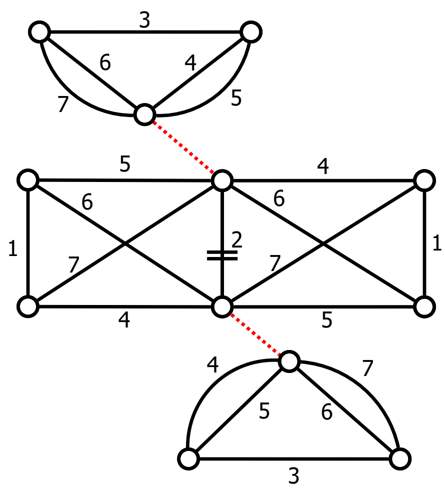


Figure 10.3.23: Double star composition of H_2' with G_3.

As before, choose one vertex of G_3 (*other* than the one with star Y), and let the two copies of that vertex be the endpoints of all edges of partition cell 2. The result is the graph G_{7^*}, with edge-partition π_{7^*}, depicted in Figure 10.3.24.

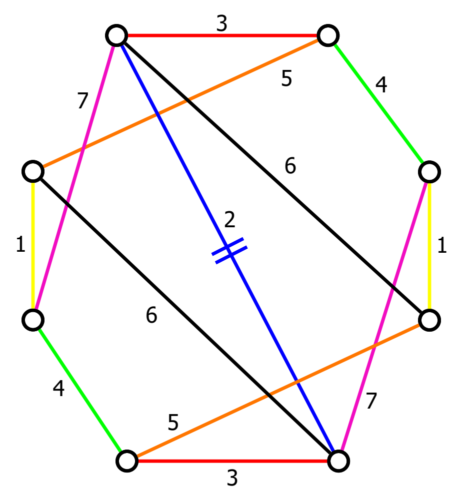


Figure 10.3.24: A graph G_{7^*} with edge-partition π_{7^*}. π_{7^*} is $\mathbb{Z}_2$-Kirchhoff with respect to F_7^*.

One may verify that edge-partition π_{7^*} is $\mathbb{Z}_2$-Kirchhoff with respect to F_7^*.

Next we demonstrate that an edge-partition that is $\mathbb{Z}_2$-Kirchhoff with respect to F_7^* can be constructed using element deletion and reintroduction, as in Section 10.3.3. Given the standard representation (10.25) of F_7^*, (10.27) is the standard binary representation of $F_7^*\backslash 7$.

$$F_7^*\backslash 7: \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{10.27}$$

Any minor of F_7*, and in particular the matroid $F_7^*\backslash 7$, is graphic. Adding row 1 of (10.27) to row 4, and adjoining a fifth row gives the matrix

$$\begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ (4^*{+}1^*) \\ (1^*{+}2^*{+}3^*{+}4^*) \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}.$$

This matrix has exactly two ones in each column, and is the incidence matrix of the graph H given in Figure 10.3.25.

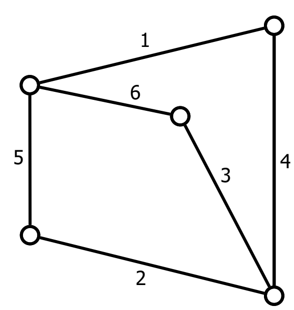


Figure 10.3.25: A graph H satisfying $M(H) = F_7^*\backslash 7$.

Once again replicate, rotate, and join a second copy of H to itself, giving a new graph H', with edge-partition π', as illustrated in Figure 10.3.26. In fact, edge-partition π' is $\mathbb{Z}_2$-Kirchhoff with respect to the matroid $F_7^*\backslash 7$.

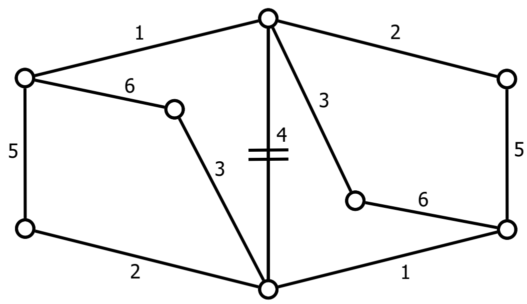


Figure 10.3.26: A graph H' with edge-partition π'. π' is $\mathbb{Z}_2$-Kirchhoff with respect to $F_7^*\backslash 7$.

Beginning with H' and π', construct a new graph with a 7-cell edge partition that is $\mathbb{Z}_2$-Kirchhoff with respect to F_7^*. Specifically, consider the graph G'_{7*}, with edge-partition π'_{7^*}, illustrated in Figure 10.3.31. G'_{7^*} and π'_{7^*} were obtained from H' and π' by adding two new edges, each contained in cell 7.

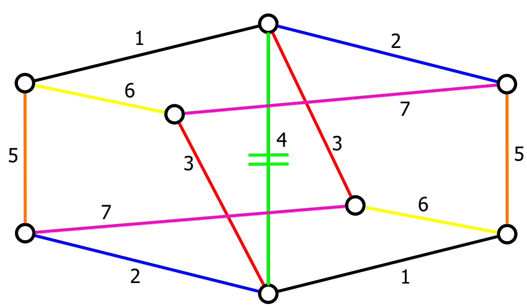


Figure 10.3.27: A graph G'_{7^*} with edge-partition π'_{7^*}. π'_{7^*} is $\mathbb{Z}_2$-Kirchhoff with respect to F_7^*.

For any vertex, v, and any cycle, C, of G'_{7^*}, one may check that

$$\lambda_{\pi'_{7^*}}(v) \cdot \chi_{\pi'_{7^*}}(C) \equiv 0 \pmod{2}.$$

Therefore edge-partition π'_{7^*} is $\mathbb{Z}_2$-Kirchhoff. Moreover, by introducing 2 edges of cell 7 in this manner, one may verify that π'_{7^*} is $\mathbb{Z}_2$-Kirchhoff with respect to F_7^*.

10.3.5 Two Further Examples of $\mathbb{Z}_2$-Kirchhoff Edge-Partitions

Sections 10.3.1 through 10.3.4 showed that although binary matroids $M^*(K_{3,3}), M^*(K_5), F_7$ and F_7^* are not graphic, they are $\mathbb{Z}_2$-Kirchhoff. Moreover, in each case $\mathbb{Z}_2$-Kirchhoff edge-partitions were constructed via methods analogous Tutte's construction algorithm. These examples, however, are minimal in the sense that deletion of a single element results in a graphic matroid. For completeness, this section considers two larger binary matroids. Each contains the Fano dual F_7^* as a nontrivial minor, and therefore is nongraphic. These examples present one distinction from those given up to this point. In Sections 10.3.1 through 10.3.4, the basic building blocks used in each construction were *graphs.* In these examples, the basic building blocks will be graphs *with edge-partitions.*

Let M' be the binary matroid with standard representation

$$\begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \end{array} \tag{10.28}$$

Notice that deleting the 8^{th} column of (10.28) gives a binary representation of F_7^*. Consider the cocircuit $Y = \{4, 5, 6, 7\}$ of M', corresponding to row 4 of (10.28). Then $M'\backslash Y$ has standard representation matrix

$$M'\backslash Y: \quad \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 8 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

$M\backslash Y$ has two elementary separators, and thus Y has two bridges, $B_1 = \{1, 3, 8\}$ and $B_2 = \{2\}$. The Y-component $M'.(B_2 \cup Y)$ has rank 2 and is therefore graphic. Figure 10.3.28 illustrates a graph G_2 for which $M(G_2) \cong M'.(B_2 \cup Y)$.

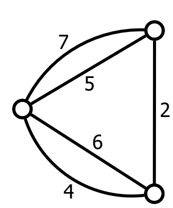


Figure 10.3.28: A graph G_2 for $M'.(B_2 \cup Y)$.

On the other hand, the Y-component $M'.(B_1 \cup Y)$ has standard representation matrix

$$M.(B_1 \cup Y) : \begin{array}{c} \\ 1^* \\ 3^* \\ 4^* \end{array} \begin{array}{c} \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \end{array},$$

which is a matrix representation of the Fano matroid. Although F_7 is not graphic, it is $\mathbb{Z}_2$-Kirchhoff. Figure 10.3.29 presents a graph G_1, with edge partition π_1, that is $\mathbb{Z}_2$-Kirchhoff with respect to $M'.(B_1 \cup Y)$.

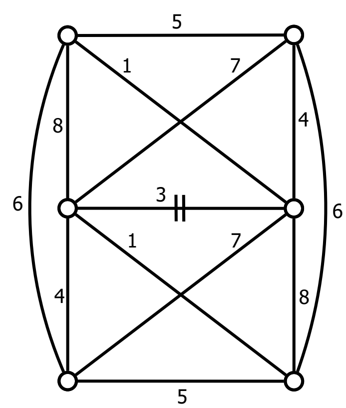


Figure 10.3.29: A graph G_1, with edge-partition π_1 that is $\mathbb{Z}_2$-Kirchhoff with respect to $M'.(B_1 \cup Y)$.

But now observe that $Y = \{4, 5, 6, 7\}$ is the star of a vertex in graph G_2, and G_1 has two vertices such that

$$\lambda_{\pi_1}(v) \equiv \Upsilon_Y \equiv \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \end{array} \pmod 2.$$

Therefore as before, take the double-star composition of G_1 with two copies of G_2. The result is a graph G', with edge-partition π', as illustrated in Figure 10.3.30. One may verify that edge-partition π' is not only $\mathbb{Z}_2$-Kirchhoff, but is $\mathbb{Z}_2$-Kirchhoff with respect to matroid M'.

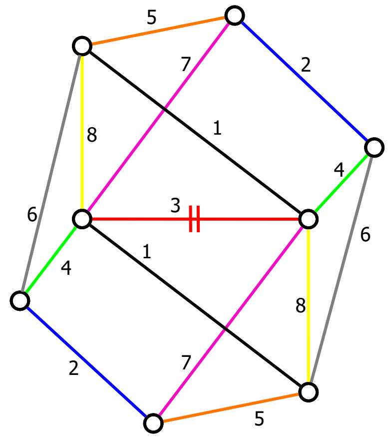


Figure 10.3.30: A graph G', with edge-partition π' that is $\mathbb{Z}_2$-Kirchhoff with respect to M'.

Finally, let M'' be the binary matroid with standard representation

$$\begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}. \qquad (10.29)$$

Observe that deleting column 8 of (10.29) gives a binary standard representation of F_7^*. Therefore as F_7^* is $\mathbb{Z}_2$-Kirchhoff, so is $M''\backslash 8$. Figure 10.3.31 presents a graph G_1, with edge-partition π_1, that is $\mathbb{Z}_2$-Kirchhoff with respect to $M''\backslash 8$.

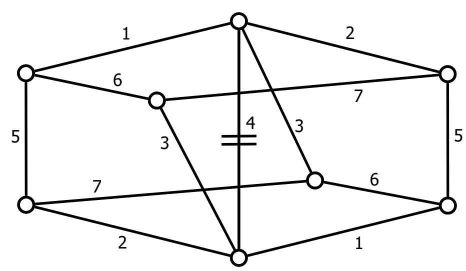


Figure 10.3.31: A graph G_1 with edge-partition π_1 that is $\mathbb{Z}_2$-Kirchhoff with respect to $M''\backslash 8$.

There is no way to introduce new edges to G_1, belonging to an 8^{th} partition cell, in a manner that produces a $\mathbb{Z}_2$-Kirchhoff edge-partition with respect to M''. Therefore we proceed as before: replicate, rotate, and join a second copy of G_1 to itself. However, G_1 has an edge-partition with all cells of cardinality 2. Therefore rather than join the copies of G_1 along a single edge, we join them along a *pair* of edges belonging to the same partition cell. The resulting graph, G_1', with edge-partition π_1', is drawn on the flat torus in Figure 10.3.32. One may verify that π_1' is $\mathbb{Z}_2$-Kirchhoff with respect to $M''\backslash 8$.

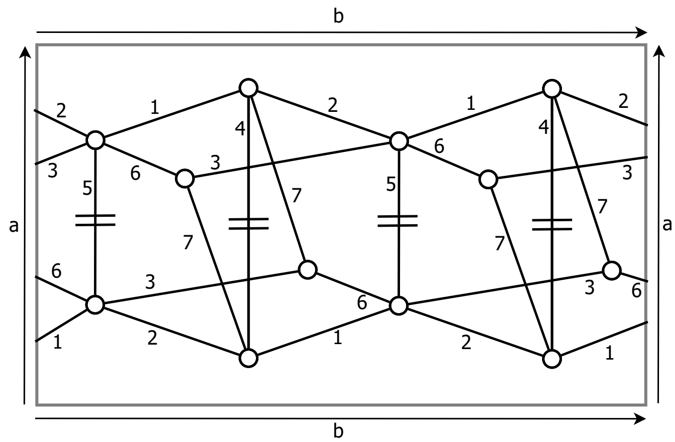


Figure 10.3.32: A graph G_1', with edge-partition π_1' that is $\mathbb{Z}_2$-Kirchhoff with respect to $M''\backslash 8$, drawn on the flat torus.

Finally introduce *four* new edges, each belonging to partition cell 8, as shown in Figure 10.3.33. The result is a graph G'', with an edge partition π''. One may verify that edge-partition π'' is not only $\mathbb{Z}_2$-Kirchhoff, but is $\mathbb{Z}_2$-Kirchhoff with respect to matroid M''.

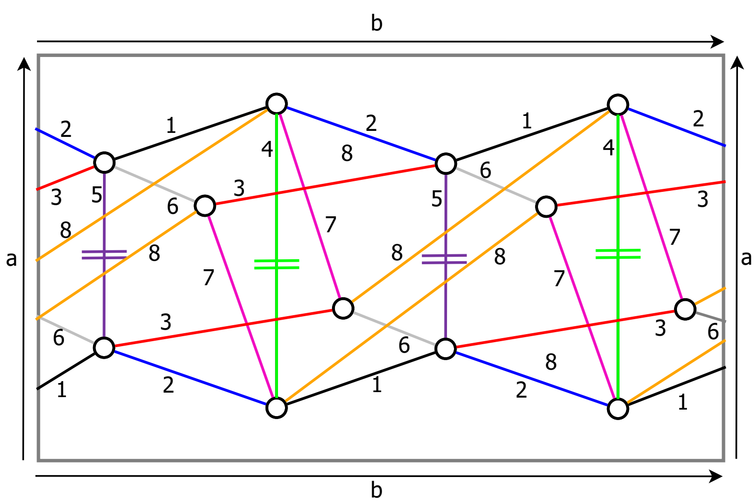


Figure 10.3.33: A graph G'', with edge-partition π'' that is $\mathbb{Z}_2$-Kirchhoff with respect to M''.

Appendix A

Algorithm Flow Charts

This chapter presents flowcharts for Tutte's algorithm. First, A.1 summarizes Tutte's recognition method (Section 9.2), and then A.2 illustrates Tutte's graph construction algorithm (Section 9.2.4).

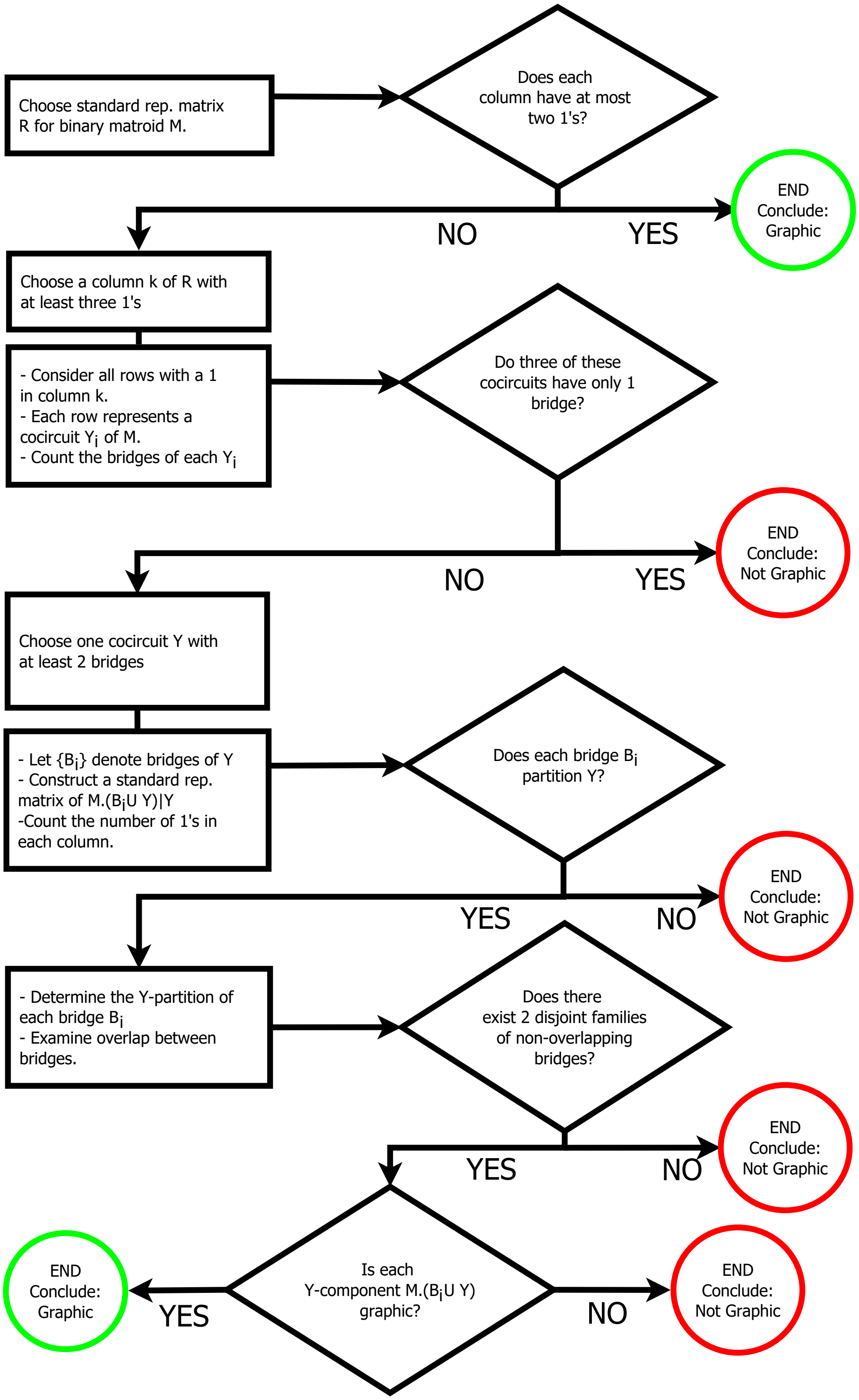


Figure A.1: Flowchart of Tutte's Algorithm

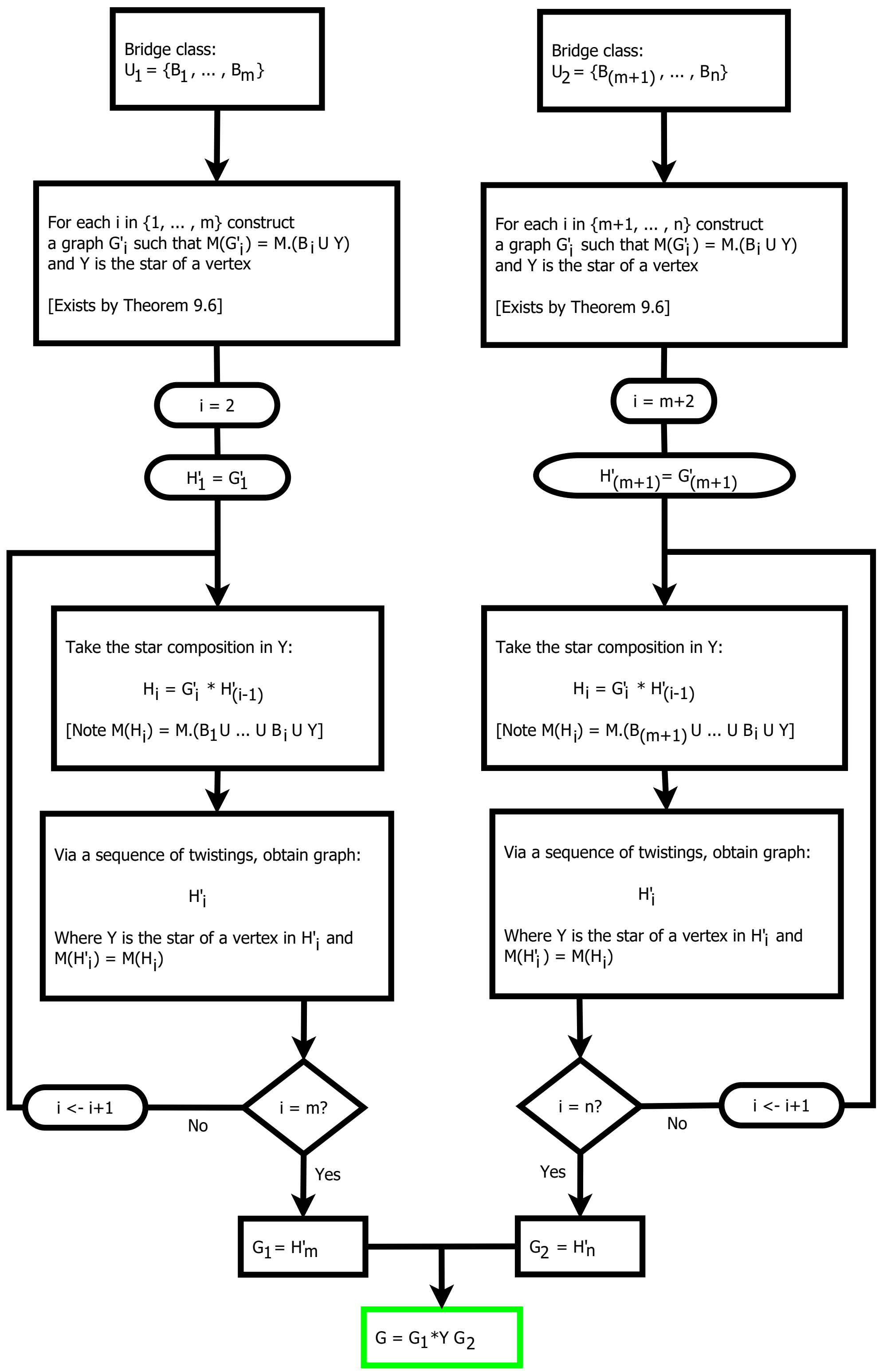


Figure A.2: Flowchart of Graph Construction

Appendix B

An Alternate Proof of Theorem 10.3

This section presents an alternate statement and proof of Theorem 10.3, in the form of Theorems B.1 and B.2. It is included for completeness, as it utilizes a number of novel−if not interesting−matrix constructions that encode the structure of edge-partitions. For ease of notation, the cells of an edge- partition $\pi = (E_1, \dots, E_k)$ will often be referred to by their integer indices. As before, for any edge e we let $\varphi(e)$ denote the subscript of the partition cell containing edge e. That is,

$$\varphi(e) = j \Leftrightarrow e \in E_j.$$

Throughout this chapter, all graphs will be undirected and may have multiple edges. Although strictly speaking these are *multi*graphs, we will frequently refer to them simply as *graphs*, denoted G, without fear of confusion.

Definition B.1. An edge-partition π is ***simple*** if any pair of edges in G sharing the same endpoints are contained in the same cell of π. In this case, we say that G is **π-simple**.

Let G be an undirected graph on n vertices (which may have multiple edges). Moreover, let π be a simple edge-partition of G. The structure of π with respect to G can be encoded by a pair of matrices..

Definition B.2. Let A be the $n \times n$ ***adjacency matrix*** of mulitgraph G. That is, the rows and columns of A are indexed by $\{v_1, \dots, v_n\}$, and

$$A(v_i, v_j) = \big|\{e \in E(G) : e = \{v_i, v_j\}\big|.$$

Definition B.3. Let B be a square matrix, with rows and columns indexed by vertices

$\{v_1, \ldots v_n\}$, such that

$$B(v_i, v_j) = \begin{cases} \varphi(\{v_i, v_j\}) & \text{if } \{v_i, v_j\} \in E(G) \\ 0 & \text{Otherwise.} \end{cases}$$

$B(v_i, v_j)$ is well-defined as G is π-simple. Call B the ***cell matrix*** of G with respect to π.

Example B.1. Figure B.1 presents a graph H, with a simple edge-partition π.

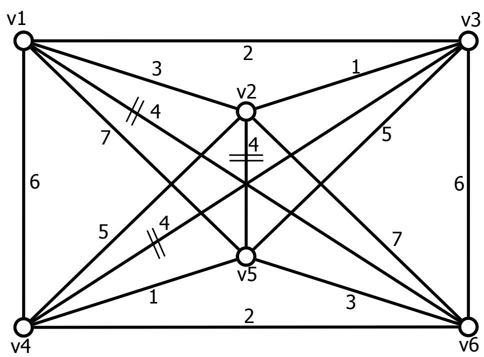


Figure B.1: A graph H with simple edge-partition π.

Accordingly, we may construct the adjacency matrix A_H, and cell matrix B_H, of H with respect to π.

$$A_H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad B_H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 3 & 2 & 6 & 7 & 4 \\ 3 & 0 & 1 & 5 & 4 & 7 \\ 2 & 1 & 0 & 4 & 5 & 6 \\ 6 & 5 & 4 & 0 & 1 & 2 \\ 7 & 4 & 5 & 1 & 0 & 3 \\ 4 & 7 & 6 & 2 & 3 & 0 \end{bmatrix} \end{array}$$

Observe that the adjacency and cell matrices are sufficient to reconstruct a graph with a simple edge-partition. That is, any graph/edge-partition pair having adjacency and cell matrices A_H and B_H must be precisely the graph H, with partition π, as depicted in Figure B.1.

Any graph G, with a simple edge-partition $\pi = (E_1, \ldots, E_K)$, is uniquely determined by its pair of adjacency and cell matrices. Therefore the π-incidence and π-cycle vectors of G can be computed from matrices A and B.

Let v_i be any vertex of G, with π-incidence vector

$$\lambda_\pi(v_i) = \left[\begin{array}{ccc} x_1 & \cdots & x_k \end{array}\right].$$

Recall that x_r is the number of times v_i occurs as the endpoint of an edge of cell r. The vertices v_j adjacent to v_i by an edge of cell r are identified by the columns of B having an r in row v_i. The number of such edges is given by $A(v_i, v_j)$. Therefore,

$$x_r = \sum_{j\,:\,B(v_i,v_j)=r} A(v_i, v_j). \tag{B.1}$$

Next let C be any cycle of G with cycle vector

$$\chi_\pi(C) = \left[\begin{array}{ccc} y_1 & \cdots & y_k \end{array}\right].$$

Recall that y_r is the the number of edges of cell r traversed by C. If $C = v_{C_1} v_{C_2} \cdots v_{C_M} v_{C_1}$, then it follows that y_r is the number of times integer r occurs in the multi-set

$$\left\{B(v_{C_1}, v_{C_2}), B(v_{C_2}, v_{C_3}), \ldots, B(v_{C_{M-1}}, v_{C_M}), B(v_{C_M}, v_{C_1})\right\}.$$

Example B.2. Consider the vertex v_1, of graph H, as given in Figure B.1. Then the v_1 rows of A_H and B_H are:

$$B_H(v_1,:) = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 3 & 2 & 6 & 7 & 4 \end{array}\right] \end{array} \quad \text{and} \quad A_H(v_1,:) = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 & 2 \end{array}\right] \end{array}.$$

Therefore,

$$\lambda_\pi(v_1) = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 1 & 2 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

Similarly, consider the cycle $C = v_1$-v_2-v_5-v_1. Then $A_H(v_1, v_2) = 3$, $A_H(v_2, v_5) = 4$, and $A_H(v_4, v_1) = 7$. Therefore

$$\chi_\pi(C) = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

Recall that an edge-partition is $\mathbb{Z}_2$-Kirchhoff if for all vertices v and all cycles C, the π-incidence vector of v is orthogonal (mod 2) to the π-cycle vector of C,

$$\lambda_\pi(v) \cdot \chi_\pi(C) \equiv 0 \pmod{2}. \tag{B.2}$$

Therefore the adjacency and cell matrices A and B can be used to determine if an edge-partition is $\mathbb{Z}_2$ Kirchhoff. Theorem B.1 is the main result we are trying to prove.

Theorem B.1. *For $n \geq 3$, let M_n^* be a $n \times 2^n - 1$ matrix whose columns are all of the non-zero vectors of $\mathbb{Z}_2^n$. Then $M[M_n^*]$ is strongly $\mathbb{Z}_2$-Kirchhoff.*

This result is a corollary of Theorem B.2, the proof of which is constructive, by induction.

Theorem B.2. *For $n \geq 1$, let M_n^* be a $n \times 2^n - 1$ matrix whose columns are all of the non-zero vectors of $\mathbb{Z}_2^n$. Then there exists a graph G_n, with simple edge-partition π_n, such that*

> *(1.) G_n has 2^n vertices, each pair of which is connected by an edge. Edge-partition π_n has $2^n - 1$ cells, indexed by $\{1, 2, \ldots, 2^n - 1\}$.*
>
> *(2.) For any partition cell k and any vertex v of G_n, v is incident to an edge of cell k.*
>
> *(3.) Edge-partition π_n is $\mathbb{Z}_2$-Kirchhoff.*
>
> *(4.) For every cycle C of G_n, if $\chi_{\pi_n}(C) \neq \mathbf{0}$ (mod 2), then it is the indicator of some dependent set of $M[M_n^*]$.*
>
> *(5.) Let X be any circuit of $M[M_n^*]$ and k any matroid element contained in X. Then for any edge e in partition cell k, there exists a cycle C in G_n, containing e, such that $\chi_{\pi_n}(C) = \Upsilon_X$ (where Υ_X is the indicator vector of X).*

Letting A_n and B_n be the adjacency and cell matrices of graph G_n with respect to partition π_n, the proof of this theorem is by induction.

Base Case $n = 1$. In this trivial case,

$$M_1^* = \begin{array}{c} 1 \\ \begin{bmatrix} 1 \end{bmatrix} \end{array}.$$

Then $M[M_1^*]$ has no circuits and one element. Consider the graph G_1 having two vertices connected by a single edge e. Let π_1 be the edge partition $\pi = (E_1)$ where $e \in E_1$. It is easy to check that G_1 and π_1 satisfy (1.)-(5.). Moreover, G_1 has adjacency and cell matrices A_1 and B_1 given by:

$$A_1 = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \qquad B_1 = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}.$$

Base Case $n = 2$. Matrix M_2^* can be written as

$$M_2^* = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{c|c|c} M_1^* & 0 & M_1^* \\ 0 & 1 & 1 \end{array}\right] \end{array}.$$

Consider the graph G_2, with simple edge-partition π_2, as illustrated in Figure B.2. One may check that G_2 and π_2 satisfy (1.)-(5.) of the theorem.

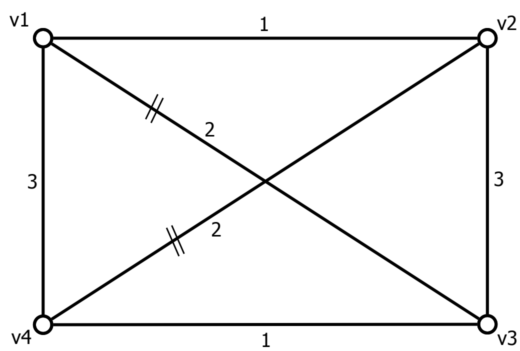


Figure B.2: A graph G_2 with edge-partition π_2 that is strongly $\mathbb{Z}_2$-Kirchhoff with respect to $M[M_2^*]$.

Graph G_2 and edge-partition π_2 have adjacency and cell matrices A_2 and B_2, where

$$A_2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{array}\right] \end{array} \qquad B_2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ \hline 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right] \end{array}.$$

Letting I_n denote the $n \times n$ identity matrix, and J_n denote the $n \times n$ ones matrix, observe that:

$$A_2 = \left[\begin{array}{c|c} A_1 & A_1 + 2I_2 \\ \hline A_1 + 2I_2 & A_1 \end{array}\right] \qquad B_2 = \left[\begin{array}{c|c} B_1 & B_1 + 2J_2 \\ \hline B_1 + 2J_2 & B_1 \end{array}\right].$$

This observation is the key step to this inductive proof.

Inductive Step. Suppose the theorem is true in case n, for some graph G_n, with simple edge-partition π_n. Let A_n and B_n be the adjacency and cell matrices of G_n with respect to π_n. Then G_n is a graph on 2^n vertices, and π_n has $2^n - 1$ cells indexed by $\{1, \ldots, 2^n - 1\}$. Matrix $M^*_{(n+1)}$ can be written in the form

$$M^*_{(n+1)} = \begin{array}{c} \begin{array}{ccccccc} 1 & \cdots & 2^n-1 & 2^n & 1+2^n & \cdots & (2^n-1)+2^n \end{array} \\ \left[\begin{array}{ccc|c|ccc} & & & 0 & & & \\ & M^*_n & & \vdots & & M^*_n & \\ & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{array}\right] \end{array}. \tag{B.3}$$

Let G_{n+1} be a graph, with simple edge-partition π_{n+1}, defined defined by the following adjacency and cell matrices, A_{n+1} and B_{n+1}.

$$A_{n+1} = \begin{array}{c} \\ v_1 \\ \vdots \\ v_{2^n} \\ v_{(1+2^n)} \\ \vdots \\ v_{(2^n+2^n)} \end{array} \begin{array}{c} \begin{array}{cc} v_1 \cdots v_{2^n} & v_{(1+2^n)} \cdots v_{(2^n+2^n)} \end{array} \\ \left[\begin{array}{c|c} A_n & A_n + 2I_{2^n} \\ \hline A_n + 2I_{2^n} & A_n \end{array}\right] \end{array}$$

$$B_{n+1} = \begin{array}{c} \\ v_1 \\ \vdots \\ v_{2^n} \\ v_{(1+2^n)} \\ \vdots \\ v_{(2^n+2^n)} \end{array} \begin{array}{c} \begin{array}{cc} v_1 \cdots v_{2^n} & v_{(1+2^n)} \cdots v_{(2^n+2^n)} \end{array} \\ \left[\begin{array}{c|c} B_n & B_n + 2^n J_{2^n} \\ \hline B_n + 2^n J_{2^n} & B_n \end{array}\right] \end{array}$$

Note that both A_{n+1} and B_{n+1} are symmetric and integer valued, thereby giving a well-defined π_{n+1}-simple graph G_{n+1}. We will demonstrate that G_{n+1} and π_{n+1} satisfy each of (1.) - (5.).

(1.) G_{n+1} has 2^{n+1} vertices, each pair of which is connected by an edge. Edge-partition π_{n+1} has $2^{n+1}-1$ cells indexed by $\{1, 2, \ldots, 2^{n+1}-1\}$. Equivalently, it suffices to show that cell matrix B_{n+1} is $2^{n+1} \times 2^{n+1}$, with all off-diagonal entries nonzero, and containing each of the integers $1, 2, \ldots, 2^{n+1} - 1$ as an entry. As G_n and π_n satisfy (1.), matrix B_n is $2^n \times 2^n$, contains entries $0, 1, 2, \ldots, 2^n - 1$, and has zeros precisely on the main diagonal. Therefore the matrix $B_n + J_{2^n}$ has entries $2^n, 2^n + 1, \ldots 2^n + 2^n - 1$. By construction,

$$B_{n+1} = \left[\begin{array}{c|c} B_n & B_n + 2^n J_{2^n} \\ \hline B_n + 2^n J_{2^n} & B_n \end{array}\right] \tag{B.4}$$

and it is clear that B_{n+1} satisfies the desired conditions.

(2.) For any partition cell k and any vertex v of G_{n+1}, v is incident to an edge of cell k. Equivalently, it suffices to show that each row of B_{n+1} contains every index of

$\{1, 2, \ldots, 2^{n+1} - 1\}$. As G_n and π_n satisfy (2.), each row of B_n contains $0, 1, 2, \ldots, 2^n - 1$ as entries. Therefore each row of $B_n + J_{2^n}$ contains $2^n, 2^n + 1, \ldots 2^{n+1} - 1$ as entries. By construction of B_{n+1} (B.4), the desired condition holds. Note that although this condition follows easily by construction, it will become important in proving (5.).

Before checking that G_{n+1} and π_{n+1} satisfy (3.), we first make a few observations. First, by construction, for $1 \leq i, j \leq 2^n$, the entries of B_{n+1} satisfy:

$$\begin{aligned} B_{n+1}\left(v_i, v_{(j+2^n)}\right) &= B_{n+1}\left(v_i, v_j\right) + 2^n \\ B_{n+1}\left(v_{(i+2^n)}, v_j\right) &= B_{n+1}\left(v_i, v_j\right) + 2^n \\ B_{n+1}\left(v_{(i+2^n)}, v_{(j+2^n)}\right) &= B_{n+1}\left(v_i, v_j\right). \end{aligned} \tag{B.5}$$

Moreover, observe that the vertices of G_{n+1} fall into a natural bipartition,

$$U = \{v_1, \ldots, v_{2^n}\} \qquad W = \left\{v_{(1+2^n)}, \ldots, v_{2^{n+1}}\right\}.$$

For any $X \subseteq V(G_{n+1})$, let $G_{n+1}[X]$ denote the induced graph X, with edge-partition induced by π_{n+1}. That is, $G_{n+1}[X]$ has vertices X and all edges of G_{n+1} with both endpoints contained in X. The edges of $G_{n+1}[X]$ can be partitioned according to the cells of π_{n+1}. Then by construction, observe that

$$G_{n+1}[U] \cong G_n \quad \text{and} \quad G_{n+1}[W] \cong G_n. \tag{B.6}$$

Proposition B.1. Every edge of G_{n+1} in a partition cell of index $\{1, \ldots, 2^n - 1\}$ lies in either $G[U]$ or $G[W]$. Any edge partitioned into a cell of index $\{2^n, 1 + 2^n, \ldots 2^{n+1} - 1\}$ has one endpoint in U and the other in W. We will call such edges UW edges. $\square$

(3.) π_{n+1} **is $\mathbb{Z}_2$-Kirchhoff.** The proof of this condition will require a number of intermediate results.

Proposition B.2. Let v be any vertex of G_{n+1}. Then the π_{n+1}-incidence vector of v satisfies:

$$\lambda_{\pi_{n+1}}(v) \equiv \begin{array}{c} \scriptstyle 1 \;\cdots\; 2^n-1 \qquad\quad 2^n \qquad\quad 1+2^n \;\cdots\; (2^n-1)+2^n \\ \left[\quad \lambda_{\pi_n}(\widetilde{v}) \quad \middle| \; 0 \; \middle| \quad \lambda_{\pi_n}(\widetilde{v}) \quad \right] \end{array} \pmod 2, \tag{B.7}$$

where $\lambda_{\pi_n}(\widetilde{v})$ is the π_n-incidence vector of some vertex $\widetilde{v}$ of G_n.

Proof. Recall that π_{n+1}-incidence vectors can be computed from the rows of matrices A_{n+1} and B_{n+1}. Observe that by (B.4) any entry of 2^n in B_{n+1} arises from a 0 entry of B_n. Since B_n satisfies (1.), these are precisely the entries

$$B_{n+1}(v_i, v_{(i+2^n)}) \text{ and } B_{n+1}(v_{(i+2^n)}, v_i)$$

for $1 \le i \le 2^n$. By construction, adjacency matrices A_n and A_{n+1} satisfy

$$A_{n+1} = \left[\begin{array}{c|c} A_n & A_n + 2I_{2^n} \\ \hline A_n + 2I_{2^n} & A_n \end{array}\right]. \tag{B.8}$$

Because $A_{n+1}(v_i, v_{(i+2^n)}) = A_{n+1}(v_{(i+2^n)}, v_i) = 2$, edges in cell 2^n always occur in pairs. Therefore by (B.1), entry 2^n of $\lambda_{\pi_{n+1}}(v)$ is 0 (mod 2). The remaining entries of (B.7) follow directly from (B.5) and (B.8). □

Proposition B.3. Every cycle C of G_{n+1} is of one of the following three types:

Type 1. All edges of C lie in $G_{n+1}[U]$.

Type 2. All edges of C lie in $G_{n+1}[W]$.

Type 3. C may be written as a sequence of the following form:

$e_1 \cdot$(Path in $G_{n+1}[U]$)$\cdot e_2 \cdot$(Path in $G_{n+1}[W]$)$\cdot e_3 \cdot$(Path in $G_{n+1}[U]$)$\cdot e_4 \cdots$(Path in $G_{n+1}[W]$)

where each e_i is a UW edge and all paths in $G_{n+1}[U]$ (respectively $G_{n+1}[W]$) are vertex-disjoint from each other, and each may be the trivial path on a single vertex. □

It suffices to prove that for any cycle C of each type in Proposition B.3, and any vertex v of G_{n+1},

$$\lambda_{\pi_{n+1}}(v) \cdot \chi_{\pi_{n+1}}(C) \equiv 0 \text{ (mod 2)}. \tag{B.9}$$

Type 1 and 2. Let C be any cycle of G_{n+1} of Type 1. Then since C is contained in $G_{n+1}[U]$, and $G_{n+1}[U] \cong G_n$, the partition cells of edges of C are identical to those of some cycle $\widetilde{C}$ in G_n. That is,

$$\chi_{\pi_{n+1}}(C) = \begin{array}{c} \scriptstyle 1 \ \cdots \ 2^n-1 \qquad 2^n \qquad 1+2^n \cdots (2^n-1)+2^n \\ \left[\begin{array}{c|c|ccc} \chi_{\pi_n}(\widetilde{C}) & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

For any vertex v of G_{n+1}, by Proposition B.2,

$$\lambda_{\pi_{n+1}}(v) = \begin{array}{c} \scriptstyle 1 \ \cdots \ 2^n-1 \qquad 2^n \qquad 1+2^n \cdots (2^n-1)+2^n \\ \left[\begin{array}{c|c|c} \lambda_{\pi_n}(\widetilde{v}) & 0 & \lambda_{\pi_n}(\widetilde{v}) \end{array}\right] \end{array}$$

where $\lambda_{\pi_n}(\widetilde{v})$ is the π_n-incidence vector of some vertex $\widetilde{v}$ of G_n. Then

$$\begin{aligned} \lambda_{\pi_{n+1}}(v) \cdot \chi_{\pi_{n+1}}(C) &\equiv \lambda_{\pi_n}(\widetilde{v}) \cdot \chi_{\pi_n}(\widetilde{C}) + 0 \cdot 0 + \lambda_{\pi_n}(\widetilde{v}) \cdot [0 \cdots 0] \\ &\equiv \lambda_{\pi_n}(\widetilde{v}) \cdot \chi_{\pi_n}(\widetilde{C}) \\ &\equiv 0 \text{ (mod 2)} \end{aligned}$$

since π_n is $\mathbb{Z}_2$-Kirchhoff. Thus for any cycle C of Type 1, and any vertex v of G_{n+1},

$$\lambda_{\pi_{n+1}}(v) \cdot \chi_{\pi_{n+1}}(C) \equiv 0 \pmod{2}.$$

Because $G_{n+1}[W] \cong G_n$ as well, the proof for cycles of Type 2 is analogous and omitted here.

Type 3. All that remains is to verify (B.9) for any cycle of Type 3. This will first require a few lemmas.

Lemma B.1. *Let C be any cycle of Type 3. Then the edge set of C can be written as the symmetric difference of a collection of edge sets of cycles of $G_{n+1}[U]$, cycles of $G_{n+1}[W]$, and triangles containing two UW edges and either one edge of $G_{n+1}[U]$ or one edge of $G_{n+1}[W]$.*

Proof. Let C be any cycle of Type 3. Then C can be written as a sequence of the form:

$$(e_1)U_1(e_2)W_1(e_3)U_2(e_4)W_2(e5)\cdots W_{k/2} \text{ (for some even } k) \tag{B.10}$$

Where each e_i is a UW edge with endpoints $u_i \in G_{n+1}[U]$ and $w_i \in G_{n+1}[W]$. Each U_j is a path in $G_{n+1}[U]$ with terminal vertices u_{2j-1} and u_{2j}, and all paths U_j are vertex-disjoint from each other. Similarly, each W_l is a path in $G_{n+1}[W]$ with terminal vertices w_{2l} and w_{2l+1} (where $w_{k+1} = w_1$), and all paths W_l are vertex-disjoint from each other.

Remark B.1. For each j, u_{2j-1} and u_{2j} need not be distinct; if not, U_j is the trivial path on one vertex. Similarly, for each l, w_{2l} and u_{2l+1} need not be distinct; if not, W_l is the trivial path on one vertex.

Recalling that all vertices of G_{n+1} are connected by an edge, let $\{v_i, v_j\}$ denote the edge between vertices v_i and v_j. Then (B.10) can be rewritten as

$$\{w_1, u_1\}U_1\{u_2, w_2\}W_1\{w_3, u_3\}U_2\{u_4, w_4\}W_2 \cdots W_{k/2}.$$

Moreover, given the labeling described above, observe that:

$$C_1 = \{w_1, w_2\}W_1\{w_3, w_4\}W_2 \cdots \{w_{k-1}, w_k\}W_{k/2}$$

traces out a cycle in $G_{n+1}[W]$. Next, let $\mathcal{C}_2$ be the collection of cycles of $G_{n+1}[U]$ of the form

$$\mathcal{C}_2 = \big\{U_{(j+1)/2} \cdot \{u_{j+1}, u_j\} : j \text{ is odd and } U_{(j+1)/2} \text{ is not the trivial path}\big\}.$$

Let $\mathcal{C}_3$ be the collection of cycles

$$\mathcal{C}_3 = \big\{e_j \cdot \{u_j, u_{j+1}\} \cdot \{u_{j+1}, w_j\} : j \text{ is odd and } U_{(j+1)/2} \text{ is not the trivial path}\big\}.$$

Note the restriction that $U_{(j+1)/2}$ is not the trivial path guarantees that each closed walk in $\mathcal{C}_3$ is a triangle: if $U_{(j+1)/2}$ were the path on a single point, then $u_j = u_{j+1}$. Moreover, observe that for each odd j, both e_j and $\{u_{j+1}, w_j\}$ are UW edges, and $\{u_j, u_{j+1}\}$ is an edge in $G_{n+1}[U]$.

Finally, let $\mathcal{C}_4$ be the collection of cycles

$$\mathcal{C}_4 = \Big\{\{w_j, u_{j+1}\} \cdot e_{j+1} \cdot \{w_{j+1}, w_j\} : \ j \text{ is odd}\Big\}.$$

Each closed walk in $\mathcal{C}_4$ is a triangle in G with one edge in $G[W]$ ($\{w_{j+1}, w_j\}$) and two UW edges (e_{j+1} and $\{w_j, u_{j+1}\}$). Letting Δ denote symmetric difference of sets, observe that for each odd j,

$$\Big\{e_j, \{u_j, u_{j+1}\}, \{u_{j+1}, w_j\}\Big\}\Delta\Big\{\{w_j, u_{j+1}\}, e_{j+1}, \{w_{j+1}, w_j\}\Big\}$$
$$\Delta\Big\{E(U_{(j+1)/2}), \{u_{j+1}, u_j\}\Big\} = \Big\{e_j, e_{j+1}, E(U_{(j+1)/2}), \{w_{j+1}, w_j\}\Big\}.$$

Moreover, for each odd j,

$$\{w_{j+1}, w_j\}, E(W_{(j+1)/2}) \in E(C_1).$$

Finally, recall that by construction

$$E(C) = \{e_1, e_2, \dots\} \cup \{E(U_1), E(U_2), \dots\} \cup \{E(W_1), E(W_2), \dots\}.$$

Therefore letting $E(\mathcal{C}_i)$ denote the set of edges that occur in cycles contained in collection $\mathcal{C}_i$, it follows that:

$$E(C) = E(C_1)\Delta E(\mathcal{C}_2)\Delta E(\mathcal{C}_3)\Delta E(\mathcal{C}_4).$$

□

Example B.3. Figure B.3 illustrates a cycle of Type 3. Note that not all vertices of G are depicted, only those contained in the cycle.

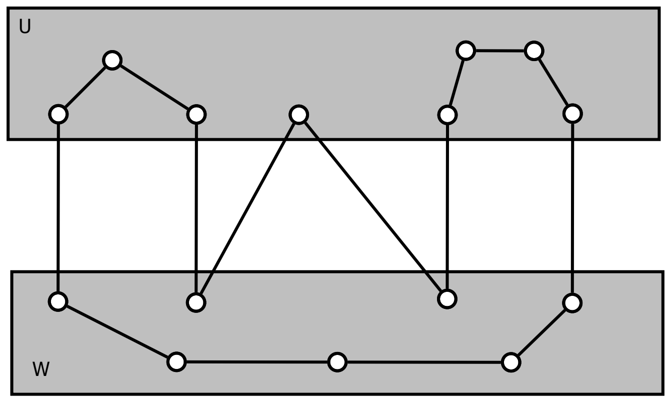


Figure B.3: A cycle of Type 3

Letting the left-most vertex (as drawn in Figure B.3) of W be w_1, this cycle is labeled as depicted in Figure B.4.

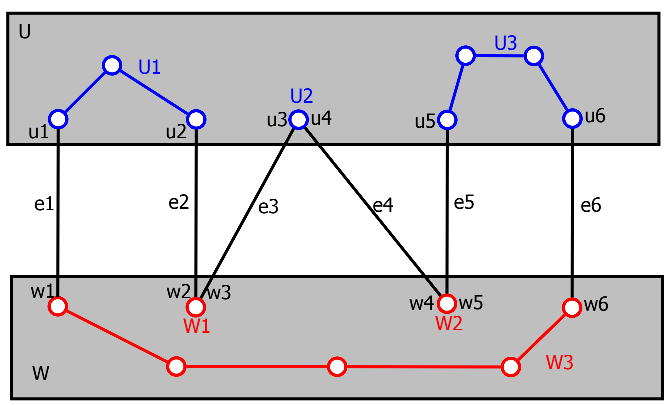


Figure B.4: Labeling a cycle of Type 3.

First, construct cycle C_1 in W. In Figure B.5, C_1 is outlined in red. The family $\mathcal{C}_2$ of cycles in U is outlined in blue. Finally we construct two families of triangles: $\mathcal{C}_3$ is outlined in green, while $\mathcal{C}_4$ is outlined in purple.

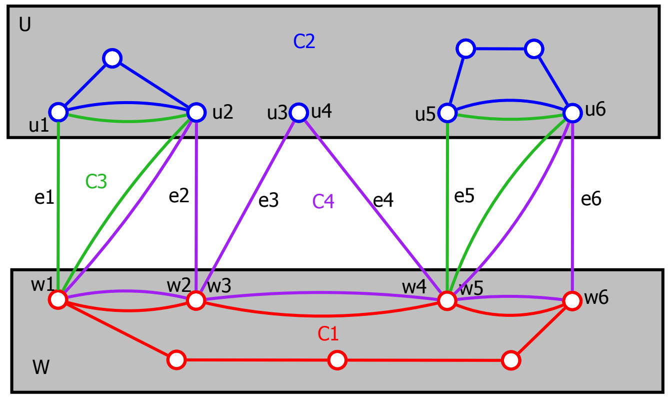


Figure B.5: Decomposing a cycle of Type 3 into 4 families of cycles. C_1 is outlined in red. The family $\mathcal{C}_2$ of cycles in U is outlined in blue. The two families of triangles, $\mathcal{C}_3$ and $\mathcal{C}_4$, are outlined in green and purple, respectively.

Observe that taking the union of all edges in these 4 families, each edge of C appears exactly once, while each edge not contained in C appears exactly twice. Therefore as desired,

$$E(C) = E(C_1)\Delta E(\mathcal{C}_2)\Delta E(\mathcal{C}_3)\Delta(\mathcal{C}_4).$$

Lemma B.2. *Let C be any triangle of G_{n+1} containing 2 UW edges, and either one edge of $G_{n+1}[U]$ or one edge of $G_{n+1}[W]$. Then for any vertex v of G_{n+1},*

$$\lambda_{\pi_{n+1}}(v) \cdot \chi_{\pi_{n+1}}(C) \equiv 0 \text{ (mod 2).}$$

Proof. Let C be any triangle of G_{n+1} with two UW edges and one edge of $G_{n+1}[U]$. Then C has vertices

$$V(C) = v_i, v_j, v_{k+2^n} \text{ for some } 1 \leq i, j, k \leq 2^n,$$

and thus the partition cells of the edges of C are

$$B_{n+1}(v_i, v_j), B_{n+1}(v_j, v_{k+2^n}), B_{n+1}(v_{k+2^n}, v_i).$$

Case 1: i, j, k are distinct. Then v_i, v_j, v_k are the vertices of a triangle $\widetilde{C}$ in $G_{n+1}[U] \cong G_n$. The partition cells of the edges in $\widetilde{C}$ are:

$$B_{n+1}(v_i, v_j), B_{n+1}(v_j, v_k), B_{n+1}(v_k, v_i).$$

But recall that by construction of B_{n+1} (B.5),

$$B_{n+1}(v_j, v_{k+2^n}) = B_{n+1}(v_j, v_k) + 2^n \text{ and } B_{n+1}(v_{k+2^n}, v_i) = B_{n+1}(v_k, v_i) + 2^n.$$

Therefore the π_{n+1}-cycle vector of C can be written as

$$\chi_{\pi_{n+1}}(C) \equiv \begin{array}{c} {\scriptstyle 1 \;\cdots\; 2^n-1 \qquad\quad 2^n \qquad 1+2^n \cdots (2^n-1)+2^n} \\ \left[\begin{array}{c|c|c} \quad\chi_S(C)\quad & 0 & \quad\chi_T(C)\quad \end{array}\right] \end{array} \pmod 2,$$

where $\chi_S(C)$ and $\chi_T(C)$ are binary vectors such that

$$\chi_S(C) + \chi_T(C) \equiv \chi_{\pi_n}(\widetilde{C}) \pmod 2.$$

Let v be any vertex of G_{n+1}. Then by Proposition B.2,

$$\lambda_{\pi_{n+1}}(v) = \begin{array}{c} {\scriptstyle 1 \;\cdots\; 2^n-1 \qquad\quad 2^n \qquad 1+2^n \cdots (2^n-1)+2^n} \\ \left[\begin{array}{c|c|c} \quad\lambda_{\pi_n}(\widetilde{v})\quad & 0 & \quad\lambda_{\pi_n}(\widetilde{v})\quad \end{array}\right] \end{array}$$

where $\lambda_{\pi_n}(\widetilde{v})$ is the π_n-incidence vector of some vertex $\widetilde{v}$ of G_n. Then

$$\begin{aligned} \lambda_{\pi_{n+1}}(v) \cdot \chi_{\pi_{n+1}}(C) &\equiv \lambda_{\pi_n}(\widetilde{v}) \cdot \chi_S(C) + 0 \cdot 0 + \lambda_{\pi_n}(\widetilde{v}) \cdot \chi_T(C) \\ &\equiv \lambda_{\pi_n}(\widetilde{v}) \cdot [\chi_S(C) + \chi_T(C)] \\ &\equiv \lambda_{\pi_n}(\widetilde{v}) \cdot \chi_{\pi_n}(\widetilde{C}) \\ &\equiv 0 \pmod 2 \text{ as } \pi_n \text{ is } \mathbb{Z}_2\text{-Kirchhoff.} \end{aligned}$$

Case 2: i, j, k are not distinct. For C to have the given form, it must have vertices

$$V(C) = v_i, v_j, v_{j+2^n} \text{ for some } 1 \le i \ne j \le 2^n.$$

Then the partition cells of C are

$$B_{n+1}(v_i, v_j), B_{n+1}(v_j, v_{j+2^n}), B_{n+1}(v_{j+2^n}, v_i).$$

However by (B.5),

$$B_{n+1}(v_j, v_{j+2^n}) = 2^n \text{ and } B_{n+1}(v_{j+2^n}, v_i) = B_{n+1}(v_i, v_j) + 2^n$$

Therefore C has binary π_{n+1}-cycle vector

$$\chi_{\pi_{n+1}}(C) \equiv \begin{array}{c} {\scriptstyle 1 \;\cdots\; 2^n-1 \qquad\quad 2^n \qquad 1+2^n \cdots (2^n-1)+2^n} \\ \left[\begin{array}{c|c|c} \quad X\quad & 1 & \quad X\quad \end{array}\right] \end{array} \pmod 2$$

for some binary vector X of size $2^n - 1$. Now let v be any vertex of G. Again, for some $\widetilde{v}$ of G_n,

$$\lambda_{\pi_{n+1}}(v) = \begin{array}{c} \begin{array}{ccc} 1 \cdots 2^n-1 & 2^n & 1+2^n \cdots (2^n-1)+2^n \end{array} \\ \left[\begin{array}{c|c|c} \quad\lambda_{\pi_n}(\widetilde{v})\quad & 0 & \quad\lambda_{\pi_n}(\widetilde{v})\quad \end{array} \right] \end{array}.$$

Therefore,

$$\begin{aligned} \lambda_{\pi_{n+1}}(v) \cdot \chi_{\pi_{n+1}}(C) &\equiv \lambda_{\pi_n}(\widetilde{v}) \cdot X + 0 \cdot 1 + \lambda_{\pi_n}(\widetilde{v}) \cdot X \\ &\equiv 2\lambda_{\pi_n}(\widetilde{v}) \cdot X \\ &\equiv 0 \pmod 2. \end{aligned}$$

Therefore for any triangle C of G_{n+1} with two UW edges and one edge of $G_{n+1}[U]$,

$$\lambda_{\pi_{n+1}}(v) \cdot \chi_{\pi_{n+1}}(C) \equiv 0 \pmod 2$$

for all vertices v of G_{n+1}. The case when C has one edge of $G_{n+1}[W]$ is analogous, and so the proof is omitted here. □

Now let C be any cycle of Type 3. Then by Lemma B.1, $E(C)$ can be written as

$$E = E(C_1)\Delta E(\mathcal{C}_2)\Delta E(\mathcal{C}_3)\Delta E(\mathcal{C}_4),$$

where C_1 is a cycle in $G_{n+1}[W]$, $\mathcal{C}_2$ a family of cycles in $G_{n+1}[U]$, and $\mathcal{C}_3$ and $\mathcal{C}_4$ are families of triangles containing two UW edges and one edge of $G_{n+1}[U]$ or $G_{n+1}[W]$ respectively. Therefore,

$$\chi_{\pi_{n+1}}(C) \equiv \chi_{\pi_{n+1}}(C_1) + \sum_{C_i \in \mathcal{C}_2} \chi_{\pi_{n+1}}(C_i) + \sum_{C_j \in \mathcal{C}_3} \chi_{\pi_{n+1}}(C_j) + \sum_{C_k \in \mathcal{C}_4} \chi_{\pi_{n+1}}(C_k) \pmod 2. \quad \text{(B.11)}$$

Thus for any vertex v of G_{n+1},

$$\begin{aligned} \chi_{\pi_{n+1}}(C) \cdot \lambda_{\pi_{n+1}}(v) \equiv\ & \chi_{\pi_{n+1}}(C_1) \cdot \lambda_{\pi_{n+1}}(v) + \sum_{C_i \in \mathcal{C}_2} \chi_{\pi_{n+1}}(C_i) \cdot \lambda_{\pi_{n+1}}(v) \\ & + \sum_{C_j \in \mathcal{C}_3} \chi_{\pi_{n+1}}(C_j) \cdot \lambda_{\pi_{n+1}}(v) + \sum_{C_k \in \mathcal{C}_4} \chi_{\pi_{n+1}}(C_k) \cdot \lambda_{\pi_{n+1}}(v) \pmod 2. \end{aligned} \quad \text{(B.12)}$$

However C_1 is a cycle of Type 2, and every cycle $C_i \in \mathcal{C}_2$ is a cycle of Type 1, so

$$\chi_{\pi_{n+1}}(C_1) \cdot \lambda_{\pi_{n+1}}(v) \equiv 0 \pmod 2 \quad \text{and} \quad \chi_{\pi_{n+1}}(C_i) \cdot \lambda_{\pi_{n+1}}(v) \equiv 0 \pmod 2.$$

Finally, by Lemma B.2, for any $C_j \in \mathcal{C}_3$ or $C_k \in \mathcal{C}_4$,

$$\chi_{\pi_{n+1}}(C_j) \cdot \lambda_{\pi_{n+1}}(v) \equiv 0 \pmod 2 \quad \text{and} \quad \chi_{\pi_{n+1}}(C_k) \cdot \lambda_{\pi_{n+1}}(v) \equiv 0 \pmod 2.$$

Therefore by (B.12),

$$\chi_{\pi_{n+1}}(C) \cdot \lambda_{\pi_{n+1}}(v) \equiv 0 \pmod 2.$$

As a result, for any vertex v of G and any cycle C of Type 1, 2, or 3,

$$\chi_{\pi_{n+1}}(C) \cdot \lambda_{\pi_{n+1}}(v) \equiv 0 \pmod 2.$$

As all cycles of G_{n+1} are of one of these three types, this completes the proof that π_{n+1} is $\mathbb{Z}_2$-Kirchhoff.

Before proving (4.) and (5.) we make a few additional observations. Recall that by induction, G_n satisfies (4.) and (5.), and moreover that M^*_{n+1} is written as

$$M^*_{(n+1)} = \begin{array}{c} \begin{array}{ccccccc} 1 & \cdots & 2^n-1 & 2^n & 1+2^n & \cdots & (2^n-1)+2^n \end{array} \\ \left[\begin{array}{ccc|c|ccc} & & & 0 & & & \\ & M^*_n & & \vdots & & M^*_n & \\ & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{array}\right] \end{array}. \tag{B.13}$$

Proposition B.4. If $i > 2^n$, by (B.13) columns i and $i - 2^n$ of M^*_{n+1} are equal in all except row $n + 1$. In row $n + 1$, column i has a 1 whereas column $i - 2^n$ has a 0. □

Corollary B.1. *Let $\boldsymbol{m}_i$ denote the i^{th} column of matrix M^*_{n+1}. If $i < 2^n$,*

$$\boldsymbol{m}_i + \boldsymbol{m}_{2^n} = \boldsymbol{m}_{i+2^n}.$$

If $i, j < 2^n$,

$$\boldsymbol{m}_i + \boldsymbol{m}_j = \boldsymbol{m}_{i+2^n} + \boldsymbol{m}_{j+2^n}.$$

For ease of notation, $\Phi(i, j)$ be the index of the partition cell containing edge $\{v_i, v_j\}$. That is,

$$\Phi(i, j) := \varphi(\{v_i, v_j\}) \equiv B_{n+1}(v_i, v_j)$$

Observe that by (B.5), for $1 \le i, j \le 2^n$,

$$\begin{aligned} \Phi(i, j + 2^n) &= \Phi(i, j) + 2^n \\ \Phi(i + 2^n, j) &= \Phi(i, j) + 2^n \\ \Phi(i + 2^n, j + 2^n) &= \Phi(i, j) \\ \Phi(i, i + 2^n) = \Phi(i + 2^n, i) &= 2^n. \end{aligned} \tag{B.14}$$

(4.) For every cycle C of G_n, $\chi_{\pi_n}(C)$ (mod 2) is the indicator of some dependent set of $M[M^*_n]$. It suffices to prove that for any cycle C of each type in Proposition B.3, there exists some dependent X of $M[M^*_{n+1}]$ such that

$$\chi_{\pi_{n+1}}(C) \equiv \Upsilon_X \pmod 2.$$

Let C be any cycle of Type 1. As previously, there exists some cycle $\widetilde{C}$ in G_n such that

$$\chi_{\pi_{n+1}}(C) = \begin{array}{c} \begin{array}{ccccccc} 1 & \cdots & 2^n-1 & 2^n & 1+2^n & \cdots & (2^n-1)+2^n \end{array} \\ \left[\begin{array}{c|c|ccc} \quad \chi_{\pi_n}(\widetilde{C}) \quad & 0 & 0 & \cdots & 0 \end{array} \right] \end{array}. \tag{B.15}$$

Since G_n satisfies (4.) for $M[M^*_n]$, there exists some dependent set X of $M[M^*_n]$ such that

$$\chi_{\pi_n}(\widetilde{C}) \equiv \Upsilon_X \pmod 2.$$

However by considering the first $2^n - 1$ columns of M^*_{n+1} in (B.13), it is clear any dependent set of $M[M^*_n]$ is dependent in $M[M^*_{n+1}]$ as well. Thus by (B.15), $\chi_{\pi_{n+1}}(C)$ is the indicator of a dependent set of M^*_{n+1}, and the result hold for cycles of Type 1. Because $G_{n+1}[W] \cong G_n$ as well, the proof for cycles of Type 2 is analogous and omitted here.

Finally, we must show that for any cycle of Type 3, there exists a dependent set X of $M[M^*_{n+1}]$ such that $\chi_{\pi_{n+1}}(C) \equiv \Upsilon_X \pmod 2$. As (4.) holds for all cycles of Type 1 or 2, by (B.11) it suffices to show that (4.) holds for any triangle T with two UW edges. Let T be any triangle of G_{n+1} with two UW edges and one edge of $G_{n+1}[U]$ (once again, the case when the third edge lies in $G_{n+1}[W]$ is analogous). Then T has vertices

$$V(T) = v_i, v_j, v_{k+2^n} \text{ for some } 1 \le i, j, k \le 2^n.$$

The partition cells of $E(T)$ are

$$\Phi(i,j), \Phi(j, k+2^n), \Phi(k+2^n, i).$$

Therefore, we must show that $\{\Phi(i,j), \Phi(j, k+2^n), \Phi(k+2^n, i)\}$ is a dependent set of $M[M^*_{n+1}]$. Recalling that $\boldsymbol{m}_i$ denotes the i^{th} column of matrix M^*_{n+1}, it suffices to show that

$$\boldsymbol{m}_{\Phi(i,j)} + \boldsymbol{m}_{\Phi(j,k+2^n)} + \boldsymbol{m}_{\Phi(k+2^n,i)} \equiv \mathbf{0} \pmod 2.$$

Case 1: i, j, k are distinct. Then v_i, v_j, v_k are the vertices of a triangle $\widetilde{T}$ in $G_{n+1}[U] \cong G_n$. Since G_n satisfies (4.), $\Phi(i,j), \Phi(j,k)$, and $\Phi(k,i)$ are a dependent set in $M[M^*_n]$. As $M[M^*_n]$ has no circuits of size 1 or 2, by (B.13) it follows that

$$\boldsymbol{m}_{\Phi(i,j)} + \boldsymbol{m}_{\Phi(j,k)} + \boldsymbol{m}_{\Phi(k,i)} \equiv \mathbf{0} \pmod 2.$$

Then by Corollary B.1,

$$\boldsymbol{m}_{\Phi(i,j)} + \boldsymbol{m}_{\Phi(j,k)+2^n} + \boldsymbol{m}_{\Phi(k,i)+2^n} \equiv \mathbf{0} \pmod 2.$$

But by (B.14),

$$\Phi(j,k) + 2^n = \Phi(j, k+2^n) \text{ and } \Phi(k,i) + 2^n = \Phi(k+2^n, i).$$

Therefore as desired,

$$\boldsymbol{m}_{\Phi(i,j)} + \boldsymbol{m}_{\Phi(j,k+2^n)} + \boldsymbol{m}_{\Phi(k+2^n,i)} \equiv \mathbf{0} \pmod 2.$$

Case 2: i, j, k are not distinct. For T to have the given form, it must have vertices

$$V(T) = v_i, v_j, v_{j+2^n} \text{ for some } 1 \le i, j \le 2^n,$$

and so the partition cells of $E(T)$ are

$$\Phi(i,j), \Phi(j, j+2^n), \Phi(j+2^n, i).$$

However by (B.14),

$$\Phi(j, j+2^n) = 2^n \text{ and } \Phi(j+2^n, i) = \Phi(i,j) + 2^n$$

Therefore by Corollary B.1,

$$\begin{aligned} \boldsymbol{m}_{\Phi(i,j)} + \boldsymbol{m}_{\Phi(j,j+2^n)} + \boldsymbol{m}_{\Phi(j+2^n,i)} &= \boldsymbol{m}_{\Phi(i,j)} + \boldsymbol{m}_{2^n} + \boldsymbol{m}_{\Phi(i,j)+2^n} \\ &= 2\boldsymbol{m}_{2^n} \\ &\equiv \mathbf{0} \pmod 2\text{, as desired.} \end{aligned}$$

This completes the proof that G_{n+1} satisfies (4.).

(5.) Let X be any circuit of $M[M_n^*]$ and k any matroid element contained in X. Then for any edge e in partition cell k, there exists a cycle C in G_n, containing e, such that $\chi_{\pi_n}(C) = \Upsilon_X$.

Proposition B.5. Every circuit X of $M[M_{n+1}^*]$ belongs to one of the following three classes:[1]

Class 1. $X \subseteq \{1, 2, \ldots, 2^n - 1\}$.

Class 2. $X \subseteq \{2^n, \ldots 2^{n+1} - 1\}$.

[1]The three circuit classes of Proposition B.5 are not correlated to the three cycle types of Proposition B.3.

Class 3. $X = Y \cup Z$ where $Y \subseteq \{1, 2, \ldots, 2^n - 1\}$, $Z \subseteq \{2^n, \ldots 2^{n+1} - 1\}$ and

$$\sum_{i \in Y} \boldsymbol{m}_i \equiv \sum_{j \in Z} \boldsymbol{m}_j \pmod 2.$$

Therefore it suffices to prove that (5.) holds for any circuit of each class in Proposition B.5.

Class 1. $X \subseteq \{1, 2, \ldots, 2^n - 1\}$. For any circuit X of class 1, X is also a circuit of $M[M_n^*]$. Let k be any element of X, and e any edge of G_{n+1} of cell k. As $k \in \{1, 2, \ldots, 2^n - 1\}$, by Proposition B.1 either $e \in G_{n+1}[U]$ or $e \in G_{n+1}[W]$. Without loss of generality, suppose $e \in G_{n+1}[U]$. By assumption, G_n satisfies (5.) for matrix M_n^*. Therefore, since $G_{n+1}[U] \cong G_n$, there exists a cycle C in $G_{n+1}[U]$, and hence in G_{n+1}, containing e such that $\chi_{\pi_{n+1}}(C) = \Upsilon_X$.

Class 2. $X \subseteq \{2^n, \ldots 2^{n+1} - 1\}$. Let X be any circuit of class 2, so $\sum_{j \in X} \boldsymbol{m}_j \equiv \mathbf{0} \pmod 2$. As columns $2^n, \ldots 2^{n+1} - 1$ of M_{n+1} all have a 1 in row $n+1$, X must have even cardinality m. Moreover, observe the following.

Proposition B.6. Suppose $X \subseteq \{2^n + 1, \ldots 2^{n+1} - 1\}$ is a circuit of $M[M_{n+1}^*]$ not containing 2^n. Then $\sum_{j \in X} \boldsymbol{m}_j \equiv \mathbf{0} \pmod 2$. However by Corollary B.1 it follows that

$$\sum_{j \in X} \boldsymbol{m}_{j - 2^n} \equiv \mathbf{0} \pmod 2.$$

Therefore $\{j - 2^n : j \in X\}$ is either a circuit of class 1 or a disjoint union of circuits of class 1. $\square$

This reduces class 2 into three cases.

Class 2a: $2^n \notin X$ and $\{j - 2^n : j \in X\}$ is a circuit of class 1. Let k be any element of X, and e any edge of G_{n+1} of cell k. As $k \in \{2^n + 1, \ldots, 2^{n+1} - 1\}$, by Proposition B.1 e must be a UW edge, say

$$e = \{v_{i_1}, v_{i_2 + 2^n}\}$$

for some $1 \leq i_1, i_2 \leq 2^n$. Then by (B.5), $\{v_{i_1}, v_{i_2}\}$ is an edge of G_{n+1} in partition cell $k - 2^n$. However, $\{i - 2^n : i \in X\}$ is a circuit of $M[M_{n+1}^*]$ of class 1 containing $k - 2^n$. Therefore there exists a cycle C in G_{n+1} containing $\{v_{i_1}, v_{i_2}\}$ such that $\chi(C)_{\pi_{n+1}} = \Upsilon_X$. By the proof of class 1, we may assume (without loss of generality) that C is contained in $G[U]$. That is there exist m distinct vertices $v_{i_1}, v_{i_2}, v_{i_3}, \ldots v_{i_m}$ (where $i_p \leq 2^n$ for all p) such that:

$$\Big\{\Phi(i_1, i_2), \Phi(i_2, i_3), \Phi(i_3, i_4), \ldots, \Phi(i_{m-1}, i_m), \Phi(i_m, i_1)\Big\} = \{j - 2^n : j \in X\}.$$

Recalling that $m = |X|$ is even, it follows from (B.14) that

$$\Big\{\Phi(i_1, i_2{+}2^n), \Phi(i_2{+}2^n, i_3), \Phi(i_3, i_4{+}2^n), \ldots, \Phi(i_{m-1}, i_m{+}2^n), \Phi(i_m{+}2^n, i_1))\Big\} = \{j : j \in X\}.$$

Therefore, letting C' be the cycle

$$C' = v_{i_1} \cdot v_{i_2+2^n} \cdot v_{i_3} \cdot v_{i_4+2^n} \cdots v_{i_{m-1}} \cdot v_{i_m+2^n} \cdot v_{i_1}$$

of G_{n+1}, C' contains edge e and

$$\chi_{\pi_{n+1}}(C') = \Upsilon_X.$$

Class 2b: $2^n \in X$**.** Then columns $\{j : j \in X \backslash 2^n\}$ of M^*_{n+1} must be a minimal set of columns summing to column 2^n. Moreover, it follows by Proposition B.4 that columns $\{j - 2^n : j \in X \backslash 2^n\}$ must be a minimally dependent set of columns. Therefore $\{j - 2^n : j \in X \backslash 2^n\}$ is a circuit of $M[M^*_{n+1}]$ of class 1. Let k be any element of X and e any edge of G_{n+1} in partition cell k. First suppose $k \neq 2^n$. As previously, e must be a UW edge, say $e = \{v_{i_1}, v_{i_2+2^n}\}$. Following as in Class 2a, there must exist $m-1$ distinct vertices $v_{i_1}, v_{i_2}, v_{i_3}, \ldots v_{i_{M-1}}$ (where $i_p \leq 2^n$ for all p) such that

$$\Big\{\Phi(i_1, i_2), \Phi(i_2, i_3), \ldots, \Phi(i_{m-2}, i_{m-1}, \Phi(i_{m-1}, i_1)\Big\} = \{j - 2^n : j \in X \backslash 2^n\}.$$

As $m - 1 = |X| - 1$ is odd, it follows from (B.14) that

$$\Big\{\Phi(i_1{+}2^n, i_1), \Phi(i_1, i_2{+}2^n), \Phi(i_2{+}2^n, i_3), \ldots, \Phi(i_{m-2}{+}2^n, i_{m-1}), \Phi(i_{m-1}, i_1{+}2^n)\Big\} = \{j : j \in X\}.$$

Therefore, letting C'' be the cycle

$$C'' = v_{i_1} \cdot v_{i_2+2^n} \cdot v_{i_3} \cdot v_{i_4+2^n} \cdots v_{i_{m-1}} \cdot v_{i_1+2^n} \cdot v_{i_1}$$

of G_{n+1}, C'' contains edge e and

$$\chi_{\pi_{n+1}}(C'') = \Upsilon_X.$$

Otherwise suppose $k = 2^n$ and let e be any edge of G_{n+1} of partition cell 2^n. Then $e = \{v_{i_1}, v_{i_1+2^n}\}$ for some $1 \leq i_1 \leq 2^n$. Let $j \in X \backslash 2^n$ be any element of circuit X other than 2^n. Then since G_{n+1} satisfies (2.), vertex v_{i_1} is incident to some edge $f = \{v_{i_1}, v_{i_2+2^n}\}$ of cell j. Proceeding as above, there exists a cycle

$$C''' = v_{i_1} \cdot v_{i_2+2^n} \cdot v_{i_3} \cdot v_{i_4+2^n} \cdots v_{i_{m-1}} \cdot v_{i_1+2^n} \cdot v_{i_1}$$

of G_{n+1}, containing edge e (and f) such that

$$\chi_{\pi_{n+1}}(C'') = \Upsilon_X.$$

Class 2c: $2^n \notin X$ and $\{j - 2^n : j \in X\}$ is a disjoint union of circuits of class 1. Lemma B.3 will be useful both in completing Class 2c here, and Class 3 which follows.

Lemma B.3. *Let X be a circuit of $M[M^*_{n+1}]$. Suppose there exists some x of $M[M^*_{n+1}]$ and a decomposition $X = X_1 \cup X_2$ such that both $X_1 \cup x$ and $X_2 \cup x$ are circuits of $M[M^*_{n+1}]$. If (5.) holds for both $X_1 \cup x$ and $X_2 \cup x$, then (5.) holds for X as well.*

Proof. Let k be any element of X; without loss of generality, assume that $k \in X_1$. Let $e = \{v_{i_1}, v_{i_2}\}$ be any edge of G_{n+1} in partition cell k. As $X_1 \cup x$ is a circuit of $M[M^*_{n+1}]$ for which (5.) holds, there is a cycle $C_1 = v_{i_1} \cdot v_{i_2} \cdots v_{i_{|X_1|+1}} \cdot v_{i_1}$ of size $|X_1| + 1$ containing e such that

$$\Big\{\Phi(i_1, i_2), \Phi(i_2, i_3), \ldots, \Phi(i_{|X_1|}, i_{|X_1|+1}), \Phi(i_{|X_1|+1}, i_1)\Big\} = X_1 \cup x.$$

In particular, some edge $f = \{v_{i_M}, v_{i_{M+1}}\}$ of C_1 is contained in the partition cell of index x. However, $X_2 \cup x$ is a circuit containing x for which (5.) holds. Then there exists a cycle $C_2 = v_{i_{M+1}} \cdot v_{i_M} \cdot v_{j_1} \cdot v_{j_2} \cdots v_{j_{|X_2|-1}} \cdot v_{i_{M+1}}$ of size $|X_2| + 1$ containing f such that

$$\Big\{\Phi(i_{M+1}, i_M), \Phi(i_M, j_1), \Phi(j_1, j_2), \ldots, \Phi(j_{|X_2|-2}, j_{|X_2|-1}), \Phi(j_{|X_2|-1}, i_{M+1})\Big\} = X_2 \cup x.$$

As a result,

$$C = v_{i_1} \cdot v_{i_2} \cdots v_{i_M} \cdot v_{j_1} \cdot v_{j_2} \cdots v_{j_{|X_2|-2}} \cdot v_{j_{|X_2|-1}} \cdot v_{i_{M+1}} \cdot v_{i_{M+2}} \cdots v_{i_{|X_1|+1}} \cdot v_{i_1}$$

is a closed walk in G_{n+1} of length $|X_1| + |X_2| = |X|$ such that

$$\Big\{\Phi(i_1, i_2), \Phi(i_2, i_3), \ldots, \Phi(i_M, j_1), \Phi(j_1, j_1), \ldots$$
$$\ldots \Phi(j_{|X_2|-1}, i_{M+1}), \Phi(i_{M+1}, i_{M+2}), \ldots, \Phi(i_{|X_1|+1}, i_1)\Big\} = X_1 \cup X_2 = X.$$

Let $E(C)$ denote the set of edges traversed in walk C, and assume for contradiction that C is not a cycle of G_{n+1}. Some proper subset of edges $F \subset E(C)$ must induce a cycle C_0 in G_{n+1}. Observe that the edges of C each belong to distinct cells of edge-partition π_{n+1}. Then

$$\chi(C_0) = \Upsilon_{\{\varphi(e):e \in F\}},$$

and $\{\varphi(e) : e \in F\}$ is a proper subset of X. However G_{n+1} satisfies (4.), meaning $\{\varphi(e) : e \in F\}$ is a dependent set of $M[M^*_{n+1}]$, contradicting that X is a circuit. Therefore C is a cycle of G_{n+1} containing edge e such that $\chi_{\pi_{n+1}}(C) = \Upsilon_X$. □

Returning to Class 2c, let Y be any subset of X such that $\{i - 2^n : i \in Y\}$ is a circuit of class 1. Then by Proposition B.6, $\sum_{i \in Y} \boldsymbol{m}_{i-2^n} \equiv \boldsymbol{0} \pmod 2$. However because X is a circuit of $M[M^*_{n+1}]$, $\sum_{i\in Y} \boldsymbol{m}_i \neq \boldsymbol{0}$ (mod 2). Therefore it must be that:

$$\sum_{i\in Y} \boldsymbol{m}_{i-2^n} = \boldsymbol{m}_{2^n} = [0, \cdots, 0, 1]^T$$

As Y was arbitrary, it follows that $\{j - 2^n : j \in X\}$ must be a disjoint union of exactly 2 circuits of class 1 (more or less than 2 would contradict that X is a circuit). Thus assume that $X = X_1 \cup X_2$ where $\{i - 2^n : i \in X_1\}$ and $\{l - 2^n : l \in X_2\}$ are both circuits of class 1. Moreover, $X_1 \cup 2^n$ and $X_2 \cup 2^n$ are both circuits of Class 2b. As (5.) holds for circuits of Class 2b, it also holds for Class 2c by Lemma B.3.

Class 3. $X = Y \cup Z$ where each $Y \subseteq \{1, 2, \ldots, 2^n - 1\}$, $Z \subseteq \{2^n, \ldots 2^{n+1} - 1\}$ and

$$\sum_{i\in Y} \boldsymbol{m}_i \equiv \sum_{j\in Z} \boldsymbol{m}_j \text{ (mod 2).}$$

First we demonstrate that (5.) holds in the case that $|Y| = 1$, and the general case of class 3 will then follow.

Proposition B.7. Let $X = y \cup Z$ be a circuit of $M[M^*_{n+1}]$ where $y \in \{1, 2, \ldots, 2^n - 1\}$, and $Z \subseteq \{2^n, \ldots, 2^{n+1} - 1\}$, so that $\sum_{j\in Z} \boldsymbol{m}_j = y$. Then (5.) holds for circuit X.

Proof. Circuits of this type can be broken down into four sub-classes.

Class 3a: $Z = \{2^n, y + 2^n\}$. That is, $X = \{y, 2^n, y + 2^n\}$. Any edge of partition cell y is either of the form $\{v_i, v_j\}$ or $\{v_{k+2^n}, v_{l+2^n}\}$ $(i, j, k, l \leq 2^n)$. In each case by (B.5), $v_i \cdot v_j \cdot v_{j+2^n} \cdot v_i$ or $v_{k+2^n} \cdot v_{l+2^n} \cdot v_l \cdot v_{k+2^n}$ are triangles with edges in the desired partition cells. Any edge of cell 2^n is of the form $\{v_i, v_{i+2^n}\}$. Since G_{n+1} satisfies (2.), v_i is incident to an edge of cell y, say $\varphi(\{v_i, v_j\}) = y$. Then $v_i \cdot v_j \cdot v_{i+2^n} \cdot v_i$ is the desired triangle. Finally any edge of cell $y+2^n$ is a UW edge, of the form $\{v_i, v_{j+2^n}\}$ $(i, j \leq 2^n)$. Once again, $v_i \cdot v_j \cdot v_{j+2^n} \cdot v_i$ is the desired triangle.

Class 3b: $Z = Z_1 \cup (y + 2^n)$, where $|Z_1| > 2$ and $\sum_{i\in Z_1} \boldsymbol{m}_i = \boldsymbol{m}_{2^n}$. Since X is a circuit, $2^n \notin Z_1$, and $Z_1 \cup 2^n$ is a circuit of Class 2. Moreover $\{y, y + 2^n, 2^n\}$ is a circuit of Class 3a, and $X = \{y, y + 2^n\} \cup Z_1$. Therefore by Lemma B.3, (5.) holds for circuits of Class 3b.

Class 3c: $Z = 2^n \cup Z_2$, where $|Z_2| > 1$ and $\sum_{j\in Z_2} \boldsymbol{m}_j = \boldsymbol{m}_{y+2^n}$. Since X is a circuit, $y + 2^n \notin Z_2$ and $Z_2 \cup (y + 2^n)$ is a circuit of Class 3b. Moreover $\{y, y + 2^n, 2^n\}$ is a circuit of Class 3a, and $X = \{y, 2^n\} \cup Z_2$. Therefore by Lemma B.3, (5.) holds for circuits of Class 3c.

Class 3d: $Z = Z_1 \cup Z_2$ **where** $|Z_1|, |Z_2| > 1$**,** $\sum_{i \in Z_1} \boldsymbol{m}_i = \boldsymbol{m}_{2^n}$**, and** $\sum_{j \in Z_2} \boldsymbol{m}_j = \boldsymbol{m}_{y+2^n}$**.** Combining the previous three cases, we see that $Z_1 \cup 2^n$ is a circuit of Class 3c and $\{y, y+2^n, 2^n\}$ is a circuit of Class 3a. By Lemma B.3, (5.) holds for $Z_1 \cup \{y, y+2^n\}$. However $Z_2 \cup y + 2^n$ is a circuit of Class 3b. Therefore again by Lemma B.3, (5.) holds for $Z_1 \cup Z_2 \cup y = Z \cup y = X$. □

Now let X be any circuit of Class 3. As X is a circuit of $M[M^*_{n+1}]$, it follows that $\sum_{i \in Y} \boldsymbol{m}_i \equiv \sum_{j \in Z} \boldsymbol{m}_j \neq 0 \pmod 2$. Moreover, because columns $1, 2, \ldots, 2^n - 1$ of M_{n+1} all have a zero in row $n+1$, it follows that

$$\sum_{i \in Y} \boldsymbol{m}_i \equiv \sum_{j \in Z} \boldsymbol{m}_j \equiv \boldsymbol{m}_P \pmod 2$$

For some $P \in \{1, 2, \ldots, 2^n - 1\} \backslash Y$. In particular, $Y \cup P$ must be a circuit of Class 1, and therefore (5.) holds. Moreover $Z \cup P$ is a circuit of the type addressed in Proposition B.7, and thus (5.) holds as well. Therefore by Lemma B.3, (5.) holds for $X = Y \cup Z$, and thus for any circuit of Class 3.

This completes the proof that (5.) holds for G_{n+1}, and thus the proof of Theorem B.2.

Example B.4. Figure B.2 illustrated a graph G_2, with simple edge-partition π_2, satisfying the conditions of Theorem B.2. In particular, π_2 is strongly $\mathbb{Z}_2$-Kirchhoff with respect to $M[M^*_2]$. Moreover, G_2 and π_2 have adjacency and cell matrices

$$A_2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{array}\right] \end{array} \qquad B_2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ \hline 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right] \end{array}.$$

Now let M^*_3 be the matrix

$$M^*_3 = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}.$$

Following the proof of Theorem B.2 we construct a graph G_3, with edge-partition π_3, that is strongly $\mathbb{Z}_2$-Kirchhoff with respect to $M[M^*_3]$. In particular, G_3 is the graph on 8 vertices with adjacency matrix A_3, and π_3 is the simple edge-partition of G_3 with cell matrix B_3 defined by

$$A_3 = \left[\begin{array}{c|c} A_2 & A_2 + 2I_4 \\ \hline A_2 + 2I_4 & A_2 \end{array}\right] \quad B_3 = \left[\begin{array}{c|c} B_2 & B_2 + 4J_4 \\ \hline B_2 + 4J_4 & B_2 \end{array}\right].$$

That is,

$$A_3 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 0 \end{bmatrix} \end{array} \quad B_3 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix} \end{array}.$$

Graph G_3, with edge-partition π_3, is illustrated in Figure B.6. By the proof of Theorem B.2, we are guaranteed that π_3 is strongly $\mathbb{Z}_2$-Kirchhoff with respect to $M[M_3^*]$.

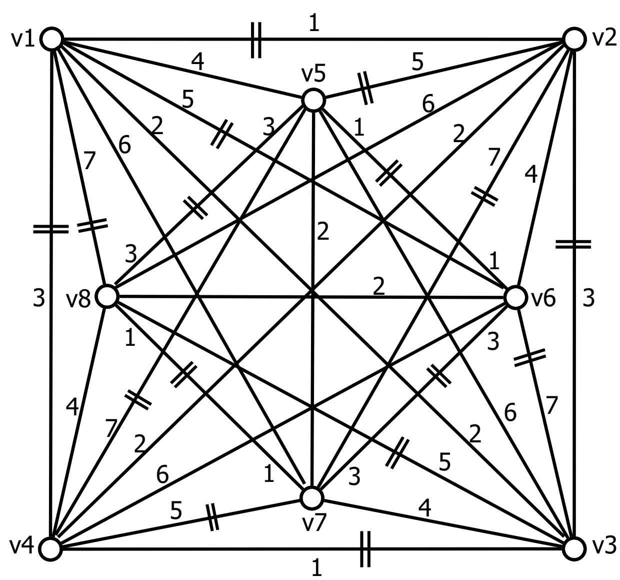


Figure B.6: The graph G_3, with edge-partition π_3, that is strongly $\mathbb{Z}_2$-Kirchhoff with respect to $M[M_3^*]$.

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