Kronecker Products and Closed Walks on Graphs

A Major Qualifying Project

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By

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Abstract

The goal of this project is to analyze families of small graphs with one or two loops at various vertices. We examine their adjacency matrices and Kronecker Products and determine their corresponding spectra. We describe features of the graphs and derive the number of closed walks. The degrees of each vertex, expressions for the spectra of each graph, and the correlating multiplicities of the eigenvalues were found through pattern recognition and implementation of the Binomial Theorem and a generating function.

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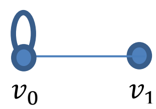
Chapter 1: Background

Matrices with entries reduced by a specified modulus have been the interest of many investigations. We consider graphs and their ***adjacency matrices***, which contain entries corresponding to the number of edges that connect two vertices of said graphs. An adjacency matrix of a simple graph is modulo 2. As our graphs are not simple, we reduce the adjacency matrices of the families of graphs to modulo 2 or 3 because it helps to identify patterns. See Figure 1. For example, 7 reduced to modulo 3 will equal 1. For our purposes, we refer to pseudo-graphs, or graphs with loops, as graphs.

The purpose of our work is to explore the properties of various graphs with the aim of finding formulas that calculate the total number of closed walks in a graph and the number of closed walks at a particular vertex. In pursuit of this objective, we mimic a paper written by Peter R. Christopher and J. W. Kennedy, *Binomial Graphs and their Spectra* (1997), while changing one key variable: the initial graph. We created many new graphs by manipulating their graph, B_1, whose graph and adjacency matrix can be seen in Figure 1; B_1 is an edge with a single loop at one vertex. We changed the number of vertices, varied the modulus, and created one or two additional loop(s) at various vertices, while keeping the graphs simple in all other respects.



$$A(B_1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Figure 1. Graph and Adjacency Matrix of B_1

We then examined the adjacency matrices of each of the respective graphs. We define an entry in an adjacency matrix to be the number of edges that connects vertex v_i to vertex v_j. Also, each loop contributes two to the total degree of a vertex. We determined the characteristic polynomials and the eigenvalues using MATLAB. A ***characteristic polynomial*** is an expression that when set equal to zero, the solutions are the eigenvalues of the adjacency matrix.

The simple programming code is displayed in Figure 2, shown below, where A is an example of an adjacency matrix.

```
syms x
A = sym([0, 1;  1, 0]);
eqn = charpoly(A, x) == 0
solx = solve(eqn, x)
```

Figure 2. MATLAB Code for Calculating the Characteristic Polynomial and its Solutions

From there, we calculated the Kronecker product of each adjacency matrix, using modulo 2 or modulo 3, depending on the matrix. If we take the Kronecker product of an $m \times n$ matrix, A, with a $p \times q$ matrix, B, the result would be an $mp \times nq$ matrix, denoted by $A \otimes B$. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$A \otimes B = \begin{bmatrix} aB & bB \\ cB & dB \end{bmatrix}.$$

Additionally, if the eigenvalues of A are λ_i where $i = 1, 2, \ldots, n$ and the eigenvalues of B are μ_j where $j = 1, 2, \ldots, k$, then the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$. We will denote a family of graphs as $Q_n = \{Q_1, Q_2, \ldots\}$, where n is one more than the number of Kronecker products taken.Suffice to say that Q_n is the graph with adjacency matrix $A(Q_1) \otimes A(Q_{n-1})$.

Given the adjacency matrices, we calculated the n^{th} Kronecker product of a graph, Q_n, by multiplying the matrix $A(Q_{n-1})$ with the initial adjacency matrix, $A(Q_1)$, as corresponding with [1]. Next, we attempted to find a relationship between these new matrices and Pascal's Triangle, the Fibonacci sequence, and other sequences with the help of the *Online Encyclopedia of Integer Sequences*.

Further, we examined if there was a correlation among a graph's eigenvalues with the next graph's eigenvalues in the family and tried to identify a pattern in the number of closed walks. We did this by finding the characteristic polynomial of the initial adjacency matrix and solving for its solutions, or eigenvalues, and their multiplicities. The multiplicity of an eigenvalue is the number of times an eigenvalue occurs as a solution to a characteristic polynomial. For example, the matrix

$$A(Z) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

has a characteristic polynomial of $\lambda^2 - 2\lambda$ and eigenvalues of $\lambda = 0$ and 2 with both values having a multiplicity of 1, or $m(0) = 1$ and $m(2) = 1$. Whereas, the matrix

$$A(Y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

has a characteristic polynomial of $\lambda^2 - 2\lambda + 1$ and eigenvalues of $\lambda = 1$ and 1 with $m(1) = 2$. From here, we determined a pattern for the eigenvalues of each adjacency matrix and their multiplicities. For the families with determinable eigenvalues and their corresponding multiplicities, we used a generating function to find the number of closed walks for said graph. A ***generating function*** is a power series where its coefficients denote an infinite sequence of numbers [2]. In our case, this infinite sequence is the number of closed walks of fixed length in a graph. If the eigenvalues were irreducible or we could not determine their multiplicities, we found an explicit formula for the number of closed walks. Similarly, we were able to formulate an equation which shows the number of closed walks at a particular vertex, v_0.

There are various fields of applications of graph theory including physics, civil engineering, chemistry, and data science. More specifically, the applications range from reducing traffic flow by developing road maps to creating a dynamic structure of molecules which help to accurately model atoms' locations.

Chapter 2: Introduction to Family C

For each non-negative integer n, we define C_n to have the vertex set $V_n = \{v_j : j = 0, 1, \dots, 2^n - 1\}$. The edge cardinality of C_n, or the number of edges, is $|E_C| = \frac{1}{4}(3^{n+1} + 1)$. Obviously, $|V_n| = 2^n$. We take $\binom{0}{0} = 1$. Thus, C_1 has two loops at v_0, but is otherwise a simple graph as shown in Figure 3.

$$A(C_1) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Figure 3. Graph and Adjacency Matrix of C_1

We used modulo 3 for $A(C_1)$ which has two vertices, as the use of modulo 2 would result in a graph too trivial to be of interest. We used modulo 2 for the adjacency matrices of the other six initial graphs, as they all have three or four vertices. However, as these other graphs have no multiple edges, meaning more than one edge incident to two vertices, or multiple loops on any one vertex, their matrices would contain only 0's and 1's no matter the modulus.

Also, for $n > 1$ and each $k = 1, \dots, n-1$, C_n has $\binom{n}{k}$ vertices of degree $3 \times 2^{k-1}$, a single vertex, v_{n-1}, of degree 1, and a single vertex, v_0, of degree $3 \times 2^{n-1} + 2$ if n is odd or $3 \times 2^{n-1} + 1$ if n is even. Thus, if n is even, the sum of the degrees of vertices in C_n is

$$1 + \sum_{k=1}^{n-1} \binom{n}{k} \times 3 \times 2^{k-1} + (3 \times 2^{n-1} + 1) = 2 + 3\sum_{k=1}^{n} \binom{n}{k} \times 2^{k-1},$$

$$= 2 - \frac{3}{2} + \frac{3}{2}\sum_{k=0}^{n} \binom{n}{k} \times 2^{k},$$

and one more if n is odd. The next step is easily shown through the Binomial Theorem,

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (x+1)^n,$$

which will equal

$$= \frac{1}{2}(3^{n+1} + 1).$$

Thus, as with all matrices, if we repeatedly take the Kronecker product of a matrix with itself, it "exhibits a self-similarity." Meaning, there is a pattern within each iteration of matrices where the matrix is mapped onto the original matrix. However, unlike any other family of graphs we will discuss, each entry in $A(C_n)$ is reduced to modulus three as mentioned above.

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 4. Adjacency Matrix of C_3

Thus, if we take $A(C_0) = [1]$, then, for each $n \geq 1$, the adjacency matrix of the family, C_n, reduced modulo three is

$$A(C_n) = \begin{bmatrix} 2A(C_{n-1}) & A(C_{n-1}) \\ A(C_{n-1}) & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(C_{n-1}) = A(C_1) \otimes A(C_{n-1}).$$

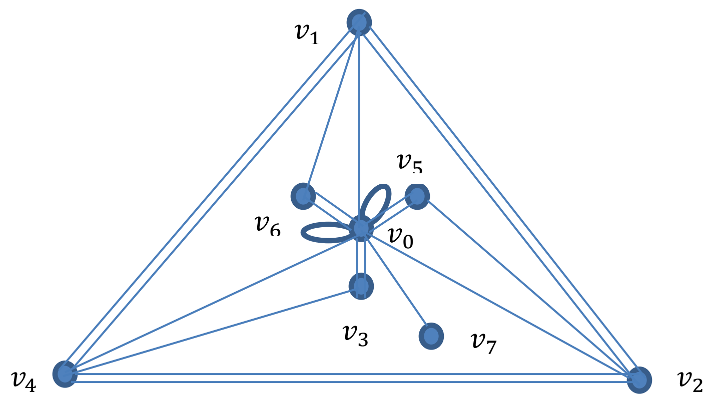


Figure 5. Graph of C_3

2.1 Spectra of Family C

Since we will take $A(C_n)$ modulo three, it is a bit more complicated than any of the other matrices we will discuss in this paper. As it happens, the set of eigenvalues, also called the spectrum of a matrix, has two distinct patterns; one when n is odd and one when n is even. Thus, we must consider these two separate cases. We denote the spectrum of a graph G as $\Lambda(G)$.

Theorem CI: When n is odd, the set of the eigenvalues at a given n is

$$\Lambda(C_n) = \{(-1)^{\frac{n+1}{2}+j}\left(a_j \pm b_j\sqrt{2}\right)\} \; where \; j = 1, \ldots, \frac{n+1}{2}.$$

The multiplicities of the eigenvalues for odd n are as follows:

$$m(\lambda_{odd\ n}) = \binom{n}{p} \; where \; p = \frac{n-1}{2} + j.$$

The constants, a_j and b_j, have a recursive relationship with each other; the succeeding term in the sequence relies on the preceding terms.

$$a_{j+1} = 3a_j + 4b_j \; where \; a_1 = 1 \; and \; b_1 = 1,$$

$$b_{j+1} = 2a_j + 3b_j \; where \; a_1 = 1 \; and \; b_1 = 1.$$

For example,

$$a_2 = 3a_1 + 4b_1, \qquad b_2 = 2a_1 + 3b_1,$$

$$a_2 = 3 + 4 = 7. \qquad b_2 = 2 + 3 = 5.$$

$$m(-1 \pm \sqrt{2}) = \binom{3}{2} = 3,$$

$$m(7 \pm 5\sqrt{2}) = \binom{3}{3} = 1.$$

Thus,

$$\Lambda(C_3) = \left\{(-1)^{\frac{3+1}{2}+j} a_j \pm b_j\sqrt{2}\right\} \; where \; j = 1, 2,$$

$$= \left\{-1 \pm \sqrt{2}^{(3)}, 7 \pm 5\sqrt{2}^{(1)}\right\}.$$

These equations can be written explicitly, where we find the value of each constant given the number of Kronecker product iterations (i.e. without relying on the preceding terms).

Lemma CI: The explicit forms of a_j and b_j for the eigenvalues of $A(C_n)$ for odd n are:

$$a_j = -\left(\frac{1-\sqrt{2}}{2}\right)\left(3+2\sqrt{2}\right)^j - \left(\frac{1+\sqrt{2}}{2}\right)\left(3-2\sqrt{2}\right)^j,$$

$$b_j = \left(\frac{2-\sqrt{2}}{4}\right)\left(3+2\sqrt{2}\right)^j + \left(\frac{2+\sqrt{2}}{4}\right)\left(3-2\sqrt{2}\right)^j.$$

Proof: From the recursive formulas stated above, we know

$$a_{j+2} = 3a_{j+1} + 4b_{j+1},$$

$$a_{j+2} = 3a_{j+1} + 4\left(2a_j + 3b_j\right),$$

$$a_{j+2} = 3a_{j+1} + 8a_j + 12\left(\frac{a_{j+1} - 3a_j}{4}\right),$$

$$a_{j+2} = 6a_{j+1} - a_j.$$

Now, we assume the solution is homogenous and of the form, $a_j = cr^j$. Thus,

$$cr^{j+2} = 6cr^{j+1} - cr^j,$$

$$0 = r^2 - 6r + 1,$$

$$r = 3 \pm 2\sqrt{2}.$$

Substituting each solution of r into the homogenous equation, $a_j = c_1 {r_1}^j + c_2 {r_2}^j$. Applying the initial conditions, we obtain

$$a_1 = c_1(3 + 2\sqrt{2}) + c_2(3 - 2\sqrt{2}) = 1,$$

$$a_2 = c_1(3 + 2\sqrt{2})^2 + c_2(3 - 2\sqrt{2})^2 = 7.$$

Now, we solve for the constants. From the first equation,

$$c_2 = \frac{1 - c_1(3 + 2\sqrt{2})}{3 - 2\sqrt{2}},$$

$$c_1(3 + 2\sqrt{2})^2 + \left(\frac{1 - c_1(3 + 2\sqrt{2})}{3 - 2\sqrt{2}}\right)(3 - 2\sqrt{2})^2 = 7,$$

$$c_1(16 + 12\sqrt{2}) = 4 + 2\sqrt{2},$$

$$c_1 = -\left(\frac{1 - \sqrt{2}}{2}\right).$$

$$-\left(\frac{1 - \sqrt{2}}{2}\right)(3 + 2\sqrt{2}) + c_2(3 - 2\sqrt{2}) = 1,$$

$$c_2(3 - 2\sqrt{2}) = \frac{1 - \sqrt{2}}{2},$$

$$c_2 = \frac{1}{2(1 - \sqrt{2})} = -\left(\frac{1 + \sqrt{2}}{2}\right).$$

Therefore,

$$a_j = -\left(\frac{1 - \sqrt{2}}{2}\right)(3 + 2\sqrt{2})^j - \left(\frac{1 + \sqrt{2}}{2}\right)(3 - 2\sqrt{2})^j.$$

We repeat the past steps to solve for b_j.

$$b_{j+2} = 2a_{j+1} + 3b_{j+1},$$

$$b_{j+2} = 2(3a_j + 4b_j) + 3b_{j+1},$$

$$b_{j+2} = 6\left(\frac{b_{j+1} - 3b_j}{2}\right) + 8b_j + 3b_{j+1},$$

$$b_{j+2} = 6b_{j+1} - b_j.$$

Just like for a_j, we assume the solution is homogenous and of the form, $b_j = dr^j$. Consequently, the solutions are the same, $r = 3 \pm 2\sqrt{2}$, but b_j has different initial conditions.

$$b_1 = d_1(3 + 2\sqrt{2}) + d_2(3 - 2\sqrt{2}) = 1,$$

$$b_2 = d_1(3 + 2\sqrt{2})^2 + d_2(3 - 2\sqrt{2})^2 = 5.$$

$$d_2 = \frac{1 - d_1(3 + 2\sqrt{2})}{3 - 2\sqrt{2}},$$

$$d_1(3 + 2\sqrt{2})^2 + \left(\frac{1 - d_1(3 + 2\sqrt{2})}{3 - 2\sqrt{2}}\right)(3 - 2\sqrt{2})^2 = 5,$$

$$d_1(16 + 12\sqrt{2}) = 2 + 2\sqrt{2},$$

$$d_1 = \frac{2 - \sqrt{2}}{4}.$$

$$\left(\frac{2 - \sqrt{2}}{4}\right)(3 + 2\sqrt{2}) + d_2(3 - 2\sqrt{2}) = 1,$$

$$d_2(3 - 2\sqrt{2}) = \frac{2 - \sqrt{2}}{4},$$

$$d_2 = \frac{2 + \sqrt{2}}{4}.$$

Therefore,

$$b_j = \left(\frac{2 - \sqrt{2}}{4}\right)(3 + 2\sqrt{2})^j + \left(\frac{2 + \sqrt{2}}{4}\right)(3 - 2\sqrt{2})^j. \blacksquare$$

Theorem CII: When n is even, the expression for the eigenvalues is as follows:

$$\Lambda(C_n) = \left\{(-1)^{\frac{n}{2}+k}\left(a_k \pm b_k\sqrt{2}\right)\right\} \textit{ where } k = 0, \ldots, \frac{n}{2}.$$

The multiplicities of the even eigenvalues are

$$m(\lambda_{even\ n}) = \binom{n}{q} \textit{ where } q = \frac{n}{2} + k.$$

Interestingly, the odd and even eigenvalues share the same recursive formulas. However, they have different explicit formulas because of different initial conditions.

$$a_{k+1} = 3a_k + 4b_k \textit{ where } a_0 = 1 \textit{ and } b_0 = 0,$$

$$b_{k+1} = 2a_k + 3b_k \; where \; a_0 = 1 \; and \; b_0 = 0.$$

For example,

$$a_2 = 3a_1 + 4b_1, \qquad b_2 = 2a_1 + 3b_1,$$

$$a_2 = 3 + 0 = 3. \qquad b_2 = 2 + 0 = 2.$$

$$m(-1) = \binom{2}{1} = 2,$$

$$m\left(3 \pm 2\sqrt{2}\right) = \binom{2}{2} = 1.$$

Thus,

$$\Lambda(C_2) = \left\{(-1)^{\frac{2}{2}+k} a_k \pm b_k\sqrt{2}\right\} \; where \; k = 0, 1,$$

$$= \left\{-1^{(2)}, 3 \pm 2\sqrt{2}^{(1)}\right\}.$$

Note, the eigenvalues of the original matrix are factors of r. This will be true for every matrix we see in the paper and will be the reason this method has difficulties when using matrices with eigenvalues that cannot be simplified to an algebraic form.

Lemma CII: The explicit forms of a_k and b_k for the eigenvalues of $A(C_n)$ for even n are:

$$a_k = \frac{1}{2}\left(3 + 2\sqrt{2}\right)^k + \frac{1}{2}\left(3 - 2\sqrt{2}\right)^k,$$

$$b_k = \frac{\sqrt{2}}{4}\left(3 + 2\sqrt{2}\right)^k - \frac{\sqrt{2}}{4}\left(3 - 2\sqrt{2}\right)^k.$$

Proof: As with Lemma CI, we assume the solution is homogenous and of the form, $a_k = c_3{r_3}^k + c_3{r_3}^k$, Thus, we apply the initial conditions and obtain

$$a_0 = c_3 + c_4 = 1,$$

$$a_1 = c_3\left(3 + 2\sqrt{2}\right) + c_4\left(3 - 2\sqrt{2}\right) = 3.$$

$$c_4 = 1 - c_3,$$

$$c_3\left(3 + 2\sqrt{2}\right) + (1 - c_3)\left(3 - 2\sqrt{2}\right) = 3,$$

$$3c_3 + 2\sqrt{2}c_3 + 3 - 2\sqrt{2} - 3c_3 + 2\sqrt{2}c_3 = 3,$$

$$4\sqrt{2}c_3 = 2\sqrt{2},$$

$$c_3 = \frac{1}{2}.$$

$$\frac{1}{2} + c_4 = 1,$$

$$c_4 = \frac{1}{2}.$$

Therefore,

$$a_k = \frac{1}{2}\left(3 + 2\sqrt{2}\right)^k + \frac{1}{2}\left(3 - 2\sqrt{2}\right)^k.$$

Again, we apply the initial conditions to b_k, and find

$$b_0 = d_3 + d_4 = 0,$$

$$b_1 = d_3\left(3 + 2\sqrt{2}\right) + d_4\left(3 - 2\sqrt{2}\right) = 2.$$

$$d_4 = -d_3,$$

$$d_3\left(3 + 2\sqrt{2}\right) - d_3\left(3 - 2\sqrt{2}\right) = 2,$$

$$3d_3 + 2\sqrt{2}d_3 - 3d_3 + 2\sqrt{2}d_3 = 2,$$

$$d_3 = \frac{\sqrt{2}}{4}.$$

$$d_4 = -\frac{\sqrt{2}}{4}.$$

Therefore,

$$b_k = \frac{\sqrt{2}}{4}\left(3 + 2\sqrt{2}\right)^k - \frac{\sqrt{2}}{4}\left(3 - 2\sqrt{2}\right)^k. \blacksquare$$

Also, the number of distinct eigenvalues is $n + 1$, and the sum of the eigenvalues at a particular n is 2^n.

2.2 Number of Closed Walks of Family C

Now that we have calculated the eigenvalues of $A(C_n)$, we are ready to compute the number of closed walks on C_n. By definition, the characteristic polynomial of C_n is

$$P(C_n; x) = \prod_{j=0}^{n}\left(x - {w_1}^j {w_2}^{n-j}\right)^{\binom{n}{j}} \ where\ w_1 = 1 + \sqrt{2}\ and\ w_2 = 1 - \sqrt{2}.$$

w_1 and w_2 are the eigenvalues. We will find the number of closed walks of length h in C_n by using Lemma CIII. In Lemma CIII, we are reproducing the result from [1] for C_n.

Lemma CIII: In the generating function, denoted by $W(G;t)$, the coefficient of t^h is the number of closed walks of length h in a graph, G.

$$W(G;t) = \frac{P'\left(G;\frac{1}{t}\right)}{tP\left(G;\frac{1}{t}\right)} \; where \; P'(G;x) = \frac{d}{dx}P(G;x) \text{ [2].}$$

Proof:

$$P(C_n;x) = \prod_{j=0}^{n}\left(x - {w_1}^j {w_2}^{n-j}\right)^{\binom{n}{j}}.$$

$$ln\big(P(C_n;x)\big) = \sum_{j=0}^{n}\binom{n}{j} \times ln\left(x - {w_1}^j {w_2}^{n-j}\right),$$

$$\frac{P'(C_n;x)}{P(C_n;x)} = \sum_{j=0}^{n}\frac{\binom{n}{j}}{x - {w_1}^j {w_2}^{n-j}},$$

$$\frac{P'\left(C_n;\frac{1}{t}\right)}{tP\left(C_n;\frac{1}{t}\right)} = \sum_{j=0}^{n}\frac{\binom{n}{j}}{1 - {w_1}^j {w_2}^{n-j}t}.$$

From here, we can see the function is an infinite geometric series as it is in the form, $\frac{a}{1-r}$, where $r = {w_1}^j {w_2}^{n-j}t$.

$$W(C_n;t) = \sum_{j=0}^{n}\sum_{h=0}^{\infty}\binom{n}{j}\left({w_1}^j {w_2}^{n-j}t\right)^h,$$

$$= \sum_{h=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}{w_1}^{jh}{w_2}^{(n-j)h}\right)t^h.$$

Because the inner summation is the Binomial Theorem,

$$\sum_{h=0}^{\infty}\left({w_1}^h + {w_2}^h\right)t^h. \blacksquare$$

Therefore, the total number of closed walks of length h in C_n is $\left(1+\sqrt{2}\right)^h + \left(1-\sqrt{2}\right)^h$. This result is consistent with the result in [1] but with a different φ as the series are similar. By manipulating ${w_1}^j {w_2}^{n-j}$, as shown below, we can see it is interchangeable with $(-1)^j\varphi^{n-2j}$ where $\varphi = 1 - \sqrt{2}$.

$$w_1{}^j w_2{}^{n-j} = \left(1+\sqrt{2}\right)^j \left(1-\sqrt{2}\right)^{n-j},$$

$$= \left(-\frac{1}{1-\sqrt{2}}\right)^j \left(1-\sqrt{2}\right)^{n-j},$$

$$= (-1)^j \left(\frac{1}{1-\sqrt{2}}\right)^j \left(1-\sqrt{2}\right)^{n-j},$$

$$= (-1)^j \left(1-\sqrt{2}\right)^{n-2j} = (-1)^j \varphi^{n-2j}.$$

Along with the total number of closed walks in C_n, we determined the total number of closed walks in C_1 at v_0.

Theorem CIII: The number of closed walks at v_0 of length h in C_1.

$$a_h = \left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{h+1} - \left(\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{h+1},$$

and the recursive formula is $a_{h+1} = 2a_h + a_{h-1}$ where $a_0 = 1$ and $a_1 = 2$.

Proof: As the solution is homogeneous by definition, it's of the form $a_h = cr^h$. Thus,

$$cr^{h+1} = 2cr^h + cr^{h-1},$$

$$0 = r^2 - 2r - 1,$$

$$r = 1 \pm \sqrt{2}.$$

Notice the solutions of r are eigenvalues of C_1.

$$a_h = c_1\left(1+\sqrt{2}\right)^h + c_2\left(1-\sqrt{2}\right)^h.$$

Using the initial conditions we can see

$$c_1 + c_2 = 1,$$

$$c_1\left(1+\sqrt{2}\right) + c_2\left(1-\sqrt{2}\right) = 2.$$

$$c_2 = 1 - c_1,$$

$$c_1\left(1+\sqrt{2}\right) + (1-c_1)\left(1-\sqrt{2}\right) = 2,$$

$$c_1 + \sqrt{2}c_1 + 1 - \sqrt{2} - c_1 + \sqrt{2}c_1 = 2,$$

$$2\sqrt{2}c_1 = 1 + \sqrt{2},$$

$$c_1 = \frac{1+\sqrt{2}}{2\sqrt{2}}.$$

$$c_2 = 1 - \left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right),$$

$$c_2 = -\left(\frac{1-\sqrt{2}}{2\sqrt{2}}\right).$$

Therefore, the final formula for the number of closed walks of length h in C_1 at v_0 is

$$a_h = \left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)\left(1+\sqrt{2}\right)^h - \left(\frac{1-\sqrt{2}}{2\sqrt{2}}\right)\left(1-\sqrt{2}\right)^h.$$

$$a_h = \left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{h+1} - \left(\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{h+1}. \blacksquare$$

We will next show an inductive proof that shows ${A^L}_{ij}$ gives the number of walks of length L from i to j. These are well-known results of linear algebra.

Proof: A^1 is trivial since it is the adjacency matrix. Assume it holds for A^k.

$${A^{k+1}}_{ij} = \left(A^k \cdot A\right)_{ij} = \sum_{m=1}^{N} {A^k}_{im}\, A_{mj},$$

$$= {A^k}_{i1}\, A_{1j} + {A^k}_{i2}\, A_{2j} + \cdots + {A^k}_{iN}\, A_{Nj},$$

$$= the\ number\ of\ walks\ from\ i\ to\ j\ of\ length\ k+1. [8]$$

Theorem CIV: The number of closed walks of length L in arbitrary family Q is

$$\left(\sum_{i=1}^{N} {A^L}_{ii}\right)^n = \left(\sum_{i=1}^{N} \lambda_i^L\right)^n. [8]$$

Proof: The number of closed walks of length L in Q_1 is

$$\sum_{i=1}^{N} {A^L}_{ii} = tr(A^L) = tr((\mathrm{V}\Lambda\mathrm{V}^{-1})^L) = tr(\mathrm{V}\Lambda^L\mathrm{V}^{-1}) = tr(\mathrm{V}^{-1}\mathrm{V}\Lambda^L) = tr(\Lambda^L) = \sum_{i=1}^{N} \lambda_i^L.$$

We can diagonalize A into $\mathrm{V}\Lambda\mathrm{V}^{-1}$ because the matrix is symmetric. Then we use the cyclic property of the trace. Now, we will show the number of closed walks of length L in Q_n.

$$\sum_{i=1}^{N} {(A \otimes A \otimes \dots \otimes A)^L}_{ii} = tr[A \otimes A \otimes \dots \otimes A]^L,$$

$$= tr[(A \otimes A \otimes \dots \otimes A) \times (A \otimes A \otimes \dots \otimes A) \times \dots \times (A \otimes A \otimes \dots \otimes A)].$$

By the mixed product property of matrices,

$$= tr[(A \times A \times \ldots \times A) \otimes (A \times A \times \ldots \times A) \otimes \ldots \otimes (A \times A \times \ldots \times A)].$$

$$= tr[A^L \otimes A^L \otimes \ldots \otimes A^L].$$

By the trace product of the Kronecker product,

$$= tr(A^L) \times tr(A^L) \times \ldots \times tr(A^L),$$

$$= [tr(A^L)]^n,$$

$$= \left(\sum_{i=1}^{N} \lambda_i^L\right)^n. \blacksquare$$

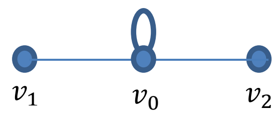
The number of closed walks from v_0 to v_0 is of the same form. Therefore, the formula for the number of closed walks of length h in C_n at v_0 is

$$(a_h)^n = \left(\left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{h+1} - \left(\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{h+1}\right)^n. \blacksquare$$

Please note, it is important to distinguish h from n; h is the number of times we use matrix multiplication, and n is the number of times we take the Kronecker product.

Chapter 3: Introduction to Family D

We define D_n to have the vertex set $V_n = \{v_j : j = 0, 1, \dots, 3^n - 1\}$ and $|V_n| = 3^n$. The edge cardinality of D_n is $|E_D| = \frac{1}{2}(5^n + 1)$.



$$A(D_1) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Figure 6. Graph and Adjacency Matrix of D_1

For each $k = 0, 1, \dots, n-1$, D_n has $\binom{n}{k} 2^{n-k}$ vertices of degree 3^k and a single vertex of degree $3^n + 1$. Thus, the sum of the degrees of vertices in D_n is

$$\sum_{k=0}^{n-1} \binom{n}{k} 2^{n-k} \times 3^k + (3^n + 1) = 1 + \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \times 3^k,$$

$$= 1 + 2^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{3}{2}\right)^k.$$

By implementing the Binomial Theorem,

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n,$$

it will equal

$$= 5^n + 1.$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 7. Adjacency Matrix of D_2

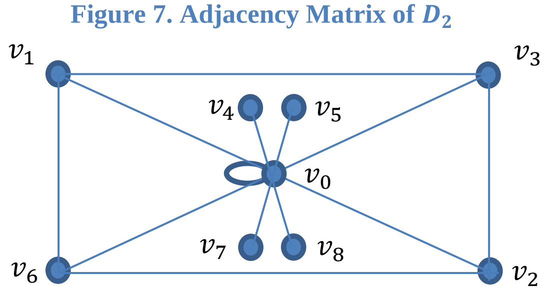


Figure 8. Graph of D_2

3.1 Spectra of Family D

D_1 is similar to B_1, but it adds an edge and vertex to v_0. D_1's eigenvalues are $-1, 0$, and 2, and $A(D_n)$ has $n+2$ distinct eigenvalues. Theorem DI shows the set of eigenvalues but does not include zero. We know the matrix $A \otimes B$ has eigenvalues that are a combination of the eigenvalues of the matrices A and B. Because zero is an eigenvalue of both and multiplying any number by zero is zero, zero will always be an eigenvalue. Thus, Theorem DI shows the non-zero eigenvalues.

Theorem DI: Let $\Lambda'(D_n)$ denote the non-zero spectrum of $A(D_n)$. Then

$$\Lambda'(D_n) = \{(-1)^k \times 2^{n-k}\}\, for\ k = 0, \dots, n.$$

Proof: Using induction, we first show the statement is true for $n = 1$.

$$\Lambda'(D_1) = (-1)(-2)^{1-k}\, for\ k = 0, 1.$$

When $k = 0$, $\lambda_1 = 2$, and when $k = 1$, $\lambda_1 = -1$, which are the eigenvalues of D_1. We assume $\Lambda'(D_n) = (-1)^k \times 2^{n-k}$ for $k = 0, \dots, n$. Next, we multiply $\Lambda'(D_n)$ with the initial non-zero eigenvalues to find the eigenvalues of the following matrix, $A(D_{n+1})$.

$$(-1) \times (-1)^k \times 2^{n-k} = (-1)^{k+1}(2)^{n-k},$$

$$= (-1)^{k+1}(-2)^{(n+1)-(k+1)}\, for\ k+1 = 1, \dots, n+1.$$

Also,

$$(2) \times (-1)^k \times 2^{n-k} = (-1)^k(2)^{n-k+1},$$

$$= (-1)^k(-2)^{(n+1)-k}\, for\ k = 0, \dots, n+1.$$

Note the bounds of k. Multiplying $\Lambda'(D_n)$ by 2 creates $\Lambda'(D_{n+1})$ with the bounds of $k = 0, \dots, n+1$, which means the formula holds for $n+1$. In addition, multiplying $\Lambda'(D_n)$ by -1 increases the multiplicity of the eigenvalues of $\Lambda'(D_{n+1})$ for $k = 0, \dots, n$. Hence, the newest eigenvalue terms are derived from multiplying $\Lambda'(D_n)$ by 2 and have multiplicities of 1. Therefore, $\Lambda'(D_n) = (-1)^k \times 2^{n-k}$ for $k = 0, \dots, n$ holds for D_n.■

The multiplicities of the non-zero eigenvalues of D_n are one layer of Pascal's Triangle in order of magnitude. For example, when $n = 2$, the eigenvalues of $A(D_2)$ are $1, -2$, and 4, with multiplicities of 1, 2, and 1, respectively. The eigenvalues of the next matrix when $n = 3$, the eigenvalues of $A(D_3)$ are -1, $2, -4$, and 8, with multiplicities of 1, 3, 3, and 1, respectively. Finally, 0 will always appear as an eigenvalue with a multiplicity of $3^n - 2^n$.

3.2 Number of Closed Walks of Family D

From the eigenvalues and their respective multiplicities, we can obtain the characteristic polynomial of D_n.

$$P(D_n; x) = \prod_{j=0}^{n} [x - ((-1)^j \times 2^{n-j})]^{\binom{n}{j}} \times x^{3^n - 2^n}$$

$$ln[P(D_n; x)] = \sum_{j=0}^{n} \binom{n}{j} ln[x - ((-1)^j \times 2^{n-j})] + (3^n - 2^n) ln[x],$$

$$\frac{P'(D_n; x)}{P(D_n; x)} = \sum_{j=0}^{n} \frac{\binom{n}{j}}{x - ((-1)^j \times 2^{n-j})} + \frac{3^n - 2^n}{x},$$

$$\frac{P'\left(D_n; \frac{1}{t}\right)}{tP\left(D_n; \frac{1}{t}\right)} = \sum_{j=0}^{n} \frac{\binom{n}{j}}{1 - ((-1)^j \times 2^{n-j})t} + (3^n - 2^n),$$

$$W(D_n; t) = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{\infty} [((-1)^j \times 2^{n-j})t]^k + (3^n - 2^n),$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{n} \left[\binom{n}{j} ((-1)^{jk} \times 2^{(n-j)k})\right] t^k + (3^n - 2^n).$$

From these calculations,

$$W(D_n; t) = \sum_{k=0}^{\infty} \sum_{j=0}^{n} \left[\binom{n}{j} [(-1)^k]^j \times [2^k]^{(n-j)}\right] t^k + (3^n - 2^n).$$

By the Binomial Theorem,

$$W(D_n; t) = \sum_{k=0}^{\infty} [(-1)^k + (2)^k]^n t^k + (3^n - 2^n).$$

From here, we can see the coefficient of t^k is the number of closed walks of length k in D_n. ■

Theorem DII: The number of closed walks from v_0 of length k in D_1.

$$a_{k+1} = a_k + 2a_{k-1} \text{ where } a_0 = 1 \text{ and } a_1 = 1.$$

Proof: Assume the solution is homogeneous and of the form $a_k = cr^k$. Thus,

$$cr^{k+1} = cr^k + 2cr^{k-1},$$

$$0 = r^2 - r - 2,$$

$$r = -1, 2.$$

The solutions of r are eigenvalues.

$$a_k = c_1(-1)^k + c_2(2)^k.$$

$$c_1 + c_2 = 1,$$

$$-c_1 + 2c_2 = 1.$$

Solving for the constants, we find $c_1 = \frac{1}{3}$ and $c_2 = \frac{2}{3}$.

Therefore, the final equation for the number of closed walks of length k at v_0 in D_1 is

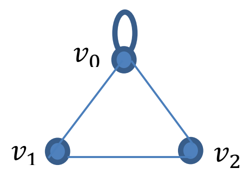
$$a_k = \frac{1}{3}\left((-1)^k + 2^{k+1}\right). \blacksquare$$

Utilizing Theorem CIV, the formula for the number of closed walks of length k in D_n at v_0 is

$$(a_k)^n = \left(\frac{1}{3}\left((-1)^k + 2^{k+1}\right)\right)^n. \blacksquare$$

Chapter 4: Introduction to Family E

For each non-negative integer n, we define E_n to have the vertex set $V_n = \{v_j : j = 0, 1, \dots, 3^n - 1\}$ and $|V_n| = 3^n$ for E_n. The edge cardinality of E_n is $|E_E| = \frac{1}{2}(7^n + 1)$.



$$A(E_1) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Figure 9. Graph and Adjacency Matrix of E_1

Also, for each $k = 0, 1, \dots, n-1$, E_n has $\binom{n}{k} 2^{n-k}$ vertices of degree $3^k 2^{n-k}$ and a single vertex, v_0, of degree $3^n + 1$. Thus, the sum of the degrees of vertices in E_n is

$$\sum_{k=0}^{n-1} \binom{n}{k} 2^{n-k} \times 3^k 2^{n-k} + (3^n + 1) = 1 + 4^n \left(\frac{3}{4}\right)^n + 4^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{3}{4}\right)^k.$$

Again, by the Binomial Theorem,

$$= 4^n \left(\frac{7}{4}\right)^n + 1 = 7^n + 1.$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 10. Adjacency Matrix of E_2

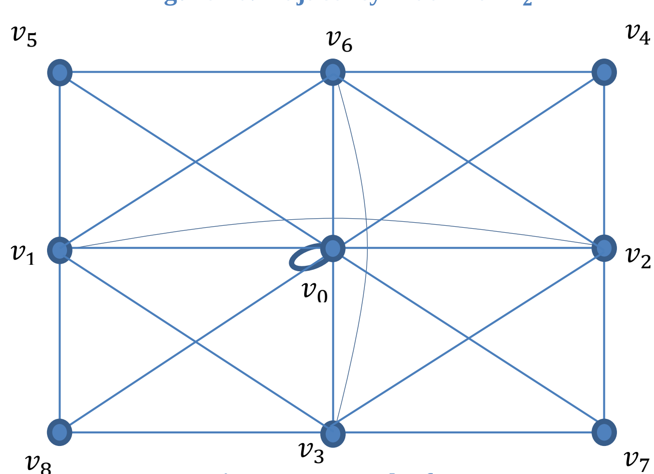


Figure 11. Graph of E_2

4.1 Spectra of Graph E

Like with D_n, we want to use the generating function to show the number of closed walks of length k on E_n. The eigenvalues of $A(E_1)$ are -1, $1 \pm \sqrt{2}$. But, we want to find a formula for the eigenvalues and their respective multiplicities for any n, which is shown in Theorem EI.

Theorem EI: The spectrum of E_n is

$$\Lambda(E_n) = \begin{Bmatrix} a_i \pm b_i\sqrt{2}\, for\ i = 0, \dots, n-2 \\ -a_i \pm b_i\sqrt{2}\, for\ i = 0, \dots, n-1 \\ a_n \pm b_n\sqrt{2} \end{Bmatrix},$$

where

$$a_i = \frac{1}{2}\left(1+\sqrt{2}\right)^i + \frac{1}{2}\left(1-\sqrt{2}\right)^i,$$

$$b_i = \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^i - \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^i.$$

The notation of $\Lambda(E_n)$ means all three cases occur simultaneously for the specified range of i.

However, two lemmas are necessary to show this.

Lemma EI: The explicit forms of a_n and b_n of E_n.

$$a_n = \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{1}{2}\left(1-\sqrt{2}\right)^n,$$

$$b_n = \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^n - \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^n.$$

Respectively, their recursive forms are

$$a_n = 2a_{n-1} + a_{n-2}\ where\ a_0 = 1\ and\ a_1 = 1,$$

$$b_n = 2b_{n-1} + b_{n-2}\ where\ b_0 = 0\ and\ b_1 = 1.$$

Proof: Given the following recursive formula holds,

$$a_n = 2a_{n-1} + a_{n-2}\ where\ a_0 = 1\ and\ a_1 = 1,$$

we assume the solution is homogenous and of the form, $a_n = cr^n$.

$$cr^n = 2cr^{n-1} + cr^{n-2},$$

$$0 = r^2 - 2r - 1,$$

$$r = 1 \pm \sqrt{2}.$$

Substituting the solutions of r into the form, $a_n = cr^n$, yields $a_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n$.

When $a_0 = 1$,

$$c_1 + c_2 = 1,$$

$$c_2 = 1 - c_1.$$

When $a_1 = 1$,

$$c_1(1+\sqrt{2}) + (1-c_1)(1-\sqrt{2}) = 1,$$

$$c_1 + c_1\sqrt{2} + 1 - c_1 - \sqrt{2} + c_1\sqrt{2} = 1,$$

$$c_1 2\sqrt{2} = \sqrt{2},$$

$$c_1 = \frac{1}{2}.$$

$$c_2 = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$a_n = \frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1-\sqrt{2})^n.$$

The same process can be applied to

$$b_n = 2b_{n-1} + b_{n-2} \; where \; b_0 = 0 \; and \; b_1 = 1.$$

Since the solutions of r are the same, we substitute them into the form, $b_n = dr^n$, which yields

$$b_n = d_1(1+\sqrt{2})^n + d_2(1-\sqrt{2})^n.$$

When $b_0 = 0$,

$$d_1 + d_2 = 0.$$

When $b_1 = 1$,

$$d_1(1+\sqrt{2}) + (-d_1)(1-\sqrt{2}) = 1.$$

$$d_1 = \frac{1}{2\sqrt{2}},$$

$$d_2 = -\frac{1}{2\sqrt{2}}.$$

Therefore,

$$b_n = \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^n - \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^n. \blacksquare$$

With a_n and b_n defined by the explicit formulas above, Lemma EII holds.

Lemma EII: Recursive equations for a_n and b_n of the eigenvalues of $A(E_n)$.

$$a_n + 2b_n = a_{n+1}, \qquad b_n + a_n = b_{n+1}, \qquad a_n - 2b_n = -a_{n-1}, \qquad b_n - a_n = -b_{n-1}.$$

Proof:

$$a_n + 2b_n = \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{1}{2}\left(1-\sqrt{2}\right)^n + \frac{1}{\sqrt{2}}\left(1+\sqrt{2}\right)^n - \frac{1}{\sqrt{2}}\left(1-\sqrt{2}\right)^n.$$

Simplifying the above equation yields

$$\frac{1+\sqrt{2}}{2}\left(1+\sqrt{2}\right)^n + \frac{1-\sqrt{2}}{2}\left(1-\sqrt{2}\right)^n = \frac{1}{2}\left(1+\sqrt{2}\right)^{n+1} + \frac{1}{2}\left(1-\sqrt{2}\right)^{n+1} = a_{n+1}.$$

The same process is applied to the other three equations:

$$b_n + a_n = \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^n - \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^n + \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{1}{2}\left(1-\sqrt{2}\right)^n,$$

$$= \frac{1+\sqrt{2}}{2\sqrt{2}}\left(1+\sqrt{2}\right)^n + \frac{1-\sqrt{2}}{2\sqrt{2}}\left(1-\sqrt{2}\right)^n = \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^{n+1} + \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^{n+1} = b_{n+1}.$$

$$a_n - 2b_n = \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{1}{2}\left(1-\sqrt{2}\right)^n - \frac{1}{\sqrt{2}}\left(1+\sqrt{2}\right)^n + \frac{1}{\sqrt{2}}\left(1-\sqrt{2}\right)^n,$$

$$= \frac{1-\sqrt{2}}{2}\left(1+\sqrt{2}\right)^n + \frac{1+\sqrt{2}}{2}\left(1-\sqrt{2}\right)^n = -\frac{1}{2}\left(1+\sqrt{2}\right)^{n-1} - \frac{1}{2}\left(1-\sqrt{2}\right)^{n-1} = -a_{n-1}.$$

$$b_n - a_n = \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^n - \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^n - \frac{1}{2}\left(1+\sqrt{2}\right)^n - \frac{1}{2}\left(1-\sqrt{2}\right)^n,$$

$$= \frac{1-\sqrt{2}}{2\sqrt{2}}\left(1+\sqrt{2}\right)^n - \left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)\left(1-\sqrt{2}\right)^n = -\frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^{n-1} + \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^{n-1} = -b_{n-1}. \blacksquare$$

Now that we have shown these two lemmas to be true, we can prove Theorem EI. As stated above,

Theorem EI: The spectrum of E_n is

$$\Lambda(E_n) = \begin{Bmatrix} a_i \pm b_i\sqrt{2}\, for\ i = 0, \dots, n-2 \\ -a_i \pm b_i\sqrt{2}\, for\ i = 0, \dots, n-1 \\ a_n \pm b_n\sqrt{2} \end{Bmatrix},$$

where

$$a_i = \frac{1}{2}\left(1+\sqrt{2}\right)^i + \frac{1}{2}\left(1-\sqrt{2}\right)^i,$$

$$b_i = \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^i - \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^i.$$

The notation of $\Lambda(E_n)$ means all three cases occur simultaneously for the specified range of i.

Proof: We need to show the theorem holds up to $n = 2$ because that is the first time all three cases occur.

When $n = 0$,

$$a_0 \pm b_0\sqrt{2} = 1.$$

When $n = 1$,

$$-a_0 \pm b_0\sqrt{2} = -1\ and\ a_1 \pm b_1\sqrt{2} = 1 \pm \sqrt{2}.$$

When $n = 2$,

$$a_0 \pm b_0\sqrt{2} = 1, -a_0 \pm b_0\sqrt{2} = -1, -a_0 \pm b_0\sqrt{2} = -1 \pm \sqrt{2}, and\ a_2 \pm b_2\sqrt{2} = 3 \pm 2\sqrt{2}.$$

Next, we assume the theorem holds for $n = k$.

$$\Lambda(E_k) = \begin{Bmatrix} a_i \pm b_i\sqrt{2}\, for\ i = 0, \dots, k-2 \\ -a_i \pm b_i\sqrt{2}\, for\ i = 0, \dots, k-1 \\ a_k \pm b_k\sqrt{2} \end{Bmatrix}.$$

We will show this is true for $n = k+1$ by multiplying $\Lambda(E_k)$ by the original eigenvalues. We will show the eigenvalues of $\Lambda(E_{k+1})$ appear in the tables below. The entries in the left column are multiplied by entries in the first row and their result is displayed at the intersection. Starting with $a_i \pm b_i\sqrt{2}$ for $i = 0, \dots, k-2$, we find

	$a_i + b_i\sqrt{2}$	$a_i - b_i\sqrt{2}$
-1	$-a_i - b_i\sqrt{2}$	$-a_i + b_i\sqrt{2}$
$1+\sqrt{2}$	$(a_i + 2b_i) + (b_i + a_i)\sqrt{2}$	$(a_i - 2b_i) - (b_i - a_i)\sqrt{2}$
$1-\sqrt{2}$	$(a_i - 2b_i) + (b_i - a_i)\sqrt{2}$	$(a_i + 2b_i) - (b_i + a_i)\sqrt{2}$

Table 1. Multiplication of $a_i \pm b_i\sqrt{2}$ for $i = 0, \dots, k-2$ by the Original Eigenvalues of E_n

Table 1 can be simplified to Table 2 by Lemma EII.

	$a_i + b_i\sqrt{2}$	$a_i - b_i\sqrt{2}$
-1	$-a_i - b_i\sqrt{2}$	$-a_i + b_i\sqrt{2}$
$1+\sqrt{2}$	$a_{i+1} + b_{i+1}\sqrt{2}$	$-a_{i-1} + b_{i-1}\sqrt{2}$
$1-\sqrt{2}$	$-a_{i-1} - b_{i-1}\sqrt{2}$	$a_{i+1} - b_{i+1}\sqrt{2}$

Table 2. Simplification of Table 1

Though Table 2 contains some of the values we are searching for, none of the values have the full range of the values in $\Lambda(E_{k+1})$. But this repetition of eigenvalues is how the multiplicities increase. We will see the full range of the values of $\Lambda(E_{k+1})$ in Table 3 and Table 4 where we have already simplified by Lemma EII.

	$-a_i + b_i\sqrt{2}$	$-a_i - b_i\sqrt{2}$
-1	$a_i - b_i\sqrt{2}$	$a_i + b_i\sqrt{2}$
$1+\sqrt{2}$	$a_{i-1} - b_{i-1}\sqrt{2}$	$-a_{i+1} - b_{i+1}\sqrt{2}$
$1-\sqrt{2}$	$-a_{i+1} + b_{i+1}\sqrt{2}$	$a_{i-1} + b_{i-1}\sqrt{2}$

Table 3. Multiplication of $-a_i \pm b_i\sqrt{2}$ for $i = 0, \dots, k-1$ by the Original Eigenvalues of E_n

	$a_k + b_k\sqrt{2}$	$a_k - b_k\sqrt{2}$
-1	$-a_k - b_k\sqrt{2}$	$-a_k + b_k\sqrt{2}$
$1+\sqrt{2}$	$a_{k+1} + b_{k+1}\sqrt{2}$	$a_{k-1} - b_{k-1}\sqrt{2}$
$1-\sqrt{2}$	$a_{k-1} + b_{k-1}\sqrt{2}$	$a_{k+1} - b_{k+1}\sqrt{2}$

Table 4. Multiplication of $a_k \pm b_k\sqrt{2}$ by the Original Eigenvalues of E_n

We can see that $a_i \pm b_i\sqrt{2}$ holds for $i = 0, \dots, k-1$ from the information provided in Table 3 (highlighted in green), $-a_i \pm b_i\sqrt{2}$ for $i = 0, \dots, k$ from Table 3 and Table 4 (highlighted in blue), and $a_{k+1} \pm b_{k+1}\sqrt{2}$ from Table 4 (highlighted in orange). ■

As with D_n, in order to use the generating function for E_n, we must find the multiplicities of the eigenvalues. However, it was difficult to determine a formulaic solution to this problem. Thus, we started by displaying the values graphically. The first eigenvalue graph of E_n is

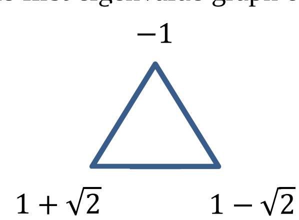


Figure 12. Graph of the Eigenvalues of E_1

As it happens, the multiplicities directly correlate with the Binomial Pyramid. The Binomial Pyramid is the Binomial (or Pascal's) Triangle in the third dimension. Each of the three sides of the pyramid is Pascal's Triangle. The interior of the structure relies on the same principle: the value of a term is the sum of the previous terms above it. Unlike the outer terms, each interior term is the sum of three terms. For example, let us calculate the fourth layer of the pyramid from the third.

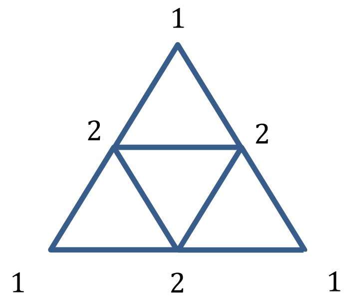


Figure 13. Graph of the Third Level of Pascal's Pyramid

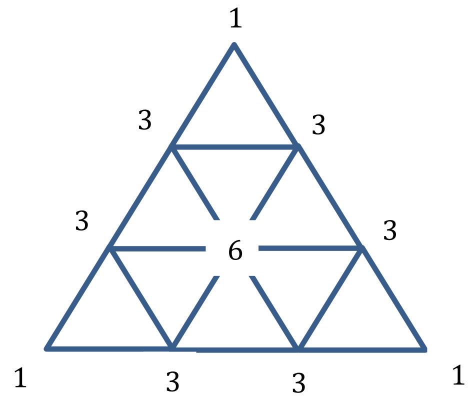


Figure 14. Graph of the Fourth Level of Pascal's Pyramid

We can see the first interior term, 6, is located in the fourth layer, where it is the sum of the three 2's located in the third layer. Looking back, the first eigenvalue graph of E_n corresponds with the second layer of the Binomial Pyramid as the multiplicities of the first set of eigenvalues of E_n are all 1's. We define the three directions of the pyramid to be with respect to one initial eigenvalue. For instance, if we move from left to right once, we are multiplying by $1-\sqrt{2}$ and dividing by $1+\sqrt{2}$. But the number of -1's is not affected. This is true for moving in other directions. Let us take the graphic image of the eigenvalues for E_3, where $a=-1$, $b=1+\sqrt{2}$, and $c=1-\sqrt{2}$.

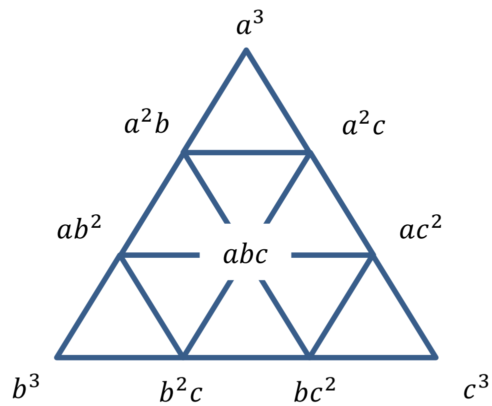


Figure 15. Graph of the Formulaic Eigenvalues of E_3

Now, we can see any eigenvalue can be represented as $a^x b^y c^z$, where x is the number of -1's, y is the number of $1+\sqrt{2}$'s, and z is the number of $1-\sqrt{2}$'s. As a result of Theorem EI,

$$\lambda_i = a^x b^y c^z = (-1)^x\left(a_j + (-1)^k b_j\sqrt{2}\right),$$

where i is the subscript to differentiate the eigenvalues, $j=|y-z|$, and $k=1$ if $z>y$ and 0 otherwise. Also, the multiplicities of a given eigenvalue can be found through the equation,

$$m(\lambda_i) = \frac{n!}{x!\,y!\,z!}.$$

We will show this by example. In $A(E_3)$, let us consider the eigenvalues $7+5\sqrt{2}$, 1, and $1-\sqrt{2}$.

The factors of $7+5\sqrt{2}$ are three $1+\sqrt{2}$'s. Thus, the multiplicity of $7+5\sqrt{2}$in $A(E_3)$ is

$$m(7+5\sqrt{2}) = \frac{3!}{0!\,3!\,0!} = 1.$$

The factors of 1 are -1, $1+\sqrt{2}$, and $1-\sqrt{2}$. Thus, the multiplicity of 1 in $A(E_3)$ is

$$m(1) = \frac{3!}{1!\,1!\,1!} = 6.$$

The factors of $1-\sqrt{2}$ are three $1+\sqrt{2}$'s. Thus, the multiplicity of $1-\sqrt{2}$ in $A(E_3)$ is

$$m(1-\sqrt{2}) = \frac{3!}{2!\,0!\,1!} = 3.$$

Notice every eigenvalues multiplicity corresponds with a location in Figure 15. One more thing to note, the multiplicity number is the number of paths from one of the initial eigenvalues to the eigenvalue being considered.

4.2 Number of Closed Walks of Family E

From Theorem EI, we know the number of closed walks of length 1, 2, and 3 of graph E_1 is $a_1=1$, $a_2=7$, and $a_3=13$, respectively. After calculating the number of closed walks of larger L, through trial and error, we found a recursive relationship shown by the sequence, $a_{L+1}=a_L+3a_{L-1}+a_{L-2}$. Note, this sequence follows from the characteristic polynomial of $A(E_1)$, $\lambda^3-\lambda^2-3\lambda-1$.

Lemma EIII:

$$a_L = (-1)^L + (1+\sqrt{2})^L + (1-\sqrt{2})^L$$

and is the explicit form of $a_L = a_{L-1}+3a_{L-2}+a_{L-3}$ where $a_1=1$, $a_2=7$, and $a_3=13$.

Proof of Lemma EIII can be found in the appendix.

Theorem EIII: The number of closed walks from v_0 of length L in E_1.

$$a_{n+1} = a_n + 3a_{n-1} + a_{n-2} \; where \; a_0 = 1, a_1 = 1, a_2 = 3$$

Proof: Assume it is homogeneous and of the form $a_n = cr^n$. Thus,

$$cr^{n+1} = cr^n + 3cr^{n-1} + cr^{n-1},$$

$$0 = r^3 - r^2 - 3r - 1,$$

$$r = -1, 1 \pm \sqrt{2}.$$

The solutions of r are eigenvalues again.

$$a_n = c_1(-1)^n + c_2\left(1+\sqrt{2}\right)^n + c_3\left(1-\sqrt{2}\right)^n.$$

Using the initial conditions, we can see

$$c_1 + c_2 + c_3 = 1,$$

$$-c_1 + \left(1+\sqrt{2}\right)c_2 + \left(1-\sqrt{2}\right)c_3 = 1,$$

$$c_1 + \left(1+\sqrt{2}\right)^2 c_2 + \left(1-\sqrt{2}\right)^2 c_3 = 3.$$

Let us use the coefficients as entries in a matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 3+2\sqrt{2} & 3-2\sqrt{2} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2+\sqrt{2} & 2-\sqrt{2} \\ 0 & 2+2\sqrt{2} & 2-2\sqrt{2} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \dfrac{2-\sqrt{2}}{2+\sqrt{2}} \\ 0 & 0 & \dfrac{-4\sqrt{2}}{\left(2+\sqrt{2}\right)\left(2+2\sqrt{2}\right)} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{2}{2+\sqrt{2}} \\ \dfrac{-2\sqrt{2}}{\left(2+\sqrt{2}\right)\left(2+2\sqrt{2}\right)} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \dfrac{2-\sqrt{2}}{2+\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{2}{2+\sqrt{2}} \\ \dfrac{1}{2} \end{bmatrix}.$$

Solving for the constants,

$$c_3 = \frac{1}{2}.$$

$$c_2 + \left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{2}{2+\sqrt{2}},$$

$$c_2 = \frac{2}{2+\sqrt{2}} - \left(\frac{1}{2}\right)\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right),$$

$$c_2 = \left(\frac{1}{2}\right)\left(\frac{4-\left(2-\sqrt{2}\right)}{2+\sqrt{2}}\right),$$

$$c_2 = \frac{1}{2}.$$

$$c_1 + \frac{1}{2} + \frac{1}{2} = 1,$$

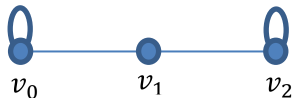
$$c_1 = 0.$$

Therefore, the final equation for the number of closed walks of length L at v_0 in E_1 is

$$a_L = \frac{1}{2}\left(\left(1+\sqrt{2}\right)^L + \left(1-\sqrt{2}\right)^L\right). \blacksquare$$

Chapter 5: Introduction to Family F

For each non-negative integer n, we define F_n to have the vertex set $V_n = \{v_j : j = 0, 1, \dots, 3^n - 1\}$ and $|V_n| = 3^n$. The edge cardinality of F_n is $|E_F| = \frac{1}{2}(6^n + 2^n)$.



$$A(F_1) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Figure 16. Graph and Adjacency Matrix of F_1

F_n has $3^n - 2^n$ vertices of degree 2^n and 2^n vertices of degree $2^n + 1$. Thus, the sum of the degrees of vertices in F_n is

$$(3^n - 2^n) \times\ 2^n + 2^n \times (\ 2^n + 1) = 2^n(3^n + 1),$$

$$= 6^n + 2^n.$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Figure 17. Adjacency Matrix of F_2

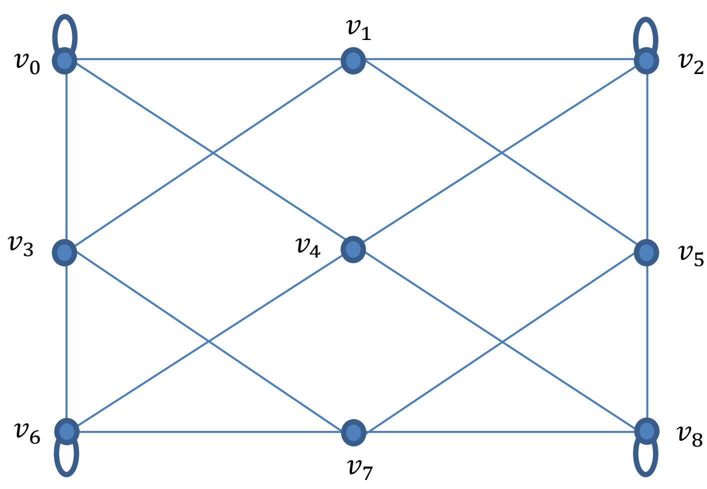


Figure 18. Graph of F_2

5.1 Spectra of Graph F

$A(F_n)$ has $n+2$ distinct eigenvalues and they are very similar to $A(D_n)$. They both have -1 and 2 as eigenvalues, but instead of the third initial value being 0, it is 1. Theorem FI is the equation for the spectrum of F_n.

Theorem FI: The spectrum of F_n.

$$\Lambda(F_k) = \begin{Bmatrix} -2^k for\ k = 0, \dots, n-1 \\ 2^k for\ k = 0, \dots, n \end{Bmatrix}.$$

Proof: As with $A(D_n)$ and $A(E_n)$, we multiply the eigenvalues of $A(F_1)$ to $\Lambda(F_k)$. Thus,

$$-2^k\ for\ k = 0, \dots, n-1, \qquad 2^k\ for\ k = 0, \dots, n-1,$$

$$-2^{k+1}\ for\ k = 0, \dots, n-1, \qquad 2^k\ for\ k = 0, \dots, n,$$

$$-2^k\ for\ k = 0, \dots, n, \qquad 2^{k+1}\ for\ k = 0, \dots, n.$$

We adjust the second case of negative values and the third case of positive values where the exponents are $k+1$ by changing the bounds of k.

$$-2^k\ for\ k = 1, \dots, n, \qquad 2^k\ for\ k = 1, \dots, n+1.$$

Now, we can see the eigenvalues of $A(F_n)$ hold for $n+1$ since

$$\Lambda(F_{k+1}) = \begin{Bmatrix} -2^k for\ k = 0, \dots, n \\ 2^k for\ k = 0, \dots, n+1 \end{Bmatrix}.$$

The multiplicities of the eigenvalues of $A(F_n)$ are not as simple. The multiplicity of an eigenvalue has the same multiplicity as its negative eigenvalue, except for the iteration the value is introduced. For example, when $n = 3$, $m(1) = 4$ and $m(-1) = 4$, $m(2) = 6$ and $m(-2) = 6$, and $m(4) = 3$ and $m(-4) = 3$, while $m(8) = 1$ and $m(-8) = 0$. The new eigenvalue, 2^n, always has a multiplicity of 1 and its negative is always 0.

Additionally, the multiplicity of an eigenvalue of one iteration relies on the multiplicity of itself, its negative, and its half from the previous iteration. For example, the multiplicity of 2 for $n = 3$, relies on the multiplicities of 2, -2, and 1 of $n = 2$. Thus the formula for the multiplicity of an eigenvalue is

$$m(\lambda_i)_n = m(\lambda_i)_{n-1} + m(-\lambda_i)_{n-1} + m\left(\frac{1}{2}\lambda_i\right)_{n-1}.$$

Following the example,

$$m(2)_3 = m(2)_2 + m(-2)_2 + m\left(\frac{1}{2} \times 2\right)_2,$$

$$m(2)_3 = 2 + 2 + 2 = 6.$$

The exceptions to this rule are ± 1 where each relies on itself and its negative, and $m(\pm 1) = 2^{n-1}$.

5.2 Number of Closed Walks of Family F

Like graph E_n, we found a way to calculate the number of closed walks of length L of F_1. However, the formula changes for odd and even L.

Theorem FII: The number of closed walks of L_{odd} of F_1.

$$a_L = 2^L,$$

and the recursive formula is

$$a_{L+2} = 4a_L \textit{ where } a_1 = 2. \blacksquare$$

Therefore, the number of closed walks of L_{odd} of F_n is

$$(a_L)^n = (2^L)^n.$$

Theorem FIII: The number of closed walks of L_{even} of F_1.

$$a_L = 2^L + 2,$$

and the recursive formula is

$$a_{L+2} = 4a_L - 6 \textit{ where } a_0 = 3. \blacksquare$$

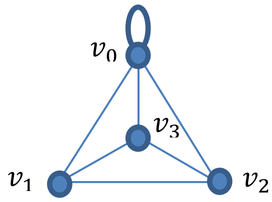
Therefore, the number of closed walks of L_{even} of F_n is

$$(a_L)^n = (2^L + 2)^n.$$

We will not be showing proofs for either of the above theorems as the process is practically identical to earlier calculations.

Chapter 6: Number of Closed Walks of Family G

Initially, we were only going to research graphs with two or three vertices. However, we happened to stumble upon a possible pattern of complete graphs with a single loop at one vertex. A complete graph is a graph where every vertex is adjacent to every other vertex except itself. In [1], we see B_n, the complete graph of two vertices with a single loop at one vertex, correlates to the Binomial Triangle. Our graph, E_n, correlates to the Binomial Pyramid, as shown above. We propose the n^{th} complete graph with a single loop at one vertex would correlate to the n^{th} level of the binomials. Thus, we wanted to consider the complete graph of four vertices with a single loop.



$$A(G_1) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure 19. Graph and Adjacency Matrix of G_1

Unfortunately, the pattern we hypothesized for the multiplicities of complete graphs with a single loop was incorrect. We hoped the multiplicities of the eigenvalues would be the Binomial Triangle in the fourth dimension. Meaning, the multiplicities of the eigenvalues of the initial adjacency matrix would all be 1. However, the multiplicities turned out to be 1, 1, and 2. We did manage to calculate the number of closed walks of length L at v_0.

Theorem G: The number of closed walks of length L from v_0 of G_1.

$$a_{L+1} = 3a_L + a_{L-1} \text{ where } a_0 = 1 \text{ and } a_1 = 1.$$

The recurrence relation was formed from experimentation.

Proof: Assume it is homogeneous and of the form $a_L = cr^L$.

$$cr^{L+1} = 3cr^L + cr^{L-1},$$

$$0 = r^2 - 3r - 1,$$

$$r = \frac{3 \pm \sqrt{13}}{2}.$$

$$a_L = c_1\left(\frac{3+\sqrt{13}}{2}\right)^L + c_2\left(\frac{3-\sqrt{13}}{2}\right)^L.$$

Using the initial conditions, we find

$$c_1 + c_2 = 1,$$

$$c_1\left(\frac{3+\sqrt{13}}{2}\right) + c_2\left(\frac{3-\sqrt{13}}{2}\right) = 1,$$

Solving for the constants,

$$c_1\left(\frac{3+\sqrt{13}}{2}\right)+(1-c_1)\left(\frac{3-\sqrt{13}}{2}\right)=1,$$

$$\frac{3}{2}c_1+\frac{\sqrt{13}}{2}c_1+\frac{3-\sqrt{13}}{2}-\frac{3}{2}c_1+\frac{\sqrt{13}}{2}c_1=1,$$

$$1-\frac{3-\sqrt{13}}{2}=\sqrt{13}c_1,$$

$$c_1=\frac{13-\sqrt{13}}{26}.$$

$$c_2=1-\frac{13-\sqrt{13}}{26},$$

$$c_2=\frac{13+\sqrt{13}}{26}.$$

Therefore, the final equation for the number of closed walks of length L in G_1 at v_0 is

$$a_L=\left(\frac{13-\sqrt{13}}{26}\right)\left(\frac{3+\sqrt{13}}{2}\right)^L+\left(\frac{13+\sqrt{13}}{26}\right)\left(\frac{3-\sqrt{13}}{2}\right)^L.\ \blacksquare$$

Therefore, the number of closed walks of L in G_n at v_0 is

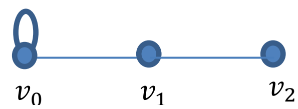
$$(a_L)^n=\left(\left(\frac{13-\sqrt{13}}{26}\right)\left(\frac{3+\sqrt{13}}{2}\right)^L+\left(\frac{13+\sqrt{13}}{26}\right)\left(\frac{3-\sqrt{13}}{2}\right)^L\right)^n.$$

Chapter 7: Discussion of Graphs H and J

As mentioned previously, we could not show an explicit formula for the number of closed walks of graphs H_n and J_n. This is because we used a method that relied on the eigenvalues of the original matrices. However, the eigenvalues of H_n and J_n could not be simplified, or in Latin, *casus irreducibilis*. In spite of this, we will show some of our findings.

7.1 Graph H

For each non-negative integer n, we define the graph H_n to have the vertex set $V_n = \{v_j : j = 0, 1, \dots, 3^n - 1\}$ and $|V_n| = 3^n$.



$$A(H_1) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Figure 20. Graph and Adjacency Matrix of H_1

Also, for each $k = 0, 1, \dots, n-1$, H_n has $\binom{n}{k}2^k$ vertices of degree 2^k, $2^n - 1$ vertices of degree 2^n, and a single vertex, v_0, of degree $2^n + 1$. The sum of the degrees of vertices in H_n is

$$\sum_{k=0}^{n-1} \binom{n}{k} 2^k \times 2^k + (2^n - 1) \times 2^n + (2^n + 1) = \sum_{k=0}^{n-1} \binom{n}{k} 4^k + 4^n - 2^n + 2^n + 1,$$

$$= \sum_{k=0}^{n} \binom{n}{k} 4^k + 1 = 5^n + 1.$$

We believe the number raised to the n^{th} degree, in this case 5, is directly related to the number of 1's in the initial matrix, H_1. This is seen in graphs D_n, E_n, and F_n as well. We believe this only occurs when the matrix is modulo two and symmetric. However, this is purely speculation from the cases discussed. An inductive proof for the sum of the degrees of vertices in H_n can be found in the appendix.

Furthermore, although D_n and H_n have different degree sets, they both have the same number of vertices, number of edges, and sum of degrees. The degree set is a sequence of numbers that corresponds with the number of edges incident to a vertex. For example, the degree set of D_1 is $1, 1, 4$ because there are two vertices that are incident to one edge and one vertex that is incident to four edges. Note, we count both ends of a loop. Perhaps any two $n \times n$ matrices with the same number of entries modulo two will have a similar result.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 21. Adjacency Matrix of H_2

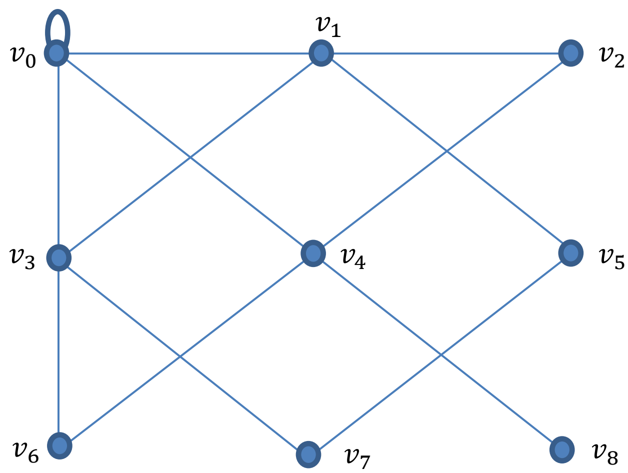
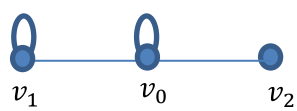


Figure 22. Graph of H_2

7.2 Graph J

For each non-negative integer n, we define the graph J_n to have the vertex set $V_n = \{v_j : j = 0, 1, \dots, 3^n - 1\}$ and $|V_n| = 3^n$.



$$A(J_1) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Figure 23. Graph and Adjacency Matrix of J_1

Also, we claim, without proof, the sum of the degrees for J_n is $6^n + 2^n$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 24. Adjacency Matrix of J_2

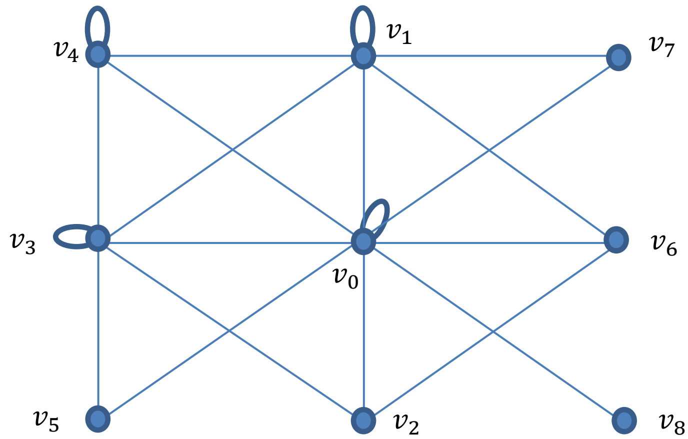


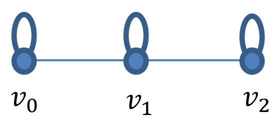
Figure 25. Graph of J_2

Chapter 8: Analysis and Discussion

A distinct feature of the graphs with determinable eigenvalues is bilateral symmetry, or symmetry in two-dimensional space. If we were to draw a line through the B_1, C_1, D_1, E_1, F_1, and G_1, we could see the left side is the mirror of the right. However, if we looked at H_1, it is possible to make the sides mirror each other, but not for J_1. Thus, our thinking is if there does not exist an orientation of a graph such that we could draw a line that would symmetrically bisect the graph, then the exact eigenvalues can be determined. But if there does exist such an orientation, then it is still unknown whether the eigenvalues can be determined. But this is purely speculation.

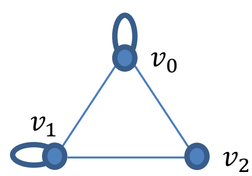
Something else to note, the C_n and D_n have the same recursive formula for the number of closed walks at v_0 but different initial conditions. This got us thinking if the recursive formulas are related, and if the recursive formula for the number of closed walks at v_0 is always the same as the recursive formula for the total number of closed walks of that graph. This is in fact the case for C_n, D_n, and E_n. But F_n and G_n do not share this similarity.

In addition, we chose the graphs we did because we wanted to remain close to the original graph B_1 from [1]. Initially, we were only going to research D_n, E_n, and H_n. However, we thought comparing more graphs would lead to more interesting findings. Two graphs we would have liked to research further are K_n and M_n. K_1 and M_1, with their corresponding adjacency matrices, can be seen below.



$$A(K_1) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Figure 26. Graph and Adjacency Matrix of K_1



$$A(M_1) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Figure 27. Graph and Adjacency Matrix of M_1

Other graphs with three vertices seemed too trivial. We also would have liked to find explicit formulas for the number of closed walks of length k at v_0 of graphs H_n and J_n.

After reading [1], we considered C_1. It has one more loop at v_0 at B_1. Thus, we would recommend considering graphs with additional loops at v_0 and complete graphs with a loop at one vertex. For complete graphs with one loop, the multiplicity of the eigenvalue, -1, seems to increase by one every time a vertex is added. The other two eigenvalues seem to have a pattern for odd and even numbers of vertices. The two non -1 eigenvalues for an even number of vertices seem to be divided by two, the integer component is equal to one less than the number of vertices, and the value under the radical is the square of that integer plus four. For example, when there are four vertices, the eigenvalues of the adjacency matrix are

$$\Lambda(G) = -1^{(2)}, \frac{3 \pm \sqrt{13}}{2}.$$

Obviously, this is closely related to the quadratic formula. The two non -1 eigenvalues for an odd number of vertices do not seem to be divided by two, the integer component is $\frac{1}{2}$ less than half the number of vertices, and the value under the radical is that integer squared plus one. For example, when there are five vertices, the eigenvalues of the adjacency matrix are

$$\Lambda(G) = -1^{(3)}, 2 \pm \sqrt{5}.$$

But that is not all. If we were to add loops to v_0 in B_1, the eigenvalues would be the non -1 eigenvalues of the complete graphs with one loop. It is possible there are even more shared characteristics that could be found.

Our project successfully demonstrated the number of closed walks of various graphs and the number of closed walks at a particular vertex in said graphs. We used [1] as our foundation, implementing similar methods to find new equations that calculate closed walks. Looking toward the future, we would recommend exploring some of the ideas mentioned above. Through this project, we were able to gain a deeper understanding and appreciation of the effort and thinking behind mathematical research in graph theory.

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Appendix

Proof of Lemma EIII:

Given this recursive formula holds,

$$a_{L+1} = a_L + 3a_{L-1} + a_{L-2} \; where \; a_1 = 1, a_2 = 7, and \; a_3 = 13,$$

we assume the solution is homogenous and of the form, $a_L = cr^L$.

$$cr^{L+1} = cr^L + 3cr^{L-1} + cr^{L-2},$$

$$0 = r^3 - r^2 - 3r - 1,$$

$$r = -1, 1 \pm \sqrt{2}.$$

Next, we substitute the solutions of r into the form, $a_L = cr^L$, which yields $a_L = c_1(-1)^L + c_2\left(1+\sqrt{2}\right)^L + c_3\left(1-\sqrt{2}\right)^L$. We then apply the initial conditions,

$$a_1 = -c_1 + c_2\left(1+\sqrt{2}\right) + c_3\left(1-\sqrt{2}\right) = 1,$$

$$a_2 = \; c_1 + c_2\left(1+\sqrt{2}\right)^2 + c_3\left(1-\sqrt{2}\right)^2 = 7,$$

$$a_3 = -c_1 + c_2\left(1+\sqrt{2}\right)^3 + c_3\left(1-\sqrt{2}\right)^3 = 13.$$

To solve for the coefficients, we put the three equations in matrix form.

$$\begin{bmatrix} -1 & 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & \left(1+\sqrt{2}\right)^2 & \left(1-\sqrt{2}\right)^2 \\ -1 & \left(1+\sqrt{2}\right)^3 & \left(1-\sqrt{2}\right)^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 13 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 1+\sqrt{2} & 1-\sqrt{2} \\ 0 & \left(1+\sqrt{2}\right)^2 + \left(1+\sqrt{2}\right) & \left(1-\sqrt{2}\right)^2 + \left(1-\sqrt{2}\right) \\ 0 & \left(1+\sqrt{2}\right)^3 - \left(1+\sqrt{2}\right) & \left(1-\sqrt{2}\right)^3 - \left(1-\sqrt{2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 12 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 1+\sqrt{2} & 1-\sqrt{2} \\ 0 & 1 & \dfrac{\left(1-\sqrt{2}\right)^2 + \left(1-\sqrt{2}\right)}{\left(1+\sqrt{2}\right)^2 + \left(1+\sqrt{2}\right)} \\ 0 & \left(1+\sqrt{2}\right)^3 - \left(1+\sqrt{2}\right) & \left(1-\sqrt{2}\right)^3 - \left(1-\sqrt{2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{8}{\left(1+\sqrt{2}\right)^2 + \left(1+\sqrt{2}\right)} \\ 12 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 1+\sqrt{2} & 1-\sqrt{2} \\ 0 & 1 & \dfrac{\left(1-\sqrt{2}\right)^2+\left(1-\sqrt{2}\right)}{\left(1+\sqrt{2}\right)^2+\left(1+\sqrt{2}\right)} \\ 0 & 0 & 4\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{8}{\left(1+\sqrt{2}\right)^2+\left(1+\sqrt{2}\right)} \\ 4\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right) \end{bmatrix},$$

Now, we solve for the coefficients.

$$4\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)c_3 = 4\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right),$$

$$c_3 = 1.$$

$$c_2 + \frac{\left(1-\sqrt{2}\right)^2+\left(1-\sqrt{2}\right)}{\left(1+\sqrt{2}\right)^2+\left(1+\sqrt{2}\right)}c_3 = \frac{8}{\left(1+\sqrt{2}\right)^2+\left(1+\sqrt{2}\right)},$$

$$c_2 = \frac{4+3\sqrt{2}}{\left(1+\sqrt{2}\right)^2+\left(1+\sqrt{2}\right)},$$

$$c_2 = 1.$$

$$-c_1 + c_2\left(1+\sqrt{2}\right) + c_3\left(1-\sqrt{2}\right) = 1,$$

$$c_1 = 1.$$

Therefore, $a_n = (-1)^n + \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ is the explicit form of $a_{n+1} = a_n + 3a_{n-1} + a_{n-2}$ where $a_1 = 1$, $a_2 = 7$, and $a_3 = 13$. ■

Inductive proof of determining the sum of the degrees of vertices in H_n:

When $n = 1$,

$$\sum_{k=0}^{1} \binom{1}{k} 4^k + 1 = 5^1 + 1.$$

Assume:

$$\sum_{k=0}^{n} \binom{n}{k} 4^k + 1 = 5^n + 1.$$

Show:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} 4^k + 1 = 5^{n+1} + 1.$$

Thus,

$$\sum_{k=0}^{n+1} \binom{n+1}{k} 4^k + 1 = \binom{n+1}{0} 4^0 + \sum_{k=1}^{n} \binom{n+1}{k} 4^k + \binom{n+1}{n+1} 4^{n+1} + 1,$$

$$= 1 + \sum_{k=1}^{n} \binom{n}{k-1} 4^k + \sum_{k=1}^{n} \binom{n}{k} 4^k + 4^{n+1} + 1,$$

$$= 1 + \sum_{k=0}^{n-1} \binom{n}{k} 4^{k+1} + \sum_{k=1}^{n} \binom{n}{k} 4^k + 4^{n+1} + 1,$$

$$= 1 + \left(\sum_{k=0}^{n} \binom{n}{k} 4^{k+1} - 4^{n+1} \right) + \left(\sum_{k=0}^{n} \binom{n}{k} 4^k - 1 \right) + 4^{n+1} + 1,$$

$$= \sum_{k=0}^{n} \binom{n}{k} 4^{k+1} + \sum_{k=0}^{n} \binom{n}{k} 4^k + 1,$$

$$= 5 \sum_{k=0}^{n} \binom{n}{k} 4^k + 1,$$

$$= 5^{n+1} + 1. \blacksquare$$