Finite Energy Functional Spaces on Unbounded Domains with a Cut

by

Will Owens

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APPROVED:
Dr. Darko Volkov, Major Advisor
Dr. Bogdan Vernescu, Head of Department

Abstract

We study in this thesis functional spaces involved in crack problems in unbounded domains. These spaces are defined by closing spaces of Sobolev H^1 regularity functions (or vector fields) of bounded support, by the L^2 norm of the gradient. In the case of linear elasticity, the closure is done under the L^2 norm of the symmetric gradient. Our main result states that smooth functions are in this closure if and only if their gradient, (respectively symmetric gradient for the elasticity case), is in L^2. We provide examples of functions in these newly defined spaces that are not in L^2. We show however that some limited growth in dimension 2, or some decay in dimension 3 must hold for functions in those spaces: this is due to Hardy's inequalities.

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Introduction

Let d be 2 or 3 and U an open and bounded subset of $\mathbb{R}^d$. If $d = 2$ let Γ be a smooth open curve strictly included in U. It is assumed that $\overline{\Gamma}$ is a smooth curve, non self intersecting, with two end points, see Figure 1; if $d = 3$, Γ is assumed to be open and bounded in U and to be a smooth manifold with a smooth boundary. In each case we assume that a normal vector $\mathbf{n}$ to Γ is well defined.

Consider the following crack problem

$$\text{find a function } u \text{ in } U \text{ such that}$$

$$\Delta u = 0 \text{ in } U \setminus \Gamma \tag{1}$$

$$u = 0 \text{ on } \partial U \tag{2}$$

$$[\partial_{\mathbf{n}} u] = 0 \text{ across } \Gamma \tag{3}$$

$$[u] = g \text{ across } \Gamma \tag{4}$$

where $[u]$ is the jump of u across Γ, which is defined as

$$[u](x) = \lim_{h \to 0+} u(x + h\mathbf{n}) - u(x - h\mathbf{n}), \quad x \in \Gamma$$

Problem (1-4) can model at least two physical situations. The first one, in electrostatics, u is an electric potential forced to be zero on the boundary of U. Γ is a charged crack in U. The electric field is continuous across Γ.
The second physical situation relates to two dimensional elastostatics: in this case we assume that $d = 2$ and that U is a cross section of an infinite domain that is linear in the third dimension. Displacements u all occur along that third dimension. It is then assumed that the boundary of U is immobile, a slip (along the third dimension) is imposed on Γ and the elastic domain deforms accordingly. The strain vector is assumed to be continuous across the crack Γ. u measures the displacements at each point.

It is well known that problem (1-4) is well posed for the unknown u in the Sobolev space $H^1(U \setminus \Gamma)$ and the forcing term g in $\tilde{H}^{\frac{1}{2}}(\Gamma)$, where this latter space is defined as follows: if Γ' is a closed smooth curve if $d = 2$ (respectively a closed smooth surface if $d = 3$) which strictly contains Γ, $\tilde{H}^{\frac{1}{2}}(\Gamma)$ is the space of functions in $H^{\frac{1}{2}}(\Gamma')$ supported in Γ, see [9]. Problem (1-4) can be easily solved from the following variational formulation. Let w be in $H^1(U \setminus \Gamma)$ such that $[w] = g$, and w is zero away from a neighborhood

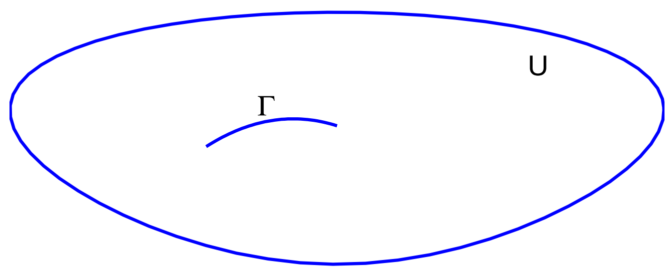

Figure 1 – *The bounded domain U cut by the crack Γ*

of Γ. The existence of w may be explained by potential theory. Let

$$G(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}$$

be the free space Green's function for the Laplacian in $\mathbb{R}^2$. Let φ be a function in $C_c^\infty(U)$ equal to 1 in a neighborhood of Γ. We may define w as

$$w(x) = -\varphi(x) \int_\Gamma \partial_{n(y)} G(x,y) g(y) dy$$

Set $\tilde{u} = u - w$. Solving for u is equivalent to solving the following variational problem for $\tilde{u}$:

$$\begin{gathered}\text{find a function } \tilde{u} \text{ in } H_0^1(U) \text{ such that}\\ \int_U \nabla \tilde{u} \cdot \nabla v = -\int_U \nabla w \cdot \nabla v, \quad \forall v \in H_0^1(U)\end{gathered}$$

or by solving the following minimization problem

$$\begin{gathered}\text{find a function } \tilde{u} \text{ in } H_0^1(U) \text{ minimizing}\\ \frac{1}{2}\int_U |\nabla v|^2 + \int_U \nabla w \cdot \nabla v, \quad v \in H_0^1(U)\end{gathered}$$

In this thesis we address the following questions: how can we formulate problem (1-4) in unbounded domains? Which functional space will need to be considered to obtain a proper formulation? What can be said in the case of full three dimensional elasticity equations? We will be particularly interested in the case where the unbounded domain of interest is the half space $x_d < 0$ cut by Γ. Focusing on this particular case is driven by applications to geophysics. In the case where the dimension d is 2, we will define a functional space, V , which was crucial in a prior study, [3]. That paper is actually about an eigenvalue problem derived from (1-4). The analog of this eigenvalue problem in the bounded set U is

$$\begin{aligned} &\text{find a function } u \text{ in } U \text{ such that} \\ \Delta u &= 0 \text{ in } U \setminus \Gamma \\ u &= 0 \text{ on } \partial U \\ [\partial_{\mathbf{n}} u] &= 0 \text{ across } \Gamma \\ \beta[u] &= \partial_{\mathbf{n}} u \text{ across } \Gamma, \end{aligned}$$

where β is the eigenvalue. This problem can be solved from the following variational formulation:

$$\text{find a function } u \text{ in } H^1(U \setminus \Gamma) \text{ such that } u = 0 \text{ on } \partial U \text{ and}$$
$$\int_U \nabla u \cdot \nabla v = \beta \int_\Gamma [u][v], \quad \forall v \in H^1(U \setminus \Gamma) \text{ such that } v = 0 \text{ on } \partial U$$

or by solving the following minimization problem

$$\text{find a function } u \text{ in } H^1(U \setminus \Gamma) \text{ such that}$$
$$u = 0 \text{ on } \partial U, \quad [u] \neq 0 \text{ and minimizing } \frac{\int_U |\nabla v|^2}{\int_\Gamma [v]^2}$$

The space V was actually defined in [3] and it was stated, without proof, that V contains strictly the space $H^1(\mathbb{R}^{2-} \setminus \Gamma)$, where $\mathbb{R}^{2-}$ is the half plane $x_2 < 0$. In this thesis, we prove this fact and we go even further: we are able to fully characterize functions in V as functions which have local Sobolev regularity H^1 and whose gradients are square integrable in $\mathbb{R}^{2-}$.
In three dimensions we are chiefly interested in linear elasticity and the corresponding elastic energy, rather than the Laplace operator and the L^2 norm of the gradient. Introducing the correct space for the analog of problem (1-4) for the equations of linear elasticity in unbounded domains was crucial in [8]. Yet again it was mentioned in that paper that this correct space contains vector fields of Sobolev H^1 regularity; no detailed discussion on how

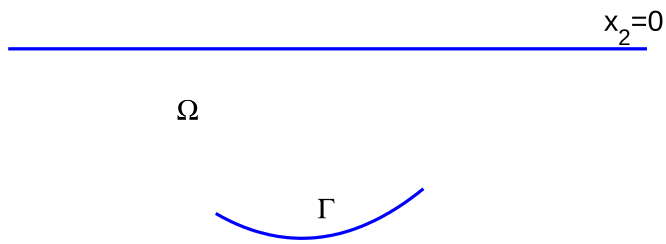

Figure 2 – *The unbounded domain Ω: Ω is the lower half space cut by the crack Γ*

to characterize vector fields in this space has been published.
Interestingly it turns out that results for the Laplace case are quite useful to studying the elastic case: this is due to a strikingly powerful family of inequalities comparing gradients to symmetric gradients in unbounded domains. These were shown by Kondrat'ev and Oleinik in [5].
We now state the main two results for this thesis. In the following two statements, if $d = 2$, Ω is the open lower half plane $x_2 < 0$ minus the open curve Γ, see Figure 2; if $d = 3$, Ω is the open lower half space $x_3 < 0$ minus the open surface Γ. We define $V(\mathbb{R}^d)$ to be the closure of the space $C_c^\infty(\mathbb{R}^d)$ under the norm defined by

$$\|u\| = (\int_{\mathbb{R}^d} |\nabla u|^2)^{\frac{1}{2}}.$$

Theorem 0.1. *A function u is in $V(\mathbb{R}^d)$ if and only if it has local H^1 regularity and $\int_{\mathbb{R}^d} |\nabla u|^2$ is finite.*

In particular the space $V(\mathbb{R}^d)$ contains strictly $H^1(\mathbb{R}^d)$. More precisely, we have the following specific example of a function in $V(\mathbb{R}^2)$ and not in $H^1(\mathbb{R}^2)$.

Proposition 0.1. *The radial function $f(r) = \ln(\ln(r^2+2))$ tends to infinity uniformly as r tends to infinity. f is not in $H^1(\mathbb{R}^2)$ but f is in $V(\mathbb{R}^2)$.*

In the case of the unbounded domain Ω we define the functional space $V(\Omega)$ as follows: we set it to be the closure of the space of functions in $H^1(\Omega)$ of bounded support under the norm defined by

$$\|u\| = (\int_\Omega |\nabla u|^2)^{\frac{1}{2}}.$$

We will prove the following:

Theorem 0.2. *A function u is in $V(\Omega)$ if and only if it has local H^1 regularity in Ω and $\int_\Omega |\nabla u|^2$ is finite. The space $V(\Omega)$ strictly contains the space $H^1(\Omega)$.*

We now define precisely what is meant by elastic energy. Let $\mathbf{u}$ be a smooth vector field in $\mathbb{R}^3$. Let $\lambda > 0$ and $\mu > 0$ be the Lamé coefficients for this elastic medium. We will denote the stress and strain tensors as follows,

$$\sigma_{ij}(\mathbf{u}) = \lambda \operatorname{div} \mathbf{u}\, \delta_{ij} + \mu\, (\partial_i u_j + \partial_j u_i),$$
$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i).$$

For vector fields $\mathbf{u}, \mathbf{v}$ whose gradient is square integrable we introduce the bilinear product

$$B(\mathbf{u}, \mathbf{v}) = \int_{\mathbb{R}^3} \lambda \operatorname{tr}(\nabla \mathbf{u}) \operatorname{tr}(\nabla \mathbf{v}) + 2\mu \operatorname{tr}(\epsilon(\mathbf{u})\epsilon(\mathbf{v})), \tag{5}$$

where tr is the trace and $\cdot^t$ marks transposition. See [1] for a thorough account of how B relates to the equations of elasticity. We define the space $W(\mathbb{R}^3)$ to be the closure of $C_c^\infty(\mathbb{R}^3)^3$ under the norm defined by

$$\|\mathbf{u}\| = B(\mathbf{u}, \mathbf{u})^{\frac{1}{2}}. \tag{6}$$

Explaining why (6) is a norm on $C_c^\infty(\mathbb{R}^3)^3$ will be part of our work. We will then show that the space $W(\mathbb{R}^3)$ is in fact quite similar to $V(\mathbb{R}^3)$.

Theorem 0.3. *The functional spaces $W(\mathbb{R}^3)$ and $V(\mathbb{R}^3)^3$ are identical. A vector field $\boldsymbol{u}$ is in $W(\mathbb{R}^3)$ if and only if it has local H^1 regularity and $B(\boldsymbol{u}, \boldsymbol{u})$ is finite.*

Finally we state our most interesting result: it pertains to linear elasticity in half space. We define the functional space $W(\Omega)$ as follows: we set it to be the closure of the space of vector fields in $H^1(\Omega)^3$ of bounded support under the norm defined by

$$\|\mathbf{u}\| = B^-(\mathbf{u},\mathbf{u})^{\frac{1}{2}},$$

where B^- is given by

$$B^-(\mathbf{u},\mathbf{v}) = \int_\Omega \lambda\,\mathrm{tr}(\nabla\mathbf{u})\mathrm{tr}(\nabla\mathbf{v}) + 2\mu\,\mathrm{tr}(\epsilon(\mathbf{u})\epsilon(\mathbf{v})),$$

We will prove the following:

Theorem 0.4. *A vector field u is in $W(\Omega)$ if and only if it has local H^1 regularity in Ω and $B^-(\boldsymbol{u},\boldsymbol{u})$ is finite.*

Throughout this thesis will use the following notation: S^{d-1} will denote the unit sphere in $\mathbb{R}^d$ and ω will be the variable on S^{d-1}. With these notations spherical coordinates are given by (r,ω) and the volume element by $r^{d-1}drd\omega$.

1 Spaces of functions of finite energy for the gradient: proofs of Theorems 0.1 and 0.2

1.1 Inclusion of $H^1(\mathbb{R}^d)$ in $V(\mathbb{R}^d)$

Lemma 1.1. *The functional space $V(\mathbb{R}^d)$ contains $H^1(\mathbb{R}^d)$.*

Proof: Let u be a fixed function in $H^1(\mathbb{R}^d)$. We need to find a function v in $C_c^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |\nabla u - \nabla v|^2$$

is arbitrarily small. We first fix a plateau function p in $C_c^\infty(\mathbb{R}^d)$ such that

$$p \text{ is radial} \tag{7}$$
$$0 \le p \le 1 \tag{8}$$
$$p(x) = 1 \text{ if } |x| \le 1 \tag{9}$$
$$p(x) = 0 \text{ if } |x| \ge 2 \tag{10}$$

The plateau function p can be constructed by convolution, see [2]. We then define

$$p_n(x) = p(\frac{x}{n}). \tag{11}$$

We now show that $p_n u$ tends to u in $V(\mathbb{R}^d)$. Fix $\varepsilon > 0$. For all n greater or equal that some N large enough,

$$(\int_{|x|\geq n} |u|^2)^{\frac{1}{2}} + (\int_{|x|\geq n} |\nabla u|^2)^{\frac{1}{2}} < \varepsilon.$$

Next for $n > N$,

$$\begin{aligned}(\int_{\mathbb{R}^d} |\nabla(u - p_n u)|^2)^{\frac{1}{2}} &= (\int_{|x|\geq n} |\nabla(u - p_n u)|^2)^{\frac{1}{2}} \\ &\leq (\int_{|x|\geq n} |\nabla u|^2)^{\frac{1}{2}} + (\int_{|x|\geq n} |\nabla(p_n u)|^2)^{\frac{1}{2}} \\ \leq 2(\int_{|x|\geq n} |\nabla u|^2)^{\frac{1}{2}} + \max|p'|\frac{1}{n}(\int_{|x|\geq n} |u|^2)^{\frac{1}{2}} &\leq 2\varepsilon + \max|p'|\varepsilon\end{aligned}$$

Next we set

$$\rho(x) = \frac{p(x)}{\int_{\mathbb{R}^d} p(x)dx},$$

and we define

$$\rho_n(x) = n^d \rho(nx). \tag{12}$$

If w is any compactly supported function in $H^1(\mathbb{R}^d)$, the convolution product $\rho_n * w$ is a function is a function in $C_c^\infty(\mathbb{R}^d)$ and the sequence $\rho_n * w$ has limit w in $H^1(\mathbb{R}^d)$, see [2]. Therefore for some m large enough

$$(\int_{\mathbb{R}^d} |\nabla(p_N u - \rho_m * (p_N u))|^2)^{\frac{1}{2}} \leq \varepsilon$$

thus

$$(\int_{\mathbb{R}^d} |\nabla(u - \rho_m * (p_N u))|^2)^{\frac{1}{2}} \leq 3\varepsilon + \max|p'|\varepsilon,$$

which proves the Lemma. $\square$

1.2 Proof of Proposition 0.1

First we notice that since f tends uniformly to infinity at infinity, f is not in $L^2(\mathbb{R}^2)$; hence f is not in $H^1(\mathbb{R}^2)$. Next we show that the gradient of f is square integrable. We have

$$\nabla f = \frac{2r}{\ln(2+r^2)(2+r^2)}$$

But

$$\int_2^\infty (\frac{2r}{\ln(2+r^2)(2+r^2)})^2 r dr$$

is finite if and only if

$$\int_2^\infty \frac{1}{r\ln^2(r)} dr$$

is finite, which is well known to hold. Fix a positive ε. Pick N such that

$$\int_{|x|\geq N} |\nabla f|^2 \leq \varepsilon$$

We now define the sequence $f_n = (f - \ln(\ln(n^2)))p_n(r)$, where p_n was defined in (11). We note that

$$\begin{aligned}(\int_{\mathbb{R}^2} |\nabla(f-f_N)|^2)^{\frac{1}{2}} &= (\int_{|x|\geq N} |\nabla(f-f_N)|^2)^{\frac{1}{2}} \\ &\leq \varepsilon + (\int_{N\leq|x|\leq 2N} |\nabla f_N|^2)^{\frac{1}{2}}.\end{aligned}$$

There now remains to show that $\int_{N\leq|x|\leq 2N} |\nabla f_N|^2$ tends to zero as N tends to infinity. To do so we split it in two pieces

$$\begin{aligned}(\int_{N\leq|x|\leq 2N} |\nabla f_N|^2)^{\frac{1}{2}} &\leq (\int_{N\leq|x|\leq 2N} |\nabla f|^2 p_N^2)^{\frac{1}{2}} + (\int_{N\leq|x|\leq 2N} |f-\ln(\ln(n^2))|^2 \frac{1}{N^2} p'(\frac{r}{N}))^{\frac{1}{2}} \\ &\leq \varepsilon + \max|p'|(\frac{1}{N^2}\int_N^{2N} \ln(\frac{\ln(2+r^2)}{\ln N^2}) r dr)^{\frac{1}{2}}\end{aligned}$$

At this stage it suffices to prove that $\frac{1}{N^2}\int_N^{2N} \ln(\frac{\ln(2+r^2)}{\ln N^2}) r dr$ tends to zero as N tends to infinity. This is easily done:

$$\frac{1}{N^2}\int_N^{2N} \ln(\frac{\ln(2+r^2)}{\ln N^2}) r dr \leq \frac{1}{N^2}\frac{4N^2-N^2}{2}\ln(\frac{\ln(2+4N^2)}{\ln N^2}),$$

but by L' Hôpital's rule $\lim_{N\to\infty} \frac{\ln(2+4N^2)}{\ln N^2} = 1$, thus $\lim_{N\to\infty} \ln \frac{\ln(2+4N^2)}{\ln N^2} = 0$. The proof of Proposition 0.1 is now complete. □

Remark: In view of the properties of function f defined in Proposition 0.1, one might wonder what is the fastest growth at infinity of functions in $V(\mathbb{R}^2)$. The answer is given in the following Proposition. A byproduct of this Proposition is that for any f in $V(\mathbb{R}^2)$, $\frac{f}{\sqrt{(1+r^2)}\ln(r^2+2)}$ is in $L^2(\mathbb{R}^2)$.

Proposition 1.2. *There is a positive constant C such that given any f in $V(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} \frac{f^2}{(1+r^2)\ln(r^2+2)^2} \leq C \int_{\mathbb{R}^2} |\nabla f|^2 \tag{13}$$

Proof: The proof relies on Hardy's inequalities. A derivation of Hardy's inequalities can be found in [6], but only for functions supported away from zero. Let g be a smooth function in $\mathbb{R}^2$ whose support is strictly between the two spheres of radius a and b and centered at the origin. We use integration by parts and Cauchy Schwartz inequality in the following

$$\int_a^b (r^{-1}(\ln r)^{-1}g)^2 r dr = \int_a^b r^{-1}(\ln r)^{-2}g^2 dr = 2|\int_a^b (\ln r)^{-1} g \partial_r g dr| \tag{14}$$

$$= 2|\int_a^b g(\ln r)^{-1} r^{-\frac{1}{2}} \partial_r g \, r^{\frac{1}{2}} dr| \tag{15}$$

$$\leq \frac{1}{\eta^2}|\int_a^b g^2(\ln r)^{-2} r^{-1} dr| + \eta^2 |\int_a^b (\partial_r g)^2 r dr|, \tag{16}$$

for any $\eta > 0$. Choosing $\eta^2 = \frac{1}{2}$ and re arranging we find that,

$$|\int_a^b (r^{-1}(\ln r)^{-1}g)^2 r dr| \leq 4 \int_a^b (\partial_r g)^2 r dr. \tag{17}$$

It is clear that for some constant C

$$\frac{1}{(1+r^2)\ln(r^2+2)^2} \leq C(r^{-1}(\ln r)^{-1})^2,$$

for all positive r. Thus (17) becomes

$$\int_a^b \frac{g^2}{(1+r^2)\ln(r^2+2)^2} r dr \leq 4C \int_a^b (\partial_r g)^2 r dr, \tag{18}$$

for all smooth functions g in $\mathbb{R}^2$ whose supports are strictly between the two spheres of radius a and b and centered at the origin. Let now f be in $C_c^\infty(\mathbb{R}^2)$. If we apply the integration by part from (14) to f, we obtain the extra term $|f^2(a)(\ln a)^{-1}|$, and eventually

$$\int_a^b \frac{f^2}{(1+r^2)\ln(r^2+2)^2} r dr \leq 4C \int_a^b (\partial_r f)^2 r dr + 2C|f^2(a)(\ln a)^{-1}|$$

As C is independent of a and b, we may let a tend to zero to obtain

$$\int_0^b \frac{f^2}{(1+r^2)\ln(r^2+2)^2} r dr \leq 4C \int_0^b (\partial_r f)^2 r dr$$

After integration if the polar angle ω, inequality (13) becomes clear for all f in $C_c^\infty(\mathbb{R}^2)$. It then must hold by density for all f in $V(\mathbb{R}^2)$. $\square$

1.3 Proof of Theorem 0.1

The only if part of the statement of Theorem 0.1 is already proved due to Lemma 1.1. To prove the if part, we look at the proof of Proposition 0.1 for guidance. Our argument will consist of subtracting an appropriate constant to the function u in $V(\mathbb{R}^d)$. The constant is provided to us by this particularly adequate version of Poincaré' s lemma:

Lemma 1.3. *Let U be an open and bounded subset in $\mathbb{R}^d$. For every u in the space $H^1(U)$ we have*

$$\int_U |u-\alpha|^2 \, dx \leq C^2 \int_U |\nabla u|^2 \, dx \tag{19}$$

where $\alpha = \dfrac{1}{meas(U)} \displaystyle\int_U u(x)dx$ and C is a constant depending only on U. Denoting U_n the set $\{nx : x \in U\}$ the previous estimate can be rescaled to

$$\int_{U_n} |u-\alpha|^2 \, dx \leq C^2 n^2 \int_{U_n} |\nabla u|^2 \, dx \tag{20}$$

Proof: A proof for estimate (19) can be found in [2]. The estimate (20) can be then inferred from (19) by setting $x' = nx$, $dx' = n^d dx$, $\nabla_{x'} = \dfrac{1}{n}\nabla_x$.
$\square$

We first prove the following intermediate result:

Proposition 1.4. *If f is in $C^\infty(\mathbb{R}^d)$ and ∇f is in $L^2(\mathbb{R}^d)$, then f is in $V(\mathbb{R}^d)$.*

Proof: Let f be in $C^\infty(\mathbb{R}^d)$ such that ∇f is in $L^2(\mathbb{R}^d)$. Define

$$\alpha_n = \frac{1}{meas(\{x : n \le |x| \le 2n\})} \int_{n\le|x|\le 2n} f dx.$$

Set $f_n = (f - \alpha_n)p_n$, where p_n was defined in (11). Clearly f_n is in $C_c^\infty(\mathbb{R}^d)$; we now want to show that $\|\nabla f - \nabla f_n\|_{L^2(\mathbb{R}^d)}$ can be made arbitrarily small. Let $\varepsilon > 0$ be given. There is a number N such that

$$\int_{|x|\ge N} |\nabla f|^2 \le \epsilon$$

We note that

$$(\int_{\mathbb{R}^d} |\nabla(f - f_N)|^2)^{\frac{1}{2}} = (\int_{|x|\ge N} |\nabla(f - f_N)|^2)^{\frac{1}{2}} \quad (21)$$

$$\le \varepsilon + (\int_{N\le|x|\le 2N} |\nabla f_N|^2)^{\frac{1}{2}} \quad (22)$$

$$\le \varepsilon + (\int_{N\le|x|\le 2N} |\nabla f|^2 p_N|^2)^{\frac{1}{2}} + (\int_{N\le|x|\le 2N} |f - \alpha_N|^2 \frac{1}{N^2}(p'(\frac{r}{N}))^2)^{\frac{1}{2}} \quad (23)$$

$$\le 2\varepsilon + \frac{1}{N} \max|p'|(\int_{N\le|x|\le 2N} |f - \alpha_N|^2)^{\frac{1}{2}} \quad (24)$$

To bound the term $\int_{N\le|x|\le 2N} |f-\alpha_N|^2$ we apply Lemma 1.3, where $\{x : 1 \le |x| \le 2\}$ plays the role of the bounded set U

$$\int_{N\le|x|\le 2N} |f - \alpha_N|^2 \le C^2N^2 \int_{N\le|x|\le 2N} |f|^2 \le C^2N^2\varepsilon^2 \quad (25)$$

Combining (24) to (25) we obtain

$$(\int_{\mathbb{R}^d} |\nabla(f - f_N)|^2)^{\frac{1}{2}} \le 2\varepsilon + C \max|p'|\varepsilon,$$

which ends this proof as ε is arbitrary. □

We are now ready to prove Theorem 0.1. Recall the definition of $H^1_{loc}(\mathbb{R}^d)$:

it is the space of functions u defined on $\mathbb{R}^d$ such that if U is any bounded open set in $\mathbb{R}^d$, the restriction of u to U is in $H^1(U)$. We need to show the the following: if u is in $H^1_{loc}(\mathbb{R}^d)$ and ∇u is in $L^2(\mathbb{R}^d)$, then there is a function v in $C_c^\infty(\mathbb{R}^d)$ such that $\|\nabla u - \nabla v\|_{L^2(\mathbb{R}^d)}$ is arbitrarily small. Fix a positive ε. First we observe that if we repeat the arguments from the proof of Proposition 1.4, we can find a function w in $H_0^1(\mathbb{R}^d)$ such that

$$\|\nabla u - \nabla w\|_{L^2(\mathbb{R}^d)} \leq \varepsilon \tag{26}$$

We now use the sequence of functions ρ_n defined in (12). As $w * \rho_n$ converges to w in $H^1(\mathbb{R}^d)$, for some integer m

$$\|\nabla w - \nabla(w * \rho_n)\|_{L^2(\mathbb{R}^d)} \leq \varepsilon \tag{27}$$

Combining (26) and (27)

$$\|\nabla u - \nabla(w * \rho_n)\|_{L^2(\mathbb{R}^d)} \leq 2\varepsilon, \tag{28}$$

which completes the proof of Theorem 0.1 since $w * \rho_n$ is in $C_c^\infty(\mathbb{R}^d)$ and ε is arbitrary. $\square$

Remark: Lemma 1.1 shows that $H^1(\mathbb{R}^3)$ is strictly contained in $V(\mathbb{R}^3)$. At this stage, using Theorem 0.1, it is straightforward to show that the inclusion of $H^1(\mathbb{R}^3)$ in $V(\mathbb{R}^3)$ is strict. The radial function

$$f(r) = \frac{1}{\sqrt{1+r^2}} \tag{29}$$

is clearly not in $L^2(\mathbb{R}^3)$, but ∇f is in $L^2(\mathbb{R}^3)$. We can claim due to Theorem 0.1 that f is in $V(\mathbb{R}^3)$. In view of function f defined in (29), one might wonder what is the fastest growth at infinity of functions in $V(\mathbb{R}^3)$. The answer is given in the following Proposition. A byproduct of this Proposition is that for any f in $V(\mathbb{R}^3)$, $\dfrac{f}{\sqrt{(1+r^2)}}$ is in $L^2(\mathbb{R}^3)$.

Proposition 1.5. *For all f in $V(\mathbb{R}^3)$*

$$\int_{\mathbb{R}^3} \frac{f^2}{(1+r^2)} \leq 4 \int_{\mathbb{R}^3} |\nabla f|^2 \tag{30}$$

Proof: The proof relies on Hardy's inequalities. Let g be a smooth function in $\mathbb{R}^3$ whose support is strictly between the two spheres of radius a

and b and centered at the origin. We use integration by parts and Cauchy Schwartz inequality in the following

$$\int_a^b (r^{-1}g)^2 r^2 dr = \int_a^b g^2 dr = 2|\int_a^b rg\partial_r g dr| \tag{31}$$

$$\leq \frac{1}{\eta^2}\int_a^b g^2 dr + \eta^2 \int_a^b (\partial_r g)^2 r^2 dr, \tag{32}$$

for any $\eta > 0$. Choosing $\eta^2 = \frac{1}{2}$ and re arranging we find that,

$$\int_a^b (r^{-1}g)^2 r^2 dr \leq 4\int_a^b (\partial_r g)^2 r^2 dr. \tag{33}$$

It is clear that

$$\frac{1}{(1+r^2)} \leq r^{-2},$$

for all positive r. Thus (33) becomes

$$|\int_a^b \frac{g^2}{(1+r^2)} r dr| \leq 4\int_a^b (\partial_r g)^2 r dr \tag{34}$$

Let now f be in $C_c^\infty(\mathbb{R}^3)$. If we apply the integration by part from (31) to f, we obtain the extra term $|af^2(a)|$, and eventually

$$|\int_a^b \frac{f^2}{(1+r^2)} r^2 dr| \leq 4\int_a^b (\partial_r f)^2 r^2 dr + 2|af^2(a)|$$

As C is independent of a and b, we may let a tend to zero to obtain

$$|\int_0^b \frac{f^2}{(1+r^2)} r^2 dr| \leq 4\int_0^b (\partial_r f)^2 r^2 dr$$

After integration if the spherical variable ω, inequality (30) becomes clear for all f in $C_c^\infty(\mathbb{R}^3)$. $\square$

1.4 Proof of Theorem 0.2

The only if part of the statement of Theorem 0.2 is clear. To prove the if part, we note that he main difference between proving this Proposition and proving Proposition 1.4 is the choice of the open bounded set U where Lemma 1.3 will be applied. We set

$$U_n = \{x : x_d < 0 \text{ and } n < |x| < 2n\}.$$

Understanbly, we only define U_n for n large enough to ensure that Γ does not intersect U_n. We sketched U_n in Figure 3 in the case $d = 2$ for the reader's convenience. Let f be a locally H^1 function in Ω such that ∇f is in $L^2(\Omega)$. Set $f_n = (f - \alpha_n)p_n$ where

$$\alpha_n = \frac{1}{meas(U_n)} \int_{U_n} f dx.$$

f_n is in $V(\Omega)$ by definition of that space; we now want to show that $\|\nabla f - \nabla f_n\|_{L^2(\mathbb{R}^d)}$ can be made arbitrarily small. Fix $\varepsilon > 0$ be given. There is a number N such that

$$\int_{|x| \geq N,\, x_d < 0} |\nabla f|^2 \leq \epsilon$$

We note that

$$(\int_{x_d<0} |\nabla(f - f_N)|^2)^{\frac{1}{2}} = (\int_{|x|\geq N,\, x_d<0} |\nabla(f - f_N)|^2)^{\frac{1}{2}} \tag{35}$$

$$\leq \varepsilon + (\int_{U_n} |\nabla f_N|^2)^{\frac{1}{2}} \tag{36}$$

$$\leq \varepsilon + (\int_{U_n} |\nabla f|^2 p_N|^2)^{\frac{1}{2}} + (\int_{U_n} |f - \alpha_N|^2 \frac{1}{N^2} (p'(\frac{r}{N}))^2)^{\frac{1}{2}} \tag{37}$$

$$\leq 2\varepsilon + \frac{1}{N} \max |p'| (\int_{U_n} |f - \alpha_N|^2)^{\frac{1}{2}} \tag{38}$$

To bound the term $\int_{U_n} |f - \alpha_N|^2$ we apply Lemma 1.3, where

$$\{x : 1 \leq |x| \leq 2 \text{ and } x_d < 0\}$$

plays the role of the bounded set U. This leads to

$$\int_{U_n} |f - \alpha_N|^2 \leq C^2 N^2 \int_{U_n} |f|^2 \leq C^2 N^2 \varepsilon^2 \tag{39}$$

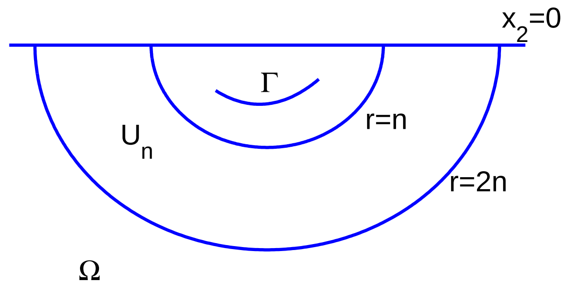

Figure 3 – *The bounded domain U_n in the half plane $x_2 < 0$*

Combining (39) to (39) we obtain

$$(\int_{x_d<0} |\nabla(f - f_N)|^2)^{\frac{1}{2}} \leq 2\varepsilon + C \max |p'|\varepsilon,$$

which ends this part of the proof as ε is arbitrary.

To show that the inclusion of $H^1(\Omega)$ in $V(\Omega)$ is strict, define the function f

$$f(x) = \begin{cases} \ln(\ln(r^2 + 2)) \text{ if } d = 2 \\ \dfrac{1}{\sqrt{1 + r^2}} \text{ if } d = 3 \end{cases}$$

Due to Proposition 0.1 and the remark from section 1.3 it is clear that f is in $V(\Omega)$ but is not in $L^2(\Omega)$. $\square$

For sake of completeness, we now prove a further density result on $V(\Omega)$. Recall that functions in $V(\Omega)$ may be non zero on the boundary $x_d = 0$ as well as on the interior boundary Γ: this certainly prohibits the space $C_c^\infty(\Omega)$ to be dense in $V(\Omega)$. We then construct another space of smooth functions of bounded support

$$C^\infty_{B,2}(\Omega) = \{f \in C^\infty(\Omega) : \text{ supp } f \text{ is bounded and } \nabla f \in L^2(\Omega)\}. \qquad (40)$$

We have following density result:

Theorem 1.1. *$C^\infty_{B,2}(\Omega)$ is dense in $V(\Omega)$.*

Proof: Let u be in $V(\Omega)$. Fix a positive ε. Due to Theorem 0.2, we can find a function w in $H^1(\Omega)$ of bounded support such that

$$\|\nabla u - \nabla w\|_{L^2(\Omega)} \leq \varepsilon \tag{41}$$

We now use the sequence of functions ρ_n defined in (12). As $w * \rho_n$ converges to w in $H^1(\Omega)$, for some integer m

$$\|\nabla w - \nabla(w * \rho_n)\|_{L^2(\Omega)} \leq \varepsilon \tag{42}$$

Combining (41) and (42)

$$\|\nabla u - \nabla(w * \rho_n)\|_{L^2(\Omega)} \leq 2\varepsilon, \tag{43}$$

which completes the proof since $w * \rho_n$ is in $C^\infty(\Omega)$ and has bounded support, and ε is arbitrary. □

2 Functional spaces of vector fields of finite elastic energy: proof of Theorem 0.3

2.1 Introductory results

In this section we review some well known results pertaining to the elastic energy defined in (5).

Lemma 2.1. *The bilinear product*

$$(P, Q) \to tr\ (PQ^t)$$

defines a dot product on the space of 3 by 3 matrices with real coefficients.

Proof: This is a well known result in Linear Algebra. To prove positiveness one has to apply the spectral theorem to the symmetric matrix PP^t. □

Lemma 2.2. *Let U be a bounded open set in $\mathbb{R}^3$ with smooth boundary. Define*

$$B_U(\boldsymbol{u},\boldsymbol{v}) = \int_U \lambda\, tr(\nabla \boldsymbol{u}) tr(\nabla \boldsymbol{v}) + 2\mu\, tr(\epsilon(\boldsymbol{u})\epsilon(\boldsymbol{v})),$$

for smooth vector fields $\mathbf{u}, \boldsymbol{v}$ in U, and where $\lambda > 0$ and $\mu > 0$. If $B_U(\boldsymbol{u},\boldsymbol{u}) = 0$ then $\mathbf{u}(x)$ is in the form $Ax + C$, where A is a constant antisymmetric 3 by 3 matrix, and C is a constant vector.

Proof: If $B_U(\mathbf{u},\mathbf{u}) = 0$ then $\epsilon(\mathbf{u}) = 0$ in U, so $\nabla\mathbf{u}$ is antisymmetric. It follows that $\partial_1 u_1 = \partial_2 u_2 = \partial_3 u_3 = 0$. We also have $\partial_2 u_1 = -\partial_1 u_2$ and $\partial_3 u_1 = -\partial_1 u_3$, so that $\partial_2^2 u_1 = \partial_3^2 u_1 = 0$. This shows u_1 is a linear function plus a constant. Similarly, u_2 and u_3 are in the same form. Thus $\mathbf{u}(x) = Ax + C$, for some matrix A and constant vector C. As $\nabla\mathbf{u} = A$, A must be antisymmetric. □

Lemma 2.3. *B_U defines a dot product on $C_c^\infty(U)^3$.*

Proof: It is clear that B_U is bilinear and non negative. To prove that it is also definite, assume that $B_U(\mathbf{u},\mathbf{u}) = 0$ for some $\mathbf{u}$ in $C_c^\infty(U)^3$. According to the previous lemma $\mathbf{u}$ must be in the form $\mathbf{u}(x) = Ax + C$. Therefore if $\mathbf{u}$ is non zero, the null set of $\mathbf{u}$ is an affine space of dimension at most 2: this can't be as $\mathbf{u}$ is required to be zero everywhere on the boundary ∂U. □

Lemma 2.4. *The two norms $B_U(u,u)^{\frac{1}{2}}$ and $\left(\int_U tr\ (\epsilon(\boldsymbol{u})\epsilon(\boldsymbol{u}))\right)^{\frac{1}{2}}$ are equivalent on $C_c^\infty(U)^3$.*

Proof: It is clear that

$$\sqrt{2\mu}\left(\int_U \mathrm{tr}\ (\epsilon(\mathbf{u})\epsilon(\mathbf{u}))\right)^{\frac{1}{2}} \leq B_U(u,u)^{\frac{1}{2}}$$

To show a reverse estimate, we first observe that for any real 3 by 3 matrix P, $\mathrm{tr}\ (P) = \dfrac{1}{2}\mathrm{tr}\ (P + P^t)$. Now as the trace is continuous, there exists a constant C such that $\text{tr } (Q) \leq C\sqrt{\text{tr } (QQ^t)}$. We thus have that $\lambda\text{tr } (\nabla \mathbf{u})^2 \leq C^2 \text{tr } (\epsilon(\mathbf{u})^2)$, which we can integrate over U:

$$\int_U \lambda\text{tr } (\nabla \mathbf{u})^2 \leq C^2 \int_U \text{tr } (\epsilon(\mathbf{u})^2)$$

$>$From here we infer that there exists a constant C such that

$$B_U(u,u)^{\frac{1}{2}} \leq C \left(\int_U \text{tr } (\epsilon(\mathbf{u})\epsilon(\mathbf{u})) \right)^{\frac{1}{2}}$$

$\square$

It is now clear that the bilinear functional B introduced in (5) defines a dot product on $C_c^\infty(\mathbb{R}^3)^3$. We need one more lemma before moving on to the proof of Theorem 0.3. Interestingly this lemma may also serve as an alternative proof for Lemma 2.3.

Lemma 2.5. *Let $\boldsymbol{u}$ be in $C_c^\infty(U)^3$. The following estimate holds:*

$$\frac{1}{2}\int_U |\nabla \boldsymbol{u}|^2 \leq \int_U tr(\epsilon(\boldsymbol{u})\epsilon(\boldsymbol{u})) \leq \int_U |\nabla \boldsymbol{u}|^2 \tag{44}$$

The norms defined by $B_U(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}$ and $(\int_U |\nabla \boldsymbol{u}|^2)^{\frac{1}{2}}$ are equivalent on $C_c^\infty(U)^3$.

Proof: Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the natural basis of $\mathbb{R}^3$. Let $\mathbf{u}$ be in $C_c^\infty(U)^3$. Denote $\mathbf{n}$ the exterior normal on ∂U. Using the divergence theorem we may write

$$0 = \int_{\partial U} (\partial_i u_j u_i \mathbf{e}_j - \partial_j u_j u_i \mathbf{e}_i)\mathbf{n}$$

$$= \int_U \partial_i u_j \partial_j u_i - \partial_j u_j \partial_i u_i$$

$$= \frac{1}{2}\int_U (\partial_i u_j + \partial_j u_i)(\partial_j u_i + \partial_i u_j) - \int_U \partial_j u_j \partial_i u_i - \frac{1}{2}\int_U (\partial_i u_j)^2 + (\partial_j u_i)^2,$$

summing over all indices i, j in $\{1,2,3\}$ we find that

$$2\int_U \text{tr}(\epsilon(\mathbf{u})\epsilon(\mathbf{u})) - \int_U (\text{div } \mathbf{u})^2 - \int_U |\nabla \mathbf{u}|^2 = 0,$$

which proves that $\frac{1}{2}\int_U |\nabla \mathbf{u}|^2 \leq \int_U \operatorname{tr}(\epsilon(\mathbf{u})\epsilon(\mathbf{u}))$. As

$$(\partial_i u_j + \partial_j u_i)^2 \leq 2(\partial_i u_j)^2 + 2(\partial_j u_i)^2,$$

$\int_U \operatorname{tr}(\epsilon(\mathbf{u})\epsilon(\mathbf{u})) \leq \int_U |\nabla \mathbf{u}|^2$ is clear. Since the Lamé coefficients λ and μ are two positive constants, the fact that the norms defined by $B_U(\mathbf{u},\mathbf{u})^{\frac{1}{2}}$ and $(\int_U |\nabla \mathbf{u}|^2)^{\frac{1}{2}}$ are equivalent on $C_c^\infty(U)^3$ can be easily derived from (44). □

2.2 Proof of Theorem 0.3

$W(\mathbb{R}^3)$ is the closure of $C_c^\infty(\mathbb{R}^3)^3$ under the norm defined by $B(\mathbf{u},\mathbf{u})^{\frac{1}{2}}$ and $V(\mathbb{R}^3)$ is the closure of $C_c^\infty(\mathbb{R}^3)^3$ under the norm defined by $(\int_U |\nabla \mathbf{u}|^2)^{\frac{1}{2}}$: it is now clear due to Lemma 2.5 that the two closures must be the same. The remainder of the proof of Theorem 0.3 is still non trivial: although it is clear due to 0.1 that if a vector field $\mathbf{u}$ has local H^1 regularity and $\int_{\mathbb{R}^d} |\nabla \mathbf{u}|^2$ is finite, $\mathbf{u}$ is necessarily in $V(\mathbb{R}^3)$, it is not clear yet how to compare B and the full gradient over the space $W(\mathbb{R}^3)$. This is where the following lemma due to Kondrat'ev and Oleinik comes into play.

Lemma 2.6. *There is a positive constant C such that for all vector fields $\boldsymbol{u}$ which are in $H^1(U)^3$ for all bounded open set U in $\mathbb{R}^3$, the following estimate holds*

$$\int_{\mathbb{R}^3} |\nabla \boldsymbol{u} - A|^2 \leq C \int_{\mathbb{R}^3} |\epsilon(\boldsymbol{u})|^2, \tag{45}$$

where A is an antisymmetric 3 by 3 matrix depending on $\boldsymbol{u}$.

Lemma 2.6 is proved in even more generality in [5].
To finish the proof of Theorem 0.3, fix a vector field $\mathbf{u}$ which which has locally H^1 regularity in $\mathbb{R}^3$ and such that $B(u,u)$ is finite. We then apply Lemma 2.6 and we set $\mathbf{v}(x) = \mathbf{u}(x) - Ax$ for the corresponding antisymmetric matrix A. Clearly $\nabla \mathbf{v}$ is in $L^2(\mathbb{R}^3)$. Theorem (0.1) can be applied to $\mathbf{v}$: there is a sequence $\mathbf{v}_n$ in $C_c^\infty(\mathbb{R}^3)$ such that $\nabla \mathbf{v} - \nabla \mathbf{v}_n$ tends to zero in $L^2(\Omega)$. But since

$$\int_{\mathbb{R}^3} \operatorname{tr}(\epsilon(\mathbf{v}-\mathbf{v}_n)\epsilon(\mathbf{v}-\mathbf{v}_n)) \leq \int_{\mathbb{R}^3} |\nabla \mathbf{v} - \mathbf{v}_n|^2,$$

we have that $B(\mathbf{v}-\mathbf{v}_n, \mathbf{v}-\mathbf{v}_n)$ tends to zero, which is the same as $B(\mathbf{u}-\mathbf{v}_n, \mathbf{u}-\mathbf{v}_n)$. $\square$

2.3 Proof of Theorem 0.4

To prove this theorem we will need the full force of Theorem 3 in [5], a pivotal paper by Kondrat'ev and Oleinik. We note that the open set $K = \{x \in \mathbb{R}^3 : x_3 < 0\}$ is a cone: for any x in K and $\alpha > 0$, αx is in K. In addition Γ is bounded. Therefore, according to Theorem 3 in [5], there is a positive constant C such that

$$\int_\Omega |\nabla \mathbf{u} - A|^2 \leq C \int_\Omega |\epsilon(\mathbf{u})|^2, \tag{46}$$

for all $\mathbf{u}$ whose restriction to all bounded open sets U included in Ω is in $H^1(U)$, and A is an antisymmetric 3 by 3 matrix depending on $\mathbf{u}$.
Fix a vector field $\mathbf{u}$ which which has local H^1 regularity in Ω and such that $B^-(u,u)$ is finite. We then apply inequality (46) and we set $\mathbf{v}(x) = \mathbf{u}(x) - Ax$ for the respective antisymmetric matrix A. Clearly $\nabla \mathbf{v}$ is in $L^2(\Omega)$. Theorem (0.2) can be applied to $\mathbf{v}$: there is a sequence $\mathbf{v}_n$ of vector fields of bounded support in $H^1(\Omega)$ such that $\nabla \mathbf{v} - \nabla \mathbf{v}_n$ tends to zero in $L^2(\Omega)$. But since

$$\int_\Omega \operatorname{tr}(\epsilon(\mathbf{v}-\mathbf{v}_n)\epsilon(\mathbf{v}-\mathbf{v}_n)) \leq \int_\Omega |\nabla \mathbf{v} - \mathbf{v}_n|^2,$$

we have that $B^-(\mathbf{v}-\mathbf{v}_n, \mathbf{v}-\mathbf{v}_n)$ tends to zero, which is the same as $B^-(\mathbf{u}-\mathbf{v}_n, \mathbf{u}-\mathbf{v}_n)$. $\square$

For sake of completeness, we now prove a further density result on $W(\Omega)$. Recall that vector fields in $W(\Omega)$ may be non zero on the boundary $x_3 = 0$ as well as on the interior boundary Γ: this certainly prohibits the space $C_c^\infty(\Omega)^3$ to be dense in $W(\Omega)$. Recall the definition of the space $C_{B,2}^\infty(\Omega)$ given by (40). We have following density result:

Theorem 2.1. *$C_{B,2}^\infty(\Omega)^3$ is dense in $W(\Omega)$.*

Proof: Let $\mathbf{u}$ be in $W(\Omega)$. Fix a positive ε. Due to Theorem 0.2, we can find a vector field $\mathbf{w}$ in $H^1(\Omega)^3$ and of bounded support such that

$$B^-(\mathbf{u}-\mathbf{w}, \mathbf{u}-\mathbf{w})^{\frac{1}{2}} \leq \varepsilon \tag{47}$$

We now use the sequence of functions ρ_n defined in (12). As $\mathbf{w} * \rho_n$ converges to $\mathbf{w}$ in $H^1(\Omega)^3$, for some integer m

$$B^-(\mathbf{w} - \mathbf{w} * \rho_n, \mathbf{w} - \mathbf{w} * \rho_n)^{\frac{1}{2}} \leq \varepsilon \tag{48}$$

Combining (47) and (48)

$$B^-(\mathbf{u} - \mathbf{w} * \rho_n, \mathbf{u} - \mathbf{w} * \rho_n)^{\frac{1}{2}} \leq 2\varepsilon, \tag{49}$$

which completes the proof since $w * \rho_n$ is in $C^\infty(\Omega)^3$ and has bounded support, and ε is arbitrary. $\square$

2.4 Example of a vector field u in $W(\Omega)$ which is not affine, not in $L^2(\Omega)^3$ and whose gradient is not in $L^2(\Omega)^9$

This example illustrates two facts:
- $H^1(\Omega)^3$ is strictly included in $W(\Omega)$,
- $V(\Omega)$ is strictly included in $W(\Omega)$.

we set

$$\mathbf{u}(x) = x_2 e_1 - x_1 e_2 + \frac{1}{\sqrt{1+r^2}} e_3$$

Note that the linear part of $\mathbf{u}$, that is, $x_2 e_1 - x_1 e_2$ is antisymmetric: its symmetric part is therefore zero. The other part, that is, $\frac{1}{\sqrt{1+r^2}}$ was explained earlier not to be in $L^2(\Omega)$ and to have its gradient in $L^2(\Omega)$.

3 Discussion and directions for future work

3.1 Decay at infinity of solutions to crack problems and integral representations using adequate Green's functions

If $d = 2$ or 3, consider the following crack problem in the unbounded domain Ω

$$\begin{aligned}
&\text{find a function } u \text{ in } \Omega \text{ such that} \\
&\Delta u = 0 \text{ in } \Omega \setminus \Gamma && (50)\\
&\partial_{\mathbf{n}} u = 0 \text{ on } x_d = 0 && (51)\\
&[\partial_{\mathbf{n}} u] = 0 \text{ across } \Gamma && (52)\\
&[u] = g \text{ across } \Gamma && (53)
\end{aligned}$$

It is natural to assume that the forcing term g is in $\tilde{H}^{\frac{1}{2}}(\Gamma)$ and to express problem (50-53) as a variational problem in the space $V(\Omega)$. Let w be in $H^1(\Omega)$ such that w has bounded support and $[w] = g$. Set $\tilde{u} = u - w$. Solving for u is equivalent to solving the following variational problem for $\tilde{u}$:

$$\text{find a function } \tilde{u} \text{ in } V(\mathbb{R}^{2-}) \text{ such that} \tag{54}$$

$$\int_{\Omega} \nabla \tilde{u} \cdot \nabla v = -\int_{\Omega} \nabla w \cdot \nabla v, \quad \forall v \in V(\mathbb{R}^{2-}) \tag{55}$$

It is clear that any constant solves problem (50-53) if $g = 0$, and the variational problem (54-55) has a unique solution in $V(\mathbb{R}^{2-})$.
Of particular interest is the following issue: how "fast" does u solution to (50-53) decay to infinity? It should be fairly simple to address this question using potential theory: the solution u could be expressed as an integral over the crack Γ of the adequate Green's function for problem (50-53) times a correct density.
The case of three dimensional elasticity is even more interesting, due to important applications in geophysics, see [4]. For $d = 3$, consider the following crack problem in the unbounded domain Ω, where the stress tensor is defined

by $\sigma(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda \text{ div } \mathbf{u}I_3$,

$$\text{find a vector field } \mathbf{u} \text{ in } \Omega \text{ such that}$$
$$\text{div } (\sigma(\mathbf{u})) = 0 \text{ in } \Omega \tag{56}$$
$$\sigma(\mathbf{u})\mathbf{n} = 0 \text{ on } x_3 = 0 \tag{57}$$
$$[\sigma(\mathbf{u})] = 0 \text{ across } \Gamma \tag{58}$$
$$[\mathbf{u}] = \mathbf{g} \text{ across } \Gamma \tag{59}$$

It is natural to assume that the forcing term $\mathbf{g}$ is in $\tilde{H}^{\frac{1}{2}}(\Gamma)^3$ and to express problem (56-59) as a variational problem in the space $W(\Omega)$. Let $\mathbf{w}$ be in $H^1(\Omega)^3$ such that $[\mathbf{w}] = \mathbf{g}$, $[\sigma_{\mathbf{n}}(\mathbf{w})] = 0$, and $\mathbf{w}$ is compactly supported in a neighborhood of Γ. The existence of $\mathbf{w}$ can be argued by use of an appropriate Green's tensor and use of a smooth cut off function. Set $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{w}$.

$$\text{find a function } \tilde{\mathbf{u}} \text{ in } V(\mathbb{R}^{3-}) \text{ such that} \tag{60}$$
$$B^-(\tilde{\mathbf{u}}, \mathbf{v}) = -B^-(\mathbf{w}, \mathbf{v}), \quad \forall \mathbf{v} \in V(\mathbb{R}^{3-}), \tag{61}$$

where the space $V(\mathbb{R}^{3-})$ is the closure of smooth vector fields in $\mathbb{R}^{3-}$ of bounded support for the norm defined by $B^-(\mathbf{v}, \mathbf{v})^{\frac{1}{2}}$. It is clear that any constant vector solves problem (56-59) if $\mathbf{g} = 0$, and the variational problem (60-61) has a unique solution in $W(\Omega)$.
Of particular interest is the following issue: how "fast" does $\mathbf{u}$ solution to (56-59) decay to infinity? The solution $\mathbf{u}$ could be expressed as an integral over the crack Γ of the adequate Green's function for problem (50-53) times a correct density. Now, if more generally we allow solutions to (56-59) to grow at infinity while having finite B^- energy, what is the general form of these solutions? We would have to examine which antisymmetric linear vector fields solve (56-59) if $\mathbf{g} = 0$; the answer might also depend on the geometry of Γ.

3.2 Decay at infinity of solutions to crack problems and adequate artificial boundary conditions for numerical solutions in truncated domains

If $d = 2$, a numerical solution to problem (50-53) can be obtained by solving a boundary integral equation on Γ. Although the related integral operator is hypersingular, a very efficient and accurate numerical scheme can be devised, see [3]. If $d = 3$ integral equation methods become much more involved since Γ is in that case a surface; there is no simple way of treating the hypersingular character of the integral operator and matters become even more complicated close to the boundary of Γ. A simpler way of obtaining a numerical solution to problem (50-53) is to employ a finite element package. However an additional issue arises: the computational domain must be bounded and an artificial boundary condition has to be devised. This idea was implemented for the elasticity case, namely for an eigenvalue problem related to (56-59): this was done in [8]. Numerical convergence was observed. It was noted in [8] that this numerical method is reliable since it was able to reproduce with great accuracy for the two dimensional case a numerical solution obtained by use of integral equations. However no theoretical argument was given in [8] to validate the domain truncation method. Denote

$$\Omega_n = \{x \in \Omega : |x| < n\}$$

the truncated domain, and S_n the part of the boundary of Ω_n where $|x| = n$. If $\mathbf{u}$ is in $W(\Omega)$ and solves (56-59), the following integral representation holds

$$\mathbf{u}(x) = \int_\Gamma \tilde{K}(x, y)\mathbf{g}(y)dy, \tag{62}$$

where $\tilde{K}$ is the adequate Green's tensor for problem (56-59), see [7]. Due to the asymptotic form of $\tilde{K}$ at infinity the following estimates hold for $\mathbf{u}$

$$\mathbf{u}(x) = O(\frac{1}{|x|^2}) \tag{63}$$

$$\partial_r \mathbf{u}(x) = O(\frac{1}{|x|^3}) \tag{64}$$

$$-\frac{2}{|x|}\mathbf{u}(x) + \partial_r \mathbf{u}(x) = O(\frac{1}{|x|^4}) \tag{65}$$

Let $\mathbf{u}_n$ be the solution to equations (56-59) in the truncated domain Ω_n:

$$\text{find a vector field } \mathbf{u}_n \text{ in } H^1(\Omega_n) \text{ such that}$$

$$\text{div } (\sigma(\mathbf{u}_n)) = 0 \text{ in } \Omega_n \tag{66}$$

$$\sigma(\mathbf{u}_n)\mathbf{n} = 0 \text{ on } x_3 = 0 \tag{67}$$

$$[\sigma(\mathbf{u}_n)] = 0 \text{ across } \Gamma \tag{68}$$

$$[\mathbf{u}_n] = \mathbf{g} \text{ across } \Gamma \tag{69}$$

$$\mathbf{u}_n \text{ satisfies a boundary condition on } S_n \tag{70}$$

Keeping in mind (63-65) it is natural to choose the boundary condition on S_n (70) to be one of the following three:

$$\mathbf{u}_n(x) = 0 \tag{71}$$

$$\partial_r \mathbf{u}_n(x) = 0 \tag{72}$$

$$-\frac{2}{|x|}\mathbf{u}(x) + \partial_r \mathbf{u}(x) = 0 \tag{73}$$

We would then like to explore the study of the convergence of $\mathbf{u}_n$ solution to (66-70) to $\mathbf{u}$ solution to (56-59) for each of the three possible boundary conditions on S_n, (71) to (73).

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