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GABRIEL'S HORN

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By

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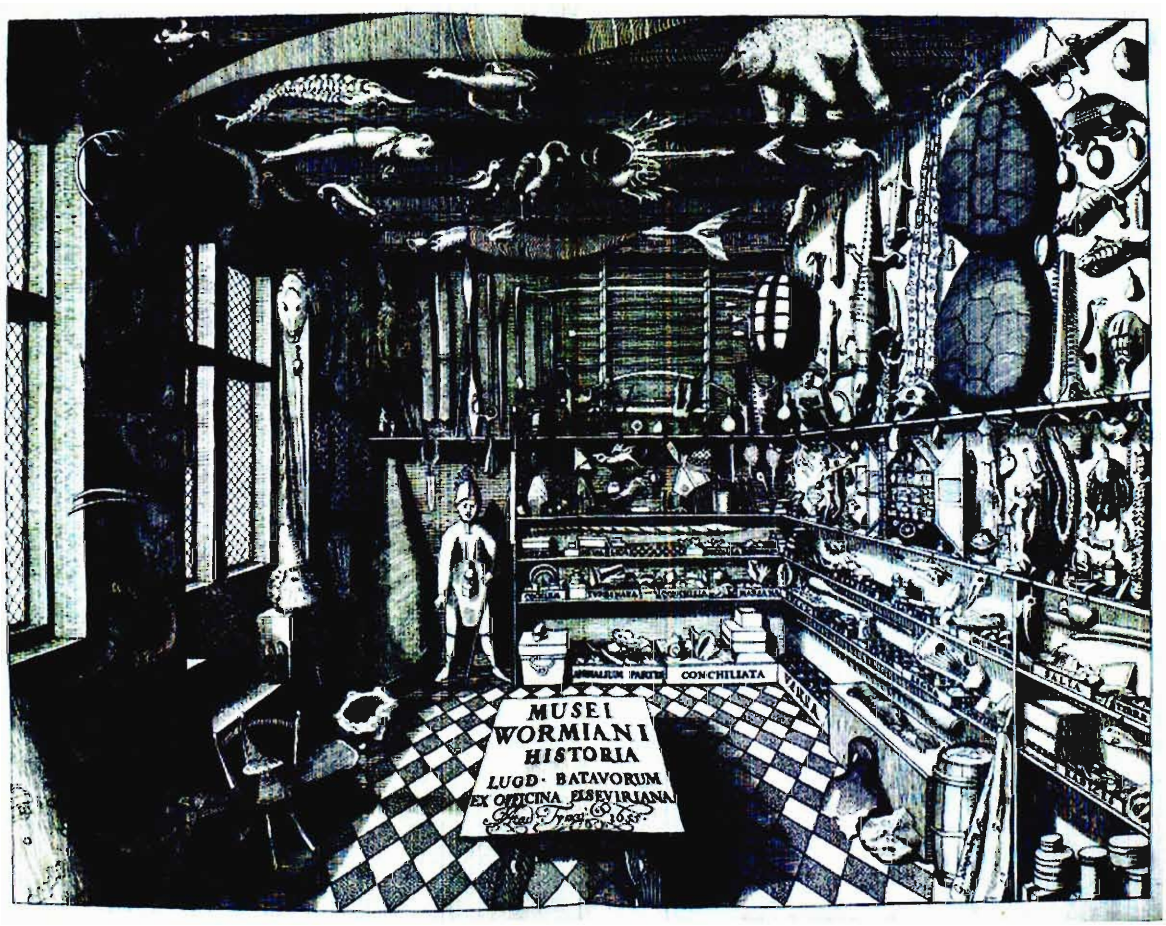
Abstract

Though Cavalieri is well known for the *Method of Indivisibles*, the ideas underlying this method are generally not. We explore Cavalieri's methods with links to Euclidian theory, Messrs. Galileo and Torricelli and show how his method differs from those using infinitesimals. Finally, we a recreate a classic proof using techniques borrowed from Cavalieri's method.

GABRIEL'S HORN: Indivisibles, Paradoxical Solids and Other Ponderables from the Cabinet of Curiosities in 17th Century Mathematics.



Detail from: Ole Worm, *Museum Wormianum* [Worm's Museum], Leiden, 1655. Rare Books Division.

"Learned gentlemen should build a goodly, huge cabinet, wherein whatsoever the hand of man by exquisite art or engine has made rare in stuff, form or motion; whatsoever singularity, chance, and the shuffle of things hath produced; whatsoever Nature has wrought in things that want life and may be kept; shall be sorted and included."

Francis Bacon

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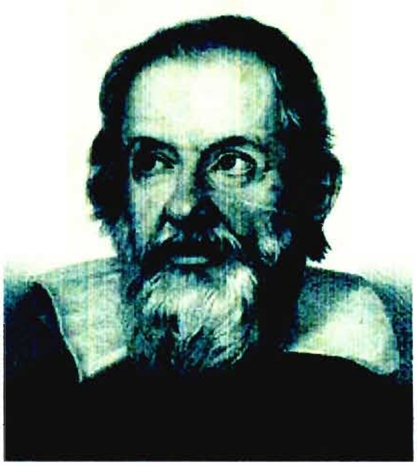
1 Outline

1.1 The Galileo Code

In Two New Sciences a dialogue published toward the end of his life in 1638, Galileo addressed the philosophical problem of mathematical infinity. Essentially taking the Aristotelian position, Simplicio says the following to Salviati:



Galileo Galilei
1564-1642
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Simplicio; "From this [the problems posed by the mathematics of indivisibles] immediately arises a doubt that seems to me unresolvable. It is that we certainly do find lines of which one may say that one is greater than another; each containing an infinite number of points, we are forced to admit that, within one and the same class, we may have something greater than infinity, because the infinity of points in the long line is greater than the infinity of points in the short line.

Bonaventura
Francesco Cavalieri
1598 - 1647
Archivio Fotografio
dei Civici Musei,
Milano

This assigning to an infinite quantity a value greater than infinity is quite beyond my comprehension."

These remarks were specifically addressed to one, Bonaventura Cavalieri, father of the Method of Indivisibles; a new means of finding areas and volumes of figures. Correspondence between the two reveals that Cavalieri had sought desperately to win Galileo's approval for his new method.[1] It never came.

Salviati's reply to Simplicio's comments on infinities, (arguably Galileo's own position on the matter), goes as follows:

> *Salviati; "This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another."*[2]

By the early 1600's Galileo had began to show signs of a modern attitude toward the infinite, when he proposed, (*using Salviati as his mouthpiece*), that: "infinity should obey a different arithmetic than finite numbers." That Galileo may have been overly cautious in putting forward his views on the topic is understandable. He no doubt was keenly aware of Giordano Bruno's own fate at the hand of the Inquisition.[3]

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[1] Cavalieri corresponded with many mathematicians and scientists of his day, including Galileo, Mersenne, Renieri, Rocca, Torricelli and Viviani. His correspondence with Galileo includes at least 112 letters. Of Cavalieri, Galileo is reportedly to have written; "few, if any, since Archimedes, have delved as far and as deep into the science of geometry."

[2] Two New Sciences; Galileo Galilei translated by Stillman Drake pp.39

[3] Bruno, a dabbler in astronomy and developer of a rather complex system of mnemonics, had argued the case that the universe was infinite and that the stars were distant suns. This ruffled the feathers of Church authorities, who held quite the opposite view. For his troubles, Bruno was tortured for nine years in an attempt to make him recant his heretical views. Stubborn to the end, he was burned at the stake in 1600. Having once taught briefly at the University of Padua in 1591, Bruno had ambitions for a permanant position, but the chair he sought went instead to Galileo Galilei. Bruno was denounced to the Inquisition a year later.

The dialogue neatly encapsulates the era in which Cavalieri worked; sandwiched between a revival of ancient Greek methods and the birth of modern notions of infinitesimals, with which Newton and Leibniz were to develop the calculus a half century later. While Newton and Leibniz made use of arguments based upon intuitive notions of infinitesimals which were appealing and produced correct results, their arguments were not mathematically rigorous. Lack of rigor was something Cavalieri felt he could ill afford, as any novel method would most certainly suffer the arrows of critical attack if it could not be demonstrated using accepted first principles.

If you ask any first year college student what is their most troublesome course, the answer most likely; "Calculus." The basic idea of the calculus is actually quite simple; though quite counterintuitive to minds cultivated by experience with discreet numbers and finite sets. Essentially the calculus involves doing mathematics on sets of things which are allowed to become both, infinitely large in number and infinitely small in size, such that some combination of the two remains finite and meaningful. Students who resist such fancies find themselves in good company.

Up to the 17th century infinities and infinitesimals had no proper place in mathematical discourse. Within the mathematical principle set for by Euclid, infinitesimals cannot be "parts" of the mathematical object they belong to- because comparability and the quality of "being a part," go together. Thus, when Cavalieri developed his method of 'indivisibles', (not to be confused with infinitesimals) he allowed himself only the sort of indivisibles that classical geometry and philosophical tradition knew of.

Though Cavalieri is well known for the method of indivisibles, the ideas underlying this method are generally little known. This almost paradoxical situation is mainly caused by the fact that authors dealing with the general development of analysis in the 17th century take Cavalieri as a natural starting point, but do not discuss his rather special method in detail, because their aim is to trace the ideas about infinitesimals. In fact, Cavalieri's notions were utterly misrepresented by the second half of the 17th century, and Torricelli[4]-one of Cavalieri's keenest followers- may have been particularly to blame for this misrepresentation. Examples showing how the word "indivisible" was misused in the second half of the 17th century could easily be multiplied as other projected their own ideas into Cavalieri's theory.

Evangelista Torricelli
1608-1647

We propose to show that Cavalieri's method was not an early form of integration as most modern authors mistakenly describe it. Far from being unconcerned

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[4] Evangelista Torricelli: Italian mathematician and physicist, born at Faenza, 15 October, 1608; held a three month post as assistant and secretary to Galileo, before the old master died. Torricelli would later be appointed to succeed Galileo as the court mathematician to Grand Duke Ferdinando II of Tuscany. Despite all his mathematics, Torricelli is best known today as the inventor of the barometer; conceived after pumpmakers of the Grand Duke attempting to raise water to a height of more than forty feet, found that thirty-two feet was the limit to which it would rise with their suction pump- with no explanation as to why.

with lack of rigor, Cavalieri attempted to demonstrate his method followed accepted principals of his time, largely inherited from Greek sources. We shall recreate, using Cavalieri methods, the ancient proof that the ratio of a cylinder to a cone of the same base and equal altitude is 3:1. Cavalieri sought Galileo's approval for his new method. Indeed he delayed publication of his discovery for over eight years pursuant to this effort. Instead, Galileo submitted veiled criticism in novel form- the "Paradox of The Soup Dish;" which we shall present. Finally we show that Evangelista Torricelli made his own contributions to the method of indivisibles. We include his counterintuitive demonstration concerning a certain solid of rotation; today know as "Gabriel's Horn."

All three men are inextricably linked to this little known period of pre-infinitesimal methods of the early 17th century which would be eclipsed by more powerful and popular methods within three decades.

1.2 Our program:

In deciding on the order by which material should be presented, it seemed only fitting to give pride of place to Galileo and in particular a rather cleverly crafted demonstration he inserted into *Two New Sciences*, meant to draw attention to the perils of using the "method of indivisibles" in geometry.

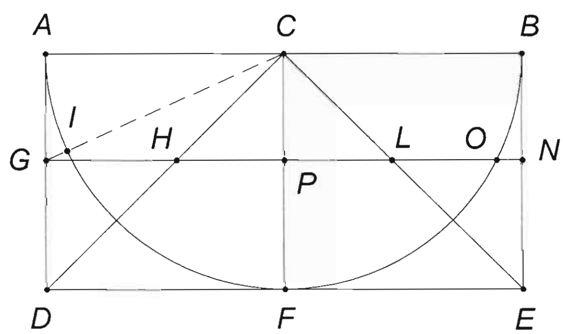
We shall then go from the more familiar to the unfamiliar using the following outline:

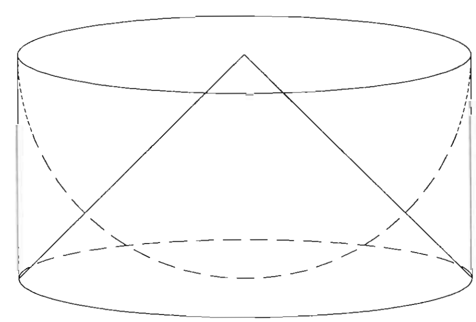
1) We shall first provide a proof (using the integral calculus) of what has been referred to as Gabriel's Horn, a rather counterintuitive figure discovered by Torricelli.
2) Introduce the Eudoxian: The Method of Exhaustion: *(The method of the Ancients; Including a classic demonstration proving that the ratio of a cylinder to inscribed cone is 3:1.)*
3) Introduce background material for Cavalieri's Geometry of Indivisibles; developed as a means to counter the limitations of the Method of Exhaustion.
4) Introduce Cavalieri's *Omnes* Concept.
5) Apply the Method of Indivisibles to show the ratio of a cylinder to inscribed cone is 3:1.
6) Then coming full circle, reproduce Torricelli's own demonstration using his version of "indivisibles" to show that a certain solid of revolution though infinite in length has surprisingly, only a finite volume.

2 Galileo: The Paradox of the Soup Dish

(2.1) In *Two New Sciences*, Galileo presents a demonstration which shows, in the words of Salviati; "how a single point can be understood to be equal to a line."

The demonstration goes as follows:





> *"Take a semicircle ◡AFB whose center is C, and around it the rectangular parallelogram □ADEB; from the center to points D and E draw the straight lines CD and CE. Next imagine the whole figure rotated around the fixed radius CF, perpendicular to the straight lines AB and DE. It is manifest that the cylinder will be described by the rectangle ADEB, a hemisphere by the semicircle ◡AFB, and a cone by the triangle ΔCDE. We now suppose the hemisphere removed, leaving [intact] the cone and those remains of the cylinder which in shape resemble a soup dish, for which reason we shall call it by that name."*

After proving that the volume of the soup dish to be equal to that of the cone, Galileo continues on to the main point of the demonstration with the following;

"In the diagram drawn, angle *IPC* being a right angel, the square of the radius *IC* is equal to the two squares of the sides *IP* and *PC*."

$$(IC)^2 = (IP)^2 + (PC)^2 \qquad \textbf{[\#1]}$$

"But the radius *IC* is equal to *AC*, and this to *GP*; and *GP* is equal to *PH*."

$IC = AC = GP$ **(a)**

$PC = PH$ **(b)**

"Therefore the square of the line *GP* is equal to the two squares *IP* and *PH*, and four times the former equals four times the latter;...

Then substitute the results of **(a)** and **(b)** into **[#1]** yields:

$$(GP)^2 = (IP)^2 + (PH)^2$$

or equivalently

$$4(GP)^2 = 4(IP)^2 + 4(PH)^2 \qquad \textbf{[\#2]}$$

We note that by construction;

(A) $GN = 2(GP) \Rightarrow (GN)^2 = 4(GN)^2$

(B) $IO = 2(IP) \Rightarrow (IO)^2 = 4(IP)^2$

(C) $HL = 2\ (PH) \Rightarrow (HL)^2 = 4(PH)^2$

"...that is, the square of the diameter *GN* is equal to the two squares *IO* and *HL*."

$(GN)^2 = (IO)^2 + (HL)^2$

"And since circles are to each other as the squares of their diameters,...

Area $\otimes_D$: Area $\otimes_d = D^2 : d^2$

(Where "$\otimes_D$" shall denote the circular area of diameter D.)

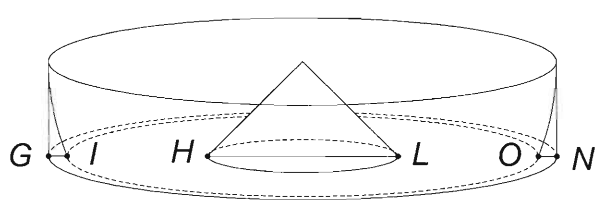
"the circle of diameter *GN* will be equal to the two circles of diameter *IO* and *HL*,"

Area $\otimes_{GN}$ =[Area $\otimes_{IO}$ + Area $\otimes_{HL}$]

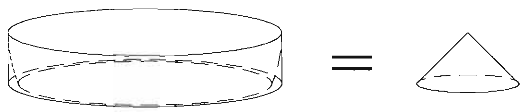
"…hence, removing the common circle whose diameter is *IO*, the remaining circle *GN* will be equal to the circle whose diameter if *HL*."

Area $\otimes_{HL}$ =[Area $\otimes_{GN}$ – Area $\otimes_{IO}$]

But, it should be notice that, this is also equal to the base of the ring whose outside diameter is *GN* and whose inside diameter is *IO*.

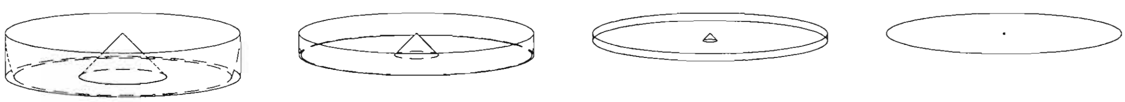


Thus, since the bases of ring and the cone are equal in area and since, by construction, the two figures have equal altitudes, then it follows that the volume of the ring is equal to the volume of the cone.



Since the line segment *GN* is purely arbitrary, Galileo argues that the volume of the ring to that of cone is always equal. Thus in diminishing the figures, (always equal), they tend to end, one in a single point and the other in a ring of any size.

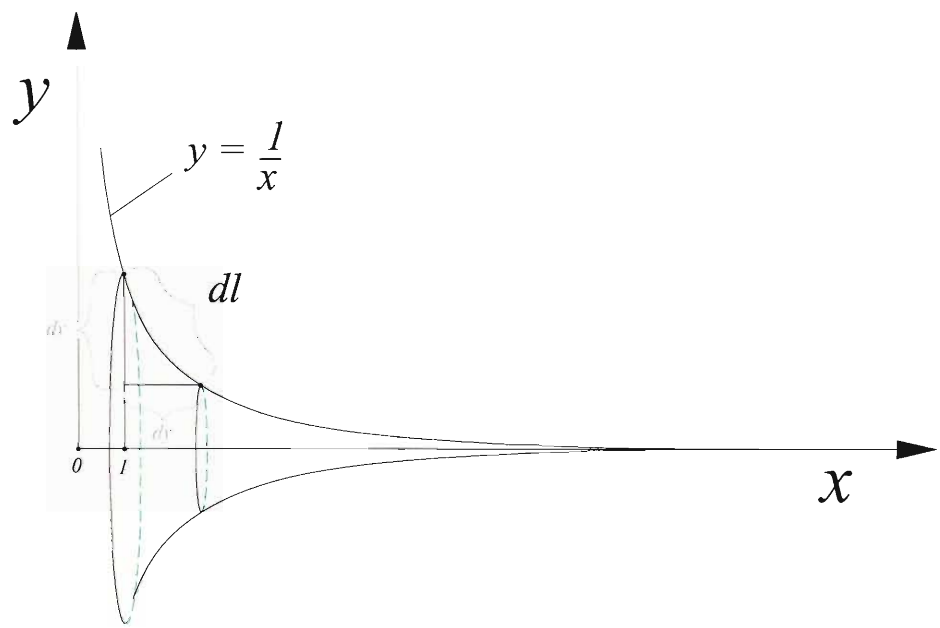
> "Now, during the diminution of the two solids, their equality was maintained right up to the end; hence it seems consistent to say that the highest and last boundaries of the reductions are still equal, rather that one is infinitely greater than the other, and so it appears that the circumference of an immense circle may be called equal to a single point!"



Galileo had sent the paradox along to Cavalieri, to caution him regarding the perils of using the "method of indivisibles" in geometry. Whether Galileo inserted the paradox as a conclusion to be accepted or simply meant to provoke careful thought, is unclear. Presumably the paradox had a double meaning here: to illustrate the nature of mathematical definitions, and to show the pitfalls of analogy in transferring the word "equal" from entities of n dimensions to their supposed counterparts of n-1 dimensions. The error here is assuming that once the equality between the solid figures has been established, that equality is maintained between the diminishing figures, even when the solids cease to be; That is, when the solid ring degenerate into a circle and the cone into a point. Indeed, Galileo seems fully aware of this. His only quandary; how one goes about charactering the precise point at which “diminution of the two solids” results in the “solids” ceasing to be solids, and the equality between the resultant figures no longer applies.

3 Background: Torricelli's Remarkable Solid of Rotation

One standard example from almost any current calculus text is the so-called Gabriel's Horn[5]. The example usually appears in the section on techniques of integration or improper integrals and is often one of the "challenging" problems at the end of the exercises. It is placed there, perhaps, as an example to show students how intuition can be fooled when infinite regions are considered.



In 1641 Evangelista Torricelli discovered that a certain solid of infinite length, which he called the "acute hyperbolic solid," had rather remarkable and counterintuitive properties. The object seemed so paradoxical and astonishing that it not only created considerable interest within mathematical and philosophical circles of the time, Torricelli himself, could scarcely believe the results.

[5] See, for example, page 539 in Calculus: Early Transcendentals Version by C. H Edwards and D.E. Penney, 6th edition, Prentice Hall, (2003).

Nowadays, we can obtain the figure which is commonly, although perhaps inappropriately known as Gabriel's Horn, by taking the graph of $y = \frac{1}{x}$, with the domain $x \geq 1$ and rotating it about the x-axis. The figure's modern-day moniker refers to the archangel Gabriel, who according to tradition serves as the messenger between the divine world and that of man; thus linking together the finite realm with that of the infinite.[6] While Torricelli made his discovery using methods which predated the invention of calculus and were somewhat laborious, today we can calculate the volume and surface area of the figure using simple integration techniques. In fact the figure is often a topic of demonstration for second semester calculus students.

The surface of the figure can be calculated as follows;

$$\begin{aligned} S &= \int_1^\infty 2\pi y\sqrt{1+(dy/dx)^2}\,dx \\ &> 2\pi\int_1^\infty y\,dx = 2\pi\int_1^\infty \frac{1}{x}\,dx = \lim_{n\to\infty}\left[2\pi\ln(x)\right]_1^n \\ &= \lim_{n\to\infty}\left[2\pi\ln(n)\right] = +\infty \end{aligned}$$

While that of its volume is found by;

$$\begin{aligned} V &= \int_1^\infty \pi y^2\,dx \\ &= \int_1^\infty \pi\frac{1}{x^2}\,dx = \lim_{n\to\infty}\left[\frac{-\pi}{x}\right]_1^n = \pi[0-(-1)] \\ &= \pi \end{aligned}$$

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[6] In the New Testament, Gabriel is the angel who reveals to Zacharias that John the Baptist will be born to Elizabeth and who visits Mary to reveal that she will give birth to Jesus. According to later legend, he is the unidentified angel in the Book of Revelation who blows the horn announcing the Judgment Day.

Hence, Torricelli’s geometric figure, though infinite in length and infinite in surface has, surprisingly, only a finite volume. This leads to the rather paradoxical consequence that while Gabriel's horn can be filled up with π cubic units of paint, an infinite number of square units of paint are needed to cover its surface. Or proposed another way; one could fill the horn with paint, but not have enough to paint the outside!

Although Gabriel's horn is an engaging example for second semester calculus, the beauty of the paradox is often obscured by the integral estimate that most students find spurious at best; simply there must be some trick that remains hidden from view, as the results run counter to common experience.

4 Eudoxus: The Method of Exhaustion

(Introducing background material for Cavalieri's Geometry of Indivisibles)

The procedure attributed to Eudoxus, which came to be called the Method of Exhaustion in the 17th century, is based upon the following proposition, (as given by Euclid.)

Proposition 1, Book XII of the *Elements*;

> *"Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that is left a magnitude greater its half, and from this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.*"

In other words, given two unequal magnitudes A and B, [Proposition 1], allows us to subdivide the greater into 2^n parts, for some n, such that each of the 2^n parts is less than the smaller given magnitude.

The process indicates that the magnitude remaining can be made as small as one pleased; however, the Greek mathematicians never considered the process as being literally carried out to an infinite number of steps; which we shall illustrate with the following example.

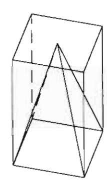
Using the Method of Exhaustion, we will seek to prove Proposition 10, from Eudoxus, which concerns the ratio of a cylinder to that of an inscribed cone of the same height and same base.

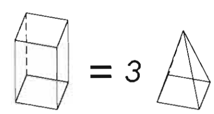
Proposition 10, Book XII of the *Elements*;

"Every cone is the third part of the cylinder that has the same base and equal height."

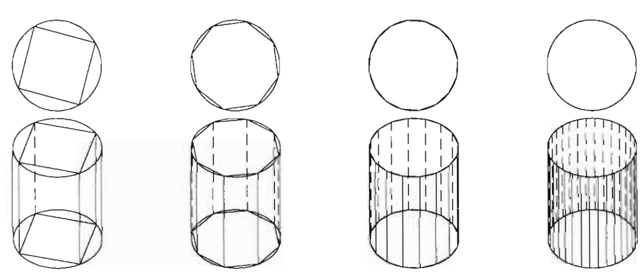
Proof.

We take for granted that any pyramid is the third part of the prism which has the same base with it and equal height. (Elements XII. Proposition 7)

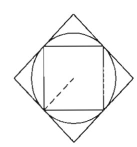




Step 1. Given a cylinder it is always possible to inscribe in it a prism, with a sufficiently large number of sides such that the difference between the cylinder and the prism is less than any arbitrary chosen magnitude.

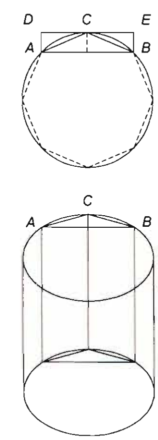


Consider, in fact, inscribed in the cylinder a square-prism; it is bigger that half the cylinder (This is readily apparent, if we notice that the area of the inscribed square is 1/2 that of the circumscribed square.) Therefore the remainder R_1 between the cylinder and the prism is less than half the inscribed

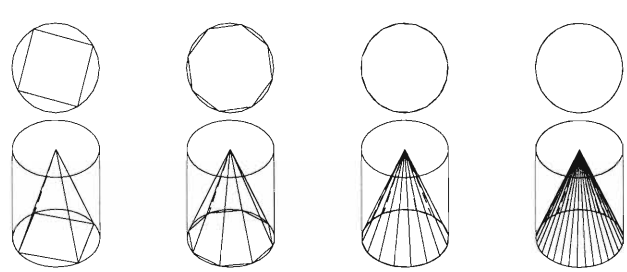


cylinder.

Now, if we divide in half the arc subtended by the side of the square and denote the resulting point by *C*, the triangle *ACB* will be bigger than half the segment *ACB*; therefore every triangular prism built on the triangle and inscribed in the cylinder is bigger than half the portion of the cylinder on *ABC*. So subtracting from the cylinder the octagon prism, we get a difference $R_2 < \frac{1}{2}R_1$. And continuing in this fashion, inscribe prisms with polygonal bases having $4{\cdot}2^2$, $4{\cdot}2^3$, ... sides, by Proposition 1 we get a prism (P_n with $4{\cdot}2^n$ sides) such that the difference with the cylinder is smaller than any pre-assigned magnitude, however small.

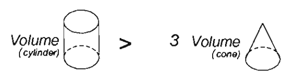


Step 2. Similarly, given a cone it is always possible to inscribe in it a pyramid with sufficiently large number of sides such that the difference between it and the cone is less than any pre-assigned magnitude, however small.



Step 3. (This is the double reductio ad absurdum.) Let *O*, *V* be the volume of the cylinder and cone respectively. If, for the sake of argument, the volume of the cylinder does not equal three parts the cone, i.e., $O \neq 3V$ than either $O>3V$ or $O<3V$. We shall show that both cases lead to a contradiction.

Case I

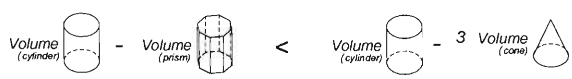


Assume: $O > 3V$.

Inscribe in the cylinder successive prisms so that we get a prism- P_n such that the difference from the cylinder is less than the difference O-3V,

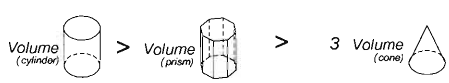
That is

$$(O - P_n) < (O - 3V)$$



or equivalently,

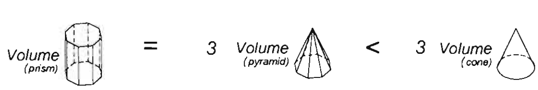
$$O > P_n > 3V \qquad \textbf{(\textit{i})}$$



But by Euclid's Proposition 7, we know that prism- P_n is triple of the pyramid- $\hat{P}_n$ with the same base and height, inscribed in the cone and this pyramid is less than V.

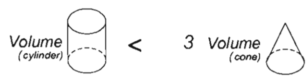
That is,

$$P_n = 3\hat{P}_n < 3V$$



But this can not be, since it contradicts the results of our initial assumption ***(i)***.

Thus, it is not the case that $O > 3V$.



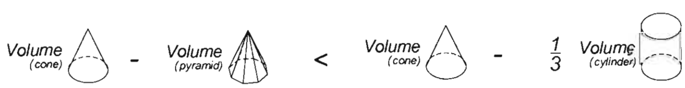
Case II

Assume: $O < 3V$, or equivalently $V > O/3$.

Let us inscribe in the cone, successive pyramids until we get a pyramid- $\hat{P}_n$ that differs from the cone by less than $(V - \frac{1}{3}O)$.

That is,

$$(V - \hat{P}_n) < (V - \frac{1}{3}O)$$



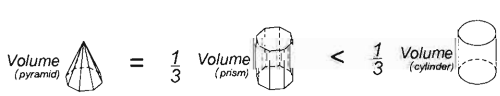
Then it follows,

$$V > \hat{P}_n > \frac{1}{3}O. \qquad \textbf{(ii)}$$

Now, by Euclid's Proposition 7, $\hat{P}_n$ is one-third of a prism with the same base and height; and this prism is less than the cylinder.

That is,

$$\hat{P}_n = \frac{1}{3}P_n < \frac{1}{3}O$$



But this can not be, since it contradicts the results of our initial assumption ***(ii)***.

Thus, it is not the case that $O < 3V$.

Since neither $O > 3V$ nor $O < 3V$ was found to be true, the conclusion from both cases is that O can only be equal to $3V$.

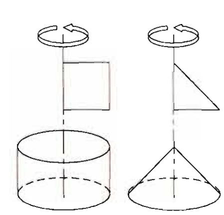
The method of exhaustion, although equivalent in many respects to the type of argument now employed in proving the existence of a limit in differential and integral calculus, does not represent the point of view involved in the passage to the limit. The Greek method of exhaustion, dealing as it did with continuous magnitude, was wholly geometrical, for there was at the time no knowledge of an arithmetic continuum, The inscribed prism could be made to approach the cylinder as nearly as desired, but it could never become the cylinder, for this would imply an end to the process of subdividing the sides. However, under the method of exhaustion it was not necessary that the two should ever coincide. By an argument based upon the reductio as absurdum, it could be shown that a ratio greater or less than that of equality was inconsistent with the principle that the difference could be made as small as desired. Thus Eudoxus avoided such unclear concepts as a prism with an infinite number of sides which ultimately coincided with the cylinder.

4.1 More Exhaustion:

(Introducing background material for Cavalieri's Geometry of Indivisibles)

(4.1.1) The method of Exhaustion was rigorous, but its reliance upon a proof by contradiction laid it open to criticism in the mathematical community of the 16th and 17th century. In many circles, proofs by contradiction were considered inferior to direct proofs, on account of their lack of causality. This view certainly had historic precedence, as no less than Aristotle had explicitly asserted that direct proofs were superior to proofs by contradiction on grounds that they clearly gave the sense of how the results had been obtained, making them, arguably, more scientific[7]

By the beginning of the 17^{th} century several attempts had been made to develop geometry along a more direct approach that would overcome both the complexities of and objections to the Exhaustion method. However, it was Cavalier's geometry of indivisibles, presented in his *Geometria indivisibilius continuorum nova quadam ratione promota*, [8] that received the most mathematical and philosophical attention. *Geometria*'s topic, the quadrature and cubature of figures was of great interest to mathematicians of the 17^{th} century; and the scarcity of publications on the subject only served to enhance its status. Though, it's questionable the number of mathematicians who actually studied the text in detail; its almost 700 pages are so difficult to follow that Maximilien Marie[9] suggested that if a prize existed for the most unreadable book, it should be awarded to Cavalieri for *Geometria*. Nevertheless, the book remained well known. Fortunately for us today, the task of studying Cavalier's method is made much easier by two excellent works on the subject, Andersen (1985) and Giusti (1980).

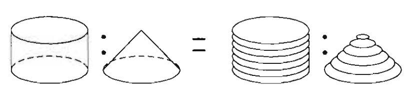


(4.1.2) In the preface to *Geometria*, Cavalieri recounts that he had been led to his method of indivisibles by reflecting on the surprising fact that solids generated around an axis from plane figures did not have the same ratio as that of the generating figures. When one considers a square and a

[7] *Posterior Analytics* I.7
[8] Geometry, advanced in a new way by the indivisibles of the continua
[9] *Histoire des sciences mathematiques et physiques physiques.* Gauthier-Villard, Paris, 1883-1888

right triangle whose legs are the base and the height of the square and whose hypotenuse is the diagonal of the square, the ratio of the square to the triangle is 2:1. However, if we rotate the square and the triangle around the height of the square, we obtain a cylinder and a cone whose ratio is 3:1. After some wrong attempts at investigating this phenomenon, Cavalieri hit upon the key idea of his method:

> *"Having thus considered the above mentioned cylinder and the cone, I found what I call in Book II, all the planes of the cylinder, to have the same ratio to all the planes of the cone as that of the cylinder to the cone."* [10]



While Cavalier's method of indivisibles was developed as a means for of quadrature and cubature of figures, the method per say, was not employed to calculate areas and volumes. Rather, Cavalieri would seek to determine areas and volumes by forming a ratio between figures. The strategy was simple; if ones aim was to determine the area or volume of some figure of interest, one would attempt to place it in ratio with some "nice" figure; namely, a figure whose area or volume was easily known – like a parallelogram.

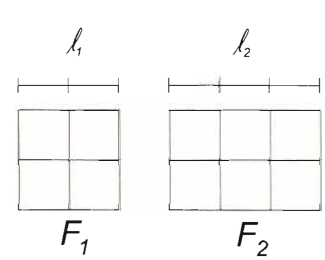
Though, Cavalieri's method was new, its foundation was not. Keenly aware that any novel approach to quadrature and cubature of figures would suffer the arrows of critical attack, Cavalieri felt compelled to justify his method conformed to classic

[10] *Geometria indivisibilius continuorum nova quadam ratione promota*. Cavalieri B. Bononiae (1635)

Greek notions of mathematics. In particular, Cavalieri meant to form a ratio and for that he needed to call upon Eudoxian theory of magnitudes.

(4.1.3) Eudoxus of Cnidus (408?-355? B.C.), introduced the concept of magnitudes in response to a crisis in Greek mathematics that occurred during the latter half of the 6^{th} century BC. Prior to this time, Pythagorean doctrine held sway over ancient Greek mathematics. That the Pythagoreans, (originating in the 6^{th} century B.C.), had distinct cult-like overtones is well known; as well as their motto "all is number." The claim that all phenomenon in the universe could be reduced to numbers or their ratios is attributed to them; by number, the Pythagoreans meant "whole" numbers. Further, a ratio between two whole numbers, such as $a:b$, was not a fraction and therefore another number, as in modern times. It was a relationship between two numbers, or possibly better put, simply an ordered pair. This is not to say that actual fractions expressing parts of a monetary unit or measure were not employed during the Classic Greek period; they were in fact a part of everyday life. But these fractions were employed in commerce and the trades by the slave class and non-citizens, thus placing them outside the pale of Greek mathematics proper and learned freemen who dismissed the practical arts in favor of the higher truths that only the philosophies could offer.

(4.1.4) Two factions were said to be proportional ($a:b = c:d$), if a is either some integral part or integral multiple of b, just as c is to d. This discrete view of numbers was also applied to geometric lengths, areas and volumes. In particular, it was believed

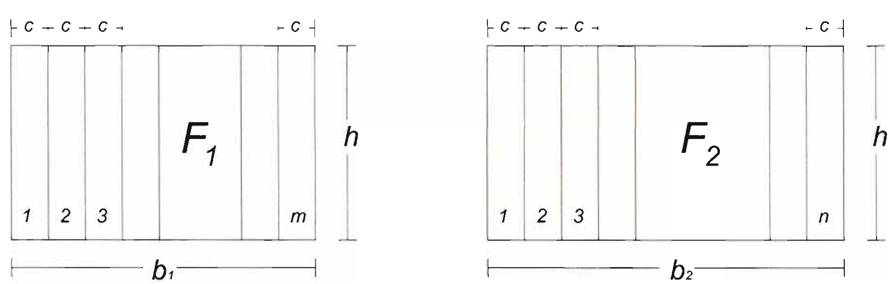


by the Pythagoreans that any two line segments were commensurable, that is, were multiples of a common unit. On this assumption, the theory of integer ratios and proportions readily extended so as to apply to lengths and areas of simple figures such as line segments and rectangles.

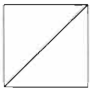
For example, the ratio of the lengths of the two line segments $l_1 : l_2$ is equal to the ratio 2:3 of integers, while the ratio of the area of the two rectangular figures $F_1 : F_2$ is equal to 4:6. Thus we can talk about proportions $l_1 : l_2 = F_1 : F_2 = 2{:}3$

Following this logic, area relationships for simple geometric figures with commeasurable dimensions, are easily established.

For example, given two rectangular figures F_1 and F_2 with commensurable bases b_1 and b_2 and equal heights h, the ratio $F_1{:}F_2$ of their areas is equal to the ratio $b_1{:}b_2$ of their bases. For if $b_1{=}m{\cdot}c$ and $b_2{=}n{\cdot}c$, (where m and n are integers), the figure F_1 consists of m sub-rectangles with base c and height h, while figure F_2 consists of n such sub-rectangles. Hence $F_1{:}F_2{=}m{:}n{=}b_1{:}b_2$.



(4.1.5) During the latter part of the 5th century B.C., the Pythagorean were startled and disturbed to discover that there exists pairs of line segments, such as the edge and diagonal of a square, that are not commensurable; that is they cannot be subdivided as integral multiples of segments of the same length. Hence the ratio of the lengths could not be expresses by the ratio of two integers.



These ratios (incommensurables) are expressed in modern mathematics by irrational numbers, but the Pythagoreans could not accept such numbers as they challenged fundamental doctrine; posing a problem that was central in Greek mathematics, namely the relationship between the discreet and the continuous. The Pythagoreans had identified number with geometry. However, the existence of incommensurable ratios shattered this identification, making the theory of integral proportions useless for the comparison of ratios of geometric quantities and thereby invalidating those geometric proofs that had utilized proportionality concepts. [11]

(4.1.6) This crisis in the foundations of geometry was resolved by Eudoxus by introducing the concept of magnitudes. Although it is not easy to determine exactly what was meant by magnitude, as there never seems to have been any explicit axiomatization of the properties that they should satisfy; clearly the idea originated from intuitive notions of extension. Magnitudes where categorized into "kinds" such as line segments, angles, areas, volumes, weights and time, which could vary continuously. This was opposite of the Greek concept of number, which following the Pythagorean school, could jump from one value to another.

[11] Legend has it that the Pythagorean philosopher Hippasus (ca. 500 BC) used geometric methods to demonstrate the irrationality of $\sqrt{2}$ while at sea and, upon notifying his comrades of his great discovery, was immediately thrown overboard by the fanatic Pythagoreans. -So much for academic freedom.

What Eudoxus accomplished was to avoid irrationals as numbers. Moreover, after the concept of magnitude was accepted, Greek mathematicians did not attempt to identify number with geometric quantities; thus, they avoided giving numeric values to length of lines, sizes of angles and other magnitudes as well as ratios of magnitudes. The question; "What is the area of a circle?" would have no meaning to the Greek geometers. But the question; "What is the ratio of the areas of two circles?" would be a legitimate one, and the answer would be expressed geometrically; "The same as that of the squares constructed on the diameters of the circles."

The unfortunate consequence of Euclid's scheme, though, was to force a sharp distinction between number and geometry, for only the latter could handle incommensurable ratios; driving mathematician into the ranks of the geometers as geometry became the basis of almost all rigorous mathematics for the next two thousand years.

In Book V of Euclid's, *The Elements*, (Based upon Eudoxus' work), the following definitions are of particular interest to the discussion of Cavalieri.

Definition 3.

"A ratio is a sort of relation in respect to size between two magnitudes of the same kind. "

Definition 4.

"Magnitudes are said to have a ratio to one another, which are capable, when multiplied, of exceeding one another."

The meaning of definition 4, is that the magnitudes $\boldsymbol{a}$ and $\boldsymbol{b}$ have a ratio if some integral multiple n of $\boldsymbol{a}$ exceeds $\boldsymbol{b}$ and some integral multiple of $\boldsymbol{b}$ exceeds $\boldsymbol{a}$. Apparently, the definition excludes the concept that the infinitely small quantity which is not zero, called the infinitesimal. Euclid's definition does not allow a ratio between two magnitudes if one is so small that some finite multiple of it does not exceed the other. The definition also excludes infinitely large magnitudes because then; no finite multiple of the smaller one will exceed the larger.

Further, Greek assumptions concerning magnitudes implied that when any two magnitudes, A and B, of the same kind, are given then:

1) A and B can be ordered so that one of the following holds:

$$A>B \text{ or } A=B \text{ or } A<B$$

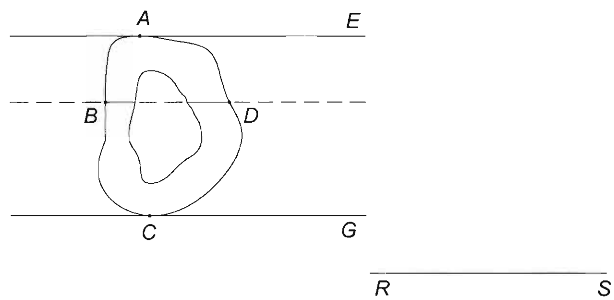
2) A and B can be added; the result denoted by $A+B$, is magnitude of the same kind as A and B.
3) If $A>B$ then B can be subtracted from A, forming the magnitude $A-B$ of the same kind as A and B.
4) A and B can form a ration $[A: B]$.

The above information should give us sufficient background to continue with the discussion of Cavalieri.

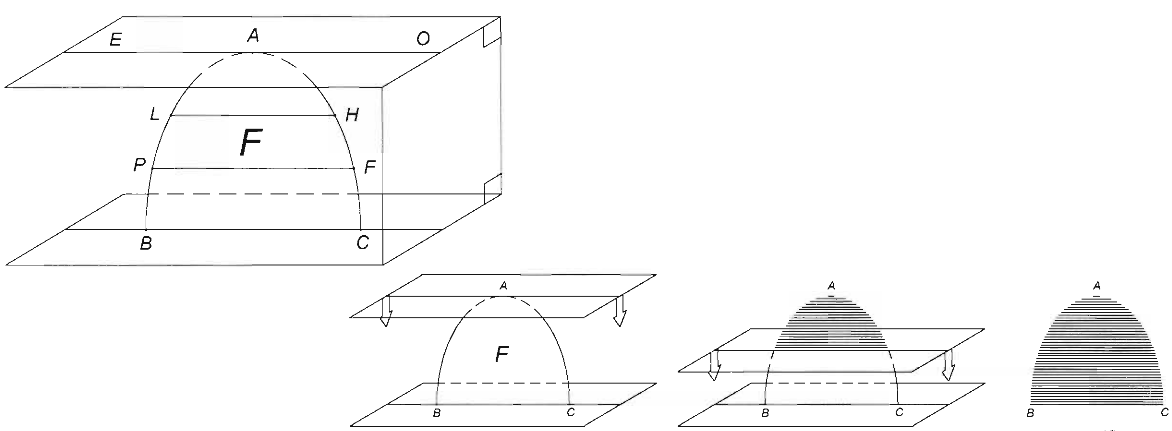
5 Cavalieri: The Omnes Concept

(5.1) On page 8 of *Geometria*, Cavalieri introduces some preliminary concepts he would later draw upon for his central definition of "all the lines."

"Given a closed plane figure, ABCD and a direction RS, called regula; the figure will have two tangents,[12] *AE and CG, parallel to the regula; moreover any line parallel to the regula situated between the two tangents for example BD, will intersect the figure line the segments, whereas any line parallel to the regula outside the tangents will have no points in common with the figure."*



(5.2) "All the lines,"-the first of Cavalier's *omnes* concepts

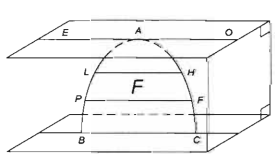


$O_F(l)$

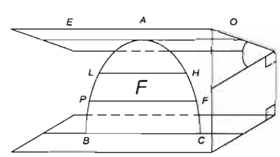
[12] Here Cavalieri's idea of tangent differs from the modern notion; "I say that a straight line touches a curve situated in the same plane as the line when it meets the curve either in a point or along a line and when the curve is either completely to the one side of the meeting line [in the case when the meeting is a point] or has no parts to the other side of it [on the case when the meeting is a point] or has no parts to the other side of it; in the case when the meeting is a line segment]."

In Book II of *Geometria*, Cavalieri introduces his concept of "all the lines" (*Omnes lineae*).[13] He explains the concept as follows;

> *"Let ABC be any plane figure, and EO and BC two opposite tangents of the plane figure, however drawn. Consider then two mutually parallel planes, indefinitely extended, drawn through EO, BC of which the one that, for example passes through EO is moved toward the plane passing through BC, always keeping parallel to it until it coincides with it. Thus, the intersections of this moving plane, or fluent, and the figure ABC, which are produced in the overall motion, taken all together, I call: all the lines of the figure ABC (some of which are the LH, PF, BC) taken with reference to one of those, such as BC: of rectilinear transit, (recti transitus), when the planes intersect the figure ABC at right angles; of oblique transit when they intersect it obliquely, (obliqui transitus)."*



recti transitus



obliqui transitus

In other words, single lines can be thought of being formed by the intersection of the moving plane to that of the given figure. When the lines are taken as a collection they are called "all the lines;" also referred to as the indivisibles of the figure. Further, Cavalieri divides collections of lines into types, dependent upon how they are "generated' by the moving plane. When the moving plane is perpendicular to the given

[13] Omnes lineae talis figurae, sumptae regula una earundem

figure, “all the lines” are referred to as *recti transitus*. However, when the moving plane is inclined to the given figure, “all the lines: are to be referred to as *obliqui transitus*. Cavalieri did not make much use of “all the lines” *obliqui transitus* and will not enter our discussion of his method.

We shall adopt the notation “$O_F(l)_{BC}$” denote the concept of “all the lines” of the figure F with reference to the regula BC (“O” stands for *omnes*). Also, when the regula is obvious or when its inclusion adds little to the discussion, we will routinely drop the subscript. Also, unless otherwise stated “all the lines” should be regarded as *recti transitus*.

In regards to the conceptual origins of Cavalier’s “all the lines,” it is quite possible that he was inspired by intuitive notions of infinitesimals. It is known he sought analogies between that of a figure to its collection of indivisibles to that of cloth composed of thread and a book assembled from pages.[14] However, in order to provide a solid foundation for his method, he had to suppress any intuitive notions of infinitesimals and keep within the Greek tradition by excluding infinities from proofs. [15]

(5.3) To deal with solid figures he introduced “all the planes;” where he imagined the one plane moving towards the other, remaining parallel to it. “All the planes” of solid figures, taken with one of the planes as regula, consists of the intersections between the solid and the moving plane.

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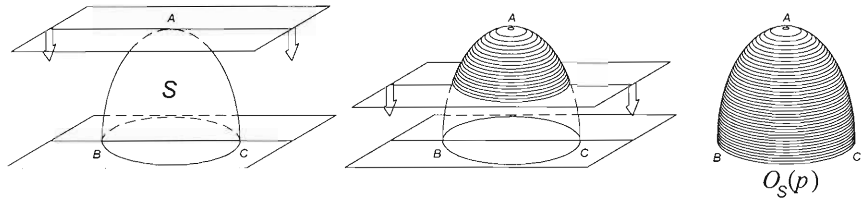
[14] “it is manifest that we can conceive of plane figures in the form of cloth woven out of parallel threads, and solids in the form of books , which are built out of parallel pages.” *Exercitatones Geometrica Sex* (Cavalieri 1647, pp3-4)]

[15] “But the treads in cloth and pages in a book are always finite and have some thickness, while in this method an indefinite number of lines in plane figures (or planes in solids) are to be supposed, without any thickness” *Exercitatones Geometrica Sex* (Cavalieri 1647, pp.3-4)]



Cavalier made two key assumptions concerning his *omnes* concept.

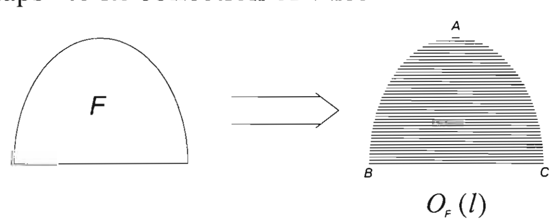
1) That each planar figure has an associated collection of lines called, "all the lines."

   Similarly, each solid figure has an associated collection of planes called, "all the planes."

In more contemporary language we might refer to this as a "transformation" or "mapping.

That is,

each planar figure "maps" to its collection of lines

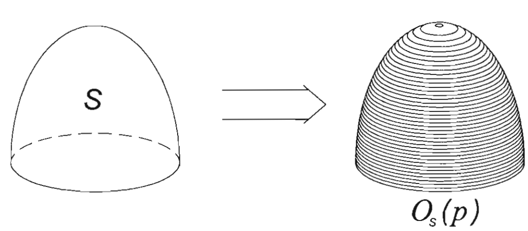


$F \rightarrow O_F(l)$,

and,

each solid figure "maps" to its collection of planes

$S \rightarrow O_S(p)$.



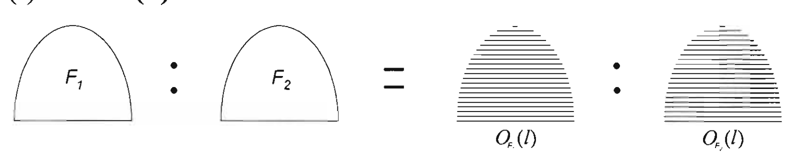
However, Cavalieri never laid out any rules which this transformation should follow; and questions concerning the composition of a continuum, most certainly, would have ruled out any explicit "point for point" type of mapping. Nonetheless, he remained content with asserting that this association between figures and their collections of lines and collections of planes, existed.

2) That $O_F(l)$- "all the lines," and $O_S(p)$- "all the planes," are magnitudes in the Eudoxian sense, which are capable of being put in a ratio.

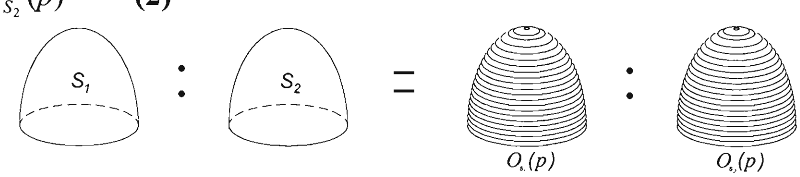
Moreover, since Cavalier's aim was to exploit collections of lines, (and collections of planes), to obtain information about their associated figures, he incorporates these two assumptions into his first fundamental theorem.

(5.4) Theorem II.3

$$F_1 : F_2 = O_{F_1}(l) : O_{F_2}(l) \qquad \textbf{(1)}$$



$$S_1 : S_2 = O_{S_1}(p) : O_{S_2}(p) \qquad \textbf{(2)}$$



(Here we assume that the lines of F_1 and F_2 are taken with the respect to same regula. Likewise, for S_1 and S_2.)

While the statements above looks innocent enough, two major problems present themselves.

A) Collections of lines would seem to be composed of an indefinite number of lines; therefore, the existence of a ratio between two such collections, which conformed to the Eudoxian theory of magnitudes, would seem problematic. After all, Eudoxian theory apparently excluded infinitesimals as magnitudes capable of being put in a ratio.

B) Since Cavalieri believed the use of infinitesimals must be rejected over foundational concerns, one assumes that any formal theory of indivisibles must be embraced as the mathematical equivalent to Atomism. But, if following accepted Aristotelian doctrine, continuous divisibility of the continuum is assumed, where does this place indivisibles and the composition of the geometric figures they are associated with? [16]

In response to the first concern, Cavalieri explained that it was not the number of lines in a collection which is used in a comparison, but;

> *"the magnitude which is equal to being, congruent with it, the space occupied by the lines."* [17]

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[16] Cavalieri never seems to have taken a definite position over whether indivisibles actually compose the figures they were associated with. His hesitation is evident in a letter to Galileo dated June 28, 1639; "*I did not dare to affirm that the composition is composed of indivisibles, but showed that between the continua there is the same proportion as between the collection of indivisibles*."
Galileo Galilei *Opere*, vol 18 p.67

[17] *Geometria,* pp. 111: magnitudinem, quae adaequatur spatio ab eisdem lineis occupato, cum illi congruat.

In other words, though a collection of lines might be infinite with respect to the number of lines, it is finite with respect to extension in space.

As to the second concern, Cavalieri left open the following two possibilities;

- If one conceives of the continuum to be compose of indivisibles, then a given plane figure and the "magnitude of all the lines" will be one in the same thing.
- However, if one assumes a continuous divisibility, then it can be readily maintained that magnitude of the individual lines consists only in terms of lengths, but when taken as a collection as in "all the lines", the lines must be considered at their actual positions. Thus, the magnitude of the collection is limited by the same limits as those of the given figure.

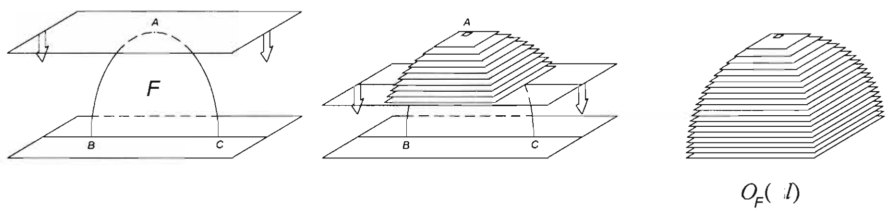
Further, Cavalieri was to argue, if the indivisibles do not make up the continuum, then the plane figure consists of "all the lines" plus something else (*aliquid aliud.*) Then the space occupied by "all the lines" is limited and the collections of lines can be added, subtracted and ordered. Hence, by Eudoxian theory, a ratio between two collections of lines is established.

(5.5) Other *omnes* concepts…

Cavalieri introduced a variety of improvised concepts to be used for the quadrature and cubature of figures.

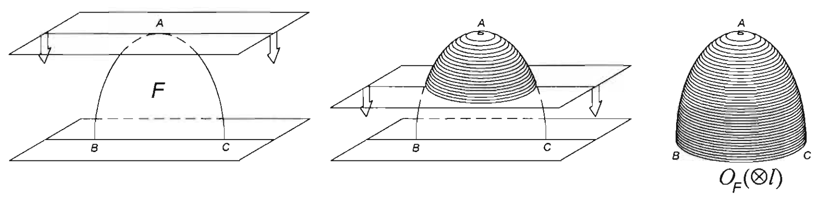
In addition to “all the lines” of planar figures, Cavalieri introduced the concept of “all the similar planes” of planar figures. Of particular interest to the discussion are “all the squares” and “all the circles” of plane figures. The symbols “ $\square l$ ” and “ $\otimes l$ ”shall denote the square and the circular disk on line l, respectively.

$F \to O_F(\square l)$,



That is, each planar figure “maps” to its collection of squares.

$F \to O_F(\otimes l)$,



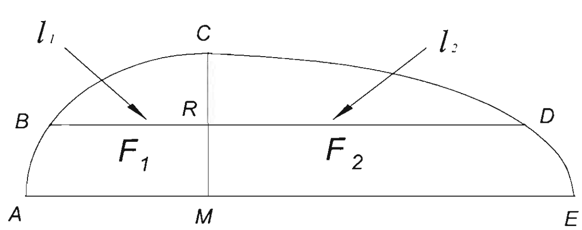
That is, each planar figure “maps” to its collection of circular disks.

(5.6) The *ut-unum* principle

Cavalieri employed the following property regarding his omnes concept, which we shall refer to as the ***ut-unum principle***.

“As one antecedent is to one consequence so are all the consequents”[18]

For plane figures the principle is illustrated as follows:



[18] *Geometria,* pp. 116: ut unum antecedentium ad unum consequentium, ita esse omnia antecedentia ad omnia consequential.

If two figures, F_1 and F_2 have their bases situated on the same line, have equal altitudes, and if each pair of corresponding line segments, l_1 (= BR) and l_2 (= RD) in $O_{F_1}(l)$ and $O_{F_2}(l)$ respectively, are in the same ratio, then $O_{F_1}(l)$ and $O_{F_2}(l)$ are also in that ratio.

Thus:

$$l_1 : l_2 = AM : ME \tag{3}$$

(for all corresponding l_1 and l_2 in figures F_1 and F_2 respectively.)

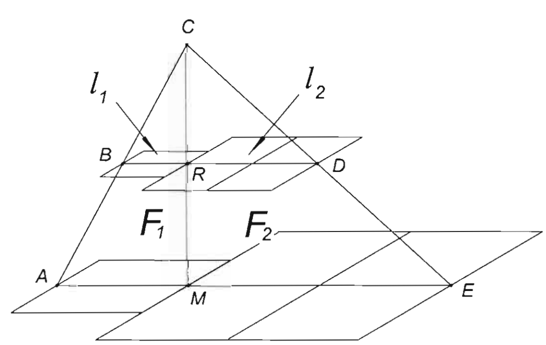
Then:

$$O_{F_1}(l) : O_{F_2}(l) = AM : ME \tag{4}$$

(5.7) Theorem II. 23 (generalized ***ut-unum*** principle);

Cavalieri generalized the ***ut-unum*** principle, so that it came to mean that "O" could be applied to relations between line segments, in which a certain consistency was maintained.

For example, consider the following figures F_1 and F_2. Assume that for each l_1 (=BR) and corresponding l_2 (=RD) that, $l_1 = 2 \cdot l_2$ for any arbitrary segment BD. Then the Generalized ***ut-unum*** principle asserts that the following relations hold:



$$O_{AMC}(l) = 2O_{MCF}(l)$$

and

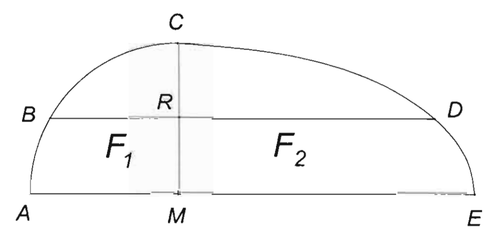
$$O_{AMC}(\square l) = 4O_{MCF}(\square l)$$

(5.8) Theorem (II.4) (often referred to as Cavalieri's principle.)

Using the ***ut-unum*** principle and Theorem II.3 he proved the following:

> *"If two planar figures have equal altitudes, and if sections made by lines parallel to the bases and at equal distances from them are always in the same ratio, then the plane figures also are in this ratio."* [19]

Thus if two plane figures like $F_1 = ACM$ and $F_2 = MCE$ have the property that for each line BD parallel to base AE the sections BR and RD, in F_1 and F_2 respectively, satisfy the relation:



$$BR : RD = AM : ME$$

Then:

$$F_1 : F_2 = AM : ME$$

(5.9) Postulate II.1

"All the planes of congruent figures are congruent."

$$F_1 \cong F_2 \Rightarrow O_{F_1}(l) \cong O_{F_2}(l) \qquad \textbf{(7)}$$

Cavalieri offered the following explanation on what he meant by congruent collections of lines.

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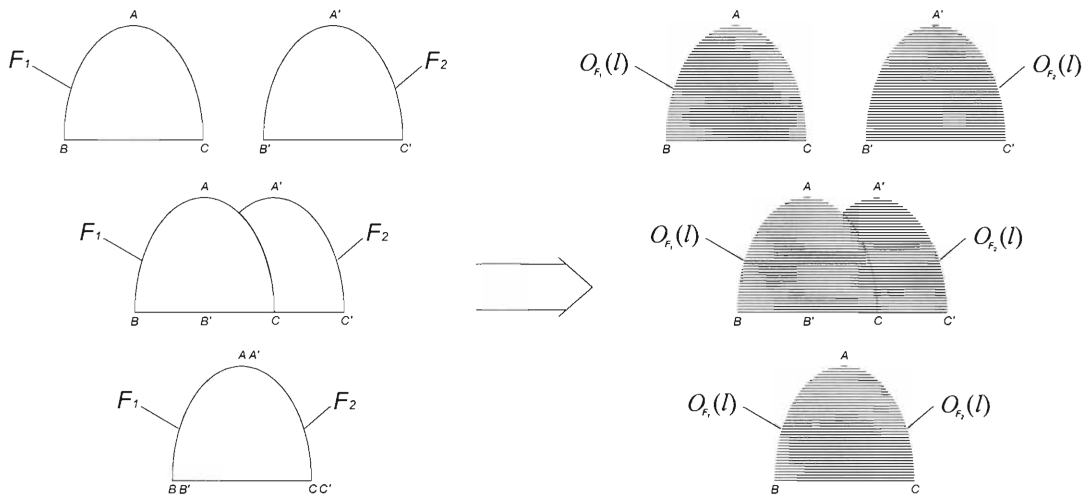
[19] *Geometria*, pp. 115

"When two congruent figures, F_1 and F_2, are placed so that they coincide, then each line in $O_{F_1}(l)$ will coincide with exactly one line in $O_{F_2}(l)$ (and visa versa), the collections of lines are called congruent." [20]



Although explicitly stated, congruency between collections of lines seems not to have played a great role in Cavalier's theory; thus in applying Postulate II.1, he use only the implication

$$F_1 \cong F_2 \Rightarrow O_{F_1}(l) = O_{F_2}(l) \qquad \textbf{(8)}$$

Cavalier's presumption that collections of lines constitute a Eudoxian magnitude meant that collections could be added, subtracted, ordered and be put in a ratio. The properties of addition and ordering were extended to his *omnes* concept as follows;

$$F \cong F_1 + F_2 \Rightarrow O_F(l) = O_{F_1}(l) + O_{F_2}(l) \qquad \textbf{(9)}$$

[20] *Exercitationes*, pp.200

and

$$F_1 > F_1 \Rightarrow O_{F_1}(l) > O_{F_2}(l) \qquad \textbf{(10)}$$

Moreover, since it was crucial to Cavalier's method that the final property, concerning Eudoxian magnitudes' ability to be put in a ratio, be extended his *omnes* concept, he found it necessary to prove the following theorem.

(5.10) Theorem II.1

"*Collections of lines are magnitudes that have a ratio to each other.*"[21]

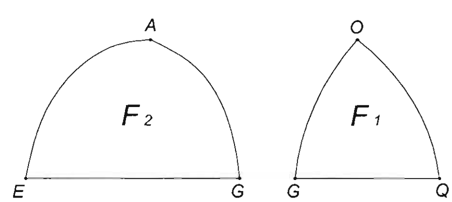
Cavalieri's aim was to show that collections of lines fulfilled Definition V.4 of Euclid's *Elements*.

Proof:

Let $O_{F_1}(l)$ and $O_{F_2}(l)$ be the collections of lines of two plane figures,

$F_1 = GOQ$

and



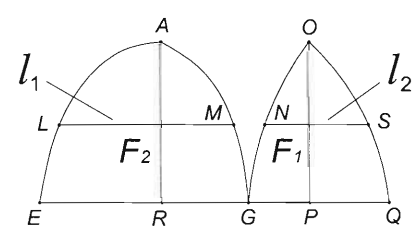
$F_2 = EAG$

[21] *Geometria* pp.13

If collections of lines were indeed Eudoxian magnitudes, which meant that Cavalieri had to show that they can be multiplied to exceed one another. That is, he had to show that there exists some n and m such that

$$\{\, \underbrace{O_{F_1}(l) + O_{F_1}(l) + \ldots O_{F_1}(l)}_{n \text{ times}} \,\} > \{\, \underbrace{\mathrm{O}_{F_2}(l) + \mathrm{O}_{F_2}(l) + \ldots \mathrm{O}_{F_2}(l)}_{m \text{ times}} \,\}$$

Case 1: $AR = OP$

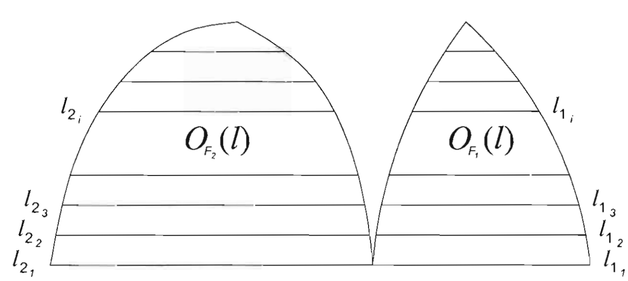


He supposed that the altitudes AR and OP of the two figures are equal.

By arguing that each l_1 (= NS) in $O_{F_1}(l)$ can be multiplied to exceed the corresponding l_2 (= LM) in $O_{F_2}(l)$ he concluded that a multiple of $O_{F_1}(l)$ greater that $O_{F_2}(l)$ exists.

That is, for each l_{1_i} and corresponding l_{2_i} in $O_{F_1}(l)$ and $O_{F_2}(l)$ respectively, there exist some n_i' such that

$$n_i'(l_{1_i}) > l_{2_i}$$



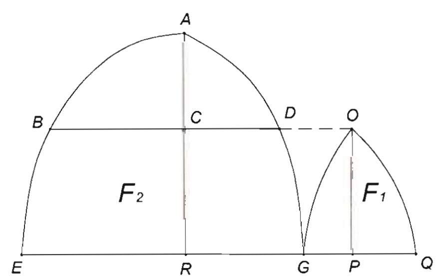
Further, there exists some $n = {}_{\max} n_i^{'}$ such that

$$nO_{F_1}(l) > O_{F_2}(l)$$

Case 2: $AR \neq OP$

Assume, with no lack of generality, that $AR > OP$.

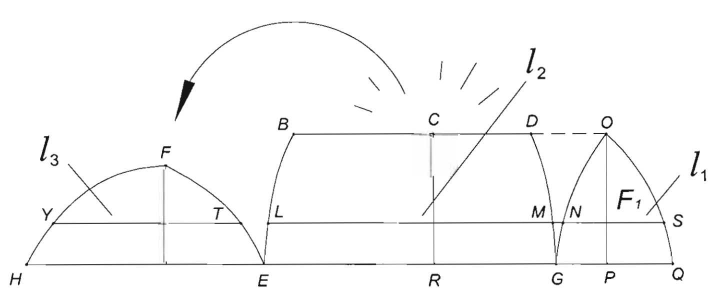
Cavalieri split the altitude- AR into parts equal to OP and a remaining part which was not greater than OP.



For the sake of simplicity he assumed that $AR = CR + AC$

(where $CR = OP$ and $AC < OP$).

Through point C he drew a line CO parallel to EG and moved the figure- BAD into figure- HFE.



As in the first part of the proof, he concluded the existence of a multiple of $O_{GOQ}(l)$ which is greater that $O_{EBDG}(l) + O_{HFE}(l)$ and hence greater than $O_{EAG}(l)$ he used the fact that each $l_l(= NS)$ in $O_{GOQ}(l)$ can be multiplied to exceed the sum of the corresponding

l_2 (= LM) in $O_{EBDG}(l)$ and l_3 (= YT) in $O_{HFE}(l)$

That is $n_i^{'}(l_{1_i}) > l_{2_i} + l_{3_i}$

then there exists some $n = {}_{\max} n_i^{'}$ such that:

$$nO_{F_1}(l) > [O_{EBDG}(l) + O_{HFE}(l)] = O_{F_2}(l)$$

Using **(9)** we can simplify the above expression to:

$$nO_{F_1}(l) > O_{F_2}(l)$$

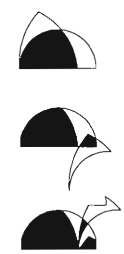
(5.11) Theorem II.2

$$\boxed{F_1 = F_2 \Rightarrow O_{F_1}(l) = \mathrm{O}_{F_2}(l) \qquad \textbf{(11)}}$$

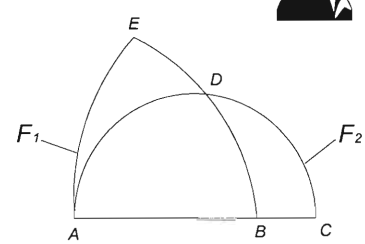
A very important element in the foundation of Cavalier's method, where he asserts that figures of equal area imply that the magnitudes of their collections are also equal.

Proof:

Cavalieri let the figures F_1 be described by AEB and figure F_2 by ADC, and assumed the figures have equal areas, that is $AEB = ADC$.



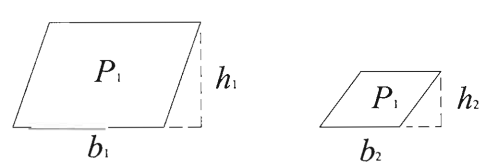
His aim is to show that $O_{AEB}(l) = O_{ADC}(l)$.



Cavalieri used superposition for this, by first placing the figures so that they had the area ADB in common. He then placed the residual of one figure over the other and continued the process "until all the residual parts have been placed over each other."

Since the two figures are split up into congruent parts, and these parts, by Theorem II.1, have equal collections of lines, then the figures also have equal collections of lines.

(5.12) Theorem II.11

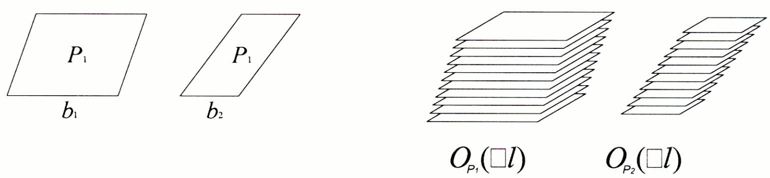


Assume P_1 and P_2 are two parallelograms with altitudes h_l and h_2 and bases b_l and b_2, then:

$$O_{P_1}(\square l) : O_{P_2}(\square l) = (\square b_1 : \square b_2)\cdot(h_1 : h_2) \qquad \textbf{(12)}$$

(The regula being parallel to the bases.)

To achieve this result Cavalieri first considered the case where $h_1=h_2$



All corresponding squares in the collections of squares of the two parallelograms are in constant ratio $\square b_1 : \square b_2$ since their altitudes are equal.

Then through the generalized ***ut-unum*** principle

$$O_{P_1}(\square l)_{AB} : O_{P_2}(\square l)_{AB} = (\square b_1 : \square b_2) \qquad \textbf{(13)}$$

Then considering the case where $h_1 \neq h_2$ and $\square b_1 = \square b_2$

Cavalieri then appealed to Book V, definition 5 of the *Elements*, which states:

"Magnitudes are said to be *in the same ratio,* the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order."

This definition says that $A{:}B = C{:}D$ if when for all whole numbers "n" and "m" it is the case that if nA is greater, equal, or less than mB, then mC is greater, equal, or less than nD, respectively, that is:

$A{:}B = C{:}D$ if for every whole numbers "n" and "m" it follows;

$$\mathrm{m}A > \mathrm{n}B \Rightarrow \mathrm{m}C < \mathrm{n}D$$

$$\mathrm{m}A = \mathrm{n}B \Rightarrow \mathrm{m}C = \mathrm{n}D$$

$$\mathrm{m}A < \mathrm{n}B \Rightarrow \mathrm{m}C < \mathrm{n}D$$

Concerning the parallelograms P_1 and P_2, Cavalieri presumed that following;

$$\mathrm{n}h_1 > \mathrm{m}h_2 \quad \Rightarrow \quad \mathrm{n}O_{P_1}(\square l)_{AB} > \mathrm{m}O_{P_2}(\square l)_{AB}$$

$$\mathrm{n}h_1 < \mathrm{m}h_2 \quad \Rightarrow \quad \mathrm{n}O_{P_1}(\square l)_{AB} < \mathrm{m}O_{P_2}(\square l)_{AB}$$

$$\mathrm{n}h_1 = \mathrm{m}h_2 \quad \Rightarrow \quad \mathrm{n}O_{P_1}(\square l)_{AB} = \mathrm{m}O_{P_2}(\square l)_{AB}$$

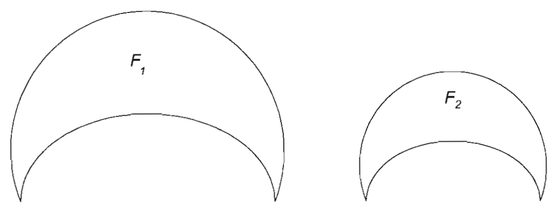
Thus, by Euclid's definition 5

$$\boxed{O_{P_1}(\Box l)_{AB} : O_{P_2}(\Box l)_{AB} = (h_1 : h_2) \qquad \textbf{(14)}}$$

The required relation **(12)** then follows from a combination of **(13)** and **(14)**.

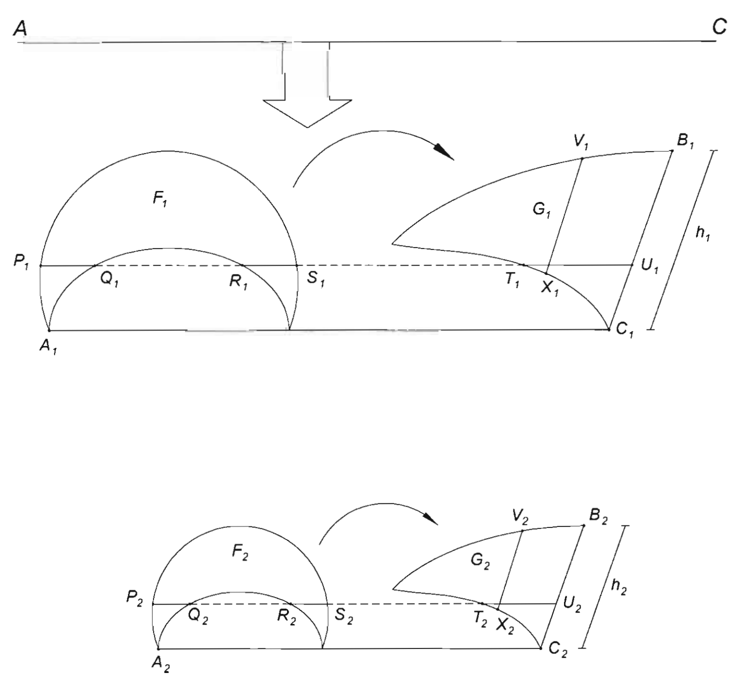
(5.13) Theorem II.5

Theorems II.5 of Geometria states that when F_1 and F_2 are similar plane figures with altitudes h_1 and h_2 and "bases" b_1 and b_2 then



$$F_1 : F_2 = (h_1 : h_2) \cdot (h_1 : h_2) = (h_1 : h_2) \cdot (b_1 : b_2)$$

("bases" here mean "horizontal altitude" of auxiliary figures - to be introduced)



For the proof Cavalier's relied upon the ***ut-unum*** principle along with an auxiliary figure with base b_1 and altitude h_2. To obtain this, he transformed the figures F_1

and F_2 in two steps into figures H_1 and H_2.

Step 1:

First he constructed figure G_i with the same altitude- h_i as F_i, (i = 1,2) such that each " l " (= T_iU_i) in $O_{G_i}(l)_{AC}$ was defined by having its one endpoint at B_iC_i and by being equal to the line segment, or to the sum of the line segments, of the corresponding " l " in $O_{F_i}(l)_{AC}$.

That is:

$$T_iU_i = P_iQ_i + R_iS_i.$$

Since by construction, $O_{F_i}(l)_{AG}$ and $O_{G_i}(l)_{AG}$ are equal, then Theorem II.2 implies the following:

$$\boxed{F_1 = G_1 \text{ and } F_2 = G_2 \quad \textbf{(1.1)}}$$

Step 2:

Cavalieri then transformed the figure G_i into H_i with altitude h_i and base b_i, (i =1, 2) by the same process as before, but now considering the collections of lines with respect to the regula BD:

Thus " l " (= K_iZ_i) in $O_{H_i}(l)_{BD}$ was defined by having the endpoint- Z_i on the segment E_iD_i and by having segment Z_iK_i equal to its corresponding segment X_iV_i.

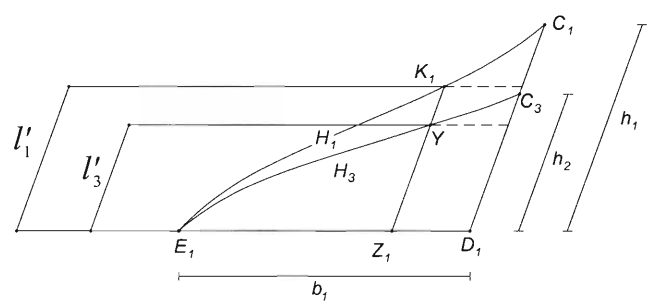
Since, by construction, $O_{G_i}(l)_{BD} = O_{H_i}(l)_{BD}$, then (Theorem II.2) implies that; $H_i = G_i$.

Combining the results above with those of (1.1), it follows:

$$\boxed{F_1 = G_1 = H_1 \text{ and } F_2 = G_2 = H_2 \quad \textbf{(2.1)}}$$

The advantage of using the figures H_1 and H_2 instead of F_1 and F_2 is that each line- " l " of $O_{H_i}(l)$, both with respect to the regula AC and the regula BC, is a line segment having one endpoint on an axis, and not a sum of arbitrarily situated line segments.

Step 3:



Cavalieri was then able to construct auxiliary figure H_3 with base $E_1D_1 = b_l$ and altitude $C_3D_1 = h_2$, by imposing the condition that for each l'_3 ($=YZ_1$) in $O_{H_3}(l)_{BD}$, to each corresponding l'_1 ($= K_1Z_1$) in $O_{H_1}(l)_{BD}$, be defined by the relation;

$$l'_1 : l'_3 = h_l : h_2;$$

Thus:

$$\boxed{K_1Z_1 : YZ_1 = C_1D_1 : C_2D_2 = h_l : h_2 \quad \textbf{(3.1)}}$$

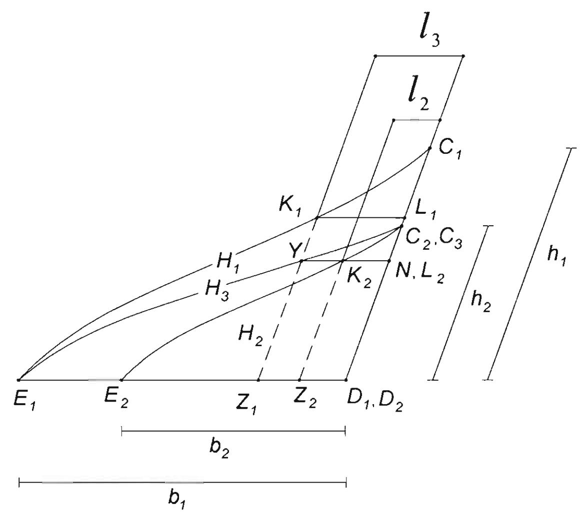
Since the above relation is true for all $K_1Z_1$and YZ_1 of figures H_1 and H_3 respectably, using Cavalieri's principle (Proposition II.4), it follows that

$$\boxed{H_1{:}H_3 = h_1{:}h_2 \quad \textbf{(3.2)}}$$

Step 4:

Next Cavalieri sought to prove that for each pair of corresponding l_3 (= YN) in $O_{H_3}(l)_{AC}$ and l_2 (= K_2L_2) in $O_{H_2}(l)_{AC}$ the following relation held;

$$l_3{:}l_2 = h_1{:}h_2.$$



He noticed that similar figures F_1 and F_2 imply that H_1 and H_2 are also similar.

He let; $ND_{1(\text{of } H_3)} = L_2D_{2\ (\text{of } H_2)}$

and denoted line segments:

$l_3 = YN$ and $l_2 = K_2L_2$

By construction, the point Y is determined by the relation;

$$YN = K_1L_1$$

and

$K_1Z_1:YZ_1 = h_1:h_2$ *The last equality from (3.1)

Since $K_1Z_1 = L_1D_1$ and $YZ_1 = L_2D_2$, Cavalieri concluded that;

$$\boxed{L_1D_1 : L_2D_2 = h_1:h_2 \quad \textbf{(4.1)}}$$

Noting that $l_3 = K_1L_1$ and $l_2 = K_2L_2$; the results- (4.1) means that the line segments l_3 and l_2 are similarly situated in the similar figures H_1 and H_2.

Thus: $l_3:l_2 = h_1:h_2$

Applying Cavalieri's principle (Theorem II.4) to the above ratio, it follows that;

$$\boxed{H_3:H_2=h_1:h_2 \quad \textbf{(4.2)}}$$

Using previous results;

$F_1=H_1$ and $F_2=H_2$	**(2.1)**
$H_1:H_3=h_1:h_2$	**(3.2)**
$H_3:H_2=h_1:h_2$	**(4.2)**

Then;

$$F_1:F_2 = H_1:H_2 = (H_1:H_2)\cdot(H_3:H_3) = (H_1:H_3)\cdot(H_3:H_2) = (h_1:h_2)\cdot(h_1:h_2)$$

Further, since F_1 and F_2 are similar figures then, by construction, G_1 and G_2 are also similar figures.

This implies that; $(h_1:h_2) = (b_1:b_2)$ and it follows that;

$$\boxed{F_1:F_2 = (h_1:h_2)\cdot(b_1:b_2)}$$

This is exactly what we wanted to prove.

6 The Method of Indivisibles Applied:

(6.1) Using the tools available to us now, (plus two other theorems to be introduced shortly), we can now outline how one can obtain, through the method of indivisibles, the result that a cone and a cylinder with the same base and equal height are in ratio to each other as 3:1.

We first establish a theorem about triangles and parallelograms.

<u>Theorem 24</u>.

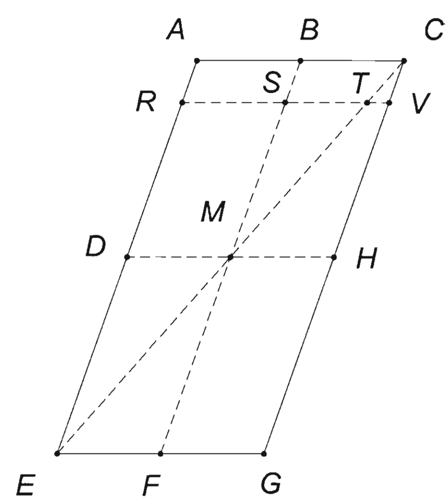
> *"Given any parallelogram and a diagonal in it, all the squares of the parallelogram are three times the squares of any of the triangles formed by the above mentioned diagonal, with a common reference one side of the sides of the parallelogram."*[22]

Proof: Let $\square ACGE$ be a parallelogram with diagonal CE.

The claim is that, with reference to EG

$$O_{ACGE}(\square l)_{EG} = 3O_{EGE}(\square l)_{EG},$$

where the triangle; ΔCGE can be replaced by the triangle; ΔAEC.



For the remainder of the demonstration the segment EG as *regula* will be understood, but omitted from print for simplicity.

[22] *Geometria*, pp. 78

Step 1:

Let B and F be the midpoints of line segments AC and EG, respectively.

Similarly, let D and H be the midpoints of AE and CG, respectively.

For any arbitrary RV parallel to EG and intersecting BF and CE in the points S and T, the following can be shown:

$$RT = (AC : AE)\cdot(RE)$$

$$TV = AC - RT = AC - (AC : AE)\cdot RE$$

$$RS = 1/2AC$$

$$\begin{aligned} TS &= 1/2\ AC - TV \\ &= 1/2\ AC - [AC - (AC : AE)\cdot RE] \\ &= (AC : AE)\cdot RE - 1/2AC \end{aligned}$$

Then:

$$\begin{aligned} \square RT + \square TV &= (RT)^2 + (TV)^2 \\ &= (AC : AE)\cdot(AC : AE)\cdot(RE)\cdot(RE) + [AC - (AC : AE)\cdot RE]\cdot[AC - (AC : AE)\cdot RE] \\ &= 2\{(AC : AE)\cdot(AC : AE)\cdot(RE)\cdot(RE) - (AC)\cdot(AC)\cdot(RE : AE) + 1/2\ (AC)\cdot(AC)\} \\ &= 2\{(AC : AE)^2\cdot(RE)^2 - (AC)^2\cdot(RE : AE) + 1/2\ (AC)^2\} \end{aligned}$$

and;

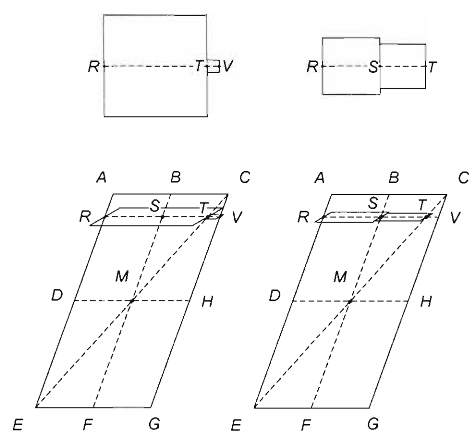
$\square RS + \square TS = (RS)^2 + (TS)^2$

$= 1/4\,(AC)\cdot(AC) + [(AC : AE)\cdot(AC : AE)\cdot RE - 1/2\,AC]\cdot[(AC : AE)\cdot(AC : AE)\cdot RE - 1/2\,AC]$

$= (AC : AE)\cdot(AC : AE)\cdot(RE)\cdot(RE) - (AC)\cdot(AC)\cdot(RE : AE) + 1/2\,(AC)\cdot(AC)$

$= (AC : AE)^2\cdot(RE)^2 - (AC)^2\cdot(RE : AE) + 1/2\,(AC)^2$

Combining the results from above, yields:



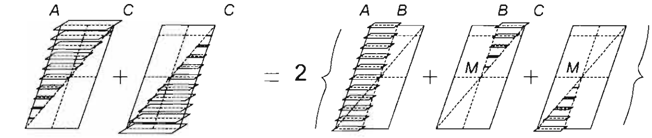
$\square RT + \square TV = 2\square RS + 2\square TS$ **(i)**

(for any arbitrary RV parallel to EG)

Where "□" denotes the square on the line segment.

Cavalier's generalized ***ut–unum*** principle- Theorem II.23, states that "*O*" can be applied to **(i)**, resulting in:

$O_{ACE}(\square l) + O_{CGE}(\square l) = 2O_{ABFE}(\square l) + 2[O_{BCM}(\square l) + O_{MEF}(\square l)]$ **(1.1)**



By using the assumption that congruent figures have equal collection of squares, then since, $\Delta ACE \cong \Delta CEG$ and $\Delta MEF \cong \Delta BCM$, we may state the following:

(Where "$\cong$" denotes congruency)

$O_{ACE}(\square l) = O_{CEG}(\square l)$ **(1.2a)** and $O_{MEF}(\square l) = O_{BCM}(\square l)$ **(1.2b)**

Step 2:

Since figures ΔCEG and ΔMEF are similar triangles, then:

$$CG : MF = EG : EF$$

Also, by construction:

$$CG = 2MF \text{ and } EG = 2EF$$

Then, we may state the following:

$O_{CEG}(\square l) : O_{MEF}(\square l)$

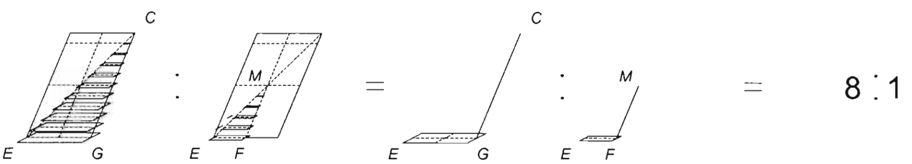
$$= CG \cdot (EG)^2 : MF \cdot (EF)^2$$

$=2MF\cdot(EG)^2 : MF\cdot(EF)^2$

$= 2(2EF)^2 : (EF)^2$

$= 8 : 1$

$O_{CEG}(\square l) : O_{MEF}(\square l) = 8 : 1$ **(2.1)**



Or equivalently;

$O_{CEG}(\square l) : [O_{BCM}(\square l) + O_{MEF}(\square l)]$

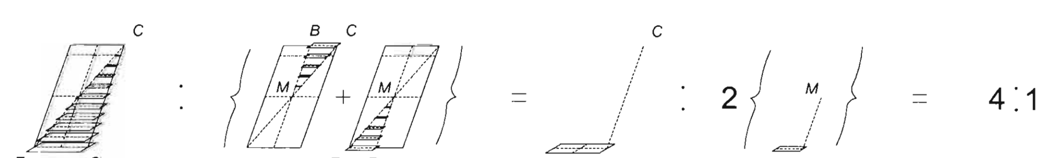
$= O_{CEG}(\square l) : 2O_{MEF}(\square 1)$ *since* $O_{MEF}(\square l) = O_{BCM}(\square l)$

$= (CG)(EG)^2 : 2(MF)(EF)^2$

$= 2(2EF)^2 : 2(EF)^2 = 8 : 2$

$= 4 : 1$

$O_{CEG}(\square l) : [O_{BCM}(\square l) + O_{MEF}(\square l)] = 4 : 1$ **(2.2)**



Step 3:

Using **(1.1)** and substituting in the results of **(1.2a)** we have:

$$O_{ACE}(\square l) + O_{CEG}(\square l) = 2O_{ABFE}(\square l) + 2[O_{BCM}(\square l) + O_{MEF}(\square l)] \quad \textbf{(1.1)}$$

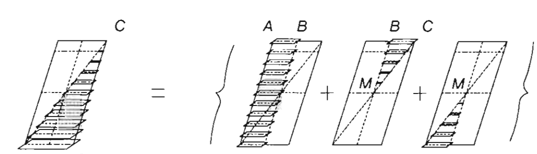
$$\Downarrow \textbf{1.2a}$$

$$O_{CEG}(\square l) + O_{CEG}(\square l) = 2O_{ABFE}(\square l) + 2[O_{BCM}(\square l) + O_{MEF}(\square l)]$$

$$2O_{CEG}(\square l) = 2O_{ABFE}(\square l) + 2[O_{BCM}(\square l) + O_{MEF}(\square l)]$$

Or simply;

$$O_{CEG}(\square l) = O_{ABFE}(\square l) + O_{BCM}(\square l) + O_{MEF}(\square l) \quad \textbf{(3.1)}$$



We can now use the ratio **(2.2)** and substitute in the results of **(3.1)**.

$$O_{CEG}(\square l) : [O_{BCM}(\square l) + O_{MEF}(\square l)] = 4 : 1 \quad \textbf{(2.2)}$$

$$\Downarrow_{\mathbf{3.1}}$$

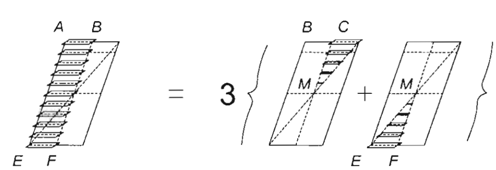
$$[O_{ABFE}(\square l) + O_{BCM}(\square l) + O_{MEF}(\square l)] : [O_{BCM}(\square l) + O_{MEF}(\square l)] = 4 : 1$$

Or equivalently;

$$[O_{ABFE}(\square l) + O_{BCM}(\square l) + O_{MEF}(\square l)] = 4[O_{BCM}(\square l) + O_{MEF}(\square l)]$$

This simplifies to:

$$O_{ABFE}(\square l) = 3[O_{BCM}(\square l) + O_{MEF}(\square l)] \qquad \textbf{(3.2)}$$



Step 4:

Applying Theorem II.11 to the following ratio:

$O_{ACGE}(\square l) : O_{ABFE}(\square l)$

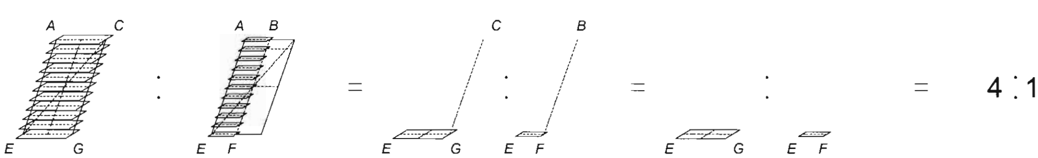
$$= (\square EG : \square EF) \cdot (CG : CG)$$

$$= (EG)^2 : (EF)^2$$

$$= (2EF)^2 : (EF)^2 \qquad \textit{*Since by construction } EG = 2EF$$

$$= 4 : 1$$

$$O_{ACGE}(\square l) : O_{ABFE}(\square l) = 4 : 1 \qquad \textbf{(4.1)}$$



Using **(4.1)** and substituting in the results from **(3.2)**.

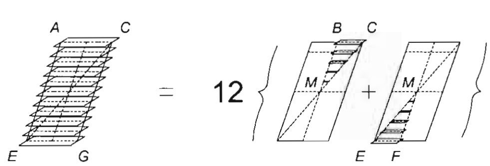
$O_{ACGE}(\square l) : O_{ABFE}(\square l)$ **(4.1)**

$\Downarrow$ **3.2**

$O_{ACGE}(\square l) : 3[O_{BCM}(\square l) + O_{MEF}(\square l)] = 4 : 1$

Or equivalently;

$O_{ACGE}(\square l) = 12[O_{BCM}(\square l) + O_{MEF}(\square l)]$ **(4.2)**



Step 5:

Finally, we may state the following; substituting in the results of **(3.1)**, **(3.2)** and **(4.2)**:

$$O_{ACGE}(\square l) : O_{CEG}(\square l)$$

$$\Downarrow_{\mathbf{3.1}}$$

$$O_{ACGE}(\square l) : [O_{ABFE}(\square l) + O_{BCM}(\square l) + O_{MEF}(\square l)]$$

$$\Downarrow_{\mathbf{3.2}}$$

$$O_{ACGE}(\square l) : \{3[O_{BCM}(\square l) + O_{MEF}(\square l)] + [O_{BCM}(\square l) + O_{MEF}(\square l)]\}$$

$$O_{ACGE}(\square l) : 4[O_{BCM}(\square l) + O_{MEF}(\square l)]$$

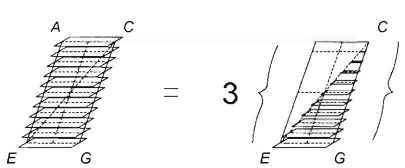
$$\Downarrow_{\mathbf{4.2}}$$

$$12[O_{BCM}(\square l) + O_{MEF}(\square l)] : 4[O_{BCM}(\square l) + O_{MEF}(\square l)]$$

$$= 3 : 1$$

Or equivalently:

$$O_{ACGE}(\square l) = 3O_{CEG}(\square l) \qquad \textbf{(5.1)}$$



This is exactly what we wanted to prove.

7 The Crowing Jewel of Cavalieri's Geometry:

Finally we appeal to Theorem 33 to finish the proof. This theorem is arguably the crowning jewel of Cavalier's geometry of indivisibles as it allows him to make powerful generalizations concerning cubatures that the ancients had only proved for specific cases.

Namely, it allowed him to use collections of planes to discover the volume of their associated solid figure.

That is, (surprisingly) Cavalier's method of indivisibles employed collections of objects which are of one dimension less than the objects to be discerned.

(7.1) **Theorem 33**.

> "Given any two plane figures, and taking an arbitrary regula in each one of them. Arbitrary solids, mutually similar, generated by the same figures according to the same references will be to each other as all the squares of the same figures taken with respect to the common referents." [23]

Although Cavalieri never explicitly states it, one can safely assume that he meant for the figures to posses equal altitudes, (in reference to the same regula), as a requirement of the theorem.

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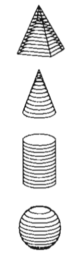
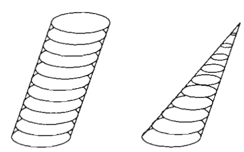
[23] *Geometria*, pp.102



Also, Cavalier's reference to "mutually similar solids" deserves a little explanation. A (single) solid figure is said to be "similar" when all its cross sections, taken with respect to the same regula, are all similar figures. Standard examples are pyramids, cones, cylinders and spheres.

Further, when two solids are found to have mutually similar cross sections, (taken with respect to the same regula), then the solids are referred to as "mutually similar."



We will prove Theorem 33 with reference to a cylinder and a cone, keeping in mind that Cavalieri proved the result for arbitrary figures. In our case we can think of the cylinder and the cone as generated by a parallelogram and a triangle, respectively.

We take the solids, S_1 and S_2 to be the cylinder and the cone, respectively, and figures F_1 and F_2 to be their generating planar figures – a parallelogram and the triangle. We take S_1 and S_2 will be mutually similar by choosing the regula such that all the cross sections of the solid figures will be circular disks.

Using the above notation, Theorem 33 makes the claim that, in the case of two mutually similar solids, (S_1 and S_2), the ratio between the solids is only a function of the squares of F_1 and F_2. That is, it does not depend on the actual profile of the cross sections.

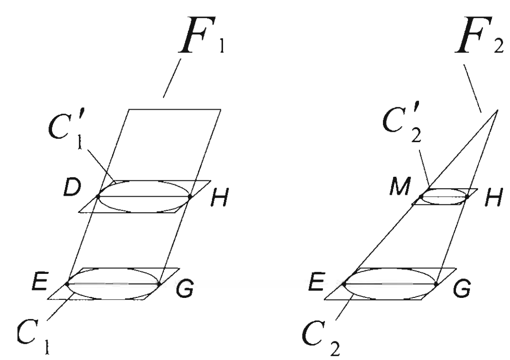
Thus;

$S_1 : S_2 = O_{F_1}(\square l) : O_{F_2}(\square l)$

The above result is established by making use of theorem II.15 and the *ut-unum principle*.

Proof:

Let line segment EG be the base of the cylinder as well as that of the cone. Also, let C_1 and C_2 be circular disks on the base EG of the cylinder and the cone respectably.



Let DH and MH be arbitrary line segments, (parallel to the regula AB), on the cylinder and the cone respectively, with similar circular disks C'_1 and C'_2 on DH and MH respectively.

Taking the cone separately:

Consider the square on EG– "$\square EG$" and the square on MH– "$\square MH$".

It should be seen that $(EG)^2 : (MH)^2 = \square EG : \square MH$

Now consider the circular disks C_1 and C_2 on EG and MH, respectively. By theorem II.15, C_2 and C'_2 stand in ratio as $(EG)^2$ is to $(MH)^2$. Thus

$C_2 : C'_2 = \square EG : \square MH$

Or, with some slight rearrangement

$C_2 : \square EG = C'_2 : \square MH$

But since DH is arbitrary, we can now use the ***ut-unum*** principle. Thus, it follows;

$C_1 : \square EG = O_{F_1}(\otimes l) : O_{F_1}(\square l)$

(Where "$O_F(\otimes l)$" and "$O_F(\square l)$" denote "all the circles" and "all the squares," respectively, of the figure– F.)

Now, considering the cylinder and using similar arguments as before

$C_2 : C'_2 = \square EG : \square MH$

Or with rearrangement

$C_2 : \square EG = C'_2 : \square MH$

But again, DH is arbitrary, so we can use the ***ut-unum*** principle to obtain

$C_2 : \square EG = O_{F_2}(\otimes l) : O_{F_2}(\square l)$

Now, since by construction $C_1 = C_2$, then;

$O_{F_1}(\otimes l) : O_{F_1}(\square l) = O_{F_2}(\otimes l) : O_{F_2}(\square l)$

Taking the results obtained by the cylinder and the cone we obtain

$O_{F_1}(\otimes l) : O_{F_2}(\otimes l) = O_{F_1}(\square l) : O_{F_2}(\square l)$ (i)

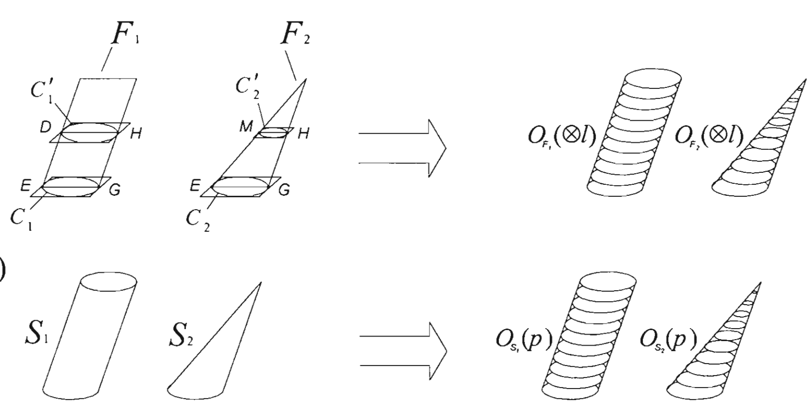
But it can be shown that $O_{F_1}(\otimes l)$ – "all the circles" of figure F_1 is congruent to

$O_{S_1}(p)$ – "all the circular planes" of solid S_1. Likewise, $O_{F_2}(\otimes l)$ to $O_{S_2}(p)$.

That is

$O_{F_1}(\otimes l) \cong O_{S_1}(p)$

$O_{F_2}(\otimes l) \cong O_{S_2}(p)$



(Were S_1 and S_2 are the cylinder and the cone, respectively, while F_1 and F_2 are their generating planar figures.)

Thus:

$$S_1 : S_2 = O_{S_1}(p) : O_{S_2}(p) = O_{F_1}(\otimes l) : O_{F_2}(\otimes l)$$

or using the results of (i), then

$$S_1 : S_2 = O_{F_1}(\square l) : O_{F_2}(\square l)$$

This is what we sought to prove.

Moerover, since we found that;

$$O_{F_1}(\square l) : O_{F_2}(\square l) = 3{:}1$$

Then it follows that;

$$S_1 : S_2 = 3{:}1$$

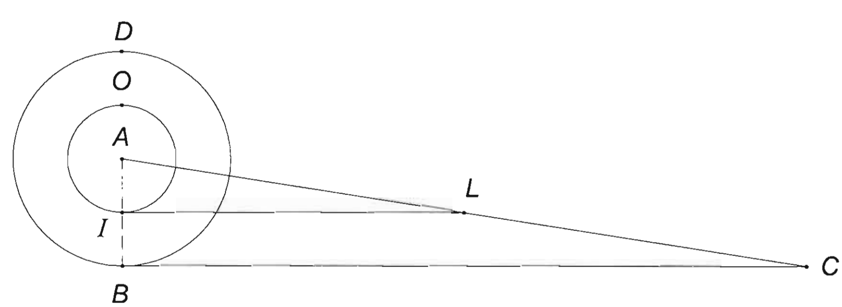
That is, we have just proven that the ratio between a cylinder and an inscribed cone with the same bases and altitudes is 3 : 1.

8 Torricelli's Exercise

Before proceeding to his main proof, Torricelli provided a "warm-up" exercise to illustrate his use of "curved indivisibles."[24] The proof is that of the Archimedean proposition concerning the measure of a circular disk, which states that the area of a circle is equal to the area of a right triangle whose legs are equal to the radius and the circumference of the circle. Torricelli's approach is novel in its stated use of curved indivisibles, thought it is unclear whether he was aware of a similar proof offered by Gerard of Brussels which dated back to the 13th century.

Warm-up exercise:

Draw the circle $\odot BDB$ with radius AB. Consider an arbitrary chosen point I on the radius AB. Let one leg of the right triangle ΔBC equal the circumference $\odot BDB$, and the other leg equal to the radius AB. If one produces a second circle with center A and radius AI one obtains the following proportions:



$$\text{Circumference } \odot BDB : \text{circumference } \odot IOI = AB : AI$$
$$= BC : IL$$

[24] De solido hyperbolico acuto, in *Opera Geometrica*, Torricelli (1644)

Hence, circumference $\odot BDB : BC$ = circumference $\odot IOI : IL$

But, since by construction the circumference $\odot BDB = BC$.

Then, as a consequence, circumference $\odot IOI = IL$

Since this is true for any arbitrarily chosen point I on the radius AB, Torricelli concluded:

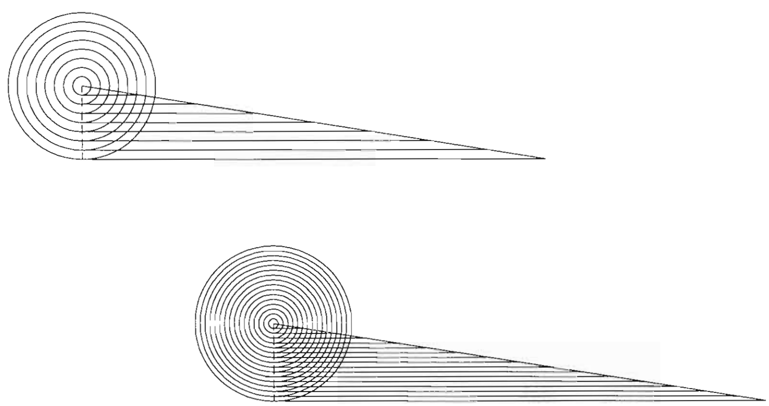
> "*all peripheries taken together are equal to all the straight lines taken together. That is, the circle $\odot BDB$ will be equal to the triangle ΔABC.*"[25]

These "peripheries" and "lines" were what Torricelli identified as the indivisibles of the figures; the techniques of which were largely adopted from Cavalier's theory of indivisibles, though Torricelli would adapt the theory in significant ways: First; he would enhance the range of application of the technique by introducing the use of curved indivisibles. Second; Torricelli would grant his indivisibles a thickness; a move that Cavalieri was reluctant to make. Third; in contrast to Cavalieri's approach, which took the geometric figure and its associated indivisibles as two separate magnitudes, Torricelli's would make a simple identification of the figure and its characterizing indivisibles.

Therefore, considering the warm-up exercise, the idea of Torricelli's proof is rather simple. Conceptually, the circular disk can be thought to be composed of the collection of curved indivisibles (of the type $\odot IOI$) and the triangle composed of the collection of linear indivisibles (of the type IL). Any arbitrarily chosen point I can be seen to determines a unique pair of individuals; that is it associates a curved indivisible

[25] De infinitis parabolis, in *Opera Geometrica*, Torricelli (1644)

from the circular disk to an indivisible from the triangle and assigns the pair a location. Moreover, it is clear that for each I on the radius AB, all such pairs are similarly placed. By appealing to what we can intuitively reason about collections of lines and that of nested circles, Torricelli concludes that since the collections of indivisibles of the two figures are equal, by the fundamental principle of the theory of indivisibles, (namely Cavalieri's Theorem II.2), the areas of the two figures will be equal as well.

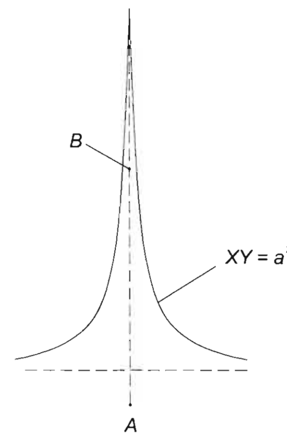


Demonstration of the paradoxical solid:

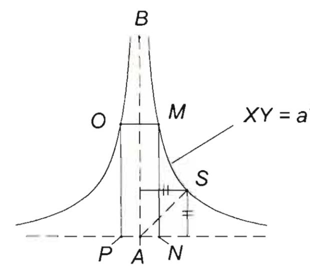
8.1 Return to Gabriel's Horn: Proof by Indivisibles

This example was introduced in Section 2.1 and analyzed there as a "standard" Calculus exercise for today's student. The construction that follows is the original construction by Torricelli and is built on five preliminary lemmas. Although none of these lemmas makes use of indivisibles, they do defined geometric relationships that Torricelli draws upon for his demonstration.

First Lemma:

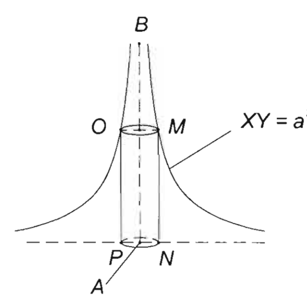


Given a hyperbole with asymptote AB, if the figure is made to rotate about the axis AB one obtains the "acute hyperbolic solid" which is infinitely long in the direction of B



Consider then, within the solid defined in this way a rectangle passing through *AB*, for example, the rectangle *POMN*. Let *AS* be the semi-axis of the hyperbole. In this way, one demonstrates that the square constructed on *AS* has the same area as every rectangle *POMN*.

Second Lemma:



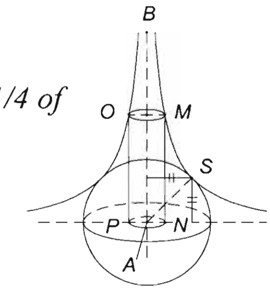
It is shown that all the cylinders inscribed in the acute solid around the common axis AB are isoperimetric (i.e. the lateral surfaces are equal)

Third Lemma:

It is shown that all these isoperimetric cylinders have volume proportional to the diameter of their base.

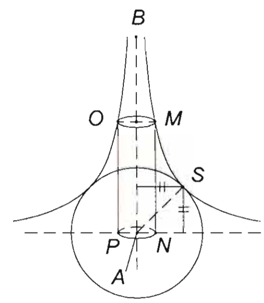
Fourth Lemma:

It is shown that the lateral surface of each cylinder POMN is 1/4 of the surface of the sphere whose radius is the semi-axis AS.



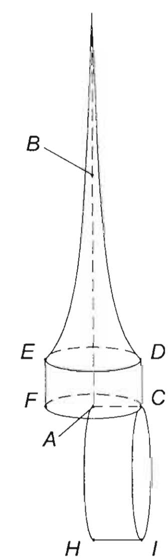
Fifth Lemma:

It is shown that the lateral surface of each cylinder POMN described in the acute solid as in the previous figure. Is equivalent to the circular disk of radius AS.

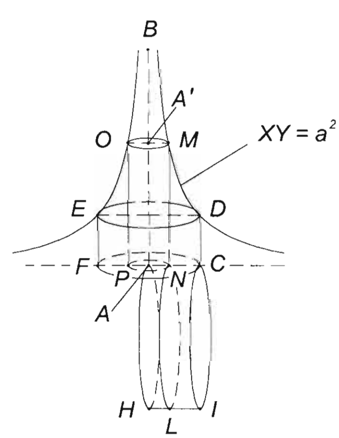


Theorem:

It is shown that the infinitely long solid made up of the acute hyperbolic solid EBD and its cylindrical base FEDC, is equivalent to the cylinder ACGH of height CA and whose base is HA=2AS.

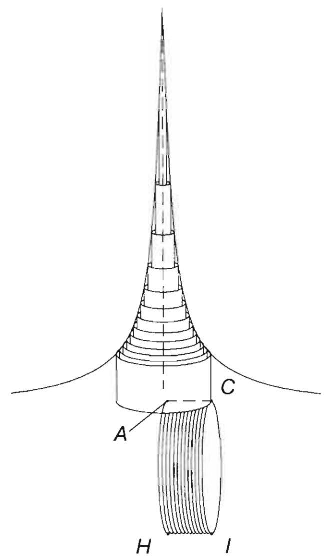


Proof: The proof follows along in a similar manner as Torricelli's warm-up exercise. The infinitely long solid can be thought to be composed of a collection of cylindrical indivisibles, (that is, lateral surfaces of the type *POMN*); and the cylinder *ACIH* can be thought to



be composed of planar indivisibles, (circular planes with diameter *AH*).

Any arbitrarily chosen point *N* determines a unique pair of indivisibles, (associating an indivisible from the infinitely long solid to an indivisible from the cylinder *ACIN*), by placing them in space. Moreover, for each *N* on the segment *AC* all such pairs are similarly placed.



By lemma 5, each lateral surface of type *POMN* is equal in area to a circular disk of radius *AS*; also, by construction, it can be seen that $AH=2AS$. Thus, each curved indivisible of the infinite solid is equal to each planar indivisible of the cylinder.

Therefore, Torricelli reasoned, since the collections of indivisibles of the two figures are equal, by the fundamental principle of the theory of indivisibles, the volumes of the two figures will be equal as well. Hence the infinitely long solid is equal is equal to the finite cylinder whose base is the circle with diameter *AH* and whose height is *AC*.

Since his proof was based upon the idea that the infinitely long solid was composed of the collection of cylindrical indivisibles, Torricelli understood that this required the filling of the solid with the lateral surfaces of the inscribed cylinders, even in the case where the innermost cylinder degenerates into an infinitely straight line. Thus, he took for granted the assertion of lemma five- that an equality between the inscribed cylinders and the circular disk (of radius *AS*) held even in this case, that is at

N=A. Indeed, Torricelli's comments published as *De infinitis parabolis*, indicate his position;

"What Galileo says of a point is equal to a line is true, and in our hyperbolic solid it is true that an infinitely long axis is equal to a circle."[26]

The reference here is to the "soup dish' paradox presented by Galileo in the *Two New Sciences*. It is surprising that Torricelli should cite the paradox to support his assumption, since prior to its publication, Galileo sent it to Bonaventura Cavalieri as a cautionary example concerning the perils of using the "method of indivisibles" in geometry. That Torricelli took the results of the paradox as a validation of his assumptions clearly goes against the Galileo's intentions.

Torricelli declared himself satisfied that the proof was compeer and stood by itself. However, concerned over its public reception led him to comment;

"I consider the previous theorem sufficiently clear in itself and more than adequately confirmed by the examples proposed at the beginning s of the book. However, in order to satisfy also the reader who is scarcely a friend of indivisibles, I shall repeat its demonstration at the end of the work with the usual demonstrative method s of ancient Geometers which, although longer, in my opinion is not for that more certain."

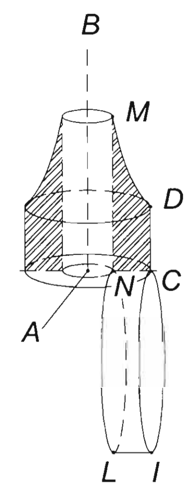
[26] *De solido hyperbolico acuto. Opera Geometrica*, Torricelli (1644)

Here Torricelli offers a proof by the method of exhaustion for those who did not favor indivisibles. Considering the criticism the method of indivisibles received from certain circles ever since its first introduction by Cavalieri back in ---, in addition to the novel contributions that Torricelli had made, no doubt he felt safer publishing both proofs.

Proof by Exhaustion:

Sixth Lemma:

Consider the solid described by the rotation of the figure *MNCD* around the axis *AB*. The hollow solid described is equal in volume to the cylinders whose height is *NC* and whose base is the circle with diameter *NL* = 2*AS*. This is true for any arbitrary chosen point *N* on *AC* different from *A*.

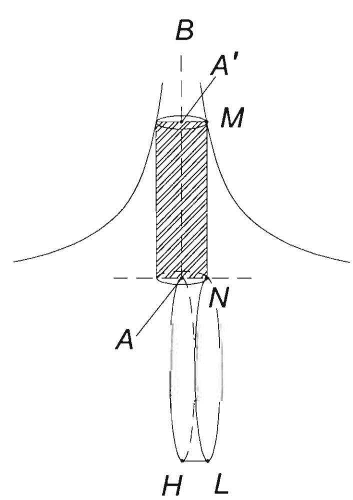


That is;

Hollow solid *FEOMDC* = cylinder *LNCI*

Seventh Lemma:

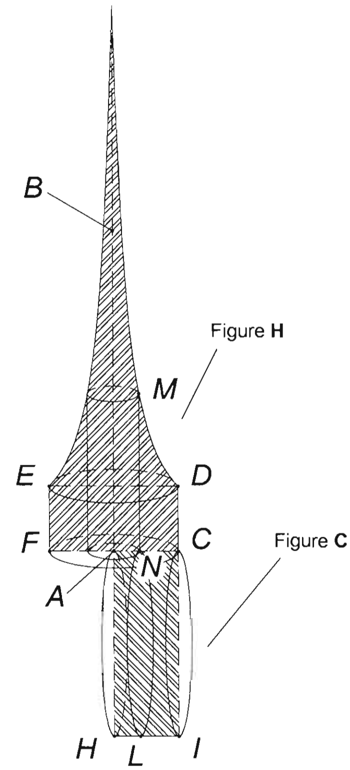
Consider the cylinder-*POMN* generated by the rotation of the rectangle *ANMA*' around *AB*. The volume of this cylinder is half the volume of the cylinder whose height is *AN* and whose base is the circle with diameter *AH* = 2*AS*. This is also hold true



for any arbitrarily chosen point N on AC different from A.

That is;

$POMN = \frac{1}{2}\, ANHL$



We shall denote the infinitely long solid, described in the theorem, by **H**. Also, the cylinder *HACI* shall be denoted as **C**. To prove the volume of **H** equaled the volume of C, Torricelli proceeded, (as was usual for the method), by contradiction.

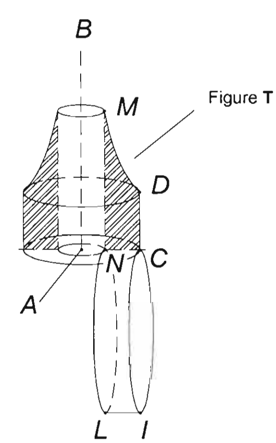
Part I:

Assume **H** < **C**. Then the volume of **H** is equal to only a portion of **C**; let us say it is equal to the cylinder *LNCI*, for all points N on the segment AC, such that $N \neq A$.

That is;

H = *LNCI* **(i)**

If we extended the line segment NL to meet the hyperbola at point M, then the solid generated by the rotation of the figure *MNCD* around the axis AB, (which we will denote as **T**), is by lemma 6, equal to the cylinder *LNCI*; and this is true for any point N on AC different from A.



That is;

T = *LNCI*

But, it can be seen that, for all *N* on *AC*, such that *N*≠*A*, the hollow solid- **T** is only a part of the figure **H**,

then it follows that

T = *LNCI* < **H**,

and this contradicts the result of our initial assumption, i.e., **(i)**.

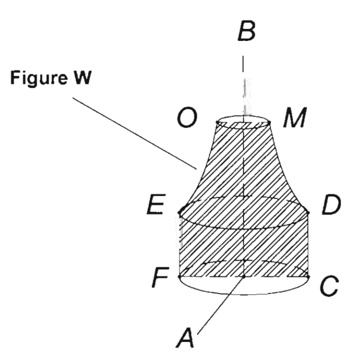
Thus, it can not be true that **H** < **C**.

Part II:

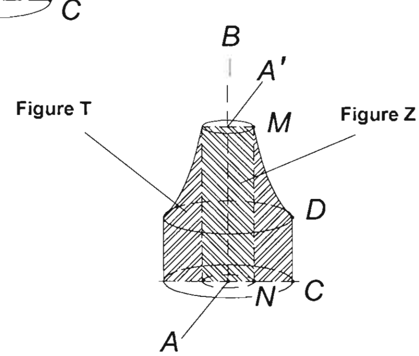
Assume then that **H** > **C**.

Since the volume of **C** is finite, it must be equal to some finite part of **H**, let us say for example, *FEOMDC*; which we shall denote as **W**. That is;

W = **C**. **(ii)**



Where **W** is the sum of the hollow solid **T**, (generated by the rotation of the figure *MNCD*), and the cylinder **Z**, (obtained by the rotation of the rectangle *ANMA'*.)



By Lemma 6, **T**=*LNCI* and by Lemma 7 **Z** =

$\frac{1}{2}(ANHL)$. Hence,

$$\mathbf{W} = \mathbf{Z} + \mathbf{T} = \tfrac{1}{2}(ANHL) + LNCI < ANHL + LNCI$$

However, by definition $\mathbf{C} = HACI = (ANHL + LNCI)$. Then, it follows;

$$\mathbf{W} < \mathbf{C}.$$

But, this contradicts the results of our initial assumption, i.e., **(ii)**. Thus, it can not be true that $\mathbf{H} > \mathbf{C}$.

Since neither $\mathbf{H} < \mathbf{C}$, nor $\mathbf{H} > \mathbf{C}$ was found to be true, the conclusion from both cases is that **H** can only equal **C**.

9 Summary and Concluding Remarks

In summary, what should we say about the Method of Indivisibles and the teachings of Cavalieri?

Cavalieri can be considered a starting point in the foundation of the theory of indivisibles during the first half of the 17th century. That his name was cited by so many who where to develop or used indivisibles over the course of the following decades, ensured that the name "Cavalieri" would become almost synonymous with the method of indivisibles. However, it seems that only Cavalieri tried to provide indivisibles with a genuine theory clearly founded on classical Greek definitions. Indeed, he strived to grant his new method a certainty "classical legitimacy" by anchoring it to the Euclidian theory of proportions. But, while Cavalieri tried to remain faithful to the canons of Greek geometry and show them proper respect in the course of his proofs, he also introduced mathematical objects completely foreign to the Classic tradition, namely "all the lines" and "all the planes." The difference in approach between that of classic geometry to that of his new method is rather striking. Consider, for example, the proof that the ratio of a cylinder to its inscribed cone is 3:1. Proof by exhaustion relies solely on an argument based upon *reductio ad absurdum*. In effect, what it shows us is that the cylinder cannot be greater than three times its inscribed cone, nor can it be less; otherwise a logical contradiction would ensue. It tells us nothing about the causes of the mathematical relationship; it simply shows that it cannot be otherwise. Cavalieri, on the other hand, looked to the relation between the two figures and tried to discover why the two figures are in a certain ratio to each other, not to logically prove that they could not be otherwise. By associating each figure with its corresponding *omnia plana*, "all the planes," Cavalieri determines a ratio through a connection with this "new kind" of magnitude.

Torricelli and later authors, such as Wallis and Leibniz, would build upon Cavalieri's methods; however they would do so with a basic misunderstanding, believing that Cavalieri considered a continuous magnitude as the sum of its indivisibles.

To the contrary, Cavalieri preferred to take no definite position concerning the composition of the continuum and remained neutral over the issue of whether indivisibles were elements which actually composed the geometric figures they were associated with. After all, there was no need for Cavalieri to take a stand. All he needed to do was compare magnitudes; through the use of indivisibles he could do so and avoid the issue entirely. In a letter to Galileo, Cavalieri describes the essence of his method: "I did not dare to affirm that the composition is composed of indivisibles, but showed that between the continua there is the same proportion as between the collection of indivisibles."[27] His refusal to take a stand on the matter must be stressed against almost all those "second-hand" versions of Cavalieri's theory which were to follow. Never in his works does he consider a volume as a sum, a surface as a collection of lines, nor does he describe a line as an aggregate of points.

However, unlike Cavalieri, Galileo sought to examine the nature of the continuum. For this, Galileo's most favored tool was the paradox. By starting with familiar geometrical relations, he pushed them to their incomprehensible limit. In the words of Galileo, though Salviati; "*that the infinite is inherently incomprehensible to us, as indivisibles are likewise; so just think what they will be when taken together!*"[28] While Galileo appreciated the difficulties with concepts and methods of infinitesimal mathematics, he never seemed to reach a decision on how to deal with them.

Perhaps this explains why, it was Cavalieri who wrote a book about indivisibles and why Galileo did not.

While the content of *Geometria* in general was very little known, the mere fact that the book existed stimulated investigation into new methods and a reworking original notions Cavalieri had set forth; And, whereas Cavalieri was reluctant to break completely with Greek tradition, others were not so reticent. Perhaps it was Torricelli, more than anyone who was responsible for popularizing the method of indivisibles. By

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[27] Galileo Galilei *Opere*, vol 18 p.76
[28] Two New Sciences; Galileo Galilei translated by Stillman Drake pp38

introducing the use of curved indivisibles he would enhance the methods range of application. However in popularizing indivisibles, he may have been most to blame for the misrepresentation of Cavalieri work. Torricelli would grant his indivisibles a thickness; a move that Cavalieri was not prepared to make. Moreover, in contrast to Cavalieri's approach, which took the geometric figure and its associated indivisibles as two separate magnitudes, Torricelli's would make a simple identification of the figure and its characterizing indivisibles.

In short, we may see Cavalieri's indivisibles as a transition from the Greek method of exhaustion to the development of infinitesimal methods that would eventually lead to the development of Integral Calculus. However, it is only through familiarity with Cavalieri's concepts and techniques that it is possible to understand how elaborate and special his method was.

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