Faraday Instability on Elastic Interfaces

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Abstract

The shape of a liquid's surface is determined by both the body force and surface force of the liquid. In this report, the body force is solely from the gravitational force. The surface force is generated from the movement of an elastic interface between the solid and liquid. To obtain the shape of the surface, both asymptotic analysis and numerical approaches are used in this report. The asymptotic analysis is applied on the potential flow. The initial conditions are chosen to be the function of the shape of the interface between the solid and liquid and the free stream velocity far away from the interface. The time dependent contributions from the fluid system are also considered. The initial condition changes according to the function of the calculated velocity potential. The numerical approach includes two parts: calculation the velocity potential and a formalism of the change of the system as time evolves. For the first part, two idealized vertical boundaries are utilized to give a unique solution of the Laplace equation. The boundary conditions are determined as the flow under linear viscosity. For the second part, the flow is first assumed to be a potential flow, and a boundary layer is considered to make the no-slip condition valid and to give a more precise approximation for the shear stress.

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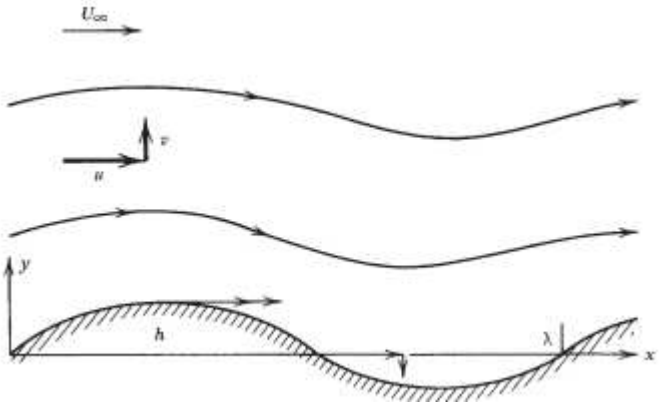
1 Introduction

Faraday waves are used to describe the non-linear standing waves in liquids. The non-linearity often comes from the vibration of the liquids' interfaces. The surface of the liquid will become unstable when the vibration exceeds a critical value. This is known as the Faraday Instability.[1] It is important to predict properties, such as the amplitude and the velocity field, of the liquid's surface. Faraday instability is not a pure mathematical problem. The starting point to model the shape of a free surface on fluid depends on particular physical situations. Different setups aim to solve different practical problems. For example,[8] the plane below is the ground which could shake due to an earthquake and cause large waves in the water, which is a tsunami. Also, although the governing behaviors of the fluid mechanics are mainly given by the Navier-Stokes equation, different assumptions can dramatically change the physical situation. Due to the complexity of the Navier-Stokes equation, the general solution for all kinds of fluid are hard to be obtained. Different applications will have different assumptions which can properly simplify the Navier-Stokes equation. The observations and boundary conditions will consequently be modified.

This report focuses on the interaction between two different phases. A brief overview of the model used in this report is given by the following figure.

Figure 1: Major Modeling Set Up



This figure gives a quick view of the model used in some of the sections. The bottom shaded wave line represents the elastic surface and the other two lines are related to the velocity of the fluid. v_∞ is the free stream velocity. Note that some of the parameters in this figure are defined differently than in the rest of this report. Taken from [4], chapter 20.

The main region of the flow will be considered as the potential flow. The most important quantity in potential flow is the velocity potential. The boundary conditions and governing equations in the potential flow are fixed. However, the initial condition can be measured or obtained from many different methods. It would be helpful to generalize a method of solving the velocity potential when given a certain set of initial conditions. This report considers the shape of the interfaces and the free stream velocity as the initial conditions. The asymptotic method is introduced to solve for the velocity potential given a particular set of initial conditions. Once the velocity potential has been found, it can be used to update the initial conditions, therefore the time stepping strategies can be used to track the evolution of the interface in time.

The shape of the free surface is found by solving the curvature force. Under the Kinetic boundary condition and dynamic boundary conditions, the stress tensor from the fluid side can be used to balance the stress tensor from the air side. For both sides, the stress tensor can be easily calculated as there is no viscid force.

The boundary layer between the liquid and the elastic surface is also discussed in this report. By considering the boundary layer, the non slip condition can be applied to make the calculation more accurate. Although the viscid force is present in the boundary layer, the Navier-Stokes equation along with boundary conditions can still be simplified. Therefore, the shape of the elastic surface can also be calculated based on the idea of balancing the stress tensor.

Numerical methods are discussed and performed in MATLAB. With proper boundary conditions, the numerical method can be used to solve general fluid as well as more complete set ups.

2 Potential flow

An important calculation in this report is to solve for pressure and displacement filed related quantities in flow when the boundary conditions are well defined. The flow in this calculation is described as the potential flow. Details describing the properties of potential flow are given in Appendix. 7.2. This section introduces the methodology of solving for the velocity potential when the boundary conditions are prescribed in two dimensional flows.

2.1 Governing Equations

The most important property for potential flow is the existence of a velocity potential. The velocity potential in fluid mechanics is a scale function whose gradient is the velocity field. By the definition, in two dimensional flow, the velocity potential, denoted by Φ, satisfies the equation

$$\frac{\partial \Phi}{\partial x} = v_x; \qquad \frac{\partial \Phi}{\partial y} = v_y; \tag{1}$$

where $\boldsymbol{v} = [v_x, v_y]$ is the velocity field of the fluid and the subscripts of v specifies the component. Potential flow should also satisfy conservation of mass. That is, the time rate of change of the mass in a selected region is zero. Therefore, the potential flow should still satisfy the continuity equation:

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho v_i}{\partial x_i} = 0, \qquad i = 1, 2. \tag{2}$$

Here, ρ is the density of the flow, $\boldsymbol{v}$ is the velocity field as before, and x_i refers the direction vector, that is, $x_1 = x$ and $x_2 = y$. The subscript i denotes the i^{th} component in its corresponding vector. For incompressible flow, the density ρ of the flow does not change with respect to both the time and the position. With this property, Eq.2 becomes

$$\sum_i \frac{\partial v_i}{\partial x_i} = 0. \tag{3}$$

Substituting Eq. 3 into Eq. 1, one obtains that the velocity potential satisfies the Laplace equation

$$\nabla^2 \Phi = 0. \tag{4}$$

The general analytic solution of the Laplace equation is well-defined. For two dimensions, the solution is

$$L(x, y) = (\alpha_1 \cos(\lambda x) + \beta_1 \sin(\lambda x)) (\alpha_2 \cosh(\lambda y) + \beta_2 \sinh(\lambda y)) . \tag{5}$$

Here, α_1, α_2 ,β_1, β_2, and λ are superposition parameters which can be determined from boundary conditions.

The other important variable in potential flow is the pressure. The pressure is a the normal components of the driving force applied on the surfaces of a subject. The pressure is a scalar function for a non-rotational surface since the normal vector remains same. In fluid mechanics, pressure is one of the main source of

the flow's motion. Also, the balance of the pressure determines the boundary condition at interfaces. Most often, one can calculate the value of pressure or use the value of pressure from the Euler equation and the Bernoulli equation. For potential flow, the Euler equation gives

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = \nabla p + \rho\boldsymbol{F}. \tag{6}$$

Here, ρ is the density of of the fluid which is a constant for incompressible flow, $\boldsymbol{v}$ is the velocity field, p is the pressure, and $\boldsymbol{F}$ is the body force. The Bernoulli equation can also relate the pressure and the velocity

$$\frac{p_1}{\rho} + \frac{{\boldsymbol{v}_1}^2}{2} + gy_1 + \frac{\partial \Phi_1}{\partial t} = \frac{p_2}{\rho} + \frac{{\boldsymbol{v}_2}^2}{2} + gy_2 + \frac{\partial \Phi_2}{\partial t}. \tag{7}$$

Bernoulli equation is derived from the conservation of energy. That is, the total energy should be the same when the liquid in a given region flow from one position to another position. In Eq. 7, the subscripts denote the corresponding position for each variable, p and $\boldsymbol{v}$ are pressure and velocity field as above, g is the gravitational acceleration, y_i is the height with respect to the reference height for each point, and Φ is the velocity potential function.

2.2 Example of Asymptotic Analysis

As stated above, the first aim of this section is to solve the velocity potential with well-defined boundary conditions. In potential flow, the word 'well-defined' means that the boundary conditions should give a unique solution to the velocity potential if the constant term is ignored. Several boundary conditions are given by assuming continuity and the no-penetration condition. A set of proper initial conditions should have both physical meaning and mathematical meaning and

allow for the determination of a unique solution. The choice of initial conditions in this section will be the function of the interface's shape between solid and liquid and the free stream velocity. These two boundary conditions are easy to detect for most of cases and clearly make sense from both the mathematical and physical perspectives. If the viscous layer between the flow and the surface were thin enough to make its effect negligible, it can be demonstrated that by applying asymptotic analysis, these two initial conditions are sufficient to obtain a unique velocity potential.

Table 1: Dimensional Analysis Table

Scalar	Description	Units
V_f	The velocity of free stream	[Length per time]
L	The length of the domain of interest	[Length]
h	The amplitude of the interface	[Length]
Dimensional function/ quantity	Description	Units
Φ	Velocity Potential	[Length Square per time]
x	The movement in horizontal direction	[Length]
y	The movement in vertical direction	[Length]

To start the asymptotic analysis, the perturbation parameter should be defined first. The common choice of the perturbation parameter in fluid mechanics is the ratio of the boundary's amplitude and the length of the interested area $\epsilon = \frac{h}{L}$. The perturbation parameter is unitless and it is used to define the shape of the interface. Here the amplitude of the interface is assumed to be much less than the length of domain. Therefore, the perturbation parameter is also much less than one.

From Table 1, the variables in the calculation process can be nondiemsionalized as

$$\hat{\Phi} = \frac{\Phi}{V_f L}, \qquad \hat{x} = \frac{x}{L}, \qquad \hat{y} = \frac{y}{L}.$$

Another quantity that should be defined before the calculation is the reference

height. This quantity can be freely chosen as the fluid and gravity are two dominant sources of potential energy. In this section, the reference height will be set to be the average amplitude of the interface between solid and liquid. As mentioned at the beginning, the initial conditions of this section's set up include the shape of the interface and the free stream velocity. The free stream velocity will be considered as a constant with respect to $\hat{x}$. Therefore, after nondimensionalization, the boundary condition of the velocity potential very far away from the reference point should be

$$\frac{\hat{\Phi}}{\hat{x}} = 1, \qquad \text{as } y \to \infty. \tag{8}$$

The other boundary condition happens at the interface between solid and liquid. The initial condition for this boundary condition is given as the function of the interface's shape. In practice, the shape might be a complicated function. This might make the calculation harder, but conceptually it follows the same solution process. Thus, it would be nice to illustrate the asymptotic analysis by providing a relatively simple function for the shape of the interface between the solid and the fluid. The shape in this set up is defined as

$$\hat{y_s} = \sin(2\pi\hat{x})\epsilon, \tag{9}$$

where $\hat{y}_s$ is the the height of the interface between the liquid and solid. Substituting the function of the interface's shape into the no-penetration condition(the fluid cannot penetrate into or leave a space between to the solid), one obtains

$$\frac{d\hat{y}_s}{dx} = \frac{\partial\hat{\Phi}}{\partial\hat{y}} \Big/ \frac{\partial\hat{\Phi}}{\partial\hat{x}}, \qquad at\ y = y_s. \tag{10}$$

Taking the derivative of Eq. 9, Eq. 10 becomes

$$2\pi\epsilon\cos(2\pi\hat{x})\frac{\partial\hat{\Phi}}{\partial\hat{x}} = \frac{\partial\hat{\Phi}}{\partial\hat{y}}. \tag{11}$$

Recall the definition of the velocity potential, the non-dimensional velocity components were given by Eq. 1. From the continuity equation of the potential flow, the velocity field should satisfy

$$\frac{\partial\hat{v}_x}{\partial x} + \frac{\partial\hat{v}_y}{\partial y} = 0. \tag{12}$$

Combine the two equations, one gets

$$\frac{\partial^2\hat{\Phi}}{\partial\hat{x}^2} + \frac{\partial^2\hat{\Phi}}{\partial\hat{y}^2} = 0. \tag{13}$$

Since the asymptotic analysis is used to solve the velocity potential, with the defined perturbation parameter, the solution is expected to have the form

$$\hat{\Phi}(\hat{x},\hat{y},\epsilon) = \sum_i \epsilon^i\hat{\Phi}^i(\hat{x},\hat{y}). \tag{14}$$

Where ϵ^i is the i^{th} power of the perturbation parameter and $\hat{\Phi}^i$ is the perturbation function of i^{th} order. The above form is valid since the velocity potential is a bounded function in a finite domain and the perturbation parameter is sufficiently small. Both the function Φ and the boundary conditions can be expanded to give an approximated solvable relation.

If the concerned order of accuracy is one, that is, the error terms of order $\mathcal{O}(\epsilon^2)$ is small enough to be dropped, the perturbation series above becomes

$$\hat{\Phi}(\hat{x},\hat{y},\epsilon) = \hat{\Phi}^0(\hat{x},\hat{y}) + \epsilon\hat{\Phi}^1(\hat{x},\hat{y}). \tag{15}$$

All the perturbation functions $\hat{\Phi}^i$ should also satisfy the continuity condition because the perturbation parameters are assumed to be independent. Therefore, we have

$$0 = \frac{\partial^2 \hat{\Phi}^0}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\Phi}^0}{\partial \hat{y}^2}; \tag{16}$$

$$0 = \frac{\partial^2 \hat{\Phi}^1}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\Phi}^1}{\partial \hat{y}^2}. \tag{17}$$

The boundary conditions of the velocity potential are given by Eq. 8 and Eq. 10. Substituting into Eq. 15, one obtains

$$\frac{\partial \hat{\Phi}}{\partial \hat{x}}(\hat{x}, \hat{y} \to \infty) + \epsilon \frac{\partial \hat{\Phi}}{\partial \hat{x}}(\hat{x}, \hat{y} \to \infty) = 1, \tag{18}$$

$$2\pi\epsilon \cos(2\pi\hat{x}) \left(\frac{\partial \hat{\Phi}}{\partial \hat{x}}(\hat{x}, \hat{y}_s) + \epsilon \frac{\partial \hat{\Phi}}{\partial \hat{x}}(\hat{x}, \hat{y}_s) \right) = \frac{\partial \hat{\Phi}}{\partial \hat{x}}(\hat{x}, \hat{y}_s) + \epsilon \frac{\partial \hat{\Phi}}{\partial \hat{x}}(\hat{x}, \hat{y}_s). \tag{19}$$

The perturbation parameter
$epsilon$ is independent of the approximated velocity potential functions. Therefore, Eq. 18 can be restated as

$$\frac{\partial \hat{\Phi}^0}{\partial \hat{x}}(\hat{x}, \hat{y} \to \infty) = 1, \tag{20}$$

$$\frac{\partial \hat{\Phi}^1}{\partial \hat{x}}(\hat{x}, \hat{y} \to \infty) = 0, \tag{21}$$

Since $\hat{y}_s$ in Eq. 18 is also a function of the perturbation parameter, one should find the zero and first order of ϵ to extract more information from Eq. 18. One can begin the analysis by using Taylor series to expand derivatives in Eq. 18 about the reference height $\hat{y} = 0$. As an example, the first derivative of of $\hat{\Phi}^0$ with respect to $\hat{x}$ is

$$\frac{\partial \hat{\Phi}^0(\hat{x}, \hat{y}_s)}{\partial \hat{x}} = \frac{\partial \hat{\Phi}^0(\hat{x}, 0)}{\partial \hat{x}} + \frac{\partial^2 \hat{\Phi}^0(\hat{x}, 0)}{\partial \hat{x} \partial \hat{y}} + \mathcal{O}(\epsilon^2). \tag{22}$$

Notice that all the higher order terms can be ignoredif one is only concerned with

the first order solution with respect to the perturbation parameter. Expanding all the derivative terms in Eq. 19 and substituing the function of $\hat{y}_s$,

$$\begin{aligned}
&\left(\frac{\hat{\Phi}^0(\hat{x},0}{\partial\hat{x}} + \frac{\hat{\Phi}^0(x,0)}{\partial\hat{x}\partial\hat{y}}\epsilon\sin(2\pi\hat{x})\right)2\pi\epsilon\cos(2\pi\hat{x}) \\
&+\left(\frac{\hat{\Phi}^1(\hat{x},0}{\partial\hat{x}}\epsilon + \frac{\hat{\Phi}^1(\hat{x},0)}{\partial\hat{x}\partial\hat{y}}\epsilon^2\sin(2\pi\hat{x})\right)2\pi\epsilon\cos(2\pi\hat{x}) \\
&-\left(\frac{\hat{\Phi}^0(\hat{x},0}{\partial\hat{y}} + \frac{\hat{\Phi}^0(\hat{x},0)}{\partial\hat{y}^2}\epsilon\sin(2\pi\hat{x}) + \frac{\hat{\Phi}^1(\hat{x},0}{\partial\hat{x}}\epsilon + \frac{\hat{\Phi}^1(x,0)}{\partial\hat{x}\partial\hat{y}}\epsilon^2\sin(2\pi\hat{x})\right) = 0
\end{aligned}$$

Dropping all the terms that have higher order than two in ϵ, one can approximate Eq. 19 as

$$\frac{\partial\hat{\Phi}^0(\hat{x},0)}{\partial\hat{x}}2\pi\cos(2\pi\hat{x}) - \frac{\partial\hat{\Phi}^0(\hat{x},0)}{\partial\hat{y}} - \frac{\partial^2\hat{\Phi}^0(\hat{x},0)}{\partial\hat{y}^2}\epsilon\sin(2\pi\hat{x}) - \frac{\partial\hat{\Phi}^0(\hat{x},0)}{\partial\hat{y}} = 0. \tag{23}$$

Similarly, one can separate the terms which do not have the perturbation parameters from other terms, that is

$$0 = \frac{\partial\hat{\Phi}^0(\hat{x},0)}{\partial\hat{y}}, \tag{24}$$

$$0 = \frac{\partial\hat{\Phi}^0(\hat{x},0)}{\partial\hat{x}}\epsilon 2\pi\sin(2\pi\hat{x}) - \frac{\partial^2\hat{\Phi}^0(x,\hat{0})}{\partial\hat{y}^2}\epsilon\sin(2\pi\hat{x}) - \frac{\partial\hat{\Phi}^1(\hat{x},0)}{\partial\hat{y}} = 0. \tag{25}$$

The boundary conditions for the zero order approximation of the velocity potential, $\hat{\Phi}^0$ are now sufficient to calculate the parameters in the solution of Laplace equation whose form is given by Eq. 5. The calculation is simple and straight forward, the solution of $\hat{\Phi}^0$ is found to be

$$\hat{\Phi}^0 = \hat{x}.$$

After having the exact form of the zero order approximation $\hat{\Phi}^0$, one can substitute $\hat{\Phi}^0$ into Eq. 25 to obtain the other boundary condition for the first order

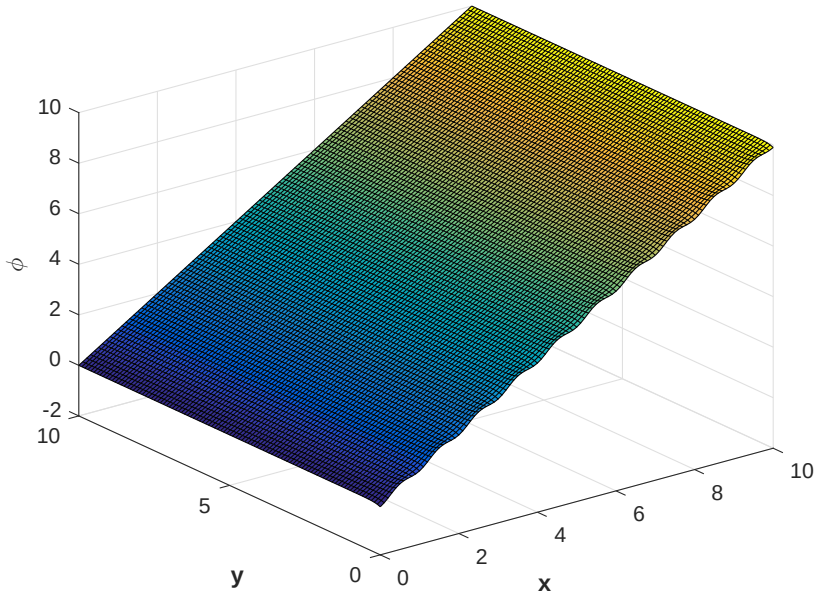
approximation $\hat{\Phi}^1$. This is solving the Laplace equation with the following setup

$$
\begin{aligned}
0 &= \frac{\partial^2 \hat{\Phi}^1}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\Phi}^1}{\partial \hat{y}^2} \\
0 &= \frac{\partial \hat{\Phi}^1(\hat{x}, 0)}{\partial \hat{y}} - 2\pi \cos(2\pi \hat{x}) \\
0 &= \frac{\partial \hat{\Phi}^1(\hat{x}, \hat{y} \to \infty)}{\partial \hat{x}}.
\end{aligned}
$$

The solution is given as

$$\hat{\Phi}^1 = -\cos(2\pi \hat{x}) \exp(2\pi \hat{y}). \tag{26}$$

Figure 2: Velocity Potential



This figure displays the velocity potential given in Eq. 27. The dominate term is clearly $\hat{x}$. For each $\hat{x}$ value, there is a small fluctuation due to the second term.

Combining the two approximations, the velocity potential is then

$$\hat{\Phi} = \hat{x} - \epsilon \cos(2\pi \hat{x}) \exp(2\pi \hat{y}). \tag{27}$$

Generally, the velocity potential can have one extra term which is any constant function of $\hat{x}$ and $\hat{y}$. The time contribution of the velocity potential is either from the extra term or behind velocity components. For example, the velocity components here are

$$\hat{v}_x = 1 + 2\pi\epsilon \sin(2\pi\hat{x}) \exp(-2\pi\hat{y}), \tag{28}$$

$$\hat{v}_y = 2\pi\epsilon \cos(2\pi\hat{x}) \exp(-2\pi\hat{y}). \tag{29}$$

As the velocity is not a trivial function, the states of the flow will continue to change. Consequently, the initial conditions will no longer hold and the boundary conditions will change accordingly. Then one will end up with new Laplace equations which will give an updated velocity potential function.

2.3 The Solution of Potential Flow as Time Evolves

The velocity potential of the fluid for a given shape of the interface was calculated in the last section. However, since the flow is a continuous quantity, the velocity potential should not be the same as time evolves. This section introduces the general idea of calculating the velocity potential, including the dependence on time.

The two initial conditions given in the previous section are the function of the interface's shape and the free stream velocity. The change from either of the initial condition will cause a change in the velocity potential of the fluid. As discussed above, the change of these two conditions is partially contributed by the velocity potential. Therefore, the shape of the interface, the free stream velocity, and the velocity potential are correlated. Consequently, from the previous set up, it is not possible to obtain the full equation of either of these quantities as the dependence of time is included. An alternative approach is to first discretize the

time dimension into small intervals and calculate the velocity potential for each interval by updating the initial conditions.

If the shape of the interface is forced to remain the same and the surface pressure is negligible, the only condition that will change is the free stream velocity. In each time interval, the free steam velocity is assumed to be a constant. Then the value in the i^{th} interval can be calculated from the $(i-1)^{th}$ interval by setting its value equal to the velocity in x direction at the height of the interface between the fluid and air. It should be emphasized that for each time step, the parameters should be non-dimensionlized independently as the scalar quantity V_f changes.

A more interesting situation is when the shape of the interface between the flow and the solid changes over time due to an applied surface force. Without losing generality, one can assume that the length of the domain is a constant. The process of updating the function of the interface's shape is more complicated than updating the free stream velocity. In order to derive the interface's shape for the i^{th} interval, the information of the displacement field of the $(i-1)^{th}$ interval has to be known. That is, we need to form a procedure to obtain the displacement field from the velocity potential.

Before starting the formalism, let us assume that all the parameters have already been properly non-dimensionlized. That is, all the variables in this section have no dimension and the hat notation will not be used. Let the time domain be discretized into $[t_0, t_1, t_2, \cdots]$.

Suppose at $t = t_0$, the shape of the interface is described by a function f_0 and the free stream velocity is given as V_f^0, that is

$$y_s(x, t_0) = f_0(x), \tag{30}$$

$$v_y(x, y \to \infty, t_0) = V_f^0. \tag{31}$$

Following the procedure introduced in the last section, the velocity potential at $t = t_0$ can be calculated and suppose the result is $\Phi(x, y, t_0) = \Phi_0$. The subscript for Φ here and later in this section will refer to the time instead of the order of the the asymptotic analysis. By definition, the velocity components at $t = t_0$ can be directly calculated by taking partial derivatives of Φ_0:

$$v_x(t_0) = \frac{\partial \Phi_0}{\partial x},$$

$$v_y(t_0) = \frac{\partial \Phi_0}{\partial y}.$$

Since we are interested in the change of variables as the time moves from t_0 to t_1, it would be ideal to have a relation which has the time involved. Fortunately, the Navier-Stokes equation has such a term

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = -\boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{v} + \rho\boldsymbol{F}. \tag{32}$$

Here, ρ is the density of the fluid, $\boldsymbol{v}$ is the velocity field, $\boldsymbol{p}$ is the pressure, μ is the viscosity, and $\boldsymbol{F}$ is the body force per unit mass. In this report, the density of the fluid is assumed to be a constant and the body force $\boldsymbol{F}$ is only attribute to the gravitational force, that is

$$F_x = 0, \qquad F_y = g.$$

For the two dimensional non-viscous fluid, the Navier-Stokes equation can be written explicitly as

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x}, \tag{33}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \rho g \tag{34}$$

The unknown in Eq. 33 and Eq. 34 is the pressure $p(x, y)$. Since the pressure

at the surface of the fluid is easy to measure, it can be used in the Bernoulli equation to calculate the pressure everywhere. Using Eq. 7, for any point k, one should expect

$$\frac{p_k}{\rho} + \frac{{\boldsymbol{v}_k}^2}{2} + gy_k + \frac{\partial \Phi_t(k)}{\partial t} = \frac{p_a}{\rho} + \frac{{\boldsymbol{v}_a}^2}{2} + gy_a + \frac{\partial \Phi_t(a)}{\partial t} \tag{35}$$

Here, the subscript a and k refer to two points, a is the any point at the average height of the surface of the fluid, which is the reference pressure position. $\Phi_t(a)$ and $\Phi_t(k)$ denote for the velocity potential at points a and k in a given time interval t.

Since in any time interval, the flow should be considered as the steady flow. Therefore, the time derivative of the velocity potential should be zero for all location k, that is

$$\frac{\partial \Phi_t(k)}{\partial t} = \frac{\partial \Phi_t(a)}{\partial t} = 0.$$

Also, the conservation of energy implies the change of the momentum is equal to the change of potential energy for a unit mass, that is, $\Delta v_y = \Delta g y$. Here y is the vertical distance with respect to the reference position and Δ denotes the difference value. Then based on Eq. 35, the difference in pressure between any layer and the surface of the flow can be expressed in terms of the ratio of their squared velocity value:

$$p_k - p_a = \frac{\rho}{2} {v_a}^2 \left(1 - \frac{{v_k}^2}{{v_a}^2}\right), \tag{36}$$

where the value of ${\boldsymbol{v}_k}^2$ is equal to $v_{k_x}^2 + v_{k_y}^2$. Since the velocity at the surface of the fluid is essentially the free stream velocity, the pressure can be derived from the velocity potential function at a given time for any point.

Going back to Navier-Stokes equations, since the goal is to find the displacement field, one should replace the time derivative of the velocity components as

the second order time derivative of the displacement field components.

$$\frac{\partial v_x}{\partial t} = \frac{\partial^2 u_x}{\partial t^2}; \tag{37}$$

$$\frac{\partial v_y}{\partial t} = \frac{\partial^2 u_y}{\partial t^2}. \tag{38}$$

Where $\boldsymbol{u}$ is the displacement field and the subscripts denotes the component in the corresponding direction. Rewriting the time derivatives of the displacement field with a second order forward numerical differentiation, one obtains

$$\frac{\partial v_x}{\partial t} = \frac{u_x(t+2\Delta t) - 2u_x(t+\Delta t) + u_x(t)}{(\Delta t)^2}; \tag{39}$$

$$\frac{\partial v_y}{\partial t} = \frac{u_y(t+2\Delta t) - 2u_y(t+\Delta t) + u_y(t)}{(\Delta t)^2}. \tag{40}$$

Where Δt is the time step size. Equation 39 Eq. 40 are valid for calculating in all the time interval except $t = t_1$. A less accurate approximation for the value of the displacement flied is to directly take the integral of the velocity components which is calculated from the velocity potential. It can of course to approximate the integral as

$$u_x(t+\Delta t) = u_x(t) + v_x(t)\Delta t; \tag{41}$$

$$u_y(t+\Delta t) = u_y(t) + v_y(t)\Delta t. \tag{42}$$

Substituting Eq. 39, Eq. 40, and Eq. 36 into the Navier-Stokes equation, one

obtains

$$\begin{aligned}
&\rho\left(\frac{u_x(t+2\Delta t)-2u_x(t+\Delta t)+u_x(t)}{(\Delta t)^2}+v_x\frac{v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)\\
=&-p_a-\frac{\rho}{2}{v_a}^2\left(1-\frac{{v_k}^2}{{v_a}^2}\right) \qquad (43)\\
&\rho\left(\frac{u_y(t+2\Delta t)-2u_y(t+\Delta t)+u_y(t)}{(\Delta t)^2}+v_x\frac{v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}\right)\\
=&-p_a-\frac{\rho}{2}{v_a}^2\left(1-\frac{{v_k}^2}{{v_a}^2}\right)+\rho g \qquad (44)
\end{aligned}$$

Here, the subscript k denotes an arbitrary position, the subscript a denotes the average y value of the fluid's surface. All the parameters have no time specification can be obtained from either $t = t$ or $t = t+\Delta t$. But the choice should be consistent for all parameters.

Rearranging these equations, one can obtain the expression for the displacement components

$$\begin{aligned}
&-\left(\frac{p_a}{\rho}-\frac{1}{2}{v_a}^2\left(1-\frac{{v_k}^2}{{v_a}^2}\right)-v_x\frac{v_x}{\partial x}-v_y\frac{\partial v_x}{\partial y}\right)(\Delta t)^2\\
=&u_x(t+2\Delta t)-2u_x(t+\Delta t)+u_x(t); \qquad (45)\\
&-\left(\frac{p_a}{\rho}-\frac{1}{2}{v_a}^2\left(1-\frac{{v_k}^2}{{v_a}^2}\right)-v_x\frac{v_x}{\partial x}-v_y\frac{\partial v_x}{\partial y}\right)(\Delta t)^2\\
=&u_y(t+2\Delta t)-2u_y(t+\Delta t)+u_y(t). \qquad (46)
\end{aligned}$$

The terms on the left hand of the above equations can all be calculated from the velocity potential. Therefore, the value of displacement in each direction can be updated from its value of the former two time interval. The function of the interface's shape can be then updated from the above procedure.

As a concrete example, let us use the set up from the former section as our

initial conditions. The velocity potential is calculated to be

$$\Phi_0 = x - \epsilon \cos(2\pi x) \exp(-2\pi y).$$

Where the subscript here denotes the time interval t_0. The velocity components are then

$$v_x(x, y, 0) = 1 + \epsilon 2\pi \sin(2\pi x) \exp(-2\pi y);$$
$$v_y(x, y, 0) = \epsilon 2\pi \cos(2\pi x) \exp(-2\pi y).$$

At the interface $y_s = \sin(2\pi x)\epsilon$, one obtain the velocity components at the interface are

$$v_x(x, y_s, 0) = 1 + \epsilon 2\pi \sin(2\pi x); \tag{47}$$
$$v_y(x, y_s, 0) = \epsilon 2\pi \cos(2\pi x). \tag{48}$$

The derivatives are

$$\frac{\partial v_x(x, y_s, 0)}{\partial x} = \epsilon 4\pi^2 \cos(2\pi x); \tag{49}$$
$$\frac{\partial v_x(x, y_s, 0)}{\partial y} = -\epsilon 4\pi^2 \sin(2\pi x); \tag{50}$$
$$\frac{\partial v_y(x, y_s, 0)}{\partial x} = -\epsilon 4\pi^2 \sin(2\pi x); \tag{51}$$
$$\frac{\partial v_y(x, y_s, 0)}{\partial y} = -\epsilon 4\pi^2 \cos(2\pi x). \tag{52}$$

With the free stream velocity being normalized to one, by dropping higher order terms of the perturbation parameter, the pressure for any point k is given by Bernoulli equation as

$$p_k - p_a = -\frac{\rho}{2} 4\pi\epsilon \sin(2\pi x) \exp(-2\pi y). \tag{53}$$

Substituting the function of the interface's shape between solid and liquid, the pressure at the interface is,

$$p_s = p_a - \frac{\rho}{2} 4\pi\epsilon \sin(2\pi x) + \mathcal{O}(\epsilon^2) \tag{54}$$

Here, the subscript s denote the interface's average height and the $\boldsymbol{O}(\epsilon^2)$ are all the terms containing high powers of the small perturbation parameters ϵ. Substituting the function of the pressure and the velocity components at the interface into the approximated Navier-Stokes equation, Eq. 45 and Eq. 46 becomes

$$\begin{aligned}
&\frac{u_x(t_2) - 2u_x(t_1) + u_x(0)}{(\Delta t)^2} \\
&= -\left(\frac{p_a}{\rho} - \frac{1}{2} 4\pi\epsilon \sin(2\pi x) - \epsilon 4\pi^2 \cos(2\pi x) + \epsilon^2 16\pi^3 \sin(2\pi x)\cos(2\pi x)\right) \\
&\approx -\left(\frac{p_a}{\rho} - \frac{1}{2} 4\pi\epsilon \sin(2\pi x) - \epsilon 4\pi^2 \cos(2\pi x)\right) \qquad (55) \\
&\frac{u_y(t_2) - 2u_y(t_1) + u_y(0)}{(\Delta t)^2} \\
&= -\left(\frac{p_a}{\rho} - \frac{1}{2} 4\pi\epsilon \sin(2\pi x) + (1 + \epsilon 2\pi \sin(2\pi x))(\epsilon 4\pi^2 \sin(2\pi x)) + \epsilon^2 8\pi^3 \cos^2(2\pi x))\right) \\
&\approx -\left(\frac{p_a}{\rho} - \frac{1}{2} 4\pi\epsilon \sin(2\pi x) + \epsilon 4\pi^2 \sin(2\pi x)\right) \qquad (56)
\end{aligned}$$

It is clear that for each time interval, the functions of the interface between solid and liquid from the above equations are a function of x only. The updating of the velocity of the free stream has been mentioned at the beginning of this section. Therefore, the initial conditions for the next time interval are now obtained. One can follow the same procedure from the former section to update the velocity potential for the next time interval.

This section loosely discusses the shape change of the interface and derived a formalism of updating the initial conditions and velocity potential for each time step. A more careful calculation will be discussed in later sections by considering

the constraints of the stress balancing.

3 Wave function of the flow at the interface with air

From previous sections, the velocity potential can be calculated based on the functions of the shape of the solid elastic surface. However, the velocity potential is a quantity that is difficult to be measured. Therefore, the result not only has limited utility for practical applications, but also is difficult to check its correctness. This then gives the motivation of deriving a formalism of the wave function of the fluid's surface from a given velocity potential.

The flow in this section will be still the potential flow to enable the existence of the velocity potential. To have the wave function at the top of the fluid, one should first consider the boundary conditions based on physical perspective.

The surface of the fluid can be treated as a thin layer which is called the free surface. As time evolves, the free surface would have a different functional form. Let us Consider particles on the free surface at a given time. Due to the surface tension, the probability that these particles follows same vertical velocity of the free surface is much larger than the probability of that other layers' particles come to top of the fluid. Therefore, these particles are assumed to remain on the surface for all time. Mathematically, this means that the change of the height of the surface over a short time period plus the normal velocity of the surface should be equal to the vertical velocity of surface particles, that is

$$\frac{\partial y_a}{\partial t} + (\boldsymbol{v}(y_a) \cdot \nabla) y_a = v_y(y_a). \tag{57}$$

Here, y_a is the function of the free surface at a given time, $\boldsymbol{v}$ is the velocity field of the fluid, $\boldsymbol{v}(y_a)$ denotes the velocity field at the free surface, and v_y is the vertical component of the velocity field. In the form of the velocity potential, the two

dimensional form of the Eq. 57 becomes

$$\frac{\partial y_a}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial y_a}{\partial x} = \frac{\partial \Phi}{\partial y} \tag{58}$$

where Φ is the velocity potential and this equation is only valid for the free surface.

Let us then take a look at the continuity equation. For an arbitrary region, applying Leibnitz's rule to the density, one can obtain

$$\frac{d}{dt}\int \rho dV = -\int \rho(v_i - w_i)n_i dS. \tag{59}$$

Since the flow is assumed to be laminar in $\hat{z}$ direction (fluid flows in parallel layers with respect to $\hat{z}$ direction), the volume integral can be reduced to an area integral. In above equation, ρ is the density, the subscript denotes the corresponding direction, v is the velocity of the fluid, $\boldsymbol{w}$ is the velocity of the interface, and $\boldsymbol{n}$ is the normal vector. Therefore, for a region containing the free surface, above equation becomes

$$\frac{d}{dt}\int \rho dV = -\int_L \mathbf{n}\cdot(\mathbf{v}-\mathbf{w})\rho dS + \int_A \mathbf{n}\cdot(\mathbf{v}-\mathbf{w})\rho dS \tag{60}$$

Here, the subscripts L and A are used to denote the liquid and air phases. Therefore, the first term on the right hand side of the equation deals with the region of liquid and the second term computes the region of air. The negative sign is due to the two regions having opposite normal direction.

As the thickness of the free surface goes to zero, the left hand side of the equation approaches zero accordingly. Thus, for any point on the free surface, one has, for the normal direction

$$\rho(v_n - w)\mid_L = \rho(v_n - w)\mid_A, \tag{61}$$

where the left hand side of the equation is for the region of liquid and the right hand side of the equation is for the region of air, and the subscript n denotes the normal direction. Here, w here is a scalar since the free surface is a function of x only at a given time.

Similarly, one can also apply the Leibnitz's rule to the momentum of for an arbitrary region and obtain

$$\frac{d}{dt}\int \rho v_i dV = -\int (\rho n_j (v_j - w_j) v_i + n_i p - n_j \tau_{ji}) dS + \int \rho F_i dV. \tag{62}$$

Here, τ_{ij} is the component of the viscid tensor. Since the thickness of the free surface is negligible, the volume integral terms vanishes in the Eq. 62 and one can then obtain

$$-\int_L (\rho(\boldsymbol{v} - w)\boldsymbol{v} + \mathbf{n}p - \boldsymbol{n} \cdot \boldsymbol{\tau}) dS + \int_L (\rho(\boldsymbol{v} - w)\boldsymbol{v} + \mathbf{n}p - \boldsymbol{n} \cdot \boldsymbol{\tau}) dS = 0 \tag{63}$$

Which then implied

$$(\rho(\boldsymbol{v} - w)\boldsymbol{v} + \mathbf{n}p - \boldsymbol{n} \cdot \boldsymbol{\tau}) \mid_L = (\rho(\boldsymbol{v} - w)\boldsymbol{v} + \mathbf{n}p - \boldsymbol{n} \cdot \boldsymbol{\tau}) \mid_A \tag{64}$$

As before, w here is a constant so it should have no attribution in the above equation. For the tangential direction, by the no slip condition, the velocity of the free surface from both sides should match. Therefore the tangential velocity terms will be canceled on both sides of the equation. Since the pressure should have no contribution in the tangential direction, the remaining term on each each side of Eq. 64 is $\boldsymbol{n} \cdot \boldsymbol{\tau}$. Since the potential flow has no viscous force, the shear stress from the viscosity at the side of the flow should be zero. Thus, in the tangential direction, the momentum equation simply states that the free surface is perfectly aligned with the stream lines of the air since $\boldsymbol{n} \cdot \boldsymbol{\tau}_A = 0$.

For the normal direction, Eq. 64 gives

$$(\rho(v_n - w)v_n) + p - \mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}) \mid_A = (\rho(v_n - w)v_n) + p - \mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}) \mid_L \tag{65}$$

Here, the left hand side of the equation is on the air side and the right hand side is on the liquid side of the free surface. As before, the velocity of the free surface should be the same for both sides of the equation. Substituting Eq. 61, one can have,

$$\mathbf{n} \cdot \boldsymbol{\tau}_\mathbf{A} \cdot \mathbf{n} - p_A \boldsymbol{I} = \mathbf{n} \cdot \boldsymbol{\tau}_\mathbf{L} \cdot \mathbf{n} - p_L \boldsymbol{I} \tag{66}$$

Where $\boldsymbol{I}$ is the identity matrix. Notice that the surface tension is ignored here, the more general condition when including the surface tension is

$$(\mathbf{n} \cdot \mathbf{T}_\mathbf{L} \cdot \mathbf{n}) - (\mathbf{n} \cdot \mathbf{T}_\mathbf{A} \cdot \mathbf{n}) = \gamma(\nabla \cdot \mathbf{n}). \tag{67}$$

Here, γ is the surface tension, $\nabla \cdot \mathbf{n}$ is the curvature force, and $\boldsymbol{T}$ is the full stress tensor which has the form $T_{ij} = -p\delta_{ij} + \tau_{ij}$, δ_{ij} is the Kronecker delta.

Since the fluid is considered as a potential flow, then, for two dimensions, Eq.67 gives

$$\Delta p = \gamma(\nabla \cdot \mathbf{n}). \tag{68}$$

This enables us to calculate the curvature force of the surface. The curvature force of the surface is defined as

$$\nabla \cdot \mathbf{n} = \left(\frac{\partial^2 y_a}{\partial x^2}\right) / \left(1 + \frac{\partial y_a}{\partial x}\right)^{\frac{3}{2}}. \tag{69}$$

Equation 69 enables one to derive the form of y_a. With specific boundary conditions, such as constraining the end points or the velocity field which can be derived from previous section, the exact form of the wave function of the free

surface can be obtained. One can perform the calculation by choosing either to calculate it calculate numerically or analytically. The latter is only feasible when dealing with simple functions and boundary conditions.

4 Boundary Layer

The boundary Layer is a small region in the fluid which has two distinct value of the velocity at the two end layers. The boundary layer can be located close to a surface. The motivation for using and studying the boundary layer is to separate the region with the viscous force and the region without the viscous force. Within the boundary layer, the other boundary condition, the no slip condition can be satisfied by the flow.

4.1 Background Information

This section will begin with an introduction of an important quantity in fluid mechanics: the Reynolds number. Reynolds number is defined as the ratio of the inertial force to viscous force, that is

$$Re = \frac{\rho \overline{v} L}{\mu} \tag{70}$$

Here, Re denotes Reynolds number, ρ is the density of the fluid, $\overline{v}$ is the velocity of the flow, L is the characteristic length, μ is the dynamic viscosity. Let us first check the unit of Reynolds number; the following table gives the units of all parameters appearing in Eq. 70. It is trivial to check that the Reynolds number is unitless. In fact, Reynolds number itself is often used as a characteristic scale in fluid mechanics. It defines the relative motion of the flow. To have a sense of the scale of Reynolds number, Table 3[16] gives values in some particular cases.

Table 2: Units Analysis of Reynolds Number

Parameter	Units
ρ	[Weight per volume]
v	[Length per time]
L	[Length]
μ	[Weight per time per Length]

By Eq. 70, the low Reynolds number indicates high viscous forces and inertia is

Table 3: Typical Value of Reynolds Number

	Reynolds Number Value
Tiny Fish	5-50
Blood Flow	10^2 to 10^3
Turbulent Flow	10^4 to 10^6
Swimming Person	$5 * 10^6$
Whale	10^8
Large Ship	$5 * 10^9$

negligible. If Re $\gg 1$, viscosity is small and inertial forces dominate. Viscosity is the quantity which measures the difference in velocity between different layers of the fluid. Until now, the viscous and the viscous force has not been included in any discussion. However, in most of practical situations, the motion of the fluid at the interface should be exactly same as the motion of the interface itself. This gives another condition besides the no penetration condition which is the no slip condition. If the no slip condition were to hold, then the non-viscous assumption will no longer be valid for all region in the flow. However, as the value of Reynolds number increase, the viscous effects can be confined to a thinner and thinner region close to the surface. This region is call the boundary layer.

Boundary layer theory is an efficient tool to enforce the fluid to satisfy the no slip condition. Outside of the boundary layer, one can still treat the fluid as the potential flow. In the boundary layer, the flow is assumed to be completely inviscid and enclosed by the existence of the surface. Also, the terms involving

the curvature of the surface will not be considered .[10] Since the flow beyond the boundary layer is potential flow, the velocity value at the top of the boundary layer would be the same scale as the free stream velocity. Therefore, the term $\frac{\partial v_x}{\partial y}$ will vanish as the thickness for the boundary layer goes to zero. This is the motivation of the boundary layer theory.

In the region of boundary layer, the continuity equation and the Navier-Stokes equation should still hold for this set up. The continuity equation gives

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{71}$$

Where as before, v_x and v_y are components of the velocity field in the x and y direction.

If the fluid were steady flow in the boundary layer, the Navier-Stokes equation is simplified to

$$\rho\left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} = -\nabla p + \mu\nabla^2\boldsymbol{v} + \rho\boldsymbol{F}. \tag{72}$$

Where ρ is the density of the flow, $\boldsymbol{v}$ is the velocity field, p is the pressure, and $\boldsymbol{F}$ is the body force per unit mass.

Before applying the Navier-Stokes equation, let us first nondimensionalize all the parameters. The dimension of each quantities appearing in the above equation and the characteristic scales are given in Table 4 and Table 5.

Table 4: Dimensional variables

Variables	Description	Units
v_x	The tangential velocity	[Length per time]
v_y	The normal velocity	[Length per time]
x	The tangential distance	[Length]
y	The normal distance	[Length]

Table 5: Scale Quantities

Parameter	Description	Units
V_f	The free stream velocity	[Length per time]
L	The characteristic length	[Length]
T	The thickness of boundary layer	[Length]
Re	The Reynolds number	[None]

The non dimensional parameters are then

$$\hat{v_x} = \frac{v_x}{V_f}, \qquad \hat{v_y} = \frac{v_y}{V_f}$$
$$\hat{x} = \frac{x}{L}, \qquad \hat{y} = \frac{y}{T}$$

Since the pressure and the viscosity does not affect the following analysis, we do not report the nondimensional pressure here.

We assume the body force has a contribution only from the gravitational force and set the reference height to be the location of the surface. Then, the Navier-Stokes equation becomes

$$\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right), \tag{73}$$
$$\rho\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right). \tag{74}$$

Let us first discuss the dominant terms in Eq .73. The order of each term in the Navier-Stokes equation is

$$\mathcal{O}\left(\frac{V_f}{L}\right) + \mathcal{O}\left(\frac{V_f}{L}\right) = \mathcal{O}\left(\frac{V_f}{L}\right) + \mathcal{O}\left(\frac{V_f\mu}{\rho L^2}\right) + \mathcal{O}\left(\frac{V_f\mu}{\rho T^2}\right) \tag{75}$$

Where $\mathcal{O}$ is the big O notation which describes the limitation of the behavior of the function. It is clear that the value of V_f and the L are of the same order. Using the characteristic scalar Reynolds number to replace $\frac{L\rho}{\mu}$, the second term

on the right hand side of the above equation can be dropped since it has zero order as Reynolds number goes to infinity. Therefore, in the boundary layer, the Navier-Stokes equation is the x direction becomes

$$\rho\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 v_y}{\partial y^2}. \tag{76}$$

Similarly, the order of each the terms in the y direction Navier-Stokes equation is

$$\mathcal{O}\left(\frac{V_f^2 T}{L^2}\right)+\mathcal{O}\left(\frac{V_f^2 T}{L^2}\right)=\mathcal{O}\left(\frac{1}{\rho T}\right)+\mathcal{O}\left(\frac{V_f T\mu}{\rho L^3}\right)+\mathcal{O}\left(\frac{V_f\mu}{LT}\right). \tag{77}$$

The dominant term in Eq. 77 is clearly the term $\mathcal{O}\left(\frac{1}{\rho T}\right)$ since the thickness of the boundary layer and the small. Thus the Navier-Stokes equation in the y direction becomes

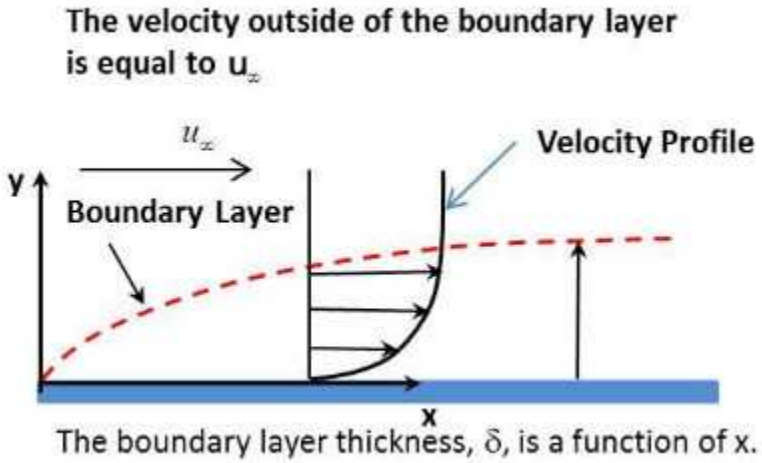
$$\frac{\partial p}{\partial y}=0. \tag{78}$$

Equation 78 simply states that the pressure is a function of only x and therefore, is a constant at each tangential layer. Then the pressure is fixed on the top of the boundary layer. The no-slip and no-penetration conditions give two boundary conditions for the boundary layer as

$$v_{lx}(x,y=y_s)=v_{sx}(x,y=y_{ss});\qquad v_{ly}(x,y=y_s)=v_{sy}(x,y=y_{ss}). \tag{79}$$

Here, the subscripts l and s denote the liquid and solid, respectively and specify the side of corresponding quantities, The height of the interface is y_{ss} . One can also assume that the velocity at the top of the boundary layer matches to the velocity from the inviscid flow at the same layer.

Figure 3: Viscosity in Boundary Layer



The velocity in x direction changes in the boundary layer due to the viscous force. Outside the boundary layer, the velocity becomes uniform since the fluid is non-viscid flow. Taken from [17].

To form a mathematically well-defined problem, one more condition is needed. A proper choice is to give the function of the velocity in the x direction at a fixed x value. Symbolically the condition can be given as $v_x(x_0, y) = f(y)$, Where x_0 is a fixed position and f is a known function.

The other important information in boundary layer theory is the thickness of the layer. The thickness of the boundary layer is defined as the distance between the surface and the point where the velocity in the boundary layer is equal to the velocity from the inviscid fluid. Let us first take a look at a simpler problem .

Suppose we have a uniform flow in the x direction on a plate which is stationary in the y direction. The continuity equation immediately gives v_y is a constant in y because of the uniformity. Since the velocity in y direction of the flow is zero at the surface, we have that v_y = 0. Thus, the Navier-Stokes equation becomes

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}.$$

The order on each side of the equation should agree with each other, thus we have

$$\mathcal{O}\left(\frac{V_f}{t}\right) = \frac{\mu}{\rho}\left(\frac{V_f}{T^2}\right). \tag{80}$$

Note that all parameters are same as defined previously. Therefore, the thickness of the boundary layer has the order

$$T \sim \sqrt{\frac{\mu t}{\rho}}. \tag{81}$$

To replace the time dependence, one can use $\frac{L}{V_f}$ to replace t. Therefore, the Thickness of the boundary layer becomes

$$T \sim \sqrt{\frac{1}{Re}}L \tag{82}$$

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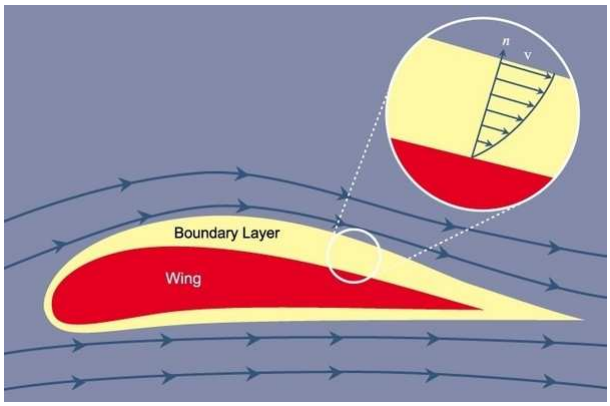
Figure 4: Example of the Boundary Layer's Thickness



This is an example of the shape of the boundary layer around a wing. The arrows outside the boundary layer denote the streamlines of the air and the zoomed in region shows the change of the velocity due to the viscous force. Taken from [18].

4.2 Stress Balance for an Elastic Surface

The elastic surface is the interface between the liquid and the solid which can move under constraints within a range of amplitude. The shape of the elastic surface should be calculated based on the equation of motion. The linear elasticity equations relates the stress to the movement of a particle. This section will follow the idea of stress balancing to calculate the function of the elastic surface.

We have applied the stress balance on the free surface in previous section. The major difference between the elastic surface and the free surface are first, the viscid force is not negligible in the boundary layer, and second, the pressure from the solid side is not a constant. However, the overall procedure should be similar.

The first condition is also that all the particles which are originally on the elastic surface should remain on the surface all the time. Therefore, similar to Eq. 57, one can obtain

$$\frac{\partial y_e}{\partial t} + (\boldsymbol{v}(y_e) \cdot \nabla) y_e = v_y(y_e). \tag{83}$$

Here, y_e is the function of the elastic surface and $\boldsymbol{v}$ is the velocity field of the fluid. y_e in parentheses specifies the value of corresponding parameters on the elastic surface. However, since the fluid is no longer a potential flow, the velocity potential is not guaranteed to exist. Therefore, the above equation will not have a form in terms of the velocity potential.

The continuity equation and the momentum equation should give exactly same general conditions for the elastic surface. That is

$$\frac{d}{dt} \int \rho dV = - \int \rho (v_i - w_i) n_i dS \tag{84}$$

and

$$\frac{d}{dt}\int \rho v_i dV = -\int (\rho n_j (v_j - w_j)v_i + n_i p - n_j \tau_{ji}) dS + \int \rho F_i dV \tag{85}$$

In Eq. 87, ρ is the density of either the liquid or the solid and it is still assumed to be a constant, v_i is the i^{th} component of the velocity field and w is the prescribed velocity of the elastic surface. The integration on the left hand side is the volume integral and the one on the right hand side is the surface integral. In Eq. 85, $\boldsymbol{n}$ is the unit normal vector(outwards) with respect to the elastic surface.

By the same analysis presented in Sec. 3, the continuity equation gives

$$\rho(v_n - w) \mid_L = \rho(v_n - w) \mid_S, \tag{86}$$

the momentum equation in tangential direction gives

$$\boldsymbol{n} \cdot \boldsymbol{\tau} \mid_S = \boldsymbol{n} \cdot \boldsymbol{\tau} \mid_L, \tag{87}$$

and the momentum equation in normal direction gives

$$(\rho(v_n - w)v_n) + p - \mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}) \mid_L = (\rho(v_n - w)v_n) + p - \mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}) \mid_S . \tag{88}$$

Here, the subscript S stands for the solid side and all the other parameters are defined the same as in Sec. 3. If the function of the elastic surface is given, then the normal vector with respect to the surface can be calculated accordingly. Therefore, the unknown terms in the above system of equations would be the shear tensor, which is related to the velocity components. Let us take a detour to introduce the shear tensor on the solid side.

In solid elastic analysis, there are important parameters called Lame's param-

eters. These two parameters are defined as

$$\lambda(K,\nu) = \frac{K\sigma}{(1+\nu)(1-2\nu)}; \tag{89}$$

$$\sigma(K,\nu) = \frac{K}{2(1+\nu)}. \tag{90}$$

Here, λ is the first Lame's parameter and σ is the second Lame's parameter. K is the bulk modulus or the incompressibility and ν is the Poisson's ratio which is defined as the transversal expansion divided by the axial compression for small deformation.

Lame's parameters are material dependent parameters and usually can be treated as a constant for a specific solid under a similar thermodynamic condition.

The summation of the two Lame's parameters is

$$\lambda + \sigma = \frac{K\nu}{(1+\nu)(1-2\nu)} + \frac{K}{2(1+\nu)} = \frac{2K\nu + K(1-2\nu)}{2(1+\nu)(1-2\nu)} = \frac{K}{2(1+\nu)(1-2\nu)}$$

The equation of motion for an elastic solid material can be derived from Newton's second law as

$$\nabla \cdot \boldsymbol{T} + \boldsymbol{F} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}. \tag{91}$$

Here, $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{F}$ is the force applied on the body per unit mass, and ρ is the density of the solid. The Cauchy stress tensor can be written in terms of the Lame's parameters

$$T_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\sigma \varepsilon_{ij} \tag{92}$$

Here, δ_{ij} is the Kronecker delta and ε_{ij} is the strain. For infinitesimal deformation,

the strain tensor $\boldsymbol{\varepsilon}$ is equal to the average of the displacement gradient tensor:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) \tag{93}$$

or more explicitly

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{94}$$

Here the subscripts denote the specific component of parameters, $\boldsymbol{u}$ is the displacement field, $\boldsymbol{x}$ is the direction vector where x_1 is the x direction and x_2 is the y direction.

Similar to the Navier-Stokes equation in fluid mechanics, the governing equation in elastic is the Elasticity equations. It is expressed as

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \frac{K}{2(1+\nu)(1-2\nu)}\nabla(\nabla \cdot \boldsymbol{u}) + \frac{K}{2(1+\nu)}\nabla^2 \boldsymbol{u} - \rho \boldsymbol{g}, \tag{95}$$

where $\boldsymbol{g}$ is the effective gravity. In terms of Lame's parameters:

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda+\sigma)\nabla(\nabla \cdot \boldsymbol{u}) + \sigma\nabla^2 \boldsymbol{u} - \rho \boldsymbol{g}. \tag{96}$$

This background information on solid elasticity is enough for us to balance the stress of the elastic surface. Going back to the boundary requirement of the elastics surface, Eq. 87 and Eq. 88 can be now written explicitly.

For the fluid side, the viscid tensor can be calculated from $\mu\nabla \boldsymbol{v}_{\boldsymbol{F}}^2$. Where μ is the viscosity of the boundary layer and $\boldsymbol{v}_{\boldsymbol{F}}$ is the velocity field in fluid. The complete set up of the problem of solving these quantities are provided in the last section. Numerically, one can solve the velocity field inside the boundary layer

for any arbitrary initial condition given in the following form

$$\begin{aligned}
&\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \\
&\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 \partial v_y}{\partial y^2}, \\
&v_x(x,0) = 0, \\
&v_y(x,0) = 0, \\
&v_x(x, y\to\infty) = V_e(x), \\
&v_x(x_0, y) = f(y).
\end{aligned}$$

Here, V_e is the velocity of the inviscid flow at the top of the boundary layer. From the Bernoulli equation, the pressure can be related to the velocity. Then the second condition can be rewritten as

$$\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = V_e\frac{\partial V_e}{\partial x} + \mu\frac{\partial^2 \partial v_y}{\partial y^2}.$$

The initial condition V_e can be found by following the procedure in Sec. 2 since outside the boundary layer the fluid is potential flow. The other initial condition $f(y)$ can be either approximated by the velocity potential by first assuming the entire domain has no viscid force or to interpolate the physical condition.

Since the boundary layer changes its thickness over the interested region, it would be nice to adjust the scale of y so that the boundary layer's thickness becomes a constant. The adjustment begins by introducing Falkner-Skan variables[5]

$$\xi = \frac{x}{L}; \qquad \eta = y\left(\frac{\mu x}{\rho V_e(x)}\right)^{\frac{1}{2}}. \tag{97}$$

Here, ξ is a normal nondimensional parameter for x and η is the nondimensional parameter for y such that the boundary layer's height is fixed. Notice now the

two velocity components are functions of Falkner-Skan variables. The nondimensional velocity components are

$$\hat{v}_x = \frac{v_x}{V_e}; \qquad \hat{v}_y = \frac{\mu}{\rho V_e}\left(\frac{\rho V_e x}{\mu}\right)^{1/2}. \tag{98}$$

Then, the initial condition becomes

$$\hat{v}_x(\xi, 0) = 0, \qquad \hat{v}_y(\xi, 0) = 0, \qquad \hat{v}_x(0, \eta) = v_x(\eta), \qquad \hat{v}_x(\xi, y \to \infty) = 1$$

To simplify the notation for later equations, let

$$\Theta \equiv \frac{x}{V_e}\frac{dV_e}{dx}, \qquad \Upsilon \equiv \hat{v}_y + \frac{1}{2}\eta \hat{v}_x(\Theta - 1).$$

Substituting the nondimensional velocities into continuity equation gives

$$\xi\frac{\partial \hat{v}_x}{\partial \xi} + \Theta \hat{v}_x + \frac{\eta}{2}(\Theta - 1)\frac{\hat{v}_x}{\partial \eta} + \frac{\partial \hat{v}_y}{\partial \eta} = 0. \tag{99}$$

The Navier-Stokes equation in the x direction of the boundary layer is given by Eq. 76. In terms of the newly defined variables,

$$\hat{v}_x \xi \frac{\hat{v}_x}{\partial \xi} + \Upsilon \frac{\partial \hat{v}_x}{\partial \eta} = (1 - \hat{v}_x^2)\Theta + \frac{\partial^2 \hat{v}_x}{\partial \eta^2}. \tag{100}$$

Notice that since the height of the boundary layer is transferred to be a constant, the domain can be treated as a rectangle and the mesh process will be straight forward. We assume that the proper mesh is done over the domain formed by ξ and η. The ξ direction is divided to have nodal points p_i and the η direction has nodal points q_i. Since the numerical method is applied to solve velocity components in the boundary layer, each derivative term should be approximated in

terms of numerical differentiation. As an example:

$$\frac{\partial \hat{v}_x}{\partial \xi} = \frac{\hat{v}_x(p_{i+1}, q) - \hat{v}_x(p_i, q)}{p_{i+1} - p_i}, \tag{101}$$

$$\frac{\partial \hat{v}_x}{\partial \eta} = \frac{\hat{v}_x(p, q_{i+1}) - \hat{v}_x(p, q_i)}{q_{i+1} - q_i}. \tag{102}$$

The form of the numerical differentiation would be different if different numerical methods are applied. From here, there is nothing new about the numerical approach of solving the velocity components in the boundary layer. Therefore, along with the pressure found from the Bernoulli equation, the stress tensor on the elastic surface can be then fully found on the side of the fluid.

For the side of the solid, the Cauchy stress tensor is given by Eq. 92. Since the strain tensor is defined in three dimensions, one should first convert it to a two dimensional form. The three dimensional strain tensor is

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}. \tag{103}$$

Where $\boldsymbol{u}$ is the displacement field. Since the set up is that the fluid flows only in the x and y direction, the z direction derivatives should be zero for all the displacement field components. We can also review the displacement in z direction as a constant, thus $\frac{\partial u_z}{\partial x} = \frac{\partial u_z}{\partial y} = 0$. Therefore, the two dimensional strain tensor becomes

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{104}$$

For the two dimensional case, the explicit form of the Cauchy stress tensor is

$$T = \frac{K\nu}{2(1+\nu(1-2\nu)} \begin{bmatrix} \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} & 0 \\ 0 & \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \end{bmatrix} + \frac{K}{1+\nu} \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} \end{bmatrix} \tag{105}$$

Now, in the four equations giving the boundary conditions of the elastic surface, Eq.83, Eq. 86, Eq. 87, and Eq.88, the unknown parameters are two velocity components in the side of solid, the pressure from the solid side, and the velocity of the interface w. Notice that w is the velocity of the the elastic surface; one can directly update the function of the interface from this quantity. The velocity components from each side of the elastic surface are not necessarily matched to each other since the density also experiences a jump at the interface.

In the last section, the stress balancing process of the free surface is aiming to use the curvature force to update the function of the surface. On the other hand, the stress balancing for the elastic surface calculates the velocity from each side and then directly uses the continuity equation to update the function of the surface. In this process, the boundary condition defined for the numerical calculation in the boundary layer would not match with the true condition since the interface itself has a velocity w. However, the small deformation is the necessary condition in determining the strain tensor for the solid side. Thus, the oscillation of the elastic surface should be small.

5 Numerical Approach

5.1 Another Approach for Velocity Potential

This section will first discuss the finite difference method. The finite difference method has three important components. First, both the domain of interest and

the boundary should be discretized, the boundary condition will be translated to values and assigned to corresponding points. Second, the derivative terms should be approximated by the function value, the adjacent nodal values, and the distance between each nodes. Third, a system of equations should be formed from the governing equations for each node.

Let us first apply this numerical method on the problem discussed in Sec. 2. However, two more boundary conditions should be given to form a unique solution for the velocity potential. Although the different added boundary conditions implies a different physical situation, the mathematical procedure should be similar. Therefore, we can specify two boundary conditions as

$$\begin{aligned} \Phi(0, y) &= f_1(y), \\ \Phi(L, y) &= f_2(y). \end{aligned} \tag{106}$$

Here, Φ is the velocity potential and L is the length of x domain. The conditions from Sec. 2 are

$$\begin{aligned} 2\pi \cos(2\pi x) \frac{\partial \Phi}{\partial x} &= \frac{\partial \Phi}{\partial y}, \qquad && y = y_s \\ \frac{\partial \Phi}{\partial x} &= 1, && y \gg 1, \end{aligned}$$

where $y_s = \epsilon \sin(2\pi x)$. The governing equations is the Laplace equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{107}$$

The domain is

$$0 \leqslant x \leqslant L; \qquad y_s \leqslant y \leqslant y_a,$$

where y_a is the free surface. The first task is to divide the domain properly. The

partition in the x direction is straight forward. One can choose to equally divide the x coordinate into m nodes and denotes them as x_i, i ranges from one to p. For the y direction, although the amplitude of both the elastic surface and the free surface are small, it would still help to not assume them to be flat. Since the length of the slice of y at each x_i is different, it would be totally p different partitions of y. The i^{th} partition of y is the slice at x_i. If all the partition of y were chosen to be equally spaced and have q nodes, then the i^{th} partition of y will have nodal coordinate x_i, y_j, where

$$y_j = y_s(x_i) + (j-1)\frac{y_a(x_i) - y_s(x_i)}{q-1}.$$

The step size in the x direction is $sx = \frac{L}{p-1}$ and the step for the i^{th} y partition is $sy(i) = \frac{y_a(x_i) - y_s(x_i)}{q-1}$.

The next step is to write the derivative terms of Φ as a function of the function values. If the forward Euler method were chosen, then one can obtain

$$\frac{\partial \Phi(x_i, y_j)}{\partial x} = \frac{\Phi(x_{i+1}, y_j) - \Phi(x_i, y_j)}{sx}, \tag{108}$$

$$\frac{\partial \Phi(x_i, y_j)}{\partial y} = \frac{\Phi(x_i, y_{j+1}) - \Phi(x_i, y_j)}{sy(i)}, \tag{109}$$

$$\frac{\partial^2 \Phi(x_i, y_j)}{\partial x^2} = \frac{\Phi(x_{i+1}, y_j) - 2\Phi(x_i, y_j) + \Phi(x_{i-1}, y_j)}{sx^2}, \tag{110}$$

$$\frac{\partial^2 \Phi(x_i, y_j)}{\partial y^2} = \frac{\Phi(x_i, y_{j+1}) - 2\Phi(x_i, y_j) + \Phi(x_i, y_{j-1})}{sy(i)^2}. \tag{111}$$

Therefore, the boundary conditions become

$$2\pi\cos(2\pi x_i)\frac{\Phi(x_{i+1},y_j)-\Phi(x_i,y_j)}{sx}=\frac{\Phi(x_i,y_{j+1})-\Phi(x_i,y_j)}{sy(i)}, \tag{112}$$

$$\frac{\Phi(x_{i+1},y_j)-\Phi(x_i,y_j)}{sx}=1, \tag{113}$$

$$\Phi(0,y_i)=f_1(y_i), \tag{114}$$

$$\Phi(L,y_i)=f_2(y_i), \tag{115}$$

and the governing equation becomes

$$\frac{\Phi(x_{i+1},y_j)-2\Phi(x_i,y_j)+\Phi(x_{i-1},y_j)}{sx^2}=\frac{\Phi(x_i,y_{j+1})-2\Phi(x_i,y_j)+\Phi(x_i,y_{j-1})}{sy(i)^2}. \tag{116}$$

Then, the number of unknown nodal values will be $(p-2)(q-2)$, whereas the number of the functions are also $(p-2)(q-2)$. Thus, this will be a unique solution for all the nodes. An alternate method of solving the Laplace equation will be provided in Sec. 8.2, which which utilizes properties of the Laplace equation..

Let us then consider the time evolution to update the function of the surface's shape. The first issue here is that Φ is not a continuous function and therefore, the velocity components cannot be obtained by taking derivatives of Φ. One solution is to apply the least square fitting on the nodal values of interested layers. The goal of a least square is to find an approximated function f based on the condition that the summation of the squared error between f and the true value of all nodes is minimized:

$$\chi^2=\sum_i\left(\frac{f^{x_i}-f_c^{x_i}}{\sigma_i}\right)^2. \tag{117}$$

Here, f_c is the calculated values, the superscripts refer to the position of these values, and σ_i is the corresponding weight for each x_i.

It is clear that the value of χ^2 would be zero for a large group of function f. However, the function f has to provide information and satisfy physical condi-

tions. Therefore, the form of the function f is normally prescribed by using shape functions. As its name describes, the shape function is chosen to have a desired shape. One can of course use the polynomial functions as the shape functions, that is, $f = \sum \alpha_i x^i$. Eq.117 then becomes the problem of solving for the parameters α_i. In the space of α_i, the extreme value of χ^2 will occur at the point such that all derivatives of χ^2 with respect to each parameter are equal to zero. For most cases, the extreme value is the minimum value since such functions consist of polynomial base functions with no upper bound. The process of solving the system of equations from taking derivatives can be naturally processed by using the matrix representation:

$$\boldsymbol{A}^T \cdot \boldsymbol{A} \cdot \boldsymbol{\alpha} = \boldsymbol{A}^T \boldsymbol{b}. \tag{118}$$

Here, the superscript T denotes the transpose of the corresponding matrix, $\boldsymbol{A}$ is the matrix containing all the values of the base function with its corresponding weight at each point:

$$A_{ij} = \frac{{x_j}^i}{\sigma_j},$$

$\boldsymbol{b}$ is the list which contains all of the experimental data values at each point with its weight:

$${b^T}_i = \frac{f_e}{\sigma_i},$$

and vector $\boldsymbol{\alpha}$ is the list of all the parameters α_i.

The issue for the polynomial interpolation is that if the least square fitting were performed from element to element, the function value of a shared nodal point is not necessarily the same for two adjacent elements. It is not acceptable since it violates the physical requirement.

The method this section used to conquer this issue is Hermite interpolation. Hermite interpolation functions also provide another nice property which is the continuity of the first derivative of the function. If the first derivative values were

known for every nodal point, the general Hermite interpolation of the function is given as

$$f_e(x) = \sum_i f_e(x_i)N_i(x) + {f_e}'(x_i)\overline{N_i}(x).$$

Here, N_i and $\overline{N_i}$ are shape functions for Hermite interpolation. Similar to polynomial functions, the Hermite interpolation function can be applied locally. If the process were performed globally, the shape functions of Hermite interpolation have to be found from element to element. An alternative method is to transfer the interval of each element to a consistent region. If the form of the shape function did not differ for elements, then the shape function will maintain the same in terms of the transformed variable. Suppose the position variable is transformed to the interval from -1 to 1 and denote the new variable as ξ. Then to make sure the continuity conditions at boundary nodes hold for each region the following properties are desired:

- $N_i(\xi_j) = \delta_{ij}; \qquad \frac{d}{d\xi}N_i(\xi_j) = 0;$
- $\overline{N_i}(\xi_j) = 0; \qquad \frac{d}{d\xi}\overline{N_i}(\xi_j) = \delta_{ij}.$

Here, ξ_i are the nodal points of each element. For the one dimensional case, there will be two nodal points for each element. It is easy to check that the nodal values and the nodal derivative values are consistent across each region and therefore continuous. For the one dimensional case, the four conditions of the shape function are sufficient to determine any kind of shape function which contains four undetermined coefficients, such as the coefficients of third order polynomials. Once the shape function is calculated, the fitting function on each element becomes

$$f_i(x) = \sum_i \left[f(x_i)N_i(\xi) + \left(\frac{df(x_i)}{dx}\right)\frac{dx}{d\xi}\overline{N_i}(\xi)\right].$$

An important fact is that the coefficients $f(x_i)$ and $\left(\frac{df(x_i)}{dx}\right)$ are not equivalent to the true value at x_i, they are determined by minimizing the error $\chi^2 = \sum_i (f_i - f_{ei})$ as the weight of all the points is assumed to be one. The positions of measured points are substituted into the shape function to form a system of equations which can be represented in the matrix form. As a concrete example, if there were five points, say $x_i, i = 1..5$, in element j. Then, the matrix $\boldsymbol{A_j}$ is

$$\boldsymbol{A_j} = \begin{bmatrix} N_1(x_1) & \overline{N_1}(x_1) & N_2(x_1) & \overline{N_2}(x_1) \\ N_1(x_2) & \overline{N_2}(x_2) & N_2(x_2) & \overline{N_2}(x_2) \\ N_1(x_3) & \overline{N_3}(x_3) & N_2(x_3) & \overline{N_2}(x_3) \\ N_1(x_4) & \overline{N_1}(x_4) & N_2(x_4) & \overline{N_2}(x_4) \\ N_1(x_5) & \overline{N_1}(x_5) & N_2(x_5) & \overline{N_2}(x_5) \end{bmatrix},$$

and the corresponding $\boldsymbol{\alpha_j}$ vector is

$$\boldsymbol{\alpha_j}^T = \left(f(x_1), \frac{df(x_1)}{dx}, f(x_5), \frac{df(x_5)}{dx}) \right).$$

Here, x_1 and x_5 are boundary positions of the j^{th} element. Denote $\boldsymbol{b_j}^T$ as the vector which consists of all the experimental data values and $\boldsymbol{\alpha_j}$ to be the parameters of each shape function. Then in element j, the error from the least square fitting function is

$$\chi_j{}^2 = (\boldsymbol{b_j} - \boldsymbol{A_j} \cdot \boldsymbol{\alpha_j})^T \cdot (\boldsymbol{b_j} - \boldsymbol{A_j} \cdot \boldsymbol{\alpha_j}). \tag{119}$$

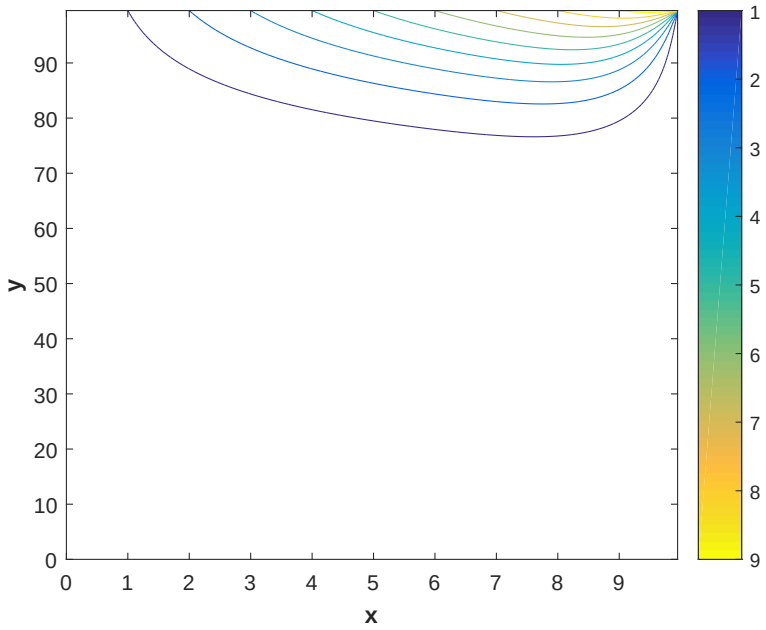
Equation 119 can be easily extended globally. Let $\boldsymbol{\alpha}$ be the vector consists of all the undetermined nodal values and their derivative values, $\boldsymbol{b}$ be the vector consisting of all the values of measured points. By carefully rewriting the system of equations in the matrix form, one can then generate a global matrix $\boldsymbol{A}$. For one dimension, the matrix $\boldsymbol{A}$ is a block-diagonal matrix. Each block in the matrix $\boldsymbol{A_j}$ is defined above. Geometrically, the problem is the same as finding the smallest

χ^2, which is same as to find the projection of $\boldsymbol{A} \cdot \boldsymbol{\alpha}$ onto $\boldsymbol{b}$, then one can solve the parameter vector $\boldsymbol{\alpha}$ from Eq.118.

After having the approximated function for the velocity potential at any layer, one can then perform the calculation process introduced in Sec 2.2.

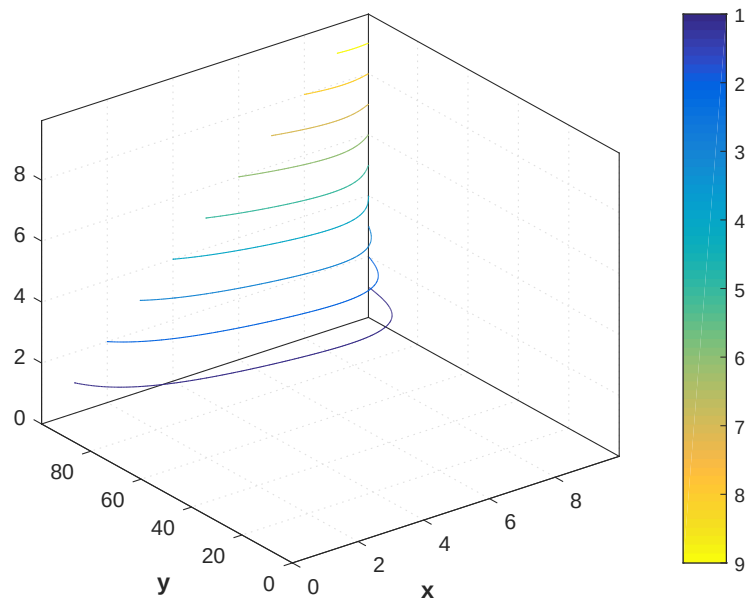
We will now compare the numerical results for the problem introduced in Sec. 2. The following figures are the plots from applying the numerical method on the same problem introduced in Sec 2.

Figure 5: Velocity Potential's Contours



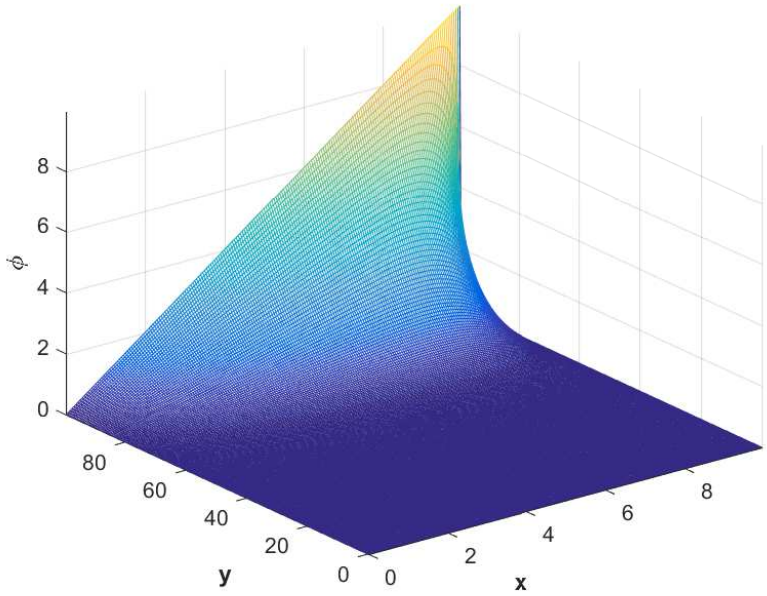
This plot shows two dimensional contours of the velocity potential.

Figure 6: Velocity Potential's Contours



This plot shows three dimensional contours of the velocity potential.

Figure 7: Velocity Potential



The velocity potential over the whole region.

In comparison to the result in Sec 2, the main different is that we restrict the values of velocity potential at $x = 0$ and $x = L$. Therefore, the dominant term is no longer simply the x coordinate. The contribution from the depth of the fluid is emphasized by the specified boundary conditions. The contour plots indicate that the most of the change of velocity potential happen close the extreme y value, which represents the height of the free surface.

Figure 8: Velocity Potential at Different Height

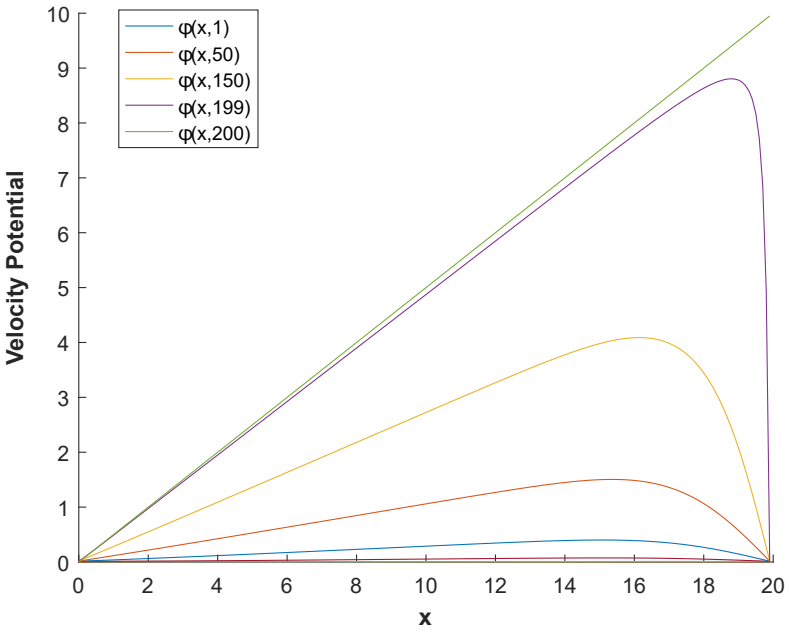


Figure 9: This plot gives the function of the velocity potential at different fixed height

At the free surface, the velocity potential is a straight line since the x derivative of the velocity potential at the free surface is equal to the free stream velocity which is set to be a constant. As Fig. 8 shows, the boundary conditions at two the ends force the velocity potential to be zero at $x = 0$ at $x = 20$, where the length of the domain is set to be 20. Also, as the value of x and y increases, both the value of Φ and its derivative value increases. This agrees with the result of Fig. 5. This indicates that a small change of the shape of the elastic surface can make the

velocity potential vary much more at the free surface.

The details of the numerical algorithm is given in Sec 8.

5.2 More on Numerical Methods

Thus far, the region of the solid phase has been considered only to be the elastic surface. What if the region has a certain height and there is force applied from the bottom of the solid region? The first idea is to use Hooke's law as the governing equation. After properly meshing the solid region, the applied force will become the nodal value for the base layer. Suppose the boundary conditions are that the particles cannot move at the bottom layer and two vertical walls are at the end of the region. Assuming the dominant terms in Hooke's law are given from the adjacent points, then from Newton's third law, the displacement can then be calculated for each node. Hence this can give the shape of the elastic surface which can then be used in the former calculation. This process can be also perform backwards as the pressure is calculated from the boundary layer or the potential flow, then the displacement of each node in the solid region can be updated. Notice that in this procedure, the change to all the particles does not happen simultaneously. However, this is also acceptable since it is close to the true physical situation. There are several other numerical methods which can deal with the whole region at the same time such as the extended finite element method.

6 Conclusion

In this report, an asymptotic method is first introduced to solve the velocity potential in a potential flow. The boundary condition is given as the no penetration condition and the initial conditions are given as the shape of the interface

between the fluid and the solid and the free stream velocity. Following the result of the velocity potential, the time contribution is then considered. The velocity potential can be used to update the free steam velocity directly by substituting the coordinates value of the fluid's surface. Since the interface is elastic, then its shape should also be a function of time. If the flow were still potential flow for the entire region, then the shape of the interface is updated from the Navier-Stokes equation via a time stepping method. These two updated initial condition can be used to calculate the velocity potential for the next time interval.

The shape of the free surface is the next discussion of this report. The procedure is to first define physical conditions and then translate these conditions to mathematical equations, and combine these equation to solve for the curvature force which can give the displacement of the free surface. The physical conditions are assumed to be the Kinetic and Dynamic boundary conditions. The Kinetic boundary condition states that the particles located on the free surface should always remain on the free surface. The Dynamic boundary conditions states that the shear tensor should be continuous across the interface. Although this report does not provide any concrete example to illustrate the calculation, this procedure is easy to follow.

This report also discusses the boundary layer. The motivation of having the boundary layer is to enforce the no-slip condition to make the analysis more accurate. The Navier-Stokes equation in the boundary layer is derived first by dropping the zero order terms. The thickness of the boundary is approximated by matching the order of the Navier-Stokes equation. With the presence of the viscous force, the stress balancing becomes more interesting and complicated. The background of the solid elasticity is introduced to give the form of the stress tensor on the solid side of the elastic surface. The stress tensor on the side of the fluid is given by solving the velocity components in the boundary layers

numerically. The x and y coordinates are transferred to keep the thickness of the boundary layer to be a constant.

Finally, the numerical method is outlined to solve more complex problems. Several approaches are introduced and these can be the direction of future efforts.

7 Appendix

This appendix introduces some of the vocabularies' definitions and mathematical notation, as well as providing derivations and explanations of important equations.

7.1 General

- **velocity field**: the velocity can be fully described as

$$\boldsymbol{v} = [v_x,\, v_y,\, v_z]$$

where v_x,v_y, and v_z are themselves functions of x, y, and z. Hence the acceleration field is

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + v_x \frac{\partial \boldsymbol{v}}{\partial x} + v_y \frac{\partial \boldsymbol{v}}{\partial y} + v_z \frac{\partial \boldsymbol{v}}{\partial z}.$$

- **Displacement field**: This is the collection of displacement vectors for all points in a certain region. One displacement vector describes the reference position of a point or a particle with contrast to its former or original position. The displacement field is usually denoted as $\boldsymbol{u}$ and used to define the body's deformation. Mathematically

$$\boldsymbol{u} = [u_x,\, u_y,\, u_z].$$

7.2 Fluid

- **Irrotational flow**: is a kind of flow which does not have vorticity. Mathematically it can be represented as $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v} = 0$

  Irrotational flow is a necessary and sufficient condition for the velocity po-

tential to exist. The velocity potential is formally defined as :

$$\phi = \int_O^P \boldsymbol{v} \cdot d\boldsymbol{x}$$

where O is an arbitrary fixed point and can be chosen freely. In most of the flow, ϕ is independent of the path between O and P, which implies that ϕ is a single-valued function of the path, thus a constant. Partial differentiation of the above integral gives:

$$\boldsymbol{v} = \nabla\phi = [\phi_x \;\; \phi_y \;\; \phi_z]$$

- **Incompressible flow**: Incompressible flow means that the density of the flow is uniformly distributed. From the conservation of total mass, one can get $\nabla \cdot \boldsymbol{v} = 0$.

- **Steady flow**: means that the properties of the flow at any position remain unchanged. That is, at all the positions of the system, the velocity field, density of the fluid, and all other properties has time derivative equal to zero.

- **Potential flow**: is also called ideal flow. It is a kind of flow which has no vorticity and no viscosity. Mathematically

$$\nabla \cdot \boldsymbol{v} = 0; \qquad \nabla \times \boldsymbol{v} = 0$$

The velocity potential is often used in potential flow. For a 2-D compressible Newtonian fluid, one can derive the continuity equation as

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$$

Where v_x and v_y are the components of the velocity field. If the velocity potential exists, by letting $v_x = \frac{\partial \Phi}{\partial y}$ and $v_y = \frac{\partial \Phi}{\partial x}$, the continuity equation can be always satisfied. For 3-D models, this continuity equation will be $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$ and this condition can still hold with selected velocity potential.

- **Incompressible Euler Equations for Inviscid Flow**: for fluid of constant density ρ

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right) = -\nabla p + \rho \boldsymbol{g}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

As its name implies, Euler equation for inviscid flow requires the flow to not be viscid. The no-slip condition from viscosity can no longer be satisfied. However, to obtain a solution, some boundary conditions have to be assumed. The proper condition for inviscid flow should be that the velocity normal to the boundary should be zero. Note: assuming viscous forces are small in comparison to inertial forces, $\mathcal{R}e \gg 1$

Note: can be integrated along a streamline to get Bernoulli equation. When the flow is the potential flow, we get Bernoulli's equation to be valid everywhere. The Euler Equation for each direction is

$$\rho g_i - \frac{\partial p}{\partial i} = \rho \left(\frac{\partial I}{\partial t} + v\frac{\partial I}{\partial x} + u\frac{\partial I}{\partial y} + w\frac{\partial I}{\partial z} \right)$$

Where $i \in (x, y, z)$ and I is the corresponding component of i.

7.3 Solid

- **Poisson's ratio**: when a material is being deformed, it changes on the axis of the applied load and on the axis perpendicular

$$\sigma = -\frac{\varepsilon_{lat}}{\varepsilon_{long}}$$

  - constant for all homogeneous, isotropic, linearly elastic materials
  - usually between 0 and 0.5
  - dimensionless, ratio of two strains that have units percentage, strain is a measure of how much a given deformation differs locally from a rigid-body deformation

- **Equation of Motion**: from Newton's second law, one can obtain

$$\nabla \boldsymbol{\tau} + \boldsymbol{F} = \rho u_{tt}.$$

  Here, $\boldsymbol{\tau}$ is the Cauchy stress tensor, $\boldsymbol{F}$ is the force applied on the body per unit volume, and ρ is the density of the solid. By applying Lame's parameters, the Cauchy stress tensor can be written as

$$\tau_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$$

  Here, δ_{ij} is the Kronecker delta and ε_{ij} is the strain which is correlated to the stress tensor by Hooke's law as $\tau_{ij} = C_{ijlk} \cdot \varepsilon_{lk}$, where C is the stiffness tensor.

- **Navier-Cauchy or Lame or Elastostatic Equations**: The governing equa-

tion for a solid is expressed as

$$\rho_e \frac{\partial^2 \boldsymbol{u}}{\partial t} = \frac{K}{2(1+\sigma)(1-2\sigma)} \nabla(\nabla \cdot \boldsymbol{u}) + \frac{K}{2(1+\sigma)} \nabla^2 \boldsymbol{u} - \rho_e \boldsymbol{g}$$

Here, $\boldsymbol{u}$ is the displacement, ρ is the density of the solid, and $\boldsymbol{g}$ is the effective gravity. In terms of Lame's parameters:

$$\rho_e \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu) \nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} - \rho_e \boldsymbol{g}.$$

As a quick demonstration of the derivation, let us take a look at the x direct. We have

$$\tau_{xx} = 2\mu \varepsilon_{xx} + \lambda(\varepsilon_{xx} \varepsilon_{yy} + \varepsilon_{zz})$$

$$\tau_{xy} = \mu(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x})$$

$$\tau_{xz} = \mu(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x})$$

Balancing the force as $\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$, then we can obtain

$$(\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \mu \left(\frac{\partial^2 u)_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right).$$

Where u_i is the component of the displacement in i direction.

7.4 Boundary Conditions

- **Boundary Conditions at a free surface**:
  - At an impermeable boundary, the flow of fluid relative to the boundary must be tangential to it. If the boundary is fixed in space, this means that the component of fluid velocity normal to the boundary must be

zero. If the boundary is moving, then the normal component of the fluid must be equal to the velocity of the boundary normal to itself. The small motion in general can be expressed as (2-D)

$$z_p + \delta z_p = \eta(x_p + \delta x_p, t + \delta t) = \eta(x_p, t) + \frac{\partial \delta}{\partial x}\delta x_p + \frac{\partial \eta}{\partial t}\delta t.$$

Here, z is the free surface elevation, η is the wave elevation, and the subscripts of x and z stands for any specific particle. Utilizing the above boundary condition ($z = \eta$) one arrives at

$$\frac{\partial z}{\partial t} = \frac{\partial \eta}{\partial x}\frac{\partial x_p}{\partial t} + \frac{\partial \eta}{\partial t}$$

$$\nabla \boldsymbol{\Phi} \cdot \hat{\boldsymbol{n}} = \boldsymbol{U} \cdot \hat{\boldsymbol{n}}$$

Here $\boldsymbol{\Phi}$ is the velocity potential of the fluid and $\boldsymbol{U}$ is the velocity of the surface.

– Dynamic boundary conditions: at any point of the interface, the normal stress tensor should be equal and in opposite direction:

$$\tau_{ij}^1 \cdot \hat{\boldsymbol{n_1}} = -\tau_{ij}^2 \cdot \hat{\boldsymbol{n_2}}$$

$$\hat{\boldsymbol{n_1}} = -\hat{\boldsymbol{n_2}}$$

For a curved free surface, there will be a pressure change due to the surface tension:

$$\Delta p = \sigma \kappa.$$

Where σ is the surface tension and κ is twice the curvature of the sur-

face. Therefore the above stress condition should be adjusted to

$$\tau_{ij}^{1} \cdot \hat{\boldsymbol{n_1}} + \Delta p \cdot \hat{\boldsymbol{n_1}} = -\tau_{ij}^{2} \cdot \hat{\boldsymbol{n_2}}$$

Normal stress balance at a free surface must be balanced by the curvature force associated with the surface tension:

$$\boldsymbol{n} \cdot \tilde{T} \cdot \boldsymbol{n} = \gamma(\nabla \cdot \boldsymbol{n})$$

where γ is surface tension, $\boldsymbol{n}$ is the unit normal vector to the surface, stress tensor is $\tilde{T} = -p\tilde{I} + \mu[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] = -p\tilde{I} + 2\mu\tilde{E}$ where $\tilde{E}$ is the deviatoric stress tensor

– tangential stress at a free surface must balance the local surface tension gradient:

$$\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{t} = \nabla\sigma \cdot \boldsymbol{t}$$

where $\boldsymbol{t}$ is unit tangent to interface

7.5 Additional important quantities and descriptions

It is normal in fluid mechanics that a scalar function has multiple dimensions, including the dimension of time. The Leibnitz's theorem offers a mutual relation between the time derivative of this function's integral over its corresponding volume, the partial time derivative, and the change of the function over its surface.

$$\frac{d}{dt}\int_{\Omega(t)} T_{ij..}(x_i, t)dV = \int_{\Omega(t)} \frac{\partial T_{ij..}}{\partial t}dV + \int_{S(t)} n_k w_k T_{ij..}dS \tag{120}$$

Here, $T_{ij..}$ is a tensor of any rank, Ω is the domain of interested and in general it is a function of time, S is the surface domain, and w_i is the Eulerian velocity of

the boundary. As an example, let the tensor T be 1, one obtains

$$\frac{dV(\Omega)}{dt} = \int_S n_k w_k dS \tag{121}$$

This well-known equation states that the rate of volume change is related to the boundary velocity in its normal direction over the surface.

Another most used idea in fluid mechanic is the material derivative. For any material moving with certain velocity, the material derivative is the rate of change of any physical quantity for this material under the continuum assumption. The Reynolds transport theorem states

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + \mathbf{v} \cdot \frac{\partial()}{\partial \boldsymbol{x}}. \tag{122}$$

Let us turn our attention to the most important physical quantity of the flow: the velocity. If the velocity were defined in Euclidean space, it should have three components. Let the velocity field be

$$v = [v_1, v_2, v_3] \tag{123}$$

where $v_i = (x_1, x_2, x_3, t)$ for each i.

Therefore, the flow at any point is not always moving straight forward. Imagine the flow is made from a lot of blocks, these block will not only move, but also rotate. This phenomena is described by the vorticity ω:

$$\omega_i = \varepsilon_{ijk}\, \partial_j v_k \tag{124}$$

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v} \tag{125}$$

7.5.1 Continuity mass equation

The idea of conservation of total mass for any chosen material gives the continuity mass equation as

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho v_i}{\partial x_i} = 0 \tag{126}$$

Using the material derivative, the continuity mass equation can be expressed as

$$\begin{aligned} \frac{D\rho}{Dt} &= -\rho\, \partial_i v_i \\ \frac{1}{\rho}\frac{D\rho}{Dt} &= -\frac{1}{V}\frac{dV}{dt} \qquad as \quad V- > 0 \end{aligned}$$

By applying Leibnitz's theorem, for any domain one can have

$$\frac{d}{dt}\int_\Omega \rho dV = -\int_\Omega \partial_i(\rho v_i)dV + \int_S n_i w_i \rho dS = -\int_S \rho(v_i - w_i)n_i dS \tag{127}$$

Notice here it is assumed that the volume flow can be expressed as

$$v_x \triangle y \triangle z$$

This expression is valid if the domain were small enough. A more precise expression should be

$$\int v_x dy dz$$

7.5.2 Continuity momentum equation

It is impossible to derive the momentum equation directly from Newton's second law. However, it still holds that the rate of momentum change equals the net force:

$$\frac{d}{dt}\int_\Omega \rho v_i dV = \int_\Omega [\frac{\partial}{\partial t}\rho v_i + \frac{\partial}{\partial x_j}\rho\, v_i v_j]dV = \int_\Omega \rho F_i dV + \int_\Omega R_i dS \tag{128}$$

Where F_i and R_i are the body force and surface force.

Substitute the stress equation

$$\int [\frac{\partial}{\partial t}\rho v_i + \frac{\partial}{\partial x_j}\rho v_j v_i - \rho F_i - \frac{\partial}{\partial x_j}T_{ji}]dV = 0 \tag{129}$$

Thus

$$\frac{\partial}{\partial t}\rho v_i + \frac{\partial}{\partial x_j}\rho v_j v_i - \rho F_i - \frac{\partial}{\partial x_j}T_{ji} = 0$$

With

$$\frac{\partial}{\partial x_j}T_{ji} = \frac{\partial}{\partial x_j}\tau_{ji} - \frac{\partial}{\partial x_i}p.$$

From the particle view point, or for a constant density fluid,

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (v\cdot\nabla)v\right] = -\nabla p + \boldsymbol{\nabla\cdot\tau} + \rho\boldsymbol{F} \tag{130}$$

Newton's second law states that

$$m\frac{\partial v_i}{\partial t} = \sum F_i$$

By Leibnitz's rule

$$\frac{d}{dt}\int_\Omega \rho v_i dV = -\int_S [\rho n_j(v_j - w_j)v_i]dS + \int_\Omega \rho F_i dV + \int_{fluid_S}(n_j\tau_{ji} - n_i p)dS + \int_{solid_S} n_j T_{ji}dS \tag{131}$$

The equivalent form in the material derivative for momentum equations is

$$\frac{Dv_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 v_i}{\partial {x_i}^2} + F_i \tag{132}$$

7.5.3 Energy equation

The energy equation for incompressible flow is

$$\frac{d}{dt}\int \rho\left(\frac{1}{2}v^2+gZ\right)dV = -\int_{ss}\rho n_i(v_i-w_i)\left(\frac{1}{2}v^2+gZ\right)dS+\int_{ss}n_iT_{ij}v_jdS \tag{133}$$

$$-\int_{fs}\rho n_i(v_i-w_i)\left(\frac{1}{2}v^2+gZ+\frac{p}{\rho}\right)dS-\int_{fs}n_iw_ipdS \tag{134}$$

$$+\int_{fs}n_i\tau_{ij}v_jdS-\int\tau_{ij}\partial_iv_jdV \tag{135}$$

Where the subscripts fs and ss stand for solid surface and fluid surface.

7.5.4 Incompressible Flow

The conditions for incompressible flow are

$$\frac{1}{\rho}\frac{D\rho}{Dt}=-\frac{\partial}{\partial x_i}v_i=\alpha\frac{Dp}{Dt}-\beta\frac{DT}{Dt}=0 \tag{136}$$

$$\rho\frac{Dv_i}{Dt}=-\frac{\partial p}{\partial x_i}+\frac{\partial}{x_j}\tau_{ji}+\rho g_i \tag{137}$$

The Newtonian viscous stress is

$$\frac{\partial}{\partial x_j}\tau_{ji}=-\frac{2}{3}\frac{\partial}{\partial x_i}(\mu\frac{\partial v_j}{\partial x_j})+2\frac{\partial}{\partial x_j}(\mu S_{ij}) \tag{138}$$

With $\frac{\partial v_i}{\partial x_i}=0$, we have

$$2\frac{\partial}{\partial x_j}S_{ji}=\frac{\partial^2 v_i}{\partial {x_j}^2}$$

and then

$$\frac{Dv_i}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+g_i+\upsilon\frac{\partial^2}{\partial x_j^2}v_i$$

Where $\upsilon = \frac{\mu}{\rho}$ the potential $g_i = -g\frac{\partial Z}{\partial x_i} = \frac{1}{\rho}\frac{\partial}{\partial x_i}(\rho g Z)$ with Z is the height above the plane. Thus we can write the pressure as

$$p = p_{kin} + p_{pot} = (p + \rho g Z) + (-\rho g Z)$$

The force given by pressure is

$$F_{p_i} = -\int_S n_i \cdot p dS = -\int_S n_i \cdot p_{kin} dS + \rho g \int_S \frac{\partial Z}{\partial x_i} dS$$

The last term is equal to $\rho g_i V$ and it is the buoyancy force.

The energy equation is

$$\rho c_p \frac{DT}{Dt} = \frac{\partial}{\partial x_i}(k\frac{\partial}{\partial x_i}T) - \frac{2}{3}\mu(\frac{\partial v_i}{\partial x_i})^2 + 2\mu S_{ij}S_{ji} \tag{139}$$

8 MATLAB code

8.1 Trivial function

G2L is the function converting the global node value to the interval [-1,1].

```
function [ LX ] = G2L(GX,a,b )
% This is the function transfer global x value to local x value
% The interval of local x is [-1,1], if one use different local
    interval,
% this function should be redefined.

LX = (2*GX-(a+b))/(b-a);
```

```
end
```

L2G is the function converting the interval [-1,1] to its global value.

```
function [ GX ] = L2G(LX,a,b)
% This is the function transfer local x value to its global value
% The interval of local x is [-1,1], if one use different local
    interval,
% this function should be redefined.
GX = (a+b)/2+(b-a)*LX*0.5;


end
```

sizeE is the function calculate the size of element (equally spaced)

```
function [ E] = sizeE( x,n )
% This function is to calculate the size of element

m = Nop(x);
E = (x(m)-x(1))/n;


end
```

Nop is the function calculate the number of points

```
function [ N] = Nop( x )
% This function is to calculate the number of point
s = size(x);
N= s(1);
end
```

Nshape is the 3rd order polynomial shape functions of hermite interpolation

```
function [ NS ] = Nshape()
% This is the file for the shape functions in normal region with
    local x
% The first column is the shape function
% The second colume is its derivative w.r.t local x
syms LX
% The local value X ranges from -1 to 1
NS(1,1) = (2-3*LX+LX^3)/4;
NS(3,1) = (2+3*LX-LX^3)/4;
NS(2,1) = (1-LX-LX^2+LX^3)/4;
NS(4,1)= (-1-LX+LX^2+LX^3)/4;
for i = 1:4
    NS(i,2) = diff(NS(i,1));
end

end
```

8.2 Velocity Potential

L2 calculates the velocity potential

```
function [ PHI ] = L2(noi,meshx,meshy,L,S,A,VF,Condition )
% This function takes boundary conditions and uses numerical
    approach
% The input variables are
% noi: the number of iteration
% mesh: the number of equally partitions for each dimension
% S: the boundary function between solid and liquid
% L: the length of the domain of x
% h: the height of the interface
% A: the function function between air and liquid
% VF: the free velocity stream
% Condition: = 1 if S&A are given, = 0 if S&A are approximated


syms x y
% specify the mesh for the domain
X =  (0:1:meshx)*L/meshx;
ySI = zeros(1,meshx);
yAI = zeros(1,meshx);
yAPI = zeros(1,meshx);
ySPI = zeros(1,meshx);
% the boundary for the domain at interface
if Condition == 1
```

```
25      for i = 1:meshx
26          ySI(i) = subs(S,X(i)); % the y value of the interfaces
27          yAI(i) = subs(A,X(i));
28          ySPI(i) = subs(diff(S,x),X(i)); % the derivative value on
                the interfaces
29          yAPI(i) = subs(diff(A,x),X(i));
30      end
31  else
32      for i = 1:meshx
33          ySI(i) = S(i,1); % the y value of the interfaces
34          yAI(i) = A(i,1);
35          ySPI(i) = S(i,2); % the derivative value on the
               interfaces
36          yAPI(i) = A(i,2);
37      end
38  end
39  % the y coordinate of the mesh
40  Y = zeros(meshy,meshx);
41  for j = 1:meshx
42      for i = 1:meshy
43          Y(i,j) = ySI(j)+(yAI(j) - ySI(j))*(i-1)/(meshy-1);
44      end
45  end
46  for i = 1:meshx
47      Y(1,i) = ySI(i);
48  end
49  % the element size in x-direction
```

```
50 hx = L/(meshx-1);
51 % the element size in y-direction
52 hy = zeros(meshx,1);
53 for i = 1:meshx
54     hy(i,1) = (Y(meshy,i)-Y(1,i))/(meshy-1);
55 end
56
57 % Define the mesh for the potential equation
58 PHI = zeros(meshy,meshx);
59 % Define the boundary conditions
60 for i=1:meshy
61     PHI(i,1) = VF*i/(Y(meshy,1)-Y(1,1));      % This is a made up
          condition
62     PHI(i,meshx) = VF*i/(Y(meshy,meshx)-Y(1,meshx)); % This is a
          made up condition
63 end
64 for i = 1:meshx
65     PHI(meshy,i) =  X(i)*VF;    % d(Phi)/dx = VF
66 end
67
68 for i = 2:meshx % This condition is from S' = [d(Phi)/dy]/[d(Phi
    )/dx]  (n_i*v_i = 0)
69     if ySPI(i) ~= 0 % In case of S' = 0, we will use previous S'
         value
70         PHI(1,i) =  (PHI(2,i-1)-PHI(1,i-1))*hx/(hy(i-1,1)*ySPI(i)
             ) + PHI(1,i-1);
```

```
71          % Since the above relation requires Phi(2,i), we need
                updata it
72          PHI(2,i) =  (PHI(1,i)-PHI(1,i-1))*hy(i,1)*ySPI(i)/hx+PHI
                (1,i);
73      else
74          PHI(1,i) =  (PHI(2,i-1)-PHI(1,i-1))*hx/(hy(i-1,1)*ySPI(i
                -1)) + PHI(1,i-1);
75          PHI(2,i) =  (PHI(1,i)-PHI(1,i-1))*hy(i,1)*ySPI(i-1)/hx+
                PHI(1,i);
76      end
77  end
78
79
80  % the idea here is that the extreme values of laplace equation
81  % have to be located on bounrdries
82  % Therefore we can evalute the adjacent points' average value
83  % As
84  for i=1:noi ; % loop for iteration
85      for j = 2:meshx-1 ;  % calculate the average of each points
            excluding boundaries.
86          for k = 2:meshy-1 ;
87              PHI(k,j) = (PHI(k-1,j)+PHI(k+1,j)+PHI(k,j-1)+PHI(k,j
                    +1))/4 ;
88          end
89      end
90  end
91
```

```
92 end
```

8.3 Least Square Fitting

LSF returns the nodal value and nodal derivative values

```
function [ NodalValue ] = LSF(shape,x,y,noe)
% Thie function will return the nodal value. (NodalValue_i, i is
    odd)
% and its corresponding first derivative. (NodalValue_j, j is
    even)
% shape: shape function, 1 = 3rd poly, 2 = other ...
% x: x values     y: y values    noe: number of element

syms GX
sizX = Nop(x);                  % how many points
ele = Arrange(x,noe);           % arrange x to proper element

% Decide which shape functions are using
if shape == 1;
    N = Nshape();
end

% choice other shape functions
A = 10;
if shape ==2;
    TMP = Base(A);
    for i = 1:4;
        N(i,:) = TMP(i,:);
    end
end
```

```
24 if shape ==3;
25     TMP = Base(A);
26     for i = 1:4;
27         N(i,:) = TMP(i+4,:);
28     end
29 end
30 if shape ==4;
31     TMP = Base(A);
32     for i = 1:4;
33         N(i,:) = TMP(i+8,:);
34     end
35 end
36
37 % Global matrix
38 % the number of row should equal to the number of point.
39 % the number of colume should be 2+2*number of element.
40 % the continuity condition is builded inside the shape function
41 % not the points' value here.
42 GA = zeros(sizX(1),2*noe+2);
43 tmprow = 1;
44 tmpcol = 1;
45 tmpx = zeros(1,1);
46 for i = 1:noe
47     tmp = ele(i,:);       % eliminate 0s for the element
48     tmp(tmp==0) = [];
49     sit = size(tmp);
50     a = tmp(1);           % the boundary x-values of each element
```

```
51      b = tmp(sit(2));
52      % Transfer each global x to local in the element
53      p = 1;
54      while p <= sit(2)
55          tmpx(p) = subs(G2L(GX,a,b),tmp(p));
56          p = p+1;
57      end
58      rowc = 1;
59      for j = tmprow:(tmprow+sit(2)-1)   % loop for each point in
            the element
60          k = tmpcol;
61          % for each point in the element, we have
62          % f(x1) = [N1*y1 + N3*y1 + N2*d(GX)/d(LX)*y1' + N4*d(GX)/
                d(LX)*y1']
63          % f(x2) = [N1*y2 + N3*y2 + N2*d(GX)/d(LX)*y2' + N4*d(GX)/
                d(LX)*y2']
64          GA(j,k) = subs(N(1,1),tmpx(rowc));
65          GA(j,k+1) = 0.5*(b-a)*subs(N(2,1),tmpx(rowc));
66          GA(j,k+2) = subs(N(3,1),tmpx(rowc));
67          GA(j,k+3) = 0.5*(b-a)*subs(N(4,1),tmpx(rowc));
68          rowc = rowc + 1;                % for next substitution
69      end
70      tmprow = tmprow+sit(2);      % The starting row of global
            matrix for next element
71      tmpcol = tmpcol+2;           % The starting column of global
            matrix for next element
72  end
```

```
73
74 % The corresponding y-value matrix
75 GB = zeros(sizX(1),1);
76 for i = 1: sizX(1)
77     GB(i) = y(i);
78 end
79
80 % Calculate the nodal values and derivatives
81 NodalValue =GA\GB;
82
83 end
```

GAssign used the value from LSF to generate the continuous function for each element

```
function [ y ] = GAssign(shape,x,noe,Ans )
% This function give the value of y as a function of global x
% shape: 1 = 3rd order poly  2 = other ...
% x: x values      noe: number of element    Ans: Nodal value and
    derivative
% The derivative can be directly calculated by diff(y) for each
    element

syms LX
% Choosing he shape function
if shape == 1
     N = Nshape();
end
% Here, the paramter is chosen to be 10, but it can vary
% Usually, For global value assign, it will use the 3rd order
    poly base
% Since the Fermi-like function is designed to solve the jump
    condition
A = 10;
if shape ==2;
     TMP = Base(A);
     for i = 1:4;
          N(i,:) = TMP(i,:);
     end
end
```

```
22 if shape ==3;
23     TMP = Base(A);
24     for i = 1:4;
25         N(i,:) = TMP(i+4,:);
26     end
27 end
28 if shape ==4;
29     TMP = Base(A);
30     for i = 1:4;
31         N(i,:) = TMP(i+8,:);
32     end
33 end
34
35 SE = sizeE(x,noe);  % The size of element
36
37 y(1) = LX;
38 for i = 1:noe        % calculate functions for each element
39     a= x(1)+(i-1)*SE;     % the boundary values of each element
40     b= x(1)+i*SE;
41     GX = G2L(LX,a,b);     % replace local variable by global
       varible
42     k(1) = Ans((i-1)*2+1)*subs(N(1,1),GX);
43     k(2) = Ans((i-1)*2+2)*0.5*(b-a)*subs(N(2,1),GX);
44     k(3) = Ans((i-1)*2+3)*subs(N(3,1),GX);
45     k(4) = Ans((i-1)*2+4)*0.5*(b-a)*subs(N(4,1),GX);
46     y(i) = sum(k);        % The function of y as global x for ith
       element
```

```
47 end
48
49 end
```

8.4 Time Evolution

UPDATA returns discrete nodal values

```
function [ UPS ] = UPDATE( dT,PHI,meshx,meshy,L,S,A )
% This function updates the shape for the interface
% This function returns nodal value and derivatives
% As equally spaced partition
% UPS(1,:) is the nodal value for interface between solid and
    liquid
% UPS(3,:) is the nodal value for interface between air and
    liquid
% UPS(2,:) and UPS(4,:) are their derivative value
% dT is the value of time step
% PHI is the current velocity potential
% meshx and meshy is the number of partition in x and y direction
% L is the length of domain
% S and A are matrix contain values of the current nodal value at
     interfaces

syms x y
% Get values for the potential velocity close to interfaces
BS1 = PHI(1,:);
BS2 = PHI(2,:);
BA1 = PHI(meshy/10,:);
BA2 = PHI(meshy/10+1,:);
% Redefine the mesh to simplify the writting
X =  (0:1:meshx)*L/meshx;
% Redefine the size of each partition
```

```
23 hx = L/(meshx-1);
24 hy = zeros(meshx,1);
25 for i = 1:meshx
26     hy(i,1) = (PHI(meshy,i)-PHI(1,i))/(meshy-1);
27 end
28
29 % Assign size for each matrix
30 DBSX = zeros(meshx,1);  %The derivative values of PHI w.r.t x on
       solid interface
31 DBSY = zeros(meshx,1);  %The derivative values of PHI w.r.t y on
       solid interface
32 DBAX = zeros(meshx,1);  %The derivative values of PHI w.r.t x on
       air interface
33 DBAY = zeros(meshx,1);  %The derivative values of PHI w.r.t y on
       air interface
34
35 % Assign values for all points on interfaces
36 % Boundary points are assumed to be fixed
37 % Therefore we only loop between the second node and the last
       second node.
38 for i = 2:meshx-1
39     DBSX(i) = (BS1(i)-BS1(i-1))/hx ;
40     DBSY(i) = (BS2(i)-BS1(i))/hy(i,1);
41     DBAX(i) = (BA1(i)-BA1(i-1))/hx ;
42     DBAY(i) = (BA2(i)-BA1(i))/hy(i,1);
43 end
44
```

```
45 % Assign size for matrix
46 US = zeros(meshx,1);          % The updated y-value for nodes on
       solid interface
47 USX = zeros(meshx,1);         % The updated x-value for nodes on
       solid interface
48 UA = zeros(meshx,1);          % The updated y-value for nodes on
       air interface
49 UAX = zeros(meshx,1);         % The updated x-value for nodes on
       air interface
50
51 % Assign values t
52 for i = 1:meshx
53     US(i) = S(i)+DBSY(i)*dT;    % uy(t+dt) = uy(t)+du/dx*dt
54     USX(i)= X(i)+DBSX(i)*dT;    % ux(t+dt) = ux(t)+du/dx*dt
55     UA(i) = A(i)+DBAY(i)*dT;
56     UAX(i)= X(i)+DBAX(i)*dT;
57 end
58
59 % Generate interpolation functions for updated coordinates
60 NVS = LSF(1,USX,US,meshx/5);       % Nodal values on the updated
       interfaces
61 NVA = LSF(1,UAX,UA,meshx/5);
62 FBS = GAssign(1,USX,meshx/5,NVS); % Global functions for the
       updated interfaces
63 FBA = GAssign(1,UAX,meshx/5,NVA);
64
65 % Assign size
```

```
66 UPS = zeros(meshx,4);
67 for i = 1:meshx
68     for j = 1:meshx/5
69         % Find the proper function for each X(i)
70         if X(i) <  X(1) + 5*L*j/meshx && X(i)>= X(1) + (j-1)*L*5/
               meshx
71             UPS(i,1) = subs(FBS(j),X(i));
72             UPS(i,2) = subs(diff(FBS(j)),X(i));
73             UPS(i,3) = subs(FBA(j),X(i));
74             UPS(1,4) = subs(diff(FBA(j)),X(i));
75         end
76     end
77 end
78 end
```

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