Arbitrage-Free Pricing of XVA for American Options in Discrete Time

by

Tingwen Zhou

A Thesis

Submitted to the Faculty

of the

WORCESTER POLYTECHNIC INSTITUTE

In partial fulfillment of the requirements for the

Degree of Master of Science

in

Financial Mathematics

May 2017

ADVISORS:

Professor Stephan Sturm

Professor Gu Wang

Abstract

Total valuation adjustment (XVA) is a new technique which takes multiple material financial factors into consideration when pricing derivatives. This paper explores how funding costs and counterparty credit risk affect pricing the American option based on no-arbitrage analysis. We review previous studies of European option pricing with different funding costs. The conclusions help to compute the no-arbitrage price of the American option in the model with different borrowing and lending rates. Another model with counterparty credit risk is set up, and this pricing approach is referred to as credit valuation adjustment (CVA). A defaultable bond issued by the counterparty is used to hedge the loss from the option's default. We incorporate these two models to assess the XVA of an American option. The collateral, which protects the option investors from default, is considered in our benchmark model. To illustrate our results, numerical experiments are designed to demonstrate the relationship between XVA and parameters, which include the funding rates, bond's rate of return, and number of periods.

Contents

List of Figures

List of Symbols

S_t	stock price at time t
$V_t(\cdot)$	the payoff of option at time t
r_b	borrowing rate
r_l	lending rate
r_c	collateral rate
u	stock price up factor
d	stock price down factor
q	probability of counterparty default
$\tilde{q}$	risk neutral measure of counterparty default
Δ_t	shares of stock in a long position hedging portfolio at time t
Δ_{-t}	shares of stock in a short position hedging portfolio at time t
M_t	shares of money market account (MMA) in a long position hedging portfolio at time t
M_{-t}	shares of MMA in a short position hedging portfolio at time t
X_t	hedging portfolio for a long position in a option at time t
X_t^*	super-hedging portfolio for a long position in a option at time t
X_{-t}	hedging portfolio for a short position in a option at time t
X_{-t}^*	super-hedging portfolio for a long position in a option at time t
$\Phi(X_t)$	value of the portfolio X at time t
E_t	European option price at time t
A_t	American option price at time t
γ	collateralization rate
P_t	no-arbitrage price of a portfolio consisted of option and collateral at time t

Chapter 1

Introduction

In the past, the traditional approaches which were adopted to price options involve several assumptions. These assumptions deny the difference between the borrowing and lending rates, or any default in the counterparty. Here, the new approach, which relaxes those assumptions when pricing options, is known as 'XVA'. It is short for value adjustment for some risk elements, which is denoted as 'X'.

The difference between borrowing and lending rates is referred to as funding spread. In this paper, we emphasize the aspects of funding spread and default of the counterparty. By comparing XVA with the traditional approaches, this new approach is more practical and realistic.

Both XVA and traditional approaches are pricing the American option under the assumption that no arbitrage opportunity exists in the market. In discrete time settings, these two methods adopt the backward induction approach in the multi-period binomial tree model [4]. The pricing process begins at the maturity date, and goes backward to the initial date by calculating the price step by step.

The paper is organized as follows. In the first Chapter, we introduce some backgrounds of the project information and provide the definition of each important term. We review and analyze some conclusions of the European option price from previous studies in Chapter 2. Some of the findings are adopted to explore the American option pricing in Chapter 3. The result reveals that different market conditions will determine the utilization of a hedging portfolio or a super-hedging portfolio at each time.

Chapters 3 - 5 discuss the no-arbitrage price of an American option. The model without funding spread and default is introduced at first. A funding spread will be added to the model next. Starting with the one-period model, we extend this to a multi-period model using the backward induction method. The second step focuses on the counterparty credit risk. Defaultable bond is introduced to replicate the payoff in this situation. After considering funding spread and credit risk separately, we incorporate these two models to compute the XVA. We divide the time interval into two different kinds of periods. Funding spread and counterparty credit risk are considered separately in these two periods. The first period allows the stock and MMA to be traded in the market. Only the defaultable bond is traded in market at the second period. To improve the applicability of the model, we add the collateral in the model we just constructed. The no-arbitrage price of an American option can be derived by the value of the portfolio consisting of option and collateral. In Chapter 6, numerical analyses are offered to demonstrate the relationship between XVA and parameters, which include the funding rates, bond's rate of return, and number of periods.

1.1 Important Terms and Concepts

- **European option:** European options are widely traded on exchanges. It can only be exercised on the maturity or expiration date T. On that day, the call option holder can buy, and the put option holder can sell the underlying asset at a specific price - the exercise price or strike price.

  Bonner and Campanelli compute the no-arbitrage European option price by considering the existence of funding spread and counterparty credit risk in discrete time settings [3]. In continuous time settings, Davis, Panas, and Zariphopoulou use the Black-Scholes model to price the European option [5]. This method considers that there is a transaction cost when selling and purchasing stocks. Bichuch, Capponi, and Sturm develop the framework to compute XVA accounting for funding spread, collateralization, and counterparty credit risk [2]. Generally, European option's price is easier to derive than an American option, but some conclusions from European options can also be applied to price American options.

- **American option:** Most of the options traded in the market are American options. Different from the European option, an America option may be exercised before or at the expiration date. The option buyer needs to optimally choose the time to exercise the option. On the other hand, the seller has the obligation to deliver the exercise payoff to the buyer when the option is exercised.

  Multiple approaches have been adopted to value American options. Amin and Bodurtha develop a discrete time model focusing on risks from currency, domestic term structure, and foreign term structure [1]. Rogers uses the Monte-Carlo simulation to price the American option with simulating the paths of the option payoff [11]. These approaches fail to consider the risk of default from counterparty and funding spread.

- **Hedging:** Hedging is a strategy used to reduce a particular risk faced by the investor. A perfect hedge is the one that completely eliminates the risk [8]. For example, using future contracts is an effective way to hedge the risk from the fluctuation of a product's price. Since a perfect hedge can mitigate all of the risks, in this circumstance, the price of the derivatives can be calculated by evaluating the price of the hedging portfolio.

  The hedging portfolio will generate the same payoff as the derivative. The value of the hedging portfolio is referred to as the hedging price. We use the stock and the money market account (MMA) to construct the hedging portfolios. When we consider the default risk from the counterparty, defaultable bonds are also involved in the hedging portfolio. Given different borrowing and lending rates, the hedging portfolio's value may not represent the derivative's value properly, therefore the super-hedging portfolio would be considered as an alternative.

- **Super-hedging:** When the market completeness breaks down, which means that the risk cannot be completely eliminated, super-hedging becomes a good way to measure the value of derivatives. This is a strategy that can at least hedge the risk of the derivative with the lowest cost. The portfolio which is constructed by the strategy of super-hedging is called the super-hedging portfolio. It can produce no less than the payoff as the derivative with the lowest price. The value of the super-hedging portfolio is referred to as the super-hedging price.

  The super-hedging portfolio is built with the same components as the hedging portfolio. Even though the super-hedging portfolio will have a better payoff at the maturity, it may not be as expensive as the hedging portfolio. Their relationship depends on different market conditions. A comparison of hedging portfolios and super-hedging portfolios will be made in Section 2.3.

- **Collateral:** "In lending agreements, collateral is a possession pledged as security for repayment of a loan to a lender, to be forfeited in the event of a default" [7]. This means that if the borrower fails to pay the principal and interest based on the lending agreements, the item acting as collateral can be forfeited to offset the loan.

  In Chapter 4, we introduce a pricing model with collateral. To eliminate parts of the risk from the counterparty's default, the hedger requires the counterparty to post cash as collateral with a collateralization rate γ. This means if the value of the derivative is C, then the amount of the cash collateral is γC. If default do not occur, the collateral receiver will give the collateral provider $r_c \gamma C$ in each period before the maturity date as an interest. r_c is called the collateral rate. At the maturity date or the time when the option is exercised, the collateral provider will receive the full amount of γC. Once the counterparty defaults on the option, this process will be terminated. The receiver will keep the cash collateral to eliminate the loss from the default.

- **Arbitrage:** An arbitrage is an investment strategy that yields with positive probability a positive profit and is not exposed to any downside risk [6]. For a portfolio X with initial value 0, $\Phi(X_t)$ is the portfolio value at time t. It is an arbitrage strategy if it satisfies the following conditions at a time t up to the maturity date T:

  1. $P(\Phi(X_t) \geq 0) = 1$.
  2. $P(\Phi(X_t) > 0) > 0$.

To compute the XVA of a derivative, we assume that the funding rates are unique to each hedger. Usually, personal interest is influenced by various factors, such as credit score and economic performance [10]. It indicates that the cost of constructing a portfolio is different. Thus, unlike classical option pricing, arbitrage strategies are no longer universal but specific to a hedger. In that way, the XVA of a derivative we derive is unique to each hedger in the market.

When we discuss the price of the American option in Chapter 3, the price will be affected by the buyer's exercising strategy. In Section 3.2, we modify the explanation of arbitrage on the basis of the definition we have mentioned above. More details will be provided.

Chapter 2

European Option

Many conclusions from European option pricing will be useful to understand no-arbitrage pricing for American options. In this chapter, we will introduce some important theorems on European options pricing from previous research. We will make a comparison between the hedging price and the super-hedging price given by different market conditions at Section 2.2. On the basis, we will compute the XVA for a European option with funding spread in a one-period model. In Section 2.4, a new model with a collateral account is generated to price European options by considering funding spread.

2.1 Important theorems of European option

This project is based on the no-arbitrage analysis in the binomial tree model. When the market consists of only the stock and the MMA, the no-arbitrage condition can be derived [3]. This can be seen in Theorem 1 below. We have an underlying asset (stock), the price of which at time t is S_t, $t = 0, 1, 2 \ldots T$. Time zero is the initial time, and time T is the maturity date. We assume that there is no dividend paid in this model. Any shares of stock and MMA are allowed in the transaction. Also, there is no transaction cost. Then at time $t+1$, the stock price has two movements, 'H'and 'T', the values of which are uS_t or dS_t respectively. We call u and d the annualized up and down factors of the stock price with $u > d$. Receptively, r_l and r_b are the annualized lending rate and borrowing rate.

Theorem 1 ***No-Arbitrage Condition:*** *In a market with stocks and MMAs, under the one-period binomial model with the length of h, there is no arbitrage in the market if and only if* $u > d$, $d < 1 + r_b$, $r_l < r_b$, *and* $1 + r_l < u$.

Adapted from: Bonner and Campanelli [3]

Since the borrowing rates and the lending rates are not the same for each individual in the market, the no-arbitrage condition is also unique for hedgers. This coincides with what we have mentioned in the definition of 'Arbitrage'.

In a discrete time setting, the no-arbitrage price of a European option in one period model, E_t, can be derived as the following Theorem 2. As noted in Section 1.1, the price of European options are related to the values of the hedging and super-hedging portfolios. Both the hedging and super-hedging portfolio consist of the underlying asset and MMA. They are constructed given by the payoffs of the European option. The difference is that hedging portfolio replicates exactly the same payoff, and the super-hedging portfolio produces at least the same payoff.

We denote the portfolio at time t as X_t. Both of these two portfolios have the length of one year. It replicates the cash flows of the option from the time period $(t, t+1)$. The superscript '*' is used to distinguish whether the portfolio is super-replicating or not. Both the hedging portfolio and super-hedging portfolio are constructed by stocks and MMAs. The subscript '$-$' means the portfolio 'X' is used in the short position.

Theorem 2 *Under the assumption of non-zero funding spread, in the one-period binomial tree model, the no-arbitrage price of the European option at time t satisfies the following condition. More than that, any prices out of this interval can construct arbitrages.*

$$max\{-\Phi(X^*_{-t}), -\Phi(X_{-t})\} \leq E_t \leq min\{\Phi(X^*_t), \Phi(X_t)\}$$

Notes: If $max\{-\Phi(X^*_{-t}), -\Phi(X_{-t})\} = -\Phi(X^*_{-t})$*, then the interval is open on the left:* $E_t > -\Phi(X^*_{-t})$*. If* $min\{\Phi(X^*_t), \Phi(X_t)\} = \Phi(X^*_t)$*, then the interval is open on the right side:* $E_t < \Phi(X^*_t)$*.*

Adapted from: Bonner and Campanelli [3]

2.2 A comparison between super-hedging price and hedging price

Under the no-arbitrage conditions in Theorem 1, we get the price of the European option in Theorem 2. We noticed that both sides of the inequality are decided by comparing the super-hedging price and the hedging price. So in this part, we use the no-arbitrage conditions to compare them.

For each node at time t, we build a one-period binomial tree model with time length h. Then the maturity date $T = t + h$. Since the pricing process is backward in a one-period model, we know the option value at time $t + h$. We can build the hedging and super-hedging portfolio based on future cash flows.

For the replication portfolio, it involves Δ shares of stock and M shares of MMA. If $M \geqslant 0$, then the investor lends money to others, then r takes the value r_l. On the other hand, if $M < 0$, the investor borrows money, then r takes the value r_b. Thus we can write r as:

$$r = r_l \mathbb{1}_{M \geqslant 0} + r_b \mathbb{1}_{M<0}.$$

In the following theorems, the proof of those in the short position shares the same approach as the long position. So we only provide the proof for the latter. For simplification, Δ is short for Δ_t, and M is short for M_t.

Theorem 3 *In the one-period model with length h at time t, If $d < 1+r_l < 1+r_b < u$, $\Phi(X_t^*) \geq \Phi(X_t)$ and $\Phi(X_{-t}^*) \geq \Phi(X_{-t})$. This means the super-hedging price is larger than or equal to the hedging price at any time t.*

Proof: Firstly, we will compute the hedging price and the super-hedging price separately. Without loss of generality, we assume the time length of the one-period model is $h = 1$.

The functions used to compute hedging price are as follows:

$$\begin{cases} \Phi(X_t) = \Delta S_t + M, \\ V_{t+1}(H) = \Delta u S_t + M(1+r), \\ V_{t+1}(T) = \Delta d S_t + M(1+r). \end{cases}$$

The functions used to compute super-hedging price are as follows:

$$\begin{cases} \Phi(X_t^*) = \Delta^* S_t + M^*, \\ V_{t+1}(H) \leq \Delta^* u S_t + M^*(1+r^*), \\ V_{t+1}(T) \leq \Delta^* d S_t + M^*(1+r^*). \end{cases}$$

Four sub-situations can be derived based on whether the hedger is borrowing or lending money in this two portfolios.

1. $M \geqslant 0$ and $M^* \geqslant 0$
   In this case, hedger will lend money in the hedging portfolio and super-hedging portfolio. Then r and r^* take the lending rate, $r = r^* = r_l$. Thus, we rewrite the previous functions.

   Hedging functions:

$$\begin{cases} \Phi(X_t) = \Delta S_t + M \\ V_{t+1}(H) = \Delta u S_t + M(1+r_l) \\ V_{t+1}(T) = \Delta d S_t + M(1+r_l) \end{cases}$$

Super-hedging functions:

$$\begin{cases} \Phi(X_t^*) = \Delta^* S_t + M^* \\ V_{t+1}(H) \leq \Delta^* u S_t + M^*(1+r_l) \\ V_{t+1}(T) \leq \Delta^* d S_t + M^*(1+r_l) \end{cases}$$

In the hedging functions, we can plug $\Phi(X_t)$ into the right hand side of $V_{t+1}(H)$ and $V_{t+1}(T)$.

$$\begin{cases} V_{t+1}(H) = u\Phi(X_t) + (1+r_l-u)M \\ V_{t+1}(T) = d\Phi(X_t) + (1+r_l-d)M \end{cases}$$

In the same way, we rewrite the super-hedging functions. By comparing those functions, we derive two inequalities as follows:

$$\begin{cases} u(\Phi(X_t) - \Phi(X_t^*)) \leq (1+r_l-u)(M^*-M), \\ d(\Phi(X_t) - \Phi(X_t^*)) \leq (1+r_l-d)(M^*-M). \end{cases} \tag{2.1}$$

We first suppose $M^* \geqslant M$ in the above inequalities. For the first inequality, since $u > 1+r_l$, we have $(1+r_l-u)(M^*-M) \leq 0$, thus $\Phi(X_t) \leq \Phi(X_t^*)$. On the other direction, we assume $M^* < M$. For the second inequality, since $d < 1+r_l$, we have $(1+r_l-u)(M^*-M) \leq 0$, thus $\Phi(X_t) < \Phi(X_t^*)$. In either cases, we can get the conclusion that the hedging price is less than or equal to the super-hedging price.

2. $M^* \geqslant 0$ and $M < 0$
In this case, the hedging portfolio has a negative position in MMAs, and the super-hedging portfolio has a non-negative position in MMAs. So, we have $r = r_b$, and $r^* = r_l$. We rewrite the functions as follows.

Hedging functions:

$$\begin{cases} \Phi(X_t) = \Delta S_t + M \\ V_{t+1}(H) = \Delta u S_t + M(1+r_b) \\ V_{t+1}(T) = \Delta d S_t + M(1+r_b) \end{cases}$$

Super-hedging functions:

$$\begin{cases} \Phi(X_t^*) = \Delta^* S_t + M^* \\ V_{t+1}(H) < \Delta^* u S_t + M^*(1+r_l) \\ V_{t+1}(T) < \Delta^* d S_t + M^*(1+r_l) \end{cases}$$

In the hedging functions, we plug $\Phi(X_t)$ into the right hand side of $V_{t+1}(H)$ and $V_{t+1}(T)$.

$$\begin{cases} V_{t+1}(H) = u\Phi(X_t) + (1+r_b-u)M \\ V_{t+1}(T) = d\Phi(X_t) + (1+r_b-d)M \end{cases}$$

In the same way, we rewrite the super-hedging functions. Next, we derive the formula as follows:

$$\begin{cases} u(\Phi(X_t)-\Phi(X_t^*)) < (1+r_l-u)M^* - (1+r_b-u)M, \\ d(\Phi(X_t)-\Phi(X_t^*)) < (1+r_l-d)M^* - (1+r_b-d)M. \end{cases} \tag{2.2}$$

Since $r_l < r_b$, the inequalities (2.2) can be transformed to (2.1). Since $M^* \geqslant 0 > M$, we have $\Phi(X_t) < \Phi(X_t^*)$, i.e., the hedging price is less than the super-hedging price.

3. $M \geqslant 0$ and $M^* < 0$
   Following the process in the case $M^* \geqslant 0$ and $M < 0$, we can draw the same conclusion.

4. $M < 0$ and $M^* < 0$
   Following the process in the case $M \geqslant 0$ and $M^* \geqslant 0$, we can draw the same conclusion.

   In conclusion, under the assumption of $d < 1+r_l < 1+r_b < u$, the super-hedging price is larger than the hedging price in a long position. This is also true for a short position.

■

Definition 2.2.1 *We say a portfolio is optimal if we cannot rebalance it with a better payoff at any time after that. A portfolio is not optimal if we can rebalance it with a better payoff at any time after that.*

A nonoptimal portfolio cannot be used as a super-hedging portfolio. A super-hedging portfolio is a portfolio which can produce at least the same payoff as the derivative does with the lowest cost. Since a nonoptimal portfolio can be rebalanced with a better payoff, we can build another portfolio which has the same payoff but less expensive. This can be a fraction of the rebalanced portfolio. Then a nonoptimal portfolio is not a super-hedging portfolio. On the other hand, either in a long position or a short position, if a hedging portfolio is nonoptimal, it is more expensive than the super-hedging portfolio.

Theorem 4 *If $u < 1+r_b$, portfolios consisted of a negative position in MMA are not optimal.*

Proof: Suppose there are two portfolios. For portfolio A, it has Δ shares of stock and M shares of MMA, where $M < 0$. For portfolio B, it only has $\Delta + \frac{M}{S_t}$ shares of stock. Thus the initial values of A and B are the same.

For portfolio A, we have $M < 0$, so the funding rate takes the borrowing rate. We can draw a table to show the payoffs of the portfolios in different situations.

	A	B
up	$\Delta uS_t + M(1+r_b)$	$\Delta uS_t + Mu$
down	$\Delta dS_t + M(1+r_b)$	$\Delta dS_t + Md$

In either case, as $d < u < 1 + r_b$, the payoff of portfolio B is better than portfolio A. Thus when $u < 1 + r_b$, portfolios consisted of a negative position in MMA are not optimal.

■

Under the condition of $u < 1 + r_b$, it is not optimal to borrow money. When borrowing money, it means to use the money borrowed to invest in the stock. But the maximum return of the stock is less than the borrowing rate, which means that it will make a loss when borrowing money. So it is better to invest less in the stock market without borrowing money. The same conclusion goes for the case when $1 + r_l < d$. In this situation, lending money is not optimal. Since the lending rate will be less than the minimum return of the stock. So using this money to invest in the stock market is a better choice. We will prove it in a mathematical way.

Theorem 5 *If $1 + r_l < d$, portfolios consisted of a positive position in MMA are not optimal.*

Proof: Suppose there are two portfolios. For portfolio A, it has Δ shares of stock and M shares of MMA, where $M > 0$. For portfolio B, it only has $\Delta + \frac{M}{S_t}$ shares of stock. Thus the initial values of A and B are the same.

For portfolio A, since $M > 0$, it means $r = r_l$. We can draw a table to show the payoff of the portfolio at different situations.

	A	B
up	$\Delta uS_t + M(1+r_l)$	$\Delta uS_t + Mu$
down	$\Delta dS_t + M(1+r_l)$	$\Delta dS_t + Md$

In either case, as $1 + r_l < d < u$, the payoff of the portfolio B is better than the portfolio A. Thus when $1 + r_l < d$, portfolios consisted of a positive position in MMA are not optimal. ■

Combining the conditions of $1 + r_l < d$, $u < 1 + r_b$, and $d < u$, we can get $1 + r_l < d < u < 1 + r_b$. At this situation, either borrowing and lending money is not optimal. So, we can draw the conclusion of the following theorem.

Theorem 6 *If $1 + r_l < d < u < 1 + r_b$, portfolios consisted of MMA are not optimal.*

When holding MMA is not optimal, and a nonoptimal portfolio is not a super-hedging portfolio, we conclude that the super-hedging portfolio will only consist of stocks. Based on that, we can make a comparison between the hedging price and the super-hedging price when $1+r_l<d<u<1+r_b$.

Theorem 7 *If $1+r_l<d<u<1+r_b$, $\Phi(X_t^*) \leq \Phi(X_t)$ and $\Phi(X_{-t}^*) \leq \Phi(X_{-t})$, which means the super-hedging price is less than or equal to the hedging price at any time.*

Proof: In the condition of $1+r_l<d<u<1+r_b$, super-hedging portfolio will not consist of MMAs. Thus $M^*=0$ in the super-hedging portfolio.

For the hedging portfolio of a long position, we can generate the following functions.

$$\begin{cases} \Phi(X_t) = \Delta S_t + M \\ V_{t+1}(H) = \Delta u S_t + M(1+r) = u\Phi(X_t) + (1+r-u)M \\ V_{t+1}(T) = \Delta d S_t + M(1+r) = d\Phi(X_t) + (1+r-d)M \end{cases}$$

By solving those functions, the hedging price is computed as follows:

$$\begin{cases} \Phi(X_t) = \Delta S_t + M, \\ \Delta = \frac{V_{t+1}(H)-V_{t+1}(T)}{S_t(u-d)}, \\ M = \frac{uV_{t+1}(T)-dV_{t+1}(H)}{(u-d)(1+r)}. \end{cases}$$

For the super-hedging portfolio of the long position, we can draw the following functions.

$$\begin{cases} \Phi(X_t^*) = \Delta^* S_t \\ V_{t+1}(H) \leq \Delta^* u S_t \\ V_{t+1}(T) \leq \Delta^* d S_t \end{cases}$$

According to the definition of super-hedging portfolio, $\Phi(X_t^*) = \max\{\frac{V_{t+1}(H)}{u}, \frac{V_{t+1}(T)}{d}\}$.

Next, we make the comparison between the hedging price and the super-hedging price.

1. $uV_{t+1}(T) - dV_{t+1}(H) > 0$
   In this case, the hedging portfolio takes a positive position in MMA, thus r takes the lending rate. For the super-hedging portfolio, since $\frac{V_{t+1}(H)}{u} < \frac{V_{t+1}(T)}{d}$, we have $\Phi(X_t^*) = \frac{V_{t+1}(T)}{d}$. The value of the hedging portfolio is as follows:

$$\Phi(X_t) = \frac{V_{t+1}(H)-V_{t+1}(T)}{u-d} + \frac{uV_{t+1}(T)-dV_{t+1}(H)}{(u-d)(1+r_l)},$$

$$\Phi(X_t) - \Phi(X_t^*) = \frac{(1+r_l-d)[dV_{t+1}(H) - uV_{t+1}(T)]}{d(1+r_l)(u-d)} > 0.$$

Then, we have $\Phi(X_t) > \Phi(X_t^*)$, i.e, the super-hedging price is less than the hedging price.

2. $uV_{t+1}(T) - dV_{t+1}(H) < 0$
In this case, the hedging portfolio takes a negative position in MMAs, thus r takes the borrowing rate. For the super-hedging portfolio, since $\frac{V_{t+1}(H)}{u} > \frac{V_{t+1}(T)}{d}$, we have $\Phi(X_t^*) = \frac{V_{t+1}(H)}{u}$. The value of the hedging portfolio is as follows:

$$\Phi(X_t) = \frac{V_{t+1}(H) - V_{t+1}(T)}{u-d} + \frac{uV_{t+1}(T) - dV_{t+1}(H)}{(u-d)(1+r_b)},$$

$$\Phi(X_t) - \Phi(X_t^*) = \frac{(1+r_b-u)[dV_{t+1}(H) - uV_{t+1}(T)]}{u(1+r_b)(u-d)} > 0.$$

Then, we have $\Phi(X_t) > \Phi(X_t^*)$, i.e, the super-hedging price is less than the hedging price.

3. $uV_{t+1}(T) - dV_{t+1}(H) = 0$
In this case, the hedging portfolio has $M = 0$. Both of the super-hedging portfolio and the hedging portfolio do not have MMA. They share the same value.

Thus, taking a long position in the derivative, the super-hedging price is less than or equal to the hedging price. This is also true for the short position. ■

2.3 XVA of European options with funding spread in a one-period model

As is shown in Theorem 2, under the no-arbitrage condition, we have found that the XVA of a European option with funding spread is:

$$\max\{-\Phi(X_{-t}^*), -\Phi(X_{-t})\} \leq E_t \leq \min\{\Phi(X_t^*), \Phi(X_t)\}.$$

While, on the left hand side, when it takes the value of the super-hedging price, it is open on the left hand side. When it takes the value of the super-hedging price in the right hand side, it is open on the right hand side.

The no-arbitrage condition in the market with the stock and the MMA is $d < 1+r_b$, and $1+r_l < u$. It can generate four sub-situations, which is $d < 1+r_l < 1+r_b < u$, $1+r_l < d < u < 1+r_b$, $d < 1+r_l < u < 1+r_b$, and $1+r_l < d < 1+r_b < u$. Under each of these situations, the option price can be simplified as follows:

1. $d < 1+r_l < 1+r_b < u$
Based on Theorem 3, the hedging price is less than the super-hedging price for any position in the European option. Thus, the no-arbitrage price of the European option now follows:

$$-\Phi(X_{-t}) \leq E_t \leq \Phi(X_t).$$

2. $1+r_l < d < u < 1+r_b$
   Based on Theorem 6, the super-hedging price is less than the hedging price for any position in the European option. Thus, the no-arbitrage price of the European option now follows:
   $$-\Phi(X_{-t}^*) < E_t < \Phi(X_t^*).$$

3. $d < 1+r_l < u < 1+r_b$
   In this situation, we have concluded that borrowing money is not optimal. So we will check whether the hedging portfolio in the long or short positions take a negative position in MMA or not.
   $$M_t = \frac{uV_{t+1}(T) - dV_{t+1}(H)}{(u-d)(1+r_v)}$$
   $$M_{-t} = \frac{-uV_{t+1}(T) + dV_{t+1}(H)}{(u-d)(1+r_{-v})}$$
   This means that whether the hedging portfolios are optimal or not depends on the value of $uV_{t+1}(T) - dV_{t+1}(H)$.

   (a) $uV_{t+1}(T) - dV_{t+1}(H) > 0$
   In this case, we have $M_t > 0$, and $M_{-t} < 0$. Thus, the hedging portfolio in the short position is not optimal. The hedging price is less than the super-hedging price in a short position. Taking a long position in the option, the super-hedging portfolio has $M_t^* > 0$. We can prove that the super-hedging price is larger than the hedging price. The reason is the same as the proof of the first part in Theorem 3. In this condition, the no-arbitrage price of the option follows:
   $$-\Phi(X_{-t}^*) < E_t \leq \Phi(X_t).$$

   (b) $uV_{t+1}(T) - dV_{t+1}(H) < 0$
   Following the same process before, the no-arbitrage price of the option follows:
   $$-\Phi(X_{-t}) \leq E_t < \Phi(X_t^*).$$

   (c) $uV_{t+1}(T) - dV_{t+1}(H) = 0$
   In this case, both of the hedging portfolios for the long position and the short position do not have MMA. But for the super-hedging portfolio, it may have MMA. We need to compare the super-hedging price and the hedging price. Following the same process in the first part of Theorem 3, the hedging portfolio is less expensive than the super-hedging portfolio. Thus we have:
   $$-\Phi(X_{-t}) \leq E_t \leq \Phi(X_t).$$

4. $1+r_l < d < 1+r_b < u$
   Following the same process as above, we get the no-arbitrage price interval of the option based on the value of $uV_{t+1}(T) - dV_{t+1}(H)$.

(a) $uV_{t+1}(T) - dV_{t+1}(H) > 0$

$$-\Phi(X_{-t}) \leq E_t < \Phi(X_t^*)$$

(b) $uV_{t+1}(T) - dV_{t+1}(H) < 0$

$$-\Phi(X_{-t}^*) < E_t \leq \Phi(X_t)$$

(c) $uV_{t+1}(T) - dV_{t+1}(H) = 0$

$$-\Phi(X_{-t}) \leq E_t \leq \Phi(X_t)$$

2.4 XVA of European options with collateral in a one-period model

We notice that an option is a contract between two counterparties. In some cases, the option will require the seller to pay a large amount of money when the buyer chooses to exercise the option. The default will occur in situations like this. To prevent great loss from default occurring, the option buyer will require the seller to post a cash collateral. If the collateral provider defaults on the option, the taker will keep the collateral to mitigate the loss from the default. If no default occurs, the taker needs to return the collateral to the provider with an extra interest.

We hedge the European option with three accounts: stock, MMA, and collateral. r_l and r_b are the returns of the lending and borrowing accounts. r_c is the return of the collateral account. We have $0 < d < 1 + r_l < 1 + r_b < u$, where u and d are the up-factor and down-factor of the stock as we have defined before. According to Theorem 3, the hedging price is smaller than the super-hedging price, and the no-arbitrage condition holds.

We build a model to price European options in a one-period binomial tree model with time length h. The initial and maturity dates are 0 and h respectively. For the hedging portfolio in the long position, $V_h(H)$ is the payoff of the derivative when the stock price goes up, and $V_h(T)$ is the payoff of the derivative on the other side. γ is the collateral rate, where $\gamma \in [0,1]$. In the short position, the hedging portfolio takes negative values of the payoffs above in either case.

In this model, we are pricing the European option under the assumption that the collateral is decided by the option value of the hedger, no matter if the hedger is the collateral taker or provider. Therefore, the no-arbitrage price of a European option is unique for each investor in this model.

2.4.1 Long position

In the long position, the buyer owns the option and receives the collateral from the counterparty. The collateral is related to the option value at the initial time with a collateral rate r_c. To hedge the payoff of the option, we construct the portfolio by holding Δ shares of stock and M shares of MMA. The option value at a long position is defined as E_0.

At the initial time, this hedging portfolio shares the same value as the combination of option and collateral.

$$E_0 + \gamma E_0 = \Delta_0 S_0 + M_0 \tag{2.3}$$

At the maturity, the buyer needs to pay the collateral, γE_0, back with an extra interest, $r_c \gamma E_0$.

$$V_h(H) + \gamma E_0(1 + r_c) = \Delta_0 u S_0 + M_0(1 + r_0) \tag{2.4}$$

$$V_h(T) + \gamma E_0(1 + r_c) = \Delta_0 d S_0 + M_0(1 + r_0) \tag{2.5}$$

r_0 is the funding rate which takes the borrowing or lending rate given by the position in MMA in the long position hedging portfolio.

$$r_0 = r_l \mathbb{1}_{M_0 \geqslant 0} + r_b \mathbb{1}_{M_0 < 0} \tag{2.6}$$

From equation (2.4) and (2.5), we can get the shares of the stock.

$$\Delta_0 = \frac{V_h(H) - V_h(T)}{(u - d)S_0}$$

By solving the rest of the functions, we can get the equations for E_0 and M_0.

$$\begin{cases} E_0 = \frac{V_h(H)(1+\gamma)(1+r_0-d) - V_h(T)(1+\gamma)(1+r_o-u)}{(u-d)[(1+r_0)(1+\gamma)+\gamma(1+r_c)]} \\ M_0 = \frac{-V_h(H)[\gamma(1+r_c)+d(1+\gamma)]+V_h(T)[\gamma(1+r_c)+u(1+\gamma)]}{(u-d)[(1+r_0)(1+\gamma)+\gamma(1+r_c)]} \end{cases}$$

Our target is to solve the value of E_0. The unknown parameter in this equation is r_0 where the r_0 is given by a function of M_0. So our priority is to determine whether M_0 is positive or not. Taking a closer look at the value of M_0 with $u > d$, we can find that the denominator part is positive and the numerator part is only related to the payoff of the derivative. By the value of payoffs, the investor can determine if the money is borrowed or lent in the hedging portfolio. Then the value of r_0 is calculated.

Once r_0 is determined, we can plug it into the equation of E_0. Thus, we have the value of the option in a long position. Using the same approach, we can get the value of the option in a short position, which is denoted as E_{-0}.

2.4.2 Short position

Taking a short position in the option, the hedger needs to post the cash collateral to the buyer. Therefore, at the initial time, the hedger needs to pay the amount $-\gamma E_0$ of cash as a collateral. To replicate a combination of collateral and option, we construct the hedging portfolio by holding Δ_{-0} shares of stock and M_{-0} shares of MMA.

At the initial time, this hedging portfolio shares the same value as the combination of the option and collateral.

$$E_{-0} + \gamma E_{-0} = \Delta_{-0} S_0 + M_{-0} \tag{2.7}$$

At the maturity, the buyer will receive the collateral, $-\gamma E_{-0}$, back with an extra interest, $-\gamma E_{-0} r_c$. Thus, in a short position, we can generate the following functions:

$$-V_h(H) + \gamma E_{-0}(1 + r_c) = \Delta_{-0} u S_0 + M_{-0}(1 + r_{-0}), \tag{2.8}$$

$$-V_h(T) + \gamma E_{-0}(1 + r_c) = \Delta_{-v} d S_0 + M_{-0}(1 + r_{-0}). \tag{2.9}$$

r_{-0} is the funding rate which takes the borrowing or lending rate given by the position in MMA in the short position hedging portfolio.

$$r_{-0} = r_l \mathbb{1}_{M_{-0} \geqslant 0} + r_b \mathbb{1}_{M_{-0} < 0} \tag{2.10}$$

From Equation (2.8) and (2.9), we can get the shares of the stock.

$$\Delta_{-0} = -\frac{V_h(H) - V_h(T)}{(u - d) S_0}$$

By solving the rest of the equations, we can get the value E_{-0} and M_{-0}.

$$\begin{cases} E_{-0} = \frac{V_h(H)(1-\gamma)(d-1-r_{-0}) - V_h(T)(1-\gamma)(u-1-r_{-0})}{(u-d)[(1+r_{-0})(1-\gamma)-\gamma(1+r_c)]} \\ M_{-0} = \frac{V_h(H)[-\gamma(1+r_c)+d(1-\gamma)] - V_h(T)[-\gamma(1+r_c)+u(1-\gamma)]}{(u-d)[(1+r_{-0})(1-\gamma)-\gamma(1+r_c)]} \end{cases}$$

As the long position, the only unknown parameter for E_{-0} and M_{-0} is the value of r_{-0}. But unlike the calculation in the long position, we need to compute the numerator part first. After that, we assume the position in MMA is negative, then we replace r_{-0} by the borrowing rate. With a given r_{-0}, we can compute the value of M_{-0}, and verify our hypothesis. This approach can help us to find the value of r_{-0}. Based on this value, we have the value of the option in a short position, E_{-0}.

The buyer takes the price of E_0, and the sellers takes the price of E_{-0}. Then the no-arbitrage price interval of the European option is $[-E_{-0}, E_0]$.

This is the no-arbitrage interval of the European option at the initial time. In the multi-period binomial tree model at time t, using the backward induction approach, replacing the payoff as the value of the option at time $t + 1$, we can compute the no-arbitrage price of the option at any time between the initial time and maturity.

Chapter 3

XVA of American Options with Funding Spread

In this chapter, we will begin the analysis of American option pricing. We will introduce a base model without funding spread and default in Section 3.1. After that, a one-period model with funding spread will be constructed in Section 3.2. Lastly. we will extend this one-period model to a multi-period model in Section 3.4.

3.1 Base model of American option pricing

Comparing with European options, the only difference for American options is that the holder can choose any time prior to maturity to exercise the option. Because of that, an American option can never be worth less than the payoff associated with immediate exercise [12]. In that way, at each node, the option buyer will make an optimal choice between exercising or holding the option to maximize the payoff.

We apply the same approach as pricing the European option in the binomial tree model. This is by working backward from the maturity to the initial date. At each node, if the investor exercises the option, then he will receive the payoff of the early exercise. On the other hand, if he chooses to hold the option, the value is given by the European option now.

In the binomial tree model, if there is no existence of funding spread, default and collateral, we define this model as the base model. The following is an example of the application of the base model.

Example 3.1.1 *In the two-period binomial tree model with a risk-free interest rate of $\frac{1}{4}$. The stock price at the initial time is 4. The up-factor, u, is 2 and the down-factor, d, is $\frac{1}{2}$. p is the possibility that the stock price goes up. We use the base model to price the American put option at initial time with maturity date $T = 2$, strike price $K = 5$.*

	Initial	H	T	HH	HT/TH	TT
Stock Price	4	8	2	16	4	1
Exercise Payoff	1	0	3	0	1	4

Table 3.1: Payoff table of the option

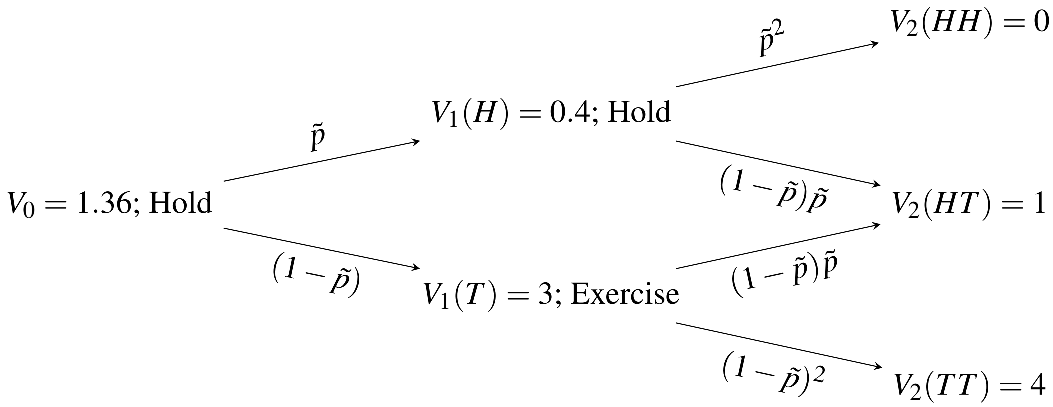

Figure 3.1: The American option pricing process in the base model.

The first step is to get the risk-neutral measure of p.

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1+1/4-1/2}{2-1/2} = \frac{1}{2}$$

In Table 3.1.1, the second row shows the stock price at each node, and the third row shows the payoff of the option if the buyer chooses to exercise the option at this node.

Next step is to find the investor's choice at each node.

At the maturity date, the investor makes a choice to exercise the option or not. Then the option value is $max\{5-S_T\}$.

At time $t = 1$*, the investor need to choose to exercise or hold the option. In this situation, the stock price can go up or down. Suppose the stock price now is* uS_0*, when the investor chooses to hold the option, the value of the option,* 0.4*, is the expected payoff at maturity:*

$$\frac{\tilde{p}V(HH)+(1-\tilde{p})V(HT)}{1+r} = \frac{1/2*1+1/2*0}{1+1/4} = 0.4.$$

The value of the option at any time is the maximum payoff. Since holding the option has a better payoff exercising early, the value of the option is 0.4.

If the stock price goes down at $t = 1$*, we can also compute the holding payoff of the option, which is 2. But the early exercise will get the payoff of 3. This means that the buyer will exercise* *the option when the stock price goes down at $t = 1$. Thus, the value of the option at this situation is 3.*

At time $t = 0$. When the investor holds the option, the option value now is the current value of the expected payoff at time $t = 1$. Then the holding value is $\frac{1/2*0.4+1/2*3}{1+1/4} = 1.36$. While if the investor chooses to exercise the option, the payoff is 1, which is less than the holding value. Thus, at the initial time, the option value is 1.36.

The pricing process is shown in the Figure 3.1. The optimal time to exercise the option before maturity is when the stock price goes down at time $t = 1$. Otherwise, the option buyer will hold the option until the maturity date.

3.2 One-period model with funding spread

Options in the market have two prices for a specific investor, the buyer's price and the seller's price, given by different positions in an option. We define the buyer's price as the maximum amount of money the buyer wants to pay for the derivative, and the seller's price as the minimum amount of money the seller wants to get for selling the derivative. Same for the European option, those two prices are not the same due to the existence of a funding spread. This is because we use the hedging or super-hedging portfolio to price the option. But the portfolios' position in the MMA is not the same on the two sides. To price the American option, we need to find those prices first.

At first, we will price the American option in a one-period binomial tree model. This model has only two time-points, the initial time and the maturity time. At the initial time, the option buyer needs to make a decision between holding or exercising the option. If the option is exercised, the buyer will receive the payoff given by the payoff function. While, if the buyer chooses to hold the option, the option will be turned to a European option.

We define an American option and a European option to have the same parameters if they share the same payoff function, initial date, maturity, and underlying asset. In a long position, if the buyer chooses to exercise the option, the payoff is $V_t(\cdot)$. If the buyer chooses to hold the option, the value of the option now is $\min\{\Phi(X_t), \Phi(X_t^*)\}$. This is equal to the long position value of the European option with the same parameters. The buyer will make the choice to maximize the value of the option. So the long position value of the option is:

$$\text{Buyer's Price} = \max\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X_t^*)\}\}.$$

In a short position, the initial value depends on the decision of the buyer at the initial time. When the buyer chooses to exercise the option, the payoff is $-V_t(\cdot)$. On the other hand, the value of the option if the buyer holds the option is $\min\{\Phi(X_{-t}), \Phi(X_{-t}^*)\}$. This is based on the short position value of the same parameters European option. The seller needs to consider the worst case of all situations, then the option value at a short position is:

$$\begin{aligned}\text{Seller's Price} &= -\min\Big\{-V_t(\cdot), \min\{\Phi(X_{-t}), \Phi(X^*_{-t})\}\Big\} \\ &= -\min\{-V_t(\cdot), \Phi(X_{-t}), \Phi(X^*_{-t})\} \\ &= \max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\}.\end{aligned}$$

Theorem 8 *Under the assumption of non-zero funding spread, in the one-period binomial tree model with a time length h, the no-arbitrage price of an American option at initial time t satisfies the following condition. Any prices out of that interval will result in arbitrage.*

$$max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\} \leq A_t \leq max\Big\{V_t(\cdot), min\{\Phi(X_t), \Phi(X^*_t)\}\Big\}$$

Notes: If $max\{V_t(\cdot), -\Phi(X^*_{-t}), -\Phi(X_{-t})\} = -\Phi(X^*_{-t})$, then the interval is open on the left side, which means $A_t > -\Phi(X^*_{-t})$. If $max\{V_t(\cdot), min\{\Phi(X_t), \Phi(X^*_t)\}\} = \Phi(X^*_t)$, then the interval is open on the right side, which means $A_t < \Phi(X^*_t)$.

Proof: Without loss of generality, we assume $h = 1$. The proof will be split into three parts. The first step is to show that the buyer's price is larger than the seller's price. The second step is to prove that any price out of that interval will result in an arbitrage opportunity. Finally, we prove that there is no arbitrage opportunity when the option price is in that interval.

Part I: We prove that the buyer's price is larger than the seller's price:

$$\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\} \leq \max\Big\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X^*_t)\}\Big\}.$$

As the analysis in Section 2.3, we discuss the above inequality under different market conditions.

(a) $d < 1 + r_l < 1 + r_b < u$
In this situation, according to Theorem 3, the hedging portfolio is cheaper than the super-hedging portfolio for either position in the American option. Thus, we need to prove that $\max\{V_t(\cdot), -\Phi(X_{-t})\} \leq \max\{V_t(\cdot), \Phi(X_t)\}$. At first, we will compare the value of $\Phi(X_t)$ and $-\Phi(X_{-t})$. In Theorem 7, we have computed the hedging price in the long position, which is expressed as follows:

$$\begin{cases}\Phi(X_t) = \frac{V(H)-V(T)}{(u-d)} + \frac{uV(T)-dV(H)}{(u-d)(1+r_t)}, \\ \Delta_t = \frac{V(H)-V(T)}{S_t(u-d)}, \\ M_{-t} = \frac{uV(T)-dV(H)}{(u-d)(1+r_t)}.\end{cases}$$

$V(H)$ and $V(T)$ are denoted as the payoff of the option at time $t+h$. Note that the time $t+h$ is removed from the subscript, and r_t is the funding rates given by the position in MMA in the long position hedging portfolio.

The hedging portfolio, X_{-t}, in the short position is constructed by Δ_{-t} shares of stock and M_{-t} shares of MMA. r_{-t} is the funding rate given by the value of M_{-t}. Likewise, the short position hedging price is calculated as follows:

$$\begin{cases} \Phi(X_{-t}) = \frac{-V(H)+V(T)}{(u-d)} + \frac{-uV(T)+dV(H)}{(u-d)(1+r_{-t})}, \\ \Delta_{-t} = \frac{-V(H)+V(T)}{S_t(u-d)}, \\ M_{-t} = \frac{-uV(T)+dV(H)}{(u-d)(1+r_{-t})}. \end{cases}$$

Analyzing the positions in stock and MMA in the hedging portfolios by the equations above, we draw the conclusion that $\Delta_t = -\Delta_{-t}$ and $M_t M_{-t} \leq 0$. A comparison between the hedging value in the long and short positions are constructed as below:

$$\Phi(X_t) + \Phi(X_{-t}) = \frac{-uV(T)+dV(H)}{(u-d)(1+r_{-t})} + \frac{uV(T)-dV(H)}{(u-d)(1+r_t)} = \frac{(uV(T)-dV(H))(r_{-t}-r_t)}{(u-d)(1+r_{-t})(1+r_t)}. \tag{3.1}$$

Whether the equation above is positive or not is given by the value of $uV(T) - dV(H)$. When $uV(T) - dV(H) > 0$, the shares of MMA in the hedging portfolios hold the conditions that $M_t > 0$ and $M_{-t} < 0$, which indicates $r_t = r_l$ and $r_{-t} = r_b$. Given that $r_b > r_l$, the equation above is non-negative. Likewise, if we assume $uV(T) - dV(H) \leq 0$, we also find that $\Phi(X_t) + \Phi(X_{-t}) \geq 0$.

The discussion above indicates that $\Phi(X_t) > -\Phi(X_{-t})$, which means that the hedging price of the option in a long position is always larger than a short position. If $\Phi(X_t) \geq V_t(\cdot)$, then the right side is $\Phi(X_t)$. Neither $-\Phi(X_{-t})$ nor $V_t(\cdot)$ is larger than $\Phi(X_t)$. This is also true when $\Phi(X_t) < V_t(\cdot)$. Thus, we have $\max\{V_t(\cdot), -\Phi(X_{-t})\} \leq \max\{V_t(\cdot), \Phi(X_t)\}$.

(b) $1 + r_l < d < u < 1 + r_b$

In this situation, according to Theorem 7, the super-hedging portfolio is less expensive than the hedging portfolio for either position in the American option. Thus, we need to prove that $\max\{V_t(\cdot), -\Phi(X^*_{-t})\} \leq \max\{V_t(\cdot), \Phi(X^*_t)\}$. We have computed the long position super-hedging price in Theorem 7, which is $\Phi(X^*_t) = \max\{\frac{V(H)}{u}, \frac{V(T)}{d}\}$.

In a short position, there is no MMA in the super-hedging portfolio in this condition. The short position super-hedging price is computed as follows:

$$\begin{cases} \Phi(X^*_{-t}) = \Delta^*_{-t} u S_t, \\ \Delta^*_{-t} u S_0 \geq -V(H), \\ \Delta^*_{-t} d S_0 \geq -V(T). \end{cases}$$

Given by the definition of super-hedging price, $-\Phi(X_{-t}^*) = \min\{\frac{V(H)}{u}, \frac{V(T)}{d}\}$. Therefore, we need to prove the following inequality:

$$\max\Big\{V_t(\cdot), \min\{\frac{V(H)}{u}, \frac{V(T)}{d}\}\Big\} \leq \max\Big\{V_t(\cdot), \max\{\frac{V(H)}{u}, \frac{V(T)}{d}\}\Big\}.$$

This is trivial, because $\max\{\frac{V(H)}{u}, \frac{V(T)}{d}\} \geq \min\{\frac{V(H)}{u}, \frac{V(T)}{d}\}$.

(c) $d < 1 + r_l < u < 1 + r_b$

(i) $uV_h(T) - dV_h(H) > 0$
In Section 2.3, we have computed the no-arbitrage price interval of a European option in this condition. The buyer takes the hedging price of the option in a long position, while the seller takes the super-hedging price in a short position. Therefore, the inequality we need to prove in Theorem 8 can be transformed to:

$$\max\{V_t(\cdot), -\Phi(X_{-t}^*)\} \leq \max\{V_t(\cdot), \Phi(X_t)\}.$$

This can be further simplified to proving $\Phi(X_t) > -\Phi(X_{-t}^*)$. The reason is that if the right hand side takes the value of $V_t(\cdot)$, which leads to $V_t(\cdot) \geq \Phi(X_t) > -\Phi(X_{-t}^*)$, the inequality becomes true. Likewise, if the right hand side takes the value of $\Phi(X_t)$, which leads to $\Phi(X_t) \geq V_t(\cdot)$ and $\Phi(X_t) > -\Phi(X_{-t}^*)$, the inequality also becomes true.

The above proof: $\Phi(X_t) > -\Phi(X_{-t}^*)$, is demonstrated as follows.

Given by the condition that $d < 1 + r_l < u < 1 + r_b$, the long position hedging portfolio and the short position super-hedging portfolio have non-negative shares of MMA. Therefore, $r_t = r_{-t} = r_l$.

The hedging price in the long position can be computed as follows:

$$\begin{cases} \Phi(X_t) = \Delta_t S_t + M_t, \\ V(H) = \Delta_t u S_t + M_t(1 + r_l), \\ V(T) = \Delta_t d S_t + M_t 1 + r_l). \end{cases}$$

The super-hedging price in the short position can be computed as follows.

$$\begin{cases} \Phi(X_{-t}^*) = \Delta_{-t}^* S_t + M_{-t}^*, \\ -V(H) < \Delta_{-t}^* u S_t + M_{-t}^*(1 + r_l), \\ -V(T) < \Delta_{-t}^* d S_t + M_{-t}^*(1 + r_l). \end{cases}$$

Combining the equations and inequalities above, we can derive that:

$$\begin{cases} u(\Phi(X_t) + \Phi(X_{-t}^*) > -(1 + r_l - u)(M_t + M_{-t}^*), \\ d(\Phi(X_t) + \Phi(X_{-t}^*) > -(1 + r_l - d)(M_t + M_{-t}^*). \end{cases}$$

Given that both M_0 and M^*_{-0} are positive and $1+r_l < u$, we have $\Phi(X_t)+\Phi(X^*_{-t}) > 0$. Furthermore, $\Phi(X_t) > -\Phi(X^*_{-t})$.

(ii) $uV_h(T) - dV_h(H) < 0$ Same as (i).

(iii) $uV_h(T) - dV_h(H) = 0$ Same as (i).

(d) $1+r_l < d < 1+r_b < u$
The proof can follow the process of (c): $d < 1+r_l < u < 1+r_b$.

Part II: We need to prove that any option price out of that interval will result in an arbitrage opportunity.

(a) $A_t \leq max\{V_t(\cdot), min\{\Phi(X_t), \Phi(X_t^*)\}\}$

We prove it by contradiction. We assume that there is no arbitrage opportunity when the option price is larger than $\max\left\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X_t^*)\}\right\}$. We prove this under one of the sub-situations, $\Phi(X_t) < \Phi(X_t^*)$. Other situation, $\Phi(X_t) \geq \Phi(X_t^*)$, can be proved by the same approach. It indicates that $A_t > V_t(\cdot)$ and $A_t > \Phi(X_t)$.

We can construct a portfolio, Y_t, by shorting one share of the option, longing one share of the hedging portfolio, and investing M_t shares of MMA, where $M_t = A_t - \Phi(X_t) > 0$.

The portfolio value at initial time is:

$$\Phi(Y_t) = -A_t + \Phi(X_t) + M_t = -A_t + \Phi(X_t) + A_t - \Phi(X_t) = 0.$$

In the following equations, 'E' and 'W' refer to the buyer exercising the option and holding the option respectively at time t. If the buyer chooses to exercise the option immediately, the portfolio value is:

$$\Phi(Y_t^E) = A_t - V_t(\cdot) > 0.$$

If the buyer chooses to hold the option to the maturity date, the option payoff is as follows. The interest rate r can either take the lending or the borrowing rates.

$$\Phi(Y_{t+1}^W(H)) = -V(H) + V(H) + M_t(1+r) > 0$$

$$\Phi(Y_{t+1}^W(T)) = -V(T) + V(T) + M_t(1+r) > 0$$

Thus, if $A_t > \max\left\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X_t^*)\}\right\}$, it will result in an arbitrage opportunity. This is a contradiction. Therefore, we have $A_t \leq \max\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X_t^*)\}\}$.

(b) $A_t \geq max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\}$

Likewise, we prove the above inequality by contradiction. Here we assume that there is a no-arbitrage opportunity if the option price is lower than $\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\}$. We prove this under one of the cases, $A_t < V_t(\cdot) < \max\{-\Phi(X_{-t}), -\Phi(X^*_{-t})\}$. The other case, $A_t < \max\{-\Phi(X_{-t}), -\Phi(X^*_{-t})\} \leq V_t(\cdot)$, can be proved by the same approach.

We can construct a portfolio at time t, Y_t, by longing one share of the option, longing one share of X_{-t}. and investing M_t shares of the MMA, where $M_t = -A_t - \Phi(X_{-t})$.

The portfolio value at initial time is:

$$\Phi(Y_t) = A_t + \Phi(X_{-t}) - A_t - \Phi(X_{-t}) = 0.$$

If the buyer chooses to exercise the option now, the portfolio value is as follows:

$$\Phi(Y_t^E) = V_t(\cdot) + \Phi(X_{-t}) - A_t - \Phi(X_{-t}) = V_t(\cdot) - A_t > 0.$$

On the other hand, if the buyer chooses to hold the option to the maturity date, the portfolio value is as follows:

$$\Phi(Y_{t+1}^W(H)) = V(H) - V(H) + M_t(1+r) > 0,$$
$$\Phi(Y_{t+1}^W(T)) = V(T) - V(T) + M_t(1+r) > 0.$$

The portfolio, Y_t, as seen in the above examples will always result in an arbitrage opportunity, and this is a contradiction. Thus, we have $A_t \geq \max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\}$.

Part III: We need to show that there is no arbitrage strategy if A_t satisfies:

$$\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\} \leq A_t \leq \max\Big\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X^*_t)\}\Big\}.$$

Let's assume that this hypothesis is true, which means any arbitrage portfolio will result in an option price out of the above interval. If we can build an arbitrage portfolio with a positive position in a derivative, this portfolio is referred to as a buying arbitrage of the derivative. On the other hand, an arbitrage portfolio, which has a negative position in a derivative, is called the selling arbitrage. For a price of the derivative, if we cannot construct a buying or a selling arbitrage portfolio, then this is one of the no-arbitrage prices of the derivative.

We assume that there exists a buying arbitrage strategy, Y_t. The portfolio is constructed at time t by longing one share of the American option, holding $-\Delta_t$ shares of stocks and $-M_t$ shares of MMA, and setting the portfolio's initial value to 0.

$$\Phi(Y_t) = A_t - \Delta_t S_t - M_t = 0$$

The portfolio's value at the maturity date depends on the decision of the buyer.

If the buyer chooses to hold the option, the investor will still keep the portfolio. But if the option is exercised at time t, then the buyer will rebalance the portfolio with $\tilde{\Delta}_t$ shares of stock and $\tilde{M}_t$ shares of MMA.

$$V_t(\cdot) - \Delta_t S_t - M_t = \tilde{\Delta}_t S_t + \tilde{M}_t$$

Thus, at maturity date, the portfolio's payoffs for the conditions {E,H} and {E,T} are:

$$\{\text{E, H}\}\colon \tilde{\Delta}_t u S_t + \tilde{M}_t(1 + r_1^*),$$
$$\{\text{E, T}\}\colon \tilde{\Delta}_t d S_t + \tilde{M}_t(1 + r_1^*),$$
$$r_1^* = r_l \mathbb{1}_{\tilde{M}_t \geq 0} + r_b \mathbb{1}_{\tilde{M}_t < 0}.$$

However, if there is no exercise occurs at time t, the portfolio's payoff for the conditions {W,H} and {W,T} are:

$$\{\text{W, H}\}\colon V_{t+1}(H) - \Delta_t u S_t - M_t(1 + r_2^*),$$
$$\{\text{W, T}\}\colon V_{t+1}(T) - \Delta_t d S_t - M_t(1 + r_2^*),$$
$$r_2^* = r_l \mathbb{1}_{M_t \geqslant 0} + r_b \mathbb{1}_{M_t < 0}.$$

According to the no-arbitrage condition, the payoff of the maturity date is at least zero. The cases {E,H} and {E,T} will result in the conclusion of $\tilde{\Delta}_t S_t + \tilde{M}_t \geq 0$. The reason is that we cannot construct an arbitrage portfolio with only the stock and MMA. So the rebalanced portfolio ($\tilde{\Delta}_t$, $\tilde{M}_t$)'s initial value is non-negative. Then we have $V_t(\cdot) - \Delta_t S_t - M_t \geq 0$, which concludes that the American option price, A_t, is as follows:

$$A_t = \Delta_t S_t + M_t \leq V_t(\cdot).$$

Considering the payoffs of the cases {W,H} and {W,T}, we have:

$$-\Delta_t u S_t - M_t(1 + r_2^*) \geq -V_{t+1}(H),$$

$$-\Delta_t d S_t - M_t(1 + r_2^*) \geq -V_{t+1}(T).$$

We can conclude from the above inequalities that $-\Delta_t S_t - M_t \geq \Phi(X_{-t}^*)$, which means $A_t \leq -\Phi(X_{-t}^*)$. Given that $A_t \leq -\Phi(X_{-t}^*)$ and $A_t \leq V_t(\cdot)$, if the portfolio Y_t is a buying arbitrage strategy, we can at least draw the conclusion that $A_t \leq \min\{-\Phi(X_{-t}^*), V_t(\cdot)\}$.

We have the fact that $\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X_{-t}^*)\} \leq A_t$. If the left side is equal to $-\Phi(X_{-t}^*)$, the equal symbol is invalid. Based on the fact that $\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X_{-t}^*)\}$ is larger than $\min\{-\Phi(X_{-t}^*), V_t(\cdot)\}$, a contradiction exists. Therefore, there is no buying arbitrage if the option price follows: $\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X_{-t}^*)\} \leq A_t \leq \max\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X_t^*)\}\}$.

We assume that there exists a selling arbitrage strategy which consists of Δ_t shares of stock, M_t shares of MMA, and negative one share of the American option.

As the seller of the American option, the portfolio's value at the maturity date is based on the price movements of the stock and the buyer's exercising strategy. p_1 is denoted as the probability that the stock price goes up, and p_2 is denoted as the probability that the buyer exercises the option.

$$(p_1, p_2) \in (0,1) \times [0,1]$$

Given that the payoff of the American option is not based on the market conditions only, we modify the definition of 'Arbitrage' at the basis what we provided in Section 1.1. We call a portfolio to be an arbitrage when its initial value is zero, and it has a positive possibility of making profits without the risk of losing money whatever the counterparties' exercising strategies are. The mathematical expression of the arbitrage is given below. P is the distribution of the portfolio's payoff at the maturity.

$$\Phi(Y_t) = 0$$

$$\inf_{(p_1,p_2)\in(0,1)\times[0,1]} P(\Phi(Y_T) \geq 0) = 1$$

$$\inf_{(p_1,p_2)\in(0,1)\times[0,1]} P(\Phi(Y_T) > 0) > 0$$

At the initial time, the portfolio value is 0.

$$\Phi(Y_t) = -A_t + \Delta_t S_t + M_t = 0$$

The payoff at different situations is based on the choice of the buyer and the stock price's movements. If the buyer chooses to exercise the option, the seller will rebalance the portfolio with $\tilde{\Delta}_t$ shares of stock and $\tilde{M}_t$ shares of MMA, which follows the condition:

$$-V_t(\cdot) + \Delta_t S_t + M_t = \tilde{\Delta}_t S_t + \tilde{M}_t.$$

However, if the buyer chooses to hold the option, the seller will keep the portfolio without any change. The payoffs of the portfolio with different market conditions and buyer's exercise strategies at the maturity date is:

$$\{\text{E, H}\}\text{: } \tilde{\Delta}_t u S_t + \tilde{M}_t(1 + r_1^*),$$

$$\{\text{E, T}\}\text{: } \tilde{\Delta}_t d S_t + \tilde{M}_t(1 + r_1^*),$$

$$r_1^* = r_l \mathbb{1}_{\tilde{M}_t \geq 0} + r_b \mathbb{1}_{\tilde{M}_t < 0};$$

$$\{\text{W, H}\}\text{: } -V_{t+1}(H) + \Delta_t u S_t + M_t(1 + r_2^*),$$

$$\{\text{W, T}\}\text{: } -V_{t+1}(T) + \Delta_t d S_t + M_t(1 + r_2^*),$$

$$r_2^* = r_l \mathbb{1}_{M_t \geqslant 0} + r_b \mathbb{1}_{M_t < 0}.$$

According to the no-arbitrage condition we mentioned above, there exists an arbitrage opportunity no matter what the buyer's strategy is. When $p_2 = 1$, it means that the buyer always exercises the option at the initial time. Without loss of generality, we assume:

$$\tilde{\Delta}_t u S_t + \tilde{M}_t(1 + r_1^*) \geq 0,$$

$$\tilde{\Delta}_t dS_t + \tilde{M}_t(1+r_1^*) > 0.$$

From the above inequality, we can conclude that $\tilde{\Delta}_t S_t + \tilde{M}_t > 0$. Therefore, the American option price, A_t, follows:

$$A_t = \Delta_t S_t + M_t > V_t(\cdot).$$

Likewise, when $p_1 = 0$, it means that the option buyer will always hold the option at the initial time. Without loss of generality, we assume:

$$-V_{t+1}(H) + \Delta_t u S_t + M_t(1+r_2^*) \geq 0,$$

$$-V_{t+1}(T) + \Delta_t d S_t + M_t(1+r_2^*) > 0.$$

From that, we can conclude: $\Delta_t S_t + M_t \geq \Phi(X_t^*)$. Thus, we have a comparison between A_t and $\Phi(X_t^*)$.

$$A_t = \Delta_t S_t + M_t \geq \Phi(X_t^*)$$

Combining the cases of $p_2 = 0$ and $p_2 = 1$, we can find that $A_t > \max\{V_t(\cdot), \Phi(X_t^*)\}$.

When $0 < p_2 < 1$, if $A_t > \max\{V_t(\cdot), \Phi(X_t^*)\}$, we can guarantee that the payoffs of the four cases above are non-negative, and at least one of them is positive. Thus, an arbitrage exists.

Hence, for the portfolio, Y_t, if it is an arbitrage, we must have $A_t > \max\{V_t(\cdot), \Phi(X_t^*)\}$. This is a contradiction. So there is no selling arbitrage if $\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X_{-t}^*)\} \leq A_t \leq \max\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X_t^*)\}\}$. This is true as long as $p_2 \in [0,1]$.

Given that, when $\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X_{-t}^*)\} \leq A_t \leq \max\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X_t^*)\}\}$, there is no selling or buying arbitrage, i.e, there is no-arbitrage. Combining the results of the three steps above, we can draw the conclusion that the price interval is the no-arbitrage price of the American option.

■

For any particular case, when we need to compute the no-arbitrage price interval of an American option, we compare the hedging price, super-hedging price and the payoff of exercising in the long or short positions. As we have discussed in Section 2.3, instead of comparing the hedging price and the super-hedging price, we only need to check the value of some market conditions. For example, when $d < 1 + r_l < u < 1 + r_b$ and $uV(T) - dV(H) > 0$, we know that the hedging price is larger than the super-hedging price in the short position, and less than the super-hedging price in the long position. In this way, the calculation process will be much simplified.

The following is an example of computing the no-arbitrage price of the American option in a one-period binomial tree model.

Example 3.2.1 *In a one-period binomial tree model, let the lending rate r_l be -0.09, and the borrowing rate r_b be 0.2 [1]. The up-factor $u = 1.5$, and the down-factor $d = 0.8$. The initial stock price*

[1] Negative interest rate does exist in the real market [9].

is 8*. What is the no-arbitrage price of a American call option with strike price* $K = 7$ *and maturity date* $T = 1$*?*

The parameters in this problem follow the relationship that $d < 1 + r_l < 1 + r_b < u$*. In Theorem 3, we have proved that the hedging portfolio is cheaper than the super-hedging portfolio in either position. So we only need to compute the hedging price of the American option when the buyer chooses to hold the option.*

At the maturity date, the payoff value of the option will be $V(H) = 5$*,* $V(T) = 0$*.*

At the initial time, when the buyer chooses to exercise the option, the payoff will be $V(0) = (8-7)^+ = 1$*. Nevertheless, if the buyer chooses to hold the option, the value of the option will be calculated by following the process in the proof of Theorem 3. The first step is to find the shares of MMA in the hedging portfolio.*

$$\Delta_0 = \frac{V(H)-V(T)}{(u-d)S_0} = \frac{5}{0.7*8} = 0.89$$

$$M_0 = \frac{uV(T)-dV(H)}{(u-d)(1+r_b)} = \frac{-4}{0.7*1.2} = 4.76$$

Therefore, if the buyer chooses to hold the option, the value will be $\Delta_0 S_o + M_0 = 2.3$*. This is larger than the exercise payoff, therefore, the buyer will choose to hold the option.*

Likewise, on the seller's side, if the buyer chooses to exercise the option, the payoff is $-V(0) = -1$*. While, if the buyer chooses to hold the option, the value of the option for the seller can be computed as follows:*

$$\Delta_{-0} = \frac{-V(H)+V(T)}{(u-d)S_0} = -\frac{5}{0.7*8} = -0.89,$$

$$M_{-0} = \frac{-uV(T)+dV(H)}{(u-d)(1+r_l)} = \frac{4}{0.7*0.91} = 6.28.$$

Thus, the short position value will be $\Delta_{-0}S_0 + M_{-0} = -0.84$*. This is less than* -1*, which means the worst case for the seller is happening when the buyer exercises the option.*

In conclusion, $\Phi(X_0) = 2.3$*,* $\Phi(X_{-0}) = -0.84$*, and* $V_0(\cdot) = 1$*, so the no-arbitrage price interval of the American option is:*

$$1 \leq A_0 \leq 2.3.$$

As we have noticed in the example, at the initial time, the buyer side chooses to hold the option, and exercising the option will cause a worse payoff for the seller. The seller side needs to consider all possible exercising decisions from the buyer because each hedger in the market has unique borrowing and lending rates. For a given option, two hedgers in a short position may make an opposite decision between exercising or holding the option.

In the next section, we will analyze the situation when the investors in the market are rational and share the same borrowing and lending rates. Under this assumption, it has no influence on the buyer, since the buyer always chooses the decision which makes the best payoff. But for the seller, the value of the option depends on the buyer's exercising strategy. If the buyer will definitely hold the option, then there is no need for the seller to prepare the situation where the buyer exercises the option. The only unknown case is when the buyer has no difference between exercising or holding the option. This problem will be discussed in the next section.

3.3 American option with funding spread in rational case

The Example 3.2.1 have given rise to a problem about the buyer's exercising strategy and the seller's worst case. In this section, we will assume that investors in the market are rational and share the same borrowing and lending rates. Under this assumption, we analyze the value of the option in two positions: long position and short position. We will discuss the base model first. In the base model, there is no funding spread. In the next part, we will discuss the buyer's exercise choice and its impact on the seller if funding spread exists in the market. When the buyer has no difference between exercising or holding the option, do the two choices have any difference for the seller?

In the base model we introduced in Section 3.1, there is no funding spread in the market. Given that, the seller's price and the buyer's price is the same in the market. Therefore, when the buyer has no difference between exercising or holding the option, the buyer's decision does not affect the payoff for the seller.

Based on the analysis in Section 2.3 about the European option pricing with funding spread, we will discuss the influence of the buyer's choice to the seller in different situations. The model we built is a one-period binomial tree from time t to $t+h$. Without loss of generality, we assume that $h=1$. The conclusion is based on the following assumptions:

1. There is no arbitrage in the market with MMA and stock.
2. Funding spread exists.
3. Lending and borrowing rates in the market are the same for hedgers in the market.
4. Hedgers in the market are rational.

From the no-arbitrage condition, we have $d < 1+r_b$ and $1+r_l < u$. Similarly, we divide this into 4 different sub-situations, $d < 1+r_l < 1+r_b < u$, $1+r_l < d < u < 1+r_b$, $d < 1+r_l < u < 1+r_b$, and $1+r_l < d < 1+r_b < u$. Under the assumption 3 and 4, we only need to check the circumstances when the buyer has no difference between exercising and holding, since the seller knows the buyer's choice well in other situations.

(a) $d < 1 + r_l < 1 + r_b < u$

Based on the condition of $d < 1 + r_l < 1 + r_b < u$, the hedging price is less than the super-hedging price for either a long or a short position in the option. At the initial time, the buyer compares the payoff from the early exercise and the long position hedging price. The buyer's price takes the maximum of those two values.

$$\text{Buyer's price} = \max\{V_t(\cdot), \Phi(X_t)\}$$

On the seller side, he knows the buyer's choice well in all situations except that the buyer has no difference between exercising or holding the option.

When the buyer has no difference between exercising or holding the option, it indicates that $V_t(\cdot) = \Phi(X_t)$. Given by the analysis of Equation 3.1, we have $\Phi(X_t) \geq -\Phi(X_{-t})$. Therefore, $-V_t(\cdot) \leq \Phi(X_{-t})$. It means that exercising the option will cause a worse payoff for the option seller. We can compute the seller's price as follows:

$$\text{Seller's price} = V_t(\cdot)\mathbb{1}_{\{\Phi(X_t) \leq V_t(\cdot)\}} + \Phi(X_{-t})\mathbb{1}_{\{\Phi(X_t) > V_t(\cdot)\}}.$$

(b) $1 + r_l < d < u < 1 + r_b$

Under the condition of $1 + r_l < d < u < 1 + r_b$, hedging price is larger than the super-hedging price for either position in the option. When the buyer has no difference between exercising or holding the option, it indicates that:

$$V_t(\cdot) = \Phi(X_t^*).$$

On the seller side, if the option is exercised at the initial date, the payoff is $-V_t(\cdot)$. While the current value of the expected payoff by holding the option is $\Phi(X_{-t}^*)$. In Section 3.2, we have computed the super-hedging price in the long and short positions.

$$\Phi(X_t^*) = \max\{\frac{V(H)}{u}, \frac{V(T)}{d}\}$$

$$\Phi(X_{-t}^*) = -\min\{\frac{V(H)}{u}, \frac{V(T)}{d}\}$$

Given that $V_t(\cdot) = \Phi(X_t^*)$ and $\Phi(X_t^*) \geq -\Phi(X_{-t}^*)$, we have $-V_t(\cdot) \leq \Phi(X_{-t}^*)$. This means that when the buyer has no difference between exercising or holding the option, the worst case for the seller is when the buyer chooses to exercise the option. Thus the option price for the buyer and the seller is:

$$\begin{cases} \text{Buyer's price} = \max\{\Phi(X_t^*), V_t(\cdot)\}, \\ \text{Seller's price} = V_t(\cdot)\mathbb{1}_{\{\Phi(X_t^*) \leq V_t(\cdot)\}} + \Phi(X_{-t}^*)\mathbb{1}_{\{\Phi(X_t^*) > V_t(\cdot)\}}. \end{cases}$$

(c) $d < 1 + r_l < u < 1 + r_b$

Given by Theorem 4, in this condition, a portfolio with negative shares of MMA is not optimal. Whether the hedging and super-hedging portfolios of the option is borrowing or lending money depends on the value of $uV(T) - dV(H)$. Therefore, we analyze the payoff of the option for the buyer and seller based on different values of $uV(T) - dV(H)$.

(i) $uV(T) - dV(H) > 0$
In this situation, the hedging portfolio in the long position has positive shares of MMA. This indicates that the long position hedging price is the value of the option when the buyer chooses to hold the option. If holding the option makes no difference for the buyer, then we have the following conclusion:

$$V_t(\cdot) = \Phi(X_t).$$

On the seller side, the exercising payoff for the option is $-V_t(\cdot)$. However, when the buyer holds the option to the maturity, the American option value for the seller is the short position price of the same parameter European option. Therefore, the seller takes the value of $\Phi(X^*_{-t})$. The part (ii) in the proof of Theorem 3.2 have shown that $\Phi(X_t) > -\Phi(X^*_{-t})$. Therefore, $-V_t(\cdot) \leq \Phi(X^*_{-t})$. This means the worse case for the seller is when the buyer chooses to exercise the option.

(ii) $uV(T) - dV(H) < 0$
The process is the same as the case $uV(T) - dV(H) > 0$. Likewise, we can get the same conclusion that when the buyer has no difference between exercising or holding the option, the worst situation for the seller is when the buyer exercises the option.

(iii) $uV(T) - dV(H) = 0$
The process is the same as the case $uV(T) - dV(H) > 0$. Likewise, we can get the same conclusion that when the buyer has no difference between exercising or holding the option, the worst situation for the seller is when the buyer exercises the option.

Under the condition of $d < 1 + r_l < u < 1 + r_b$, no matter what the value of $uV(T) - dV(H)$ is, we can draw the conclusion that when the buyer has no difference between exercising or holding the option, the seller has a worse payoff when the buyer exercises the option.

(d) $1 + r_l < d < 1 + r_b < u$
Follow the same process as the situation $d < 1 + r_l < u < 1 + r_b$, we can get the same conclusion as before.

With taking all the sub-situations of the no-arbitrage condition into consideration, we draw the following conclusion.

Theorem 9 *Under the no arbitrage condition, with funding spread, the lending and borrowing rate in the market is the same for buyer and seller, both of them are rational, when the buyer has no difference between exercising and holding the option, the worst case for the seller is always when the buyer chooses to exercise the option.*

3.4 Multi-period model with funding spread

We know how to compute the no-arbitrage price interval of an American option under the one-period binomial tree model. While in the multi-period model, the pricing process becomes more complicated. The reason is that the value of the option at each time is an interval now. But we do not know how to construct the hedging or super-hedging portfolio based on interval values.

The multi-period binomial tree model is constructed by many one-period models. Taking a one-period model from t_n to t_{n+1} as an example, suppose the option value can be represented as $A_{t_n}(\omega_1\omega_2\cdots\omega_n)$, where ω_i is the price movement state of the stock at time t_i, and $\omega_i \in \{H,T\}$. We assume that $t_{n+1} \neq T$, where T is the maturity. In this way, the option value at t_{n+1} will be an interval, and the exercise value of the option is $V_{t_n}(\omega_1\omega_2\cdots\omega_n)$. Since most variables share the same element $\omega_1\omega_2\cdots\omega_n$, we will not write this afterwards for simplification. The process from t_n to t_{n+1} is given in Figure 3.2.

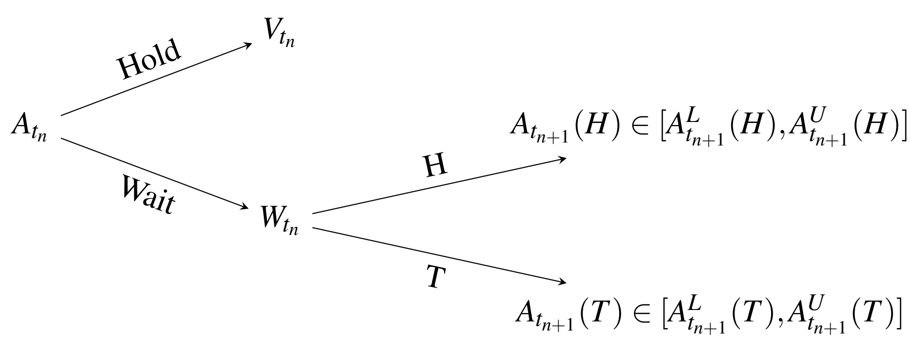

Figure 3.2: One-period American option pricing process with funding spread from t_n to t_{n+1} inside the multi-period model with funding spread.

If the price of the option at t_{n+1} is known, our target now is to compute the value of $A_{t_n}(\omega_1\omega_2\cdot\omega_n)$. We have the option's no-arbitrage price interval at t_{n+1} as follows:

$$A_{t_{n+1}}(H) \in [A^L_{t_{n+1}}(H), A^U_{t_{n+1}}(H)],$$

$$A_{t_{n+1}}(T) \in [A^L_{t_{n+1}}(T), A^U_{t_{n+1}}(T)].$$

We assume that $A_{t_n} \in [A^L_{t_n}, A^U_{t_n}]$. If we take any value $A^*_{t_{n+1}}(H) \in [A^L_{t_{n+1}}(H), A^U_{t_{n+1}}(H)]$ and $A^*_{t_{n+1}}(T) \in [A^L_{t_{n+1}}(T), A^U_{t_{n+1}}(T)]$ as the value of the option at time t_{n+1}, the option's no-arbitrage value at t_n can be calculated by Theorem 8. We denote the price interval as $[A^{L*}_{t_n}, A^{U*}_{t_n}]$.

We assume another option values $\tilde{A}^*_{t_{n+1}}(H)$ and $\tilde{A}^*_{t_{n+1}}(T)$ at time t_{n+1} as follows:

$$A^U_{t_{n+1}}(H) \geq \tilde{A}^*_{t_{n+1}}(H) \geq A^*_{t_{n+1}}(H),$$

$$A^U_{t_{n+1}}(T) \geq \tilde{A}^*_{t_{n+1}}(T) \geq A^*_{t_{n+1}}(T).$$

We can compute another no-arbitrage price interval of the option given by the payoff value $\tilde{A}^*_{t_{n+1}}(H)$ and $\tilde{A}^*_{t_{n+1}}(T)$.

$$[\tilde{A}^{L*}_{t_n}, \tilde{A}^{U*}_{t_n}]$$

Given by the conclusion in Theorem 8, the upper bound value $A^{U*}_{t_n}$ and $\tilde{A}^{U*}_{t_n}$ is represented as follows:

$$A^{U*}_{t_n} = \max\{V_{t_n}, \min\{\Phi(X_{t_n}), \Phi(X^*_{t_n})\}\},$$
$$\tilde{A}^{U*}_{t_n} = \max\{V_{t_n}, \min\{\Phi(\tilde{X}_{t_n}), \Phi(\tilde{X}^*_{t_n})\}\}.$$

Based on the fact that $\tilde{A}^*_{t_{n+1}}(H) \geq A^*_{t_{n+1}}(H)$ and $\tilde{A}^*_{t_{n+1}}(T) \geq A^*_{t_{n+1}}(T)$, we can draw the conclusion that $\min\{\Phi(X_{t_n}), \Phi(X^*_{t_n})\} \geq \min\{\Phi(\tilde{X}_{t_n}), \Phi(\tilde{X}^*_{t_n})\}$. Thus, we have:

$$\tilde{A}^{U*}_{t_n} \geq A^{U*}_{t_n}.$$

In this way, the value of $A^{U*}_{t_n}$ is strictly increasing if the values of option at time t_{n+1} increases.

Likewise, we computed the lower bounds given by Theorem 8. The value $A^{L*}_{t_n}$ and $\tilde{A}^{L*}_{t_n}$ is represented as follows:

$$A^{L*}_{t_n} = \max\{V_{t_n}, -\Phi(X_{-t_n}), -\Phi(X^*_{-t_n})\},$$
$$\tilde{A}^{L*}_{t_n} = \max\{V_{t_n}, -\Phi(\tilde{X}_{-t_n})), -\Phi(\tilde{X}^*_{-t_n})\}.$$

Based on the fact that $\tilde{A}^*_{t_{n+1}}(H) \geq A^*_{t_{n+1}}(H)$ and $\tilde{A}^*_{t_{n+1}}(T) \geq A^*_{t_{n+1}}(T)$, we have $-\tilde{A}^*_{t_{n+1}}(H) \leq -A^*_{t_{n+1}}(H)$ and $-\tilde{A}^*_{t_{n+1}}(T) \leq -A^*_{t_{n+1}}(T)$. These will be the payoffs needed to be hedged or super-hedged in a short position. Then the values of the hedging or the super-hedging portfolios follow:

$$-\Phi(X_{-t_n}) \geq -\Phi(\tilde{X}_{-t_n}),$$

$$-\Phi(X^*_{-t_n}) \geq -\Phi(\tilde{X}^*_{-t_n}).$$

Thus, we have the following conclusion:

$$\tilde{A}^{L*}_{t_n} \leq A^{L*}_{t_n}.$$

In this way, the value of $A^{L*}_{t_n}$ is strictly decreasing if the value of the option at time t_{n+1} increases.

Right now, we have an assumption of the lower and the upper bounds of the no-arbitrage American option price at time t. The upper bound is generated by the buyer's price as the payoff to construct the hedging or super-hedging portfolios. The lower bound takes the seller's price as the payoff. If we define each upper bound as the upper price, and each lower bound as the lower price, we can compute the no-arbitrage price only based on those two prices.

From the maturity date to the initial time, we need to compute each upper price. The upper prices are computed in many one-period binomial tree model. The hedging and super-hedging portfolios are constructed with the payoffs equal to the upper prices.

$$\text{Upper Price} = \max\Big\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X_t^*)\}\Big\}$$

Similarly, we can also get the lower price.

$$\text{Lower Price} = \max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X_{-t}^*)\}$$

Theorem 10 *Under the assumption of non-zero funding spread, in a one-period binomial tree model from t_n to t_{n+1} inside the multi-period model, the no-arbitrage price of an American option at time t_n satisfies the following condition. Any prices out of this interval will result in arbitrages.*

$$max\{V_{t_n}, -\Phi(X_{-t_n}), -\Phi(X_{-t_n}^*)\} \leq A_{t_n} \leq max\Big\{V_{t_n}, min\{\Phi(X_{t_n}), \Phi(X_{t_n}^*)\}\Big\}$$

Notes:

1. *X_{t_n} and $X_{t_n}^*$ are the hedging and super-hedging portfolios based on the payoff of $A_{t_{n+1}}^U(H)$ and $A_{t_{n+1}}^U(T)$ in the long position; X_{-t_n} and $X_{-t_n}^*$ are the hedging and super-hedging portfolios based on the payoff of $A_{t_{n+1}}^L(H)$ and $A_{t_{n+1}}^L(T)$ in the short position.*
2. *If $max\{V_t(\cdot), -\Phi(X_{-t}^*), -\Phi(X_{-t})\} = -\Phi(X_{-t}^*)$, then the interval is open on the left side, which means $A_t > -\Phi(X_{-t}^*)$. If $max\Big\{V_t(\cdot), min\{\Phi(X_t), \Phi(X_t^*)\}\Big\} = \Phi(X_t^*)$, then the interval is open on the right side, which means $A_t < \Phi(X_t^*)$.*

Proof: Following the same process as in Theorem 8, the first thing we need to prove is:

$$\max\{V_{t_n}, -\Phi(X_{-t_n}), -\Phi(X_{-t_n}^*)\} \leq \max\{V_{t_n}, \min\{\Phi(X_{t_n}), \Phi(X_{t_n}^*)\}\}.$$

This is trivial if we can prove:

$$\max\{-\Phi(X_{-t_n}), -\Phi(X_{-t_n}^*)\} \leq \min\{\Phi(X_{t_n}), \Phi(X_{t_n}^*)\}.$$

According to the definitions of X_{-t_n} and $X_{-t_n}^*$, these are the hedging and super-hedging portfolios in the short position with payoff $A_{t_{n+1}}^L(H)$ and $A_{t_{n+1}}^L(T)$. Based on Theorem 2, the left side is the short position value of the European option in the one-period binomial tree model, which is less than the long position value of it. Nevertheless, $\min\{\Phi(X_{t_n}), \Phi(X_{t_n}^*)\}$ is the long position value of the European option with better payoff. A better payoff will result in a larger value of the long position value. Then $\min\{\Phi(X_{t_n}),\Phi(X^*_{t_n})\}$ is even larger than $\max\{-\Phi(X_{-t_n}),-\Phi(X^*_{-t_n})\}$.

In the next step, we will prove that any price out of this no-arbitrage price interval will result in an arbitrage opportunity.

If $\max\{V_{t_n},-\Phi(X_{-t_n}),-\Phi(X^*_{-t_n})\} > A_{t_n}$, we first assume that $V_{t_n} \leq \max\{-\Phi(X_{-t_n}),-\Phi(X^*_{-t_n})\}$. Using the same approach in Section (b) of the proof in Theorem 8, we can construct a portfolio Y with longing one share option, one share of X_{-t_n}, and M_{-t_n} shares of MMA, where $M_{-t_n} = -A_{-t_n} - \Phi(X_{-t_n})$. We will find that this portfolio will always make a positive profit no matter what the buyer's exercise strategies are. This is also true when $V_{t_n} > \max\{-\Phi(X_{-t_n}),-\Phi(X^*_{-t_n})\}$. We can conclude that arbitrage strategies exist when the price is less than the lower bound of the price interval.

If $A_{t_n} > \max\Big\{V_{t_n},\min\{\Phi(X_{t_n}),\Phi(X^*_{t_n})\}\Big\}$, we can also construct an arbitrage strategy using the same method in Section (a) of the proof in Theorem 8. So, we can also conclude that any price larger than the upper bound of the price interval will result in an arbitrage.

The last step is to prove that any price in that interval is the no-arbitrage price. As we have mentioned above, for any option price at time t_{n+1}, if $A_{t_{n+1}}(H) \in [A^L_{t_{n+1}}(H), A^U_{t_{n+1}}(H)]$ and $A_{t_{n+1}}(T) \in [A^L_{t_{n+1}}(T), A^U_{t_{n+1}}(T)]$, we can construct one price interval belonging to the no-arbitrage price interval. Then the only thing we need to prove is that the no-arbitrage price fills the whole interval:

$$[\max\{V_{t_n},-\Phi(X_{-t_n}),-\Phi(X^*_{-t_n})\},\max\Big\{V_{t_n},\min\{\Phi(X_{t_n}),\Phi(X^*_{t_n})\}\Big\}].$$

We have shown that using the two no-arbitrage values in time t_{n+1} can generate two intervals, $[A^{L*}_{t_n}, A^{U*}_{t_n}]$ and $[\tilde{A}^{L*}_{t_n}, \tilde{A}^{U*}_{t_n}]$. Any other price interval can be generated with the liner combination of these two intervals. Then there is no arbitrage opportunity if the option price is in that interval.

In conclusion, $\max\{V_{t_n},-\Phi(X_{-t_n}),-\Phi(X^*_{-t_n})\} \leq A_{t_n} \leq \max\Big\{V_{t_n},\min\{\Phi(X_{t_n}),\Phi(X^*_{t_n})\}\Big\}$ is the no-arbitrage price as long as it is open on either sides when it reaches the super-hedging price. ■

Chapter 4

XVA of American Options with Funding spread and Counterparty Credit Risk

In this chapter, we will compute the total valuation adjustment of the American option. This is done by incorporating funding spread and the counterparty credit risk at one model. The first part of this chapter will focus on the counterparty credit risk only. The pricing approach, which adjusts the derivative's price by credit risk, is often referred to as the credit valuation adjustment (CVA). After that, we will combine the result in the first section and the conclusion in Chapter 3 to get the XVA of American options by considering funding spread and counterparty credit risk.

4.1 Credit valuation adjustment for American options

In this section, we assume that there is a possibility that the counterparty may default on the option, and funding spread does not exist. We use the same notation as we have mentioned before. T is the maturity of the option. We divide the time interval from the initial date to maturity into $N+1$ 'default periods', and N 'trading periods'. These two types of periods follow the conditions below:

1. For each default period, the time length is h.
2. For each trading period, the time length is g.
3. In the default period, the counterparty may default on the option with a probability of q. Once the option is defaulted, the investor will receive part of the option's exercise payoff at that time with a recovery rate α, $\alpha \in [0,1]$, and the transaction will be terminated. For instance, if the buyer chooses to exercise the option at time t, the payoff is V_t. But when default occurs on the option, the investor will only receive an amount of αV_t. On the other hand, if no default happens, it will be followed by a trading period. Stocks and MMA are not allowed to be traded during default periods.
4. At the beginning of the trading period, the option buyer must decide to exercise the option or wait. The stock and MMA will be traded in the trading period. The risk free rate interest rate in the market is r. The stock price goes up and down with the factors u and d as we defined before.

5. Because the option buyer can buy and exercise the option immediately, the investor will face the risk of default at the initial time. Given that, the investment begins with the default period. Right before the maturity date, the counterparty may default without paying the full payoff of the option. Thus, the last period for the transaction is a default period. If there are N trading periods, then the maturity date can be expressed as: $T = (N+1)h + Ng$.

To replicate the situation when the counterparty defaults on the option, the bond market is introduced in this model. This bond is issued by the counterparty with a face value of B_t at time t. The bond's rate of return is r_m, which is different from the risk free interest rate. The bond buyer will face the same risk when the counterparty defaults on the option, which means that the bond and the option are defaulted at the same time. In this situation, the bondholder will receive part of the bond's face value, which is αB_t.

We will price the American option by hedging or super-hedging it. In the trading period, the portfolio will consist of stock and MMA. But in the default period, the investor will rebalance the portfolio with stock, MMA, and bonds. Only the bond is traded in the market in the default period, which means the value of the stocks and MMA will remain the same.

The payoff of the option is described as $V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$, $0 \le n \le N$, where ω_i denotes the stock price's movement in each trading period. At the end of the default period, if the counterparty does not default on the option, the option's value is $C_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$. However, if default occurs, the buyer will receive is $\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$. At the maturity date, $T = (N+1)h+Ng$, the value of the option is the payoff, which means:

$$C_{(N+1)h+Ng}(\omega_1\omega_2\cdots\omega_N) = V_{(N+1)h+Ng}(\omega_1\omega_2\cdots\omega_N).$$

At the end of a trading period, the option value, $C_{nh+ng}(\omega_1\omega_2\cdots\omega_n)$, depends on the buyer's exercising decision and the stock price's movement. $W_{nh+ng}(\omega_1\omega_2\cdots\omega_N)$ is the value of the option when the buyer chooses to wait, which is the current value of the expected payoff in the the next time period. The option buyer chooses the exercising strategy, which will maximize the payoff.

$$C_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) = \max\{V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n), W_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)\}$$

The discount rate is 1 here, since the buyer makes the decision immediately after the default period.

Theorem 11 *The risk neutral measure of counterparty default is* $\tilde{q} = \frac{(1+r_m)^h - 1}{(1+r_m)^h - \alpha}$.

Proof: Suppose in the default period beginning at the time $nh+ng$, the hedging portfolio consists of Δ_{nh+ng} shares of stock, M_{nh+ng} shares of MMA, and B_{nh+ng} shares of bond. Based on the two situations, default or not default, the value of the portfolio at the end of default period is followed by:

$$\Delta_{nh+ng}S + M_{nh+ng} + \alpha B_{nh+ng} = \alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n),$$

$$\Delta_{nh+ng}S + M_{nh+ng} + B_{nh+ng}(1+r_m)^h = C_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n).$$

The initial value of the hedging portfolio is:

$$\begin{aligned}\Delta_{nh+ng}S + M_{nh+ng} + B_{nh+ng} = & \frac{(1+r_m)^h - 1}{(1+r_m)^h - \alpha}\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) \\ & + \frac{1-\alpha}{(1+r_m)^h - \alpha}C_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n).\end{aligned} \tag{4.1}$$

Thus, $\tilde{q} = \frac{(1+r_m)^h-1}{(1+r_m)^h-\alpha}$ is the risk neutral measure of q. It only depends on the bond's rate of return and the recovery rate. ■

Then the option value at time the beginning of the default period can be expressed as follows:

$$C_{nh+ng}(\omega_1\omega_2\cdots\omega_n) = \tilde{q}\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})C_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n). \tag{4.2}$$

The risk neutral measure for p in the market in the trading period can be calculated by:

$$\tilde{p} = \frac{(1+r)^g - d}{u-d}.$$

In this way, the option's value when the buyer chooses to hold the option is:

$$W_{nh+(n-1)g}(\omega_1\omega_2\cdots\omega_{n-1}) = \frac{1}{(1+r)^g}[\tilde{p}C_{nh+ng}(\omega_1\omega_2\cdots H) + (1-\tilde{p})C_{nh+ng}(\omega_1\omega_2\cdots T)].$$

Based on $C_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) = \max\{V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n), W_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)\}$, for $n > 1$,we can draw the following conclusion:

$$\begin{aligned}C_{nh+(n-1)g}(\omega_1\omega_2\cdots\omega_{n-1}) = \max\{ & \frac{1}{(1+r)^g}[\tilde{p}C_{nh+ng}(\omega_1\omega_2\cdots H) \\ & + (1-\tilde{p})C_{nh+ng}(\omega_1\omega_2\cdots T)], V_{nh+(n-1)g}(\omega_1\omega_2\cdots\omega_{n-1})\}.\end{aligned} \tag{4.3}$$

By plugging in the value of $C_{nh+ng}(\omega_1\omega_2\cdots H)$ and $C_{nh+ng}(\omega_1\omega_2\cdots T)$ from the Equation 4.4 to the Equation 4.5, we can get the relationship between the value of $C_{nh+(n-1)g}(\omega_1\omega_2\cdots\omega_{n-1})$ and $C_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$.

Given that the option value at the maturity date is known, we can compute any values of the option at time $nh+(n-1)g$, $C_{nh+(n-1)g}(\omega_1\omega_2\cdots\omega_{n-1})$, for $1 < n < N$. When $n = 1$, according to the conclusion in the Equation 4.4, we can calculate the price of the option at the initial time:

$$A_0 = \tilde{q}\alpha V_h(\cdot) + (1-\tilde{q})C_h(\cdot).$$

In conclusion, to price the American option with consideration of the counterparty default risk, we have the following theorem:

Theorem 12 *Under the assumption of funding spread existed in the market, in discrete time settings, the no-arbitrage price of an American option can be expressed as A_0. A_0 can be calculated by backward induction method as follows.*

$$C_{nh+ng}(\omega_1\omega_2\cdots\omega_n) = \tilde{q}\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})C_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) \quad (4.4)$$

$$C_{nh+(n-1)g}(\omega_1\omega_2\cdots\omega_{n-1}) = max\{\frac{1}{(1+r)^g}[\tilde{p}C_{nh+ng}(\omega_1\omega_2\cdots H) + (1-\tilde{p})C_{nh+ng}(\omega_1\omega_2\cdots T)], V_{nh+(n-1)g}(\omega_1\omega_2\cdots\omega_{n-1})\} \quad (4.5)$$

When $n = 1$, we compute the option price as follows:

$$A_0 = \tilde{q}\alpha V_h(\cdot) + (1-\tilde{q})C_h(\cdot)$$

The implementation of this theorem can be gone as follows.

Step 1: Compute the payoff of the option at any time.

Step 2: Based on the iteration function in Equation 4.4 and 4.5, compute the value of $C_{nh+(n-1)g}$ from maturity date to the first trading period.

Step 3: Use the function $A_0 = \tilde{q}\alpha V_h(\cdot) + (1-\tilde{q})C_h(\cdot)$ to calculate the option price at the initial time.

Example 4.1.1 *Taking $N = 1$, and following the steps we have mentioned, we calculate the price of the American option. The transaction process is given in Figure 4.1.*

In the first step, we can get the payoffs of the option at different times, these values are $V_h(\cdot)$, $V_{2h+g}(H)$, and $V_{2h+g}(T)$.

In the second step, we compute the value of $C_{h+g}(\omega_1)$ based on the Equation 4.4. The value is:

$$C_{h+g}(H) = \tilde{q}\alpha V_{2h+g}(H) + (1-\tilde{q})V_{2h+g}(H),$$

$$C_{h+g}(T) = \tilde{q}\alpha V_{2h+g}(T) + (1-\tilde{q})V_{2h+g}(T).$$

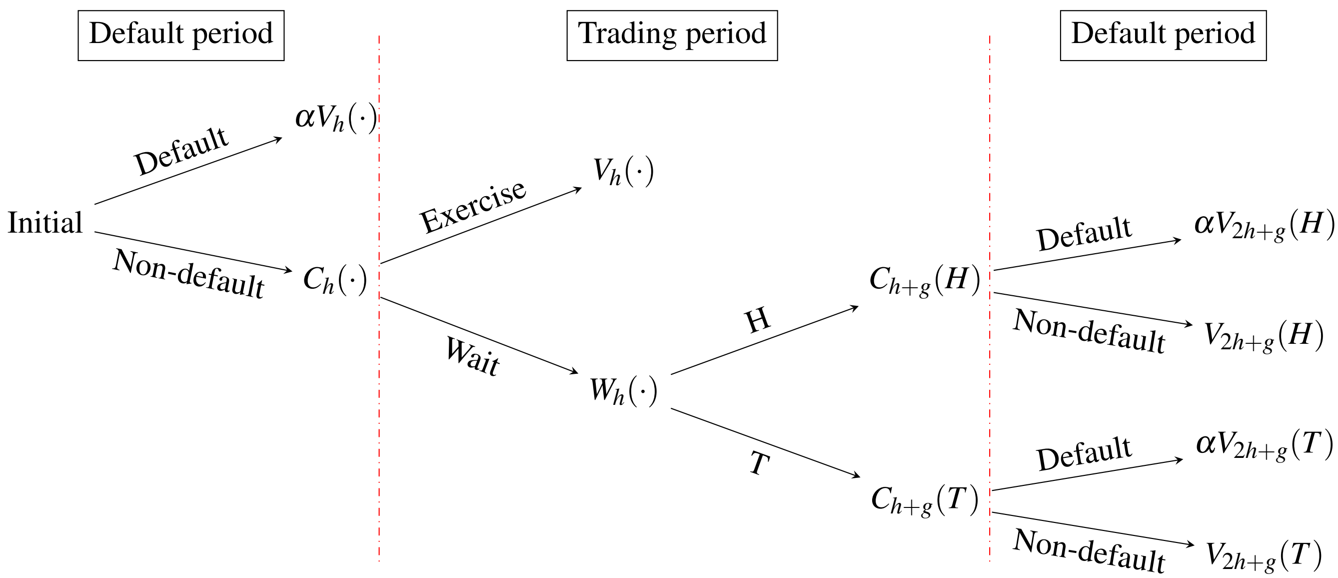

Figure 4.1: The American option pricing process in the model with counterparty credit risk when $N = 1$.

According to the Equation 4.5, we can get the value of $C_h(\cdot)$:

$$C_h(\cdot) = max\{\frac{1}{1+r}[\tilde{p}C_{h+g}(H) + (1-\tilde{p})C_{h+g}(T)], V_h(\cdot)\}.$$

At the last step, we can compute the option price directly by:

$$A_0 = \tilde{q}\alpha V_h(\cdot) + (1-\tilde{q})C_h(\cdot).$$

4.2 Multi-period XVA of American options

In Section 3.2, we have computed that the no-arbitrage price of American options with funding spread in one-period time setting. The conclusion is that the no-arbitrage price interval of an American option at initial time t is:

$$\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\} \leq A_t \leq \max\Big\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X^*_t)\}\Big\}.$$

The construction of the hedging portfolio and the super-hedging portfolio is based on the payoff of the option at the maturity.

We define $\max\Big\{V_t(\cdot), \min\{\Phi(X_t), \Phi(X^*_t)\}\Big\}$ as the upper bound of the option, U_t, and $\max\{V_t(\cdot), -\Phi(X_{-t}), -\Phi(X^*_{-t})\}$ as the lower bound of the option, L_t. Thus, the American option's value with funding spread can be rewritten as:

$$L_t \leq A_t \leq U_t.$$

The lower bound of the option price is related to the short position value of the option, while the upper bound of the option price is related to the long position value of the option. In the multi-period binomial tree model, we can compute the lower bound and the upper bound separately. And the American option value at the initial time is represented as an interval between them.

Here we incorporate two models, which are introduced in Section 3.4 and Section 4.1. The first model is pricing the American option with consideration of funding spread, and the second model is the CVA approach. The assumptions in Section 4.1 still hold. We deal with the funding spread in each trading period, and the counterparty credit risk in each default period.

In a trading period, only the stock and MMA are traded in the market with a non-zero funding spread. This is exactly the same model as the one-period binomial tree model in Section 3.2. It takes the payoffs from the next default period. If the payoff is an interval, we use the Theorem 10 to compute the no-arbitrage price interval of the American option at the beginning of the trading period. $U_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$ and $L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$ are denoted as the upper and lower bound of the no-arbitrage price interval at the beginning of a trading period.

In a default period beginning at time $nh+ng$, the option value when default occurs is $\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$. On the other hand, if the counterparty does not default on the option, the option value is the no-arbitrage price interval at time $(n+1)h+ng$, which is $[L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n), U_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)]$.

Notes: Time $nh+ng$ denotes the beginning of a default period, and time $(n+1)h+ng$ denotes the beginning of a trading period.

The no-arbitrage price interval of the American option at the beginning of the default period is denoted as $[L_{nh+ng}(\omega_1\omega_2\cdots\omega_n), U_{nh+ng}(\omega_1\omega_2\cdots\omega_n)]$. The upper and the lower bound of the no-arbitrage price interval can be calculated, respectively, by:

$$U_{nh+ng}(\omega_1\omega_2\cdots\omega_n) = \tilde{q}\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})U_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n), \quad (4.6)$$

$$L_{nh+ng}(\omega_1\omega_2\cdots\omega_n) = \tilde{q}\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n). \quad (4.7)$$

Note that the discount factor in the default period is 1. The reason is that MMA is not traded in the market at this period. At the maturity, the option value without default is equal to the payoff of the option. Therefore, when default does not occur, the upper bound and the lower bound of the no-arbitrage price at time $(N+1)h+Ng$ are equal to the option payoff.

$$U_{(N+1)h+Ng}(\omega_1\omega_2\cdots\omega_N) = L_{(N+1)h+Ng}(\omega_1\omega_2\cdots\omega_N) = V_{(N+1)h+Ng}(\omega_1\omega_2\cdots\omega_N)$$

With the payoff at the end of each trading period, we can calculated the no-arbitrage price interval at time $nh+(n-1)g$, $n = 1,2,\cdots N$. According to the conclusion of Theorem 10, the upper and lower bound of the no-arbitrage price interval without default are as follows:

$$U_{nh+(n-1)g} = \max\Big\{V_{nh+(n-1)g}, \min\{\Phi(X_{nh+(n-1)g}), \Phi(X^*_{nh+(n-1)g})\}\Big\}, \quad (4.8)$$

$$L_{nh+(n-1)g} = \max\{V_{nh+(n-1)g}, -\Phi(X_{-(nh+(n-1)g)}), -\Phi(X^*_{-(nh+(n-1)g)})\}. \quad (4.9)$$

At time $nh+(n-1)g$, for the upper bound, the hedging and super-hedging portfolio are constructed using the payoffs of $U_{nh+ng}(\omega_1\omega_2\cdots\omega_{n-1}H)$ and $U_{nh+ng}(\omega_1\omega_2\cdots\omega_{n-1}T)$. Respectively, for the lower bound, the hedging and super-hedging portfolio are constructed using the payoffs of $L_{nh+ng}(\omega_1\omega_2\cdots\omega_{n-1}H)$ and $U_{nh+ng}(\omega_1\omega_2\cdots\omega_{n-1}T)$.

Given by the iteration functions in trading periods and the default periods, we can get the no-arbitrage price interval of the American option with consideration of funding spread and counterparty credit risk at any time between the initial time and maturity.

$$L_{nh+ng} \leq A_{nh+ng} \leq U_{nh+ng}; n \in [0.N]$$

Notes: For any n, if $L_{nh+(n-1)g} = -\Phi(X^*_{-(nh+(n-1)g)})$, then the interval is open on the left side; if $U_{nh+(n-1)g} = \Phi(X^*_{nh+(n-1)g})$, then the interval is open on the right side.

Given by the analysis above, we can draw the following conclusion.

Theorem 13 *With the existence of funding spread and counterparty credit risk, in discrete time settings, the no-arbitrage price interval of an American option at the beginning of default periods or trading periods are as follows. Any price out of this interval will result in arbitrage.*

$$L_t \leq A_t \leq U_t$$

The upper bound U_t is calculated by the backward induction method as follows:

$$U_{nh+ng}(\omega_1\omega_2\cdots\omega_n) = \tilde{q}\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})U_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n),$$

$$U_{nh+(n-1)g} = max\Big\{V_{nh+(n-1)g}, min\{\Phi(X_{nh+(n-1)g}), \Phi(X^*_{nh+(n-1)g})\}\Big\}.$$

The lower bound L_t is calculated by the backward induction method as follows:

$$L_{nh+ng}(\omega_1\omega_2\cdots\omega_n) = \tilde{q}\alpha V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n),$$

$$L_{nh+(n-1)g} = max\{V_{nh+(n-1)g}, -\Phi(X_{-(nh+(n-1)g)}), -\Phi(X^*_{-(nh+(n-1)g)})\}.$$

Note that the inequality is open on the right side when $U_{nh+(n-1)g} = \Phi(X^*_{nh+(n-1)g})$ for any n. The inequality is open on the left side when $L_{nh+(n-1)g} = -\Phi(X^*_{-(nh+(n-1)g)})$ for any n.

The application of this theorem can be concluded as follows:

Step 1: Compute the payoff of the option at any time, $V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$.

Step 2: In a default period, follow the Equation 4.6 and 4.7 to compute the upper and lower bounds of the no-arbitrage price interval at the beginning of this period.

Step 3: In a trading period, follow the Equation 4.8 and 4.9 to compute the upper and lower bounds of the no-arbitrage price interval at the beginning of this period.

Step 4: Repeat the Step 2 and the Step 3 until reaching the initial time.

Chapter 5

XVA of American Options with Collateral

Collateral is widely used in the market to eliminate the loss from default. In the new model, to mitigate the loss, the option buyer will require the seller to post some cash, C, as collateral. At each time period before the maturity, if no default occurs, the collateral taker needs to pay the collateral provider with some interest based on C with an interest rate, r_c. So the payment amount is Cr_c. At the maturity date or the date when the option buyer exercises it, the collateral will be returned to the provider. But if a default occurs, the transaction will be terminated. The collateral will not be returned to the provider, and it will be used to pay the loss to the investor from the default.

The collateral provides a protection for the option buyer when default occurs. If the option buyer posts the collateral, the payoff of the option consists of two parts. The first part, C, is covered by the collateral. The rest part is uncovered, which is affected by the default.

For a long position in the option, the hedger will receive the collateral at the initial time. At time t, the option can be exercised with a payoff V_t. Once the seller defaults on the option, the covered part C is protected by the collateral. The hedger will receive the uncovered part $(V_t - C)^+$ with a recovery rate of α. Therefore, the total that the hedger receives when default occurs is expressed as follows:

$$C + \alpha(V_t - C)^+.$$

On the other hand, for a short position in the option, the hedger will give the collateral, C, to the buyer at the initial time. When the buyer exercises the option at time t, the hedger will pay the option payoff, V_t, to the buyer. Once the buyer defaults on the option, the hedger can not receive the collateral back. The hedger only needs to pay the part of the payoff which beyond the collateral with the same recovery rate α above. Therefore, the total that the hedger needs to pay when default occurs is expressed as follows:

$$C + \alpha(V_t - C)^+.$$

This is the same amount of money that the hedger will receive when default occurs for a long position in the option.

We include a collateral account in the model, which is constructed in Section 4.2. At the initial time, the option seller will post the cash collateral with collateral rate γ, $\gamma \in [0,1]$. As the model we introduced in Section 2.4, the size of the collateral depends on the option value for the hedger. Likewise, the no-arbitrage price interval of the American option in this model is unique for each hedger in the market. If the option value for the hedger is A_0, which is inside the no-arbitrage price interval of this American option, the size of the posted collateral C is:

$$C = \gamma A_0.$$

Using the same assumptions in Section 4.2, we divide the time interval $[0,T]$ to $N+1$ default periods and N trading periods. The difference is that the buyer will receive the collateral at the initial time. Collateral interest needs to be paid to the collateral provider at the end of each default period if no default occurs. If we combine the option and the collateral as a portfolio, the no-arbitrage price of this portfolio at time t is denoted as P_t.

$$P_t = A_t + C = A_t + \gamma A_0$$

We apply the same notations from the last chapter. The time interval $[nh+ng,(n+1)h+ng]$ denotes the default period, and the time interval $[(n+1)h+ng,(n+1)h+(n+1)g]$ denotes the trading period. The option is allowed to be exercised at the beginning of trading periods. The exercising payoff of the option is $V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$.

For a long position in the portfolio, if the counterparty defaults on the option, the hedger will keep the collateral and get the option payoff, then portfolio value when default occurs is $2C + \alpha(V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) - C)^+$. On the contrary, for a short position in the portfolio, if the counterparty defaults on the option, the portfolio value in this position is:$-(2C+\alpha(V_{(n+1)h+ng}(\omega_1\omega_2\cdots \omega_n - C)^+)$. Combining these two values, we can conclude that the portfolio value is $2C+\alpha(V_{(n+1)h+ng} (\omega_1\omega_2\cdots\omega_n) - C)^+$ when default occurring at time $(n+1)h+ng$.

The analysis above is analyzing the case when default occurs in the transaction. On the other hand, if the seller does not default on the option, except the last default period, the portfolio value is $P_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + Cr_c$. The first part is the option value and the principal of collateral, and the second part is the interest of the collateral. In the last default period, there is no interest paid. Therefore, the portfolio value is $V_{(N+1)h+Ng}(\omega_1\omega_2\cdots\omega_n) + C$

In each default period, the risk neutral measure of default possibility is $\tilde{q}$, this value remains the same as the value in the CVA model. The reason is that the risk neutral measure does not depend on what derivatives the market has, it is only related to the interest rate and the recovery rate in the market.

$$\tilde{q} = \frac{(1+r_m)^h - 1}{(1+r_m)^h - \alpha}$$

From the time t_{nh+ng} to $t_{(n+1)h+ng}$, this is a default period in the market. The value of the combination of the option and collateral is P_{nh+ng} at the beginning of default period. If we know the payoff of the option at the end of this default period, given by the risk neutral measure $\tilde{q}$, we can compute the no-arbitrage price of the American option at time t_{nh+ng}.

$$\begin{aligned} P_{Nh+Ng} &= \tilde{q}[2C+\alpha(V_{(N+1)h+Ng}-C)^{+}]+(1-\tilde{q})[V_{(N+1)h+Ng}+C] \\ &= (1+\tilde{q})C+\tilde{q}\alpha(V_{(N+1)h+Ng}-C)^{+}+(1-\tilde{q})V_{(N+1)h+Ng} \end{aligned} \tag{5.1}$$

$$\begin{aligned} P_{nh+ng} &= \tilde{q}[2C+\alpha(V_{(n+1)h+ng}-C)^{+}]+(1-\tilde{q})[P_{(n+1)h+ng}+Cr_c] \\ &= 2\tilde{q}C+(1-\tilde{q})Cr_c+\tilde{q}\alpha(V_{(n+1)h+ng}-C)^{+}+(1-\tilde{q})P_{(n+1)h+ng}; n<N \end{aligned} \tag{5.2}$$

In a trading period, if we know the option value at the end of this period, the no-arbitrage price of the portfolio at the beginning of this period can be calculated by Theorem 10. Since the price of a portfolio at the beginning of the trading period is also the portfolio at the end of the last trading period, we can calculate the no-arbitrage price of the American option during this trading period.

Taking $N=1$ for an example, the transaction process of this model is expressed in Figure 5.1.

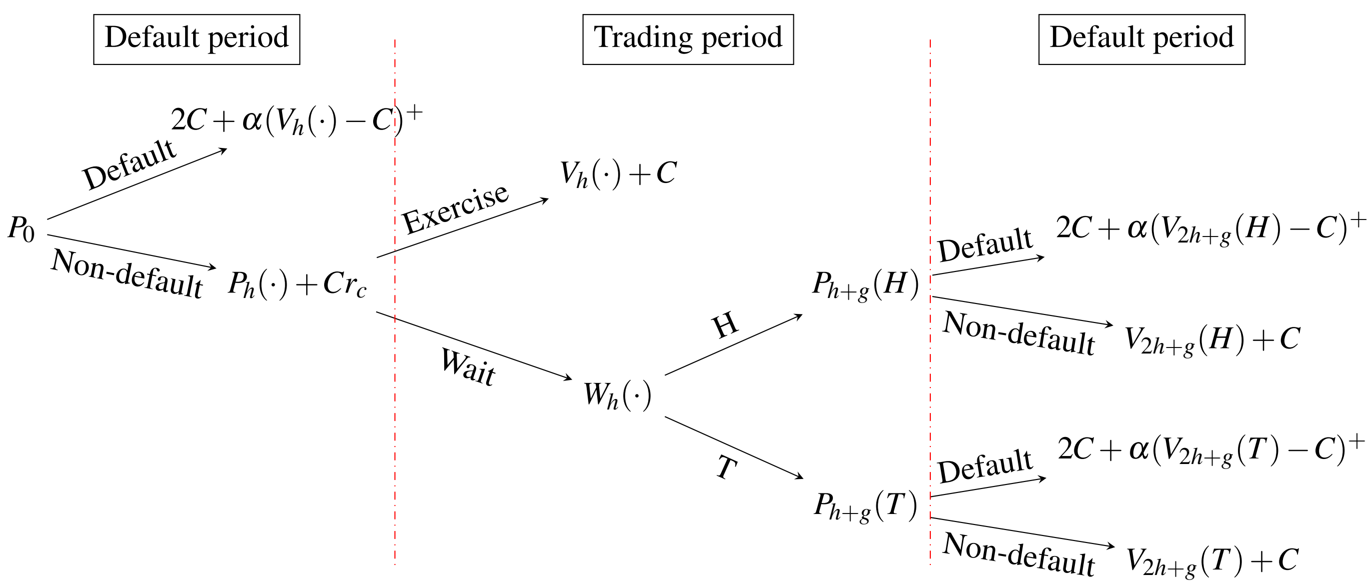

Figure 5.1: The American option transaction process with considering counter-party credit risk and funding spread. This model is incorporated with a collateral account with $N=1$.

As in the model without collateral in the last chapter, the no-arbitrage price of the option at the initial time is belonging to an interval. The same goes for the portfolio combining the option and collateral. Whether the interval is open or closed depends on the comparison between the hedging price and the super-hedging price at each step. The no-arbitrage price interval of the portfolio combining the option and collateral is as follows:

$$P_t \in [L_t, U_t].$$

Suppose P_0^* is a no-arbitrage price of the portfolio at the initial time. Given that $P_0 = A_0 + C = (1+\frac{1}{\gamma})C$, we compute the size of the collateral C. Once C is given in the whole model, we can find one no-arbitrage price interval at the initial time. This no-arbitrage price interval at the initial time is denoted as $[L_0^*, U_0^*]$. Then the union of them is equal to the no-arbitrage price at the initial time. Note that there will not have a gap in $\bigcup_{P_0^* \in [L_0, U_0]} [L_0^*, U_0^*]$. The reason is that any point in the interval $[L_0, U_0]$ can be filled with a linear combination of two no-arbitrage intervals given by different P_0^*. Thus, we have:

$$\bigcup_{P_0^* \in [L_0, U_0]} [L_0^*, U_0^*] = [L_0, U_0].$$

Theorem 14 *A larger size of collateral corresponds to higher upper and lower bounds.*

Proof: In the last default period, we have the Equation 5.1:

$$P_{Nh+Ng} = (1+\tilde{q})C + \tilde{q}\alpha(V_{(N+1)h+Ng} - C)^+ + (1-\tilde{q})V_{(N+1)h+Ng},$$

since $1+\tilde{q}-\tilde{q}\alpha > 0$, P_{Nh+Ng} is strictly increasing when C increases. Given by Theorem 10, a higher payoff will correspond to a higher value of the no-arbitrage price at the beginning of this trading period. This means $P_{Nh+(N-1)g}$ is also strictly increasing when C increases.

In any other default periods, we have the Equation 5.2:

$$P_{nh+ng} = 2\tilde{q}C + (1-\tilde{q})Cr_c + \tilde{q}\alpha(V_{(n+1)h+ng} - C)^+ + (1-\tilde{q})P_{(n+1)h+ng}; n < N,$$

we can find that $2\tilde{q} - \tilde{q}\alpha + (1-\tilde{q})r_c > 0$. If $n = N-1$, with a rising C, the first part $[2\tilde{q}C + (1-\tilde{q})Cr_c + \tilde{q}\alpha(V_{Nh+(N-1)g} - C)^+]$ will increase at the same time. Since $P_{Nh+(N-1)g}$ has a positive correlation with C, when C increases, the value of $P_{(N-1)h+(N-1)g}$ will also increase. Furthermore, a higher valued of C and $P_{(N-1)h+(N-1)g}$ will lead to a larger $P_{(N-1)h+(N-2)g}$. Keeping the analysis above, we can conclude that $P_{(n+1)h+ng}$ and P_{nh+ng} increases monotonously with increasing C. Then we can conclude that a larger size of collateral corresponds to higher upper and lower bounds. ■

With a no-arbitrage price P_0^* of the portfolio at the initial time, we can get one no-arbitrage price interval $[L_0^*, U_0^*]$. Given by our analysis above, L_0^* and U_0^* increase monotonously with increasing P_0^*. The reason is that P_0^* has a positive correlation with C. Thus, when $P_0^* = U_0$, we have $U_0^* = U_0$. Likewise, when $P_0^* = L_0$, we have $L_0^* = L_0$.

In a trading period, the upper bound is related to the long position in the portfolio, and the lower bound is related to the short position. In a default period, a higher value at the end corresponds to a higher value at the initial. Therefore, L_0^* and U_0^* correspond to the short and the long position of the portfolio respectively.

For a long position in the portfolio combining option and collateral, we denote the value is U_t. The derivatives we need to hedge involves the option itself and the collateral. At the initial time, the portfolio value is U_0. The collateral value can be calculated by:

$$C = \frac{U_0}{1+\frac{1}{\gamma}}.$$

At the end of each default period, the value of the derivatives are different because the counterparty may default on the option. The payoff of the portfolio when default occurs is denoted as $D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$, and this value is as follows:

$$D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) = 2C + \alpha(V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) - C)^+.$$

While, if there is no default occurs, the hedger pays the interest of the collateral, Cr_c, to the provider at the end of each default period.

Taking $N = 1$ for an example, the transaction process in a long position is given in Figure 5.2.

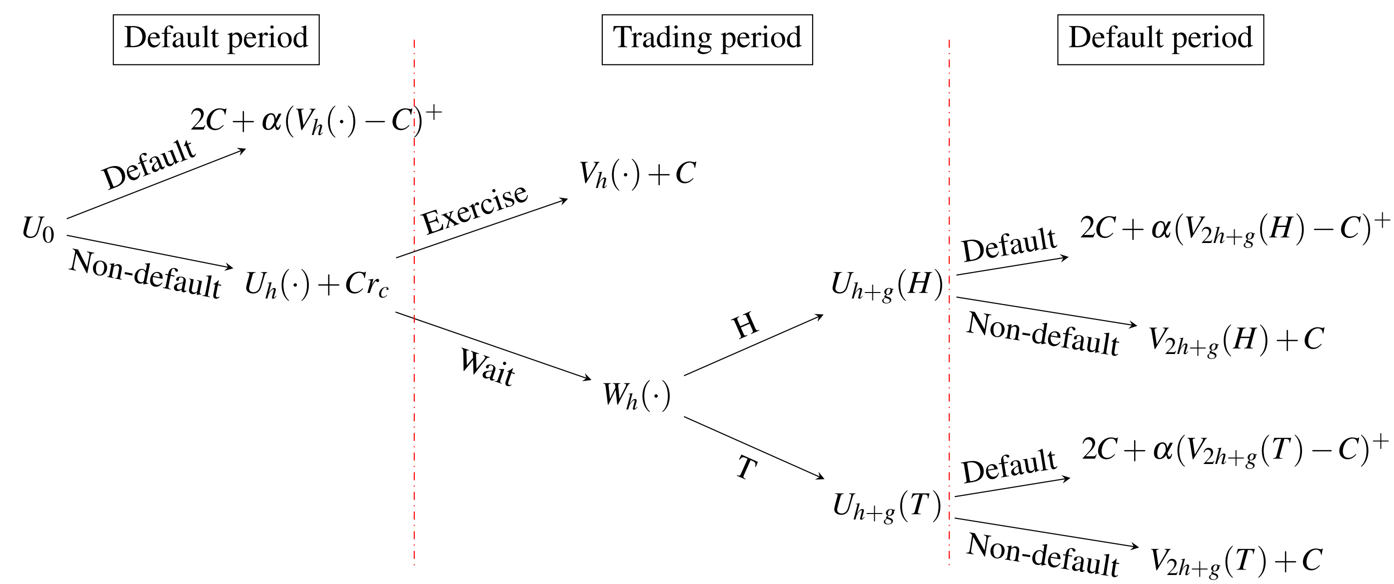

Figure 5.2: The American option transaction process with considering counter-party credit risk and funding spread. This model is incorporated with a collateral account in a long position.

If the counterparty does not default on the option, the value of the portfolio is $U_{(n+1)h+ng}(\omega_1\omega_2 \cdots\omega_n)+Cr_c$ except the last default period. If default occurs, the payoff of the portfolio is $D_{(n+1)h+ng}$

$(\omega_1\omega_2\cdots\omega_n)$, which is given above. Then the value of the option at the beginning of a default period can be expressed as follows.

$$U_{nh+ng}(\omega_1\omega_2\cdot\omega_n) = \begin{cases} \tilde{q}D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})[U_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + Cr_c] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \in [1, n-1] \\ \tilde{q}D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})[U_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + C], n = N. \end{cases} \tag{5.3}$$

In each trading period, we have the payoff at the end of this period. In this way, we can construct the hedging and the super-hedging portfolios. By comparing the hedging and the super-hedging price in a long position, we get the value of the portfolio if holding the option at this period. When exercising the option, the portfolio value is $V_{(n+1)h+ng} + C$. Then the portfolio value in a long position is the maximum of the two values above.

$$U_{(n+1)h+ng} = \max\Big\{V_{(n+1)h+ng} + C, \min\{\Phi(X_{(n+1)h+ng}), \Phi(X^*_{(n+1)h+ng})\}\Big\} \tag{5.4}$$

In the first default period, we can compute the value of the portfolio at the initial time as follows:

$$U_0 = \tilde{q}[2C + \alpha(C - V_h(\cdot))^+] + (1-\tilde{q})[U_h(\cdot) + Cr_c].$$

Given that $C = \frac{U_0}{1+\frac{1}{\gamma}}$, the only unknown parameter in this equation is U_0. By solving this equation, we compute the upper bound of no-arbitrage price interval of the portfolio consisting of the option and collateral.

For a short position in the portfolio consisting of the option and collateral, we denote the portfolio value as $-L_t$. Then L_t is a positive value here. The hedger shorts one share of the option and posts the collateral. The size of the posted collateral is calculated as follows:

$$C = \frac{L_0}{1+\frac{1}{\gamma}}.$$

At the end of each trading period, the counterparty may default on the option. In that case, the option buyer does not return the collateral to the hedger. The hedger is required to pay the uncovered part of the payoff with a recovery rate of α. Thus, the amount that the hedger needs to pay to the buyer is given below:

$$D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) = 2C + \alpha(V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) - C)^+.$$

While, if there is no default, the value of the portfolio in a short position is the negative value of $L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + Cr_c$. It involves the value of the option in a short position, the principal of the collateral, and the interest of the collateral.

In each default period, we can compute the lower bound of the portfolio value at the beginning of this period with the risk neutral measure $\tilde{q}$.

$$L_{nh+ng}(\omega_1\omega_2\cdot\omega_n) = \begin{cases} \tilde{q}D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})[L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + Cr_c] \\ \hfill n \in [1, n-1] \\ \tilde{q}D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + (1-\tilde{q})[L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + C], n = N \end{cases} \tag{5.5}$$

At a short position of the portfolio in a trading period, the lower bound of the portfolio value is calculated by Theorem 10.

$$\begin{aligned} L_{(n+1)h+ng}(\omega_1\omega_2\cdot\omega_n) &= \max\Big\{V_{(n+1)h+ng} + C, \max\{-\Phi(X_{-(n+1)h-ng}), -\Phi(X^*_{-(n+1)h-ng})\}\Big\} \\ &= \max\{V_{(n+1)h+ng} + C, -\Phi(X_{-(n+1)h-ng}), -\Phi(X^*_{-(n+1)h-ng})\} \end{aligned} \tag{5.6}$$

Note that $X_{-(n+1)h-ng}$ and $X^*_{-(n+1)h-ng}$ are the hedging and super-hedging portfolio in a short position constructed with the payoffs $L_{(n+1)h+(n+1)g}(\omega_1\omega_2\cdot\omega_n H)$ and $L_{(n+1)h+(n+1)g}(\omega_1\omega_2\cdot\omega_n T)$.

As for the long position, when $n = 0$, we get the option's short position value at the initial time.

$$L_0 = \tilde{q}[2C + \alpha(V_h(\cdot) - C)^+] + (1-\tilde{q})[L_h(\cdot)) + Cr_c]$$

Given that $C = \frac{L_0}{1+\frac{1}{\gamma}}$, the only unknown parameter in this equation is L_0. By solving this equation, we can compute the lower bound of no-arbitrage price interval of the portfolio consisting of the option and collateral.

Right now, we have computed the lower bound and the upper bound of the portfolio consisting of the option and collateral.

$$P_0 \in [L_0, U_0]$$

As $P_0 = A_0 + \gamma A_0$, the no-arbitrage price interval of the American option at initial time is as follows:

$$\frac{L_0}{\gamma+1} \leq A_0 \leq \frac{U_0}{1+\gamma}.$$

Note that interval is open on the right side when Equation 5.4 takes the super-hedging portfolio's value for any $n \in [0, N-1]$, and is open on the left side when Equation 5.6 takes the super-hedging portfolio's value for any $n \in [0, N-1]$.

Given by the analysis above, we can draw the conclusion as follows.

Theorem 15 *With the existence of funding spread and counterparty credit risk in discrete time settings, in the model with collateral, the no-arbitrage price interval of an American option at initial time 0 is as follows. Any price out of that interval will result in arbitrage.*

$$L_0 \leq A_0 \leq U_0$$

The upper bound U_t is calculated by the backward method as follows:

$$U_{nh+ng}(\omega_1\omega_2\cdot\omega_n) = \begin{cases} \tilde{q}D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) \\ ... + (1-\tilde{q})[U_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + Cr_c]; & n \in [1, n-1] \\ \tilde{q}D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) \\ ... + (1-\tilde{q})[U_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + C]; & n = N, \end{cases} \quad (5.7)$$

$$U_{(n+1)h+ng} = max\Big\{V_{(n+1)h+ng} + C, min\{\Phi(X_{(n+1)h+ng}), \Phi(X^*_{(n+1)h+ng})\}\Big\}. \quad \text{(5.8)}$$

Given by $U_0 = C(1+\frac{1}{\gamma})$, we solve the upper bound value U_0.
The upper bound U_t is calculated by the backward method as follows:

$$L_{nh+ng}(\omega_1\omega_2\cdot\omega_n) = \begin{cases} \tilde{q}D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) \\ ... + (1-\tilde{q})[L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + Cr_c]; & n \in [1, n-1] \\ \tilde{q}D_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) \\ ... + (1-\tilde{q})[L_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n) + C]; & n = N, \end{cases} \quad (5.9)$$

$$L_{(n+1)h+ng}(\omega_1\omega_2\cdot\omega_n) = max\{V_{(n+1)h+ng} + C, -\Phi(X_{-(n+1)h-ng}), -\Phi(X^*_{-(n+1)h-ng})\} \quad (5.10)$$

Given by $L_0 = C(1+\frac{1}{\gamma})$, we solve the upper bound value L_0.

Note that the inequality is open on the right side when $U_{nh+(n-1)g} = \Phi(X^*_{nh+(n-1)g})$ for any n. The inequality is open on the left side when $L_{nh+(n-1)g} = -\Phi(X^*_{-(nh+(n-1)g)})$ for any n

The implementation of this theorem can be gone as follows.

Step 1: Compute the payoff of the option at any time, $V_{(n+1)h+ng}(\omega_1\omega_2\cdots\omega_n)$.

Step 2: The upper bound and the lower bound of the portfolio consisting of the option and collateral are U_0^* and L_0^*.

Step 3: For the upper bound, given by U_0^*, the size of the posted collateral is $C = \gamma U_0^*$.

(a) Default period: follow the Equation 5.3 to compute the upper bound U_{nh+ng}.

(b) Trading period: follow the Equation 5.4 to compute the upper bound $U_{h+(n-1)g}$.

Repeat the iterations above until reaching the initial time, the upper bound value at initial time U_0 is a function of C. Given that $U_0 = U_0^*$, calculating the value U_0^*.

Step 4: For the lower bound, given by L_0^*, the size of the posted collateral is $C = \gamma L_0^*$.

(a) Default period: follow the Equation 5.5 to compute the lower bound U_{nh+ng}.

(b) Trading period: follow the Equation 5.6 to compute the lower bound $U_{h+(n-1)g}$.

Repeat the iterations above until reaching the initial time, the lower bound value at initial time L_0 is a function of C. Given that $L_0 = L_0^*$, calculating the value L_0^*.

Step 5: The no-arbitrage price interval of the American option is $[L_0^*, U_0^*]$.

Chapter 6

Numerical Analysis

Right now, we have two models to get the no-arbitrage price intervals of American options. The first model uses stocks, MMAs, and bonds to hedge or super-hedge the option's payoff. Apart from that, the second model adds a cash collateral. The latter model is more practical and makes it closer to the real financial market. In this chapter, we will use Matlab to perform some numerical analyses of these two models.

6.1 XVA of American options without collateral

In this model, we analyze the relationship between American put option's no-arbitrage price and factors, such as lending and borrowing rates, the defaultable bond's rate of return, and the number of periods. The rest of the factors are defined as follows. The stock price movement factors are $u = 1.2$, and $d = 0.8$. The initial stock price is 100. The recovery rate of the option and the bond is 0.5. The length of defaults and trading periods in a one-period model are both 1. Then the maturity $T = 2h + g = 1$. One-period model means there is one trading period. When escalating to a n-period model, it has n trading periods, and the time length for each trading and default period is $\frac{1}{n}$.

In Figure 6.1, under a 5-period binomial tree model, the lending rate ranges from 0 to 0.2, and the borrowing rate ranges from 0.2 to 0.4. We can find the relationship between the no-arbitrage price interval of an American option and the funding rates which includes the borrowing and lending rate. There exist some turning points in the surfaces of the upper bound and the lower bound. The reason is that the choice of hedging and super-hedging portfolio are changing with respect to the movement of the borrowing and lending rate. At the same time, the option price is going down when the borrowing rate increases. While, when lending rate increases, the option price will decrease. The planes of the upper bound and the lower bound intersect at a point at which the lending rate equals to the borrowing rate.

We noticed that the upper bound is not affected by the borrowing rates and the lower bound is not affected by the lending rate. This results from our choice of parameters. Our hedging portfolios in the long position are either longing or shorting the MMA, and the same goes for the hedging

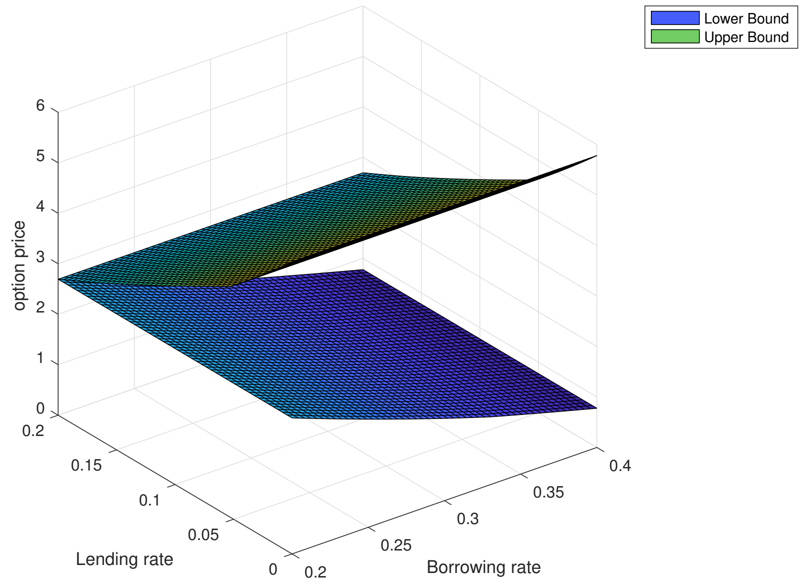

Figure 6.1: XVA of the American put option when varying the borrowing and lending rates in a 5-period binomial tree model. Other constant parameters are $u = 1.2$, $d = 0.8$, $S_0 = 100$, $K = 100$, $\alpha = 0.5$, $r_m = 0.25$, and $h = g = 1/5$.

portfolios in the short position. More importantly, the hedging portfolios in a long position and a short position have different status of borrowing or lending MMA. Therefore, the upper bound and lower bound are affected by different funding rates.

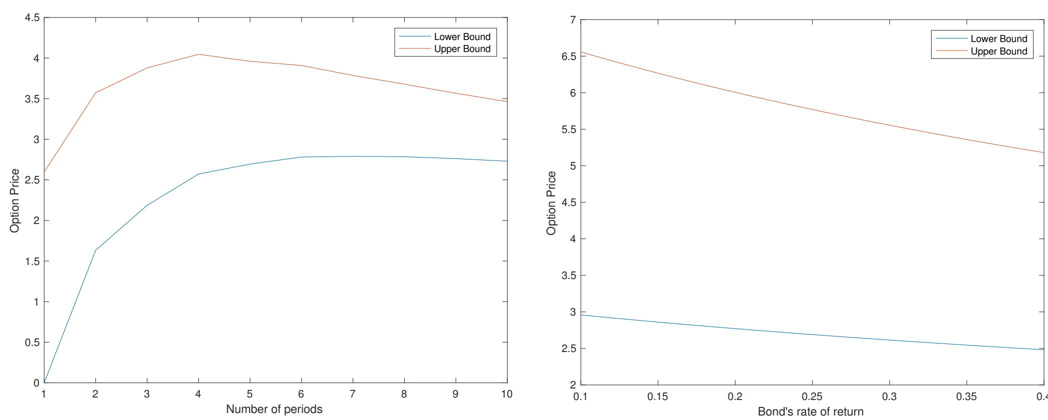

Figure 6.2: XVA of the American put option when varying the number of periods and bond's rate of return. The constant parameters in this two model are $u = 1.2$, $d = 0.8$, $S_0 = 100$, $K = 100$, $\alpha = 0.5$, $r_l = 0.1$, $r_b = 0.2$, and $h = g = 1/5$. In the left figure, bond's rate of return is $r_m = 0.25$, and the number of periods range from 1 to 10. The right figure is under a 5-period binomial tree model with the bond's rate of return ranging from 0.1 to 0.4.

In Figure 6.2, the left graph shows the relationship between the option price and the number of periods. When the value N increases, the lower bound and the upper bound of the no-arbitrage intervals increase at the same time. This is as a result of the risk neutral measure $\tilde{q}$ which is equal to $\frac{(1+r_m)^h-1}{(1+r_m)^h-\alpha}$. With increasing the number of periods, the time length of the default period decreases monotonously. Then $\tilde{q}$ is getting close to 0. It indicates that the option is less possible to be defaulted by the counterparty. Therefore, the option value will be more expensive.

The value of the rate of return for the bond can be used to measure the risk of a default in the counterparty. A larger bond's rate of return reflects that a higher default risk will result from the counterparty. Therefore, the option is less expensive. The right graph in Figure 6.2 shows this property. In our model, a higher bond's rate of return corresponds to a larger risk neutral measure $\tilde{q}$. Therefore, in each default period, the weight of the payoff when the option is defaulted becomes larger. Thus, the upper bound and the lower bound decrease.

6.2 XVA of American options with collateral

In the model with collateral, we still change the same parameters to find the common properties and differences between these two models. The unique parameters we set in this model is the collateralization rate $\gamma = 0.6$, and the collateral rate $r_c = 0.05$.

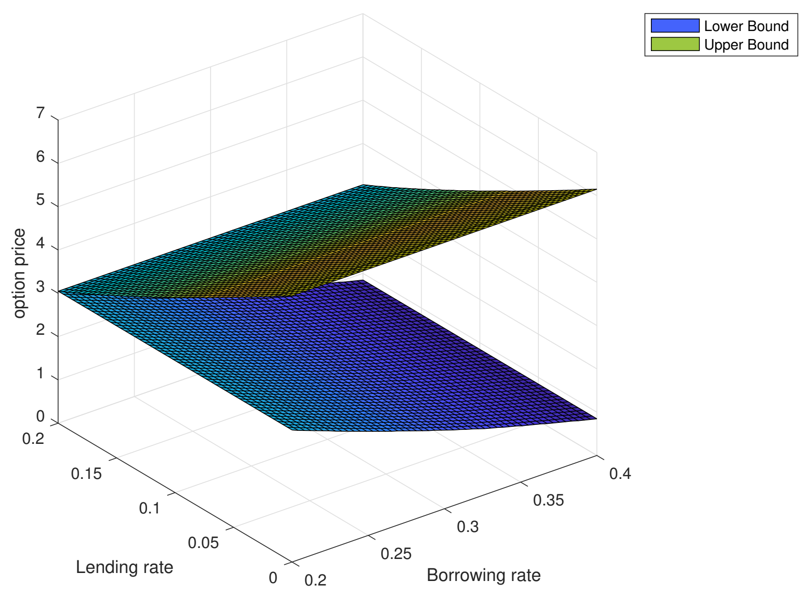

Figure 6.3: XVA of the American put option when varying the borrowing and lending rates in a 5-period binomial tree model with collateral. Other constant parameters are $u = 1.2$, $d = 0.8$, $S_0 = 100$, $K = 100$, $\alpha = 0.5$, $h = g = 1/5$, $\gamma = 0.6$, and $r_c = 0.05$.

At first, we check the relationship between the no-arbitrage price interval and the funding rates which include the borrowing and lending rates. The lending rate is varying from 0 to 0.2, and the borrowing rate is varying from 0.2 to 0.4. We observe the same laws as the model without collateral. The lower bound decreases when increasing the borrowing rate, and it is not affected by the lending rate. The upper bound increases when increasing the lending rate, and it is not affected by the borrowing rate.

Even though these two models share all the other parameters except the collateral rate and the collateralization rate, which are unique in the model with collateral, the no-arbitrage prices in these two models are very different from each other. The upper bound and the lower bound for the no-arbitrage price interval in the model with collateral is relatively larger. The reason is that the collateralized option gives the buyer the right to use the collateral to hedge the loss from default. On the other hand, the uncollateralized option is fully exposed to the risk from the counterparty credit risk.

As we noticed in the model without collateral, when we increase the number of periods, the upper bound and the lower bound will become larger. But it is not necessary in the model with collateral. A higher value of N still drives the risk neutral measure close to 1. But the payoff when default happens is not necessarily worse, since a part of the payoff is protected by the collateral. Therefore, increasing the number of periods does not lead to a larger upper bound or lower bound. This is shown in the left graph of Figure 6.4.

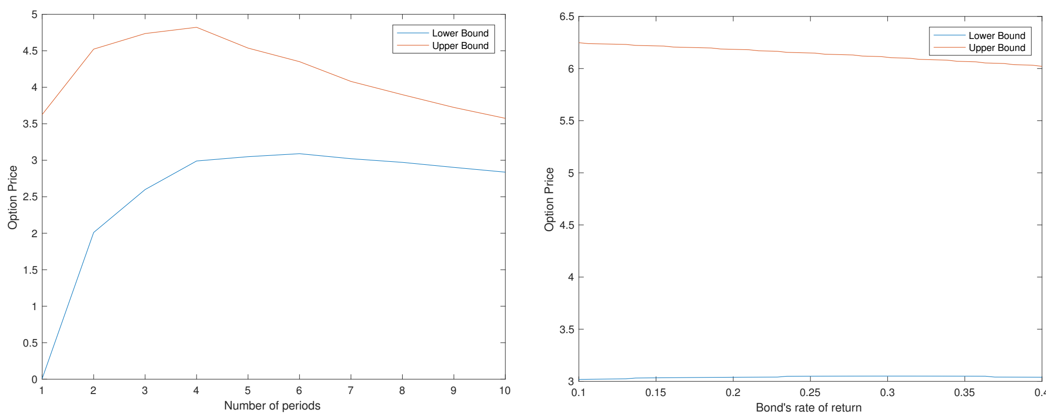

Figure 6.4: XVA of the American put option in the model with collateral when varying the number of periods and the bond's rate of return. The constant parameters in this two model are $u = 1.2$, $d = 0.8$, $S_0 = 100$, $K = 100$, $\alpha = 0.5$, $r_l = 0.1$, $r_b = 0.2$, $h = g = 1/5$, $\gamma = 0.6$, and $r_c = 0.05$. In the left figure, bond's rate of return is $r_m = 0.25$, and the number of periods range from 1 to 10. The right figure is under a 5-period binomial tree model with the bond's rate of return ranging from 0.1 to 0.4.

Bond's rate of return, collateral rate, collateralization rate all reflect the default risk of the option. So we pick up the factor bond's rate of return to find the relationship between the possibility of default and option's price. The interrelation between the bond's rate of return and the option's no-arbitrage price interval is given in the right graph of Figure 6.4. Unlike the model without collateral, the option's price is stable when the bond's rate of return increases. It indicates that the collateral has a good protection of the option from default.

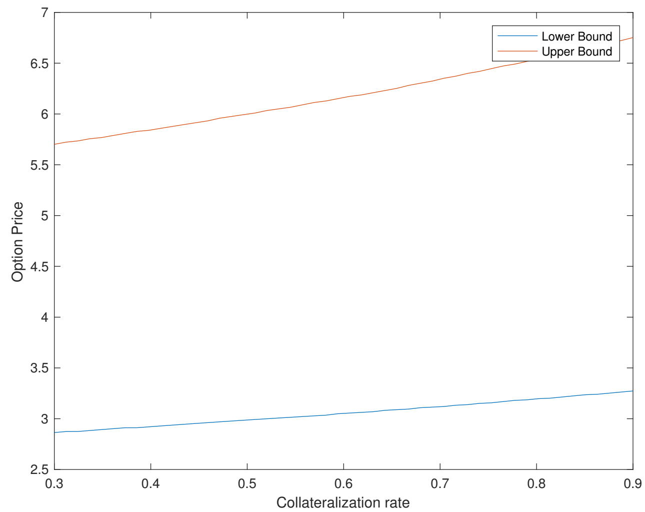

Figure 6.5: XVA of the American put option in a 5-period model with collateral when varying the collateralization rate from 0.3 to 0.9. Other constant factors are $u = 1.2$, $d = 0.8$, $S_0 = 100$, $K = 100$, $\alpha = 0.5$, $r_l = 0.1$, $r_b = 0.2$, $h = g = 1/5$, $r_m = 0.25$, and $r_c = 0.05$.

A larger collateralization rate can offer the option a better protection against the default. Figure 6.5 demonstrates the relationship between the no-arbitrage price and the collateralization rate in this model. A larger collateralization rate leads to a higher upper bound and lower bound.

Chapter 7

Conclusion

Our research has succeeded in extending the XVA from European options to American options in the discrete time settings. Unlike the base model, this new approach takes the funding spread and counterparty credit risk into consideration. In addition to the stocks and MMAs in the base model, we cover the collateral and bond to hedge or super-hedge the American option's payoff.

Before researching the American option pricing, we have thoroughly analyzed and discussed the no-arbitrage price of the European option with funding spread. We have made a comparison between hedging and super-hedging portfolios given by different market conditions and funding costs. We then started to explore the pricing model for American options.

During the research, we first built the one-period binomial tree model including the funding spread. Then, we extended it to the multi-period model. Moreover, counterparty credit risk is considered in the next step separately. This model uses the bond issued by the counterparty to hedge the risk of default. Each of these two factors was examined individually in the discussion above. Based on the results of the research, we divide the periods into trading periods and default periods in order to integrate funding spread and default into one model. Additionally, instead of calculating the option price directly, we computed the price of the portfolio consisting of the option and collateral in the last model. According to the relationship between collateral and the option value, we came up with the no-arbitrage price of the American option. These findings add substantially to our understanding of American option pricing.

Bibliography

[1] Kaushik I. Amin and James N. Bodurtha. "Discrete-time valuation of American options with stochastic interest rates". In: *Review of Financial Studies* 8.1 (1995), pp. 193–234.

[2] Maxim Bichuch, Agostino Capponi, and Stephan Sturm. "Arbitrage-free XVA". In: *Mathematical Finance* (2017). ISSN: 1467-9965. DOI: 10.1111/mafi.12146. URL: http://dx.doi.org/10.1111/mafi.12146.

[3] Catherine Bonner and Jeremiah Campanelli. "Arbitrage-Free Pricing of XVA for Options in Discrete Time". Major Qualifying Project. Worcester Polytechnic Institute, 2016.

[4] John C. Cox, Stephen A. Ross, and Mark Rubinstein. "Option pricing: A simplified approach". In: *Journal of Financial Economics* 7.3 (1979), pp. 229–263.

[5] Mark H.A. Davis, Vassilios G. Panas, and Thaleia Zariphopoulou. "European option pricing with transaction costs". In: *SIAM Journal on Control and Optimization* 31.2 (1993), pp. 470–493.

[6] Hans Föllmer and Alexander Schied. *Stochastic finance: an introduction in discrete time*. Walter de Gruyter, 2011.

[7] Joan F. Garrett. *Banks & Their Customers*. Oceana Publications, 1995.

[8] John C. Hull. *Options, futures, and other derivatives*. Pearson/Prentice Hall, 2006.

[9] *Japan's negative interest rates explained*. 2016. URL: https://www.nytimes.com/2016/09/21/business/international/japan-boj-negative-interest-rates.html.

[10] Hyun Don Lee and Sang-Bum Park. "An Empirical Study on the Factors Affecting Savings Bank Loan Interest Rates". In: *International Journal of Economics and Finance* 8.12 (2016), p. 175.

[11] L.C.G. Rogers. "Monte Carlo valuation of American options". In: *Mathematical Finance* 12.3 (2002), pp. 271–286.

[12] Steven Shreve. *Stochastic calculus for finance I: the binomial asset pricing model*. Springer Science & Business Media, 2012.