Pricing Mortgage-Backed Securities using Prepayment

Functions and Pathwise Monte Carlo Simulation.

By

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Abstract.

To value any fixed income security one needs to evaluate the discounted expected cash flows according to an arbitrage free interest rate model. In the case of mortgage-backed securities the future cash flows are uncertain due to mortgagors exercise of their prepayment options.

The present project considers prepayments which result from interest rate dependent complete refinancing of mortgages in a pool. The rate of refinancing is modeled as an arbitrary, user defined function of current and past interest rates. This enables the inclusion of refinancing rates that depend on not only on the current level of interest rates but also on the trend of the interest rates and that may also exhibit burnout effects due to past periods of low interest rates. The resulting cash flows depend on the entire past of the path that the interest rates took to get to the current level.

The Black-Derman-Toy arbitrage free binomial tree is used to model the underlying interest rates. This is a single-factor market price consistent model which also allows the specification of the observed volatilities.

Monte Carlo methodology is used to simulate random paths in the interest rate tree to evaluate the cash flows along the path.

A computer program written in MAPLE implements the entire process.

ACKNOWLEGDEMENTS

All thanks go to Professor Domokos Vermes who walked me through this project and my entire graduate studies. Professor you name spur me on!

CHAPTER 1

INTRODUCTION

1.1 Overview of the Mortgage Market

A *mortgage* is a loan secured by the guarantee of some specific real estate property and is a contractual agreement between the lender (mortgagee) and the borrower (mortgagor) that pledges the property to the lender as a security for the repayment of the loan through series of payments. The mortgage also entitles the lender the right of foreclosure on the loan if the borrower fails to make the contracted payments.

The types of real estate property that can be used as collateral (mortgaged) are divided into two broad categories: residential and non-residential properties. Residential properties include single-family structures, such as houses that accommodate one to four families, and multi-family structures, like condominiums, cooperatives, and apartments where more than four families reside. Nonresidential properties include commercial structures such as office buildings, retails malls, hotels, assisted care facilities and farm properties.

The mortgages can also be divided into two types of loan: conventional loans and non-conventional loans. A non-conventional loan is one that is backed by the full faith and guarantee of the United States Government. Such loans are provided by federal agencies such as Federal housing administration (FHA), The Veterans Administration (VA) and the Rural Development Administration (RDA). Conventional loans are those that do not carry any form of government guarantee.

The market where these funds are borrowed is called the mortgage market and it is divided into the primary and secondary mortgage market.

The primary market provides actual loans to borrowers, where as the secondary market channels liquidity into the primary market by way of purchasing packages or pools of loans from lenders[1]. Innovations have occurred in terms of design of new mortgage instruments in the primary market and the development of products that use pools of mortgages as collateral for the issuance of securities in the secondary market. Such securities are called mortgage-backed securities (MBS) and may be sold to investors either as pass-through or in structured form, known as Collateralized Mortgage Obligations (CMOs), to meet specific prepayment, maturity, and volatility trenching requirement of the investor.

The focus of this project is to look at the nature of MBS and the effective methods of valuing them in the secondary market. However before we go into the actual valuation methodology we will look further into the structure of the primary market and how it affects operations in the secondary market. The mortgage sector is the by far the largest of the debt market. The U.S. national homeownership rate is now at 67%[2] and this has led to significant developments in the mortgage industry.

[1] This project is centered on some activities of the secondary market.
[2] U.S. Department of Housing and Urban Development, April 1999.

1.2 The Mortgage Industry

The mortgage industry can be categorized into for groups: mortgage originators, mortgage servicers, mortgage insurers and mortgage investors, with the key players being commercial banks, thrift institutions, money managers, pension funds, insurance companies, security dealers, trust departments, corporate treasury departments, corporations and private investors.

Mortgage Originator: the original lender of the mortgage loan is called the mortgage originator. The three largest originators for all types of residential loan in the U.S. are commercial banks, thrifts, and mortgage bankers, originating more than 95% of annual mortgage originations. Originators make their money by basically charging what they call origination fee, which is based on some percentage basis points of the par value of the loan[3].

Mortgage Services: every mortgage loan, both securitized and non-securitized must be serviced. Servicing a loan entails the collection of monthly payments and forwarding the proceeds to owners of the loan, sending payment notices to mortgagors, reminding mortgagors when payments are overdue, maintaining records of principal balances, administering an escrow balance for real estate taxes and insurance purposes, initiating foreclosure proceedings if necessary, and furnishing tax information for mortgagors if applicable. Mortgage servicers include banks and commercially related entities and mortgage bankers. Servicers receive their revenue from several sources. The primary source is called servicing fee which is some percentage of the outstanding mortgage balance, and declines over time as the mortgage amortizes.

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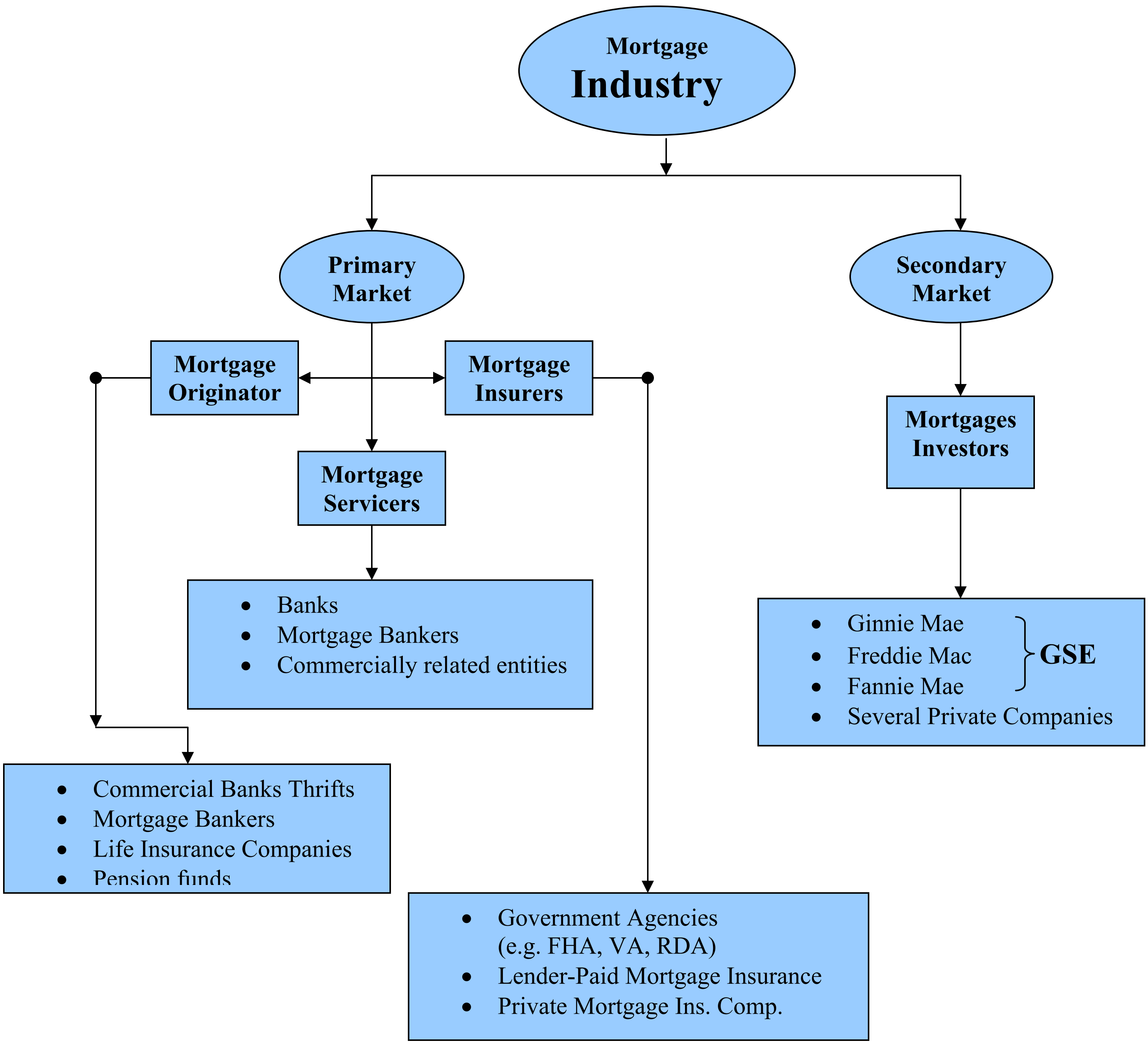
[3] For in-depth information about origination activities see Handbook Of MBS by Frank J Fabozzi-5th ed.

Mortgage Insurers: mortgage insurance protects the lender against loss in the event of default by the borrower. Hence insurance at the loan level minimizes the credit risk of the loan.

The amount of mortgage insurance varies as it is dependent on the type of loan and term of loan but it is usually required on loans with loan-to-value ratio greater than 80%. The amount insured may be some percentage of the loan and may reduce as the LTV declines. By law, as the LTV declines below 80% the mortgage insurance must be lifted. Although the lender requires the insurance its cost is borne by the borrower, usually through a high contract rate. At the loan level there are three main types of insurance: insurance provided by the government agency which applies to non-conventional loans only, regular private mortgage insurance and lender-paid mortgage insurance. When mortgages are pooled by private conduit and securities are issued, additional insurance for the pool is typically obtained to enhance the credit of the security (see footnote 3 for a complete description of mortgage insurance and the insurers.).

Figure 1 shows a diagram for the mortgage industry.

Figure 1


Mortgage
Industry
Primary
Market
Secondary
Market
Mortgage
Originator
Mortgage
Insurers
Mortgage
Servicers
Mortgages
Investors
Banks
Mortgage Bankers
Commercially related entities
Ginnie Mae
Freddie Mac
Fannie Mae
GSE
Several Private Companies
Commercial Banks Thrifts
Mortgage Bankers
Life Insurance Companies
Pension funds
Government Agencies
(e.g. FHA, VA, RDA)
Lender-Paid Mortgage Insurance
Private Mortgage Ins. Comp.

1.3 Type of Mortgage Loans

The mortgage industry has undergone a massive evolution since the great depression in the 1930s. Back then the type of mortgage loans given resembled balloon loans in which the principal was not amortized, or only partially amortized at the maturity date. There were a lot of inefficiencies in the market in that the banks that issued these loans could ask for repayment of the outstanding balance on demand or upon a short notice, even if the mortgagor was fulfilling his or her obligation. Upon establishing the FHA, new regulations regarding mortgage loans were enacted and led to the development of series of mortgage products. Since the type of mortgage loan and the cash flow it carries with it has a significant effect on the overall performance of a mortgage pool and for that matter the securities embedded with them, we will look at some very common and widely traded mortgage instruments and how their characteristics affect their cash flows.

Fixed-Rate, Level-Payment, Fully Amortized Mortgage

The basic idea behind the design of the fixed-rate, level-payment, fully amortized mortgage is that the borrower pays interest and principal in equal installments over the term of the loan. Typically payments are done monthly and by the end of the loan term the mortgage is fully amortized. Each monthly payment for a level payment mortgage is due on the first day of each month and consists of

1. Interest of 1/12 of the fixed annual interest rate times the amount of outstanding mortgage balance at the beginning of the previous month.
2. And a payment of some fraction of the principal.

The difference between the scheduled monthly payment and the scheduled interest payment, gives the portion of the principal that has being repaid.

This product is designed so that if the last scheduled monthly payment is made, the outstanding mortgage balance is zero.

To illustrate a fixed-rate, level-payment, fully amortized mortgage, we consider a 30-year (360months), $100,000 mortgage with a 2.32% mortgage rate. The monthly mortgage payment would be $18,046 (See section 1. for complete mortgage mathematics) Table 1 shows how monthly payment is divided into interest payment and principal payments (design table 1). Table 1 is called the Amortization Table. The complete cash flow of every monthly mortgage payment consists of

- The mortgage servicing fee which is the fee charged to provide services such as collecting monthly proceeds and maintaining balance records. It is the main source of revenue for mortgage servicers.
- The interest payment net of the servicing fee. The servicing fee is a portion of the mortgage rate. For instance if the mortgage rate is 6.125% and the servicing fee is 50 basis points, then the investor or the originator receives interest of 5.625%. This is called the net coupon on the loan.
- The scheduled principal payment. Borrowers can make payments in excess of the scheduled principal payments. This is called *prepayment.* We shall look more closely at prepayments later. The effect of prepayment is that the amount and timing of the mortgage cash flow is not known with certainty.

Table 1.*Amortization Table for a Fixed-rate Level Payment Mortgage*

Term	Beginning Mortgage Balance	Monthly Mortgage Payment	Interest For Month	Principal Repayment	Remaining Balance
					$100,000.000
1	$100,000.000	$840.854	$791.667	$49.188	99950.8125
2	99950.812	840.854	791.277	49.577	99901.2355
3	99901.236	840.854	790.885	49.969	99851.2661
4	99851.266	840.854	790.489	50.365	99800.9011
5	99800.901	840.854	790.090	50.764	99750.1374
6	99750.137	840.854	789.689	51.166	99698.9718
7	99698.972	840.854	789.284	51.571	99647.4011
8	99647.401	840.854	788.875	51.979	99595.4222
9	99595.422	840.854	788.464	52.390	99543.0317
10	99543.032	840.854	788.049	52.805	99,543.030
…	…	…	…	…	
...	…	…	…	…	
100	92542.947	840.854	732.632	108.223	92434.7243
101	92434.724	840.854	731.775	109.079	92325.6450
102	92325.645	840.854	730.911	109.943	92215.7021
103	92215.702	840.854	730.041	110.813	92104.8889
104	92104.889	840.854	729.164	111.690	92,050.320
…	…	…	…	…	
…	…	…	…	…	
200	76372.161	840.854	604.613	236.241	76135.9194
201	76135.919	840.854	602.743	238.112	75897.8079
202	75897.808	840.854	600.858	239.997	75657.8114
203	75657.811	840.854	598.958	241.897	75415.9149
204	75415.915	840.854	597.043	243.812	75172.1033
205	75172.103	840.854	595.112	245.742	74926.3616
206	74926.362	840.854	593.167	247.687	74678.6744
207	74678.674	840.854	591.206	249.648	74429.0264
208	74429.026	840.854	589.230	251.624	74177.4020
209	74177.402	840.854	587.238	253.616	
…	…	…	…	…	
…	…	…	…	…	
…	…	…	…	…	
352	7276.639	840.854	57.607	783.247	6493.3914
353	6493.391	840.854	51.406	789.448	5703.9432
354	5703.943	840.854	45.156	795.698	4908.2452
355	4908.245	840.854	38.857	801.997	4106.2480
356	4106.248	840.854	32.508	808.346	3297.9016
357	3297.902	840.854	26.108	814.746	2483.1558
358	2483.156	840.854	19.658	821.196	1661.9599
359	1661.960	840.854	13.157	827.697	834.2629
360	834.263	840.854	6.605	834.250	0.0000

Adjustable-Rate Mortgages

An adjustable rate mortgage (ARM) is a loan in which the mortgage rate is retuned periodically in accordance with some appropriate chosen reference rate. This instrument was specifically developed to deal with mismatch between mortgage durations and other liabilities in a high interest rate environment. ARMs usually start with lower interest rates and are reset in accordance with some index rate, such as the U.S. Treasury securities, London Interbank Offered Rate (LIBOR), the Eleventh (11th) District Cost of Funds (COFI or ECOFI), or the prevailing prime rate. To encourage borrowers to accept ARM rather than fixed rate mortgages, originators generally offer an initial contract rate that is less than prevailing market mortgage rate. A one-year ARM typically offers 100 basis points spread over the index rate. For example suppose the index rate is 5.625%, then the initial contract rate for the ARM is 6.625%. However, the originator might set the initial contract rate at 6.125, a rate 50bps below the current value of the reference rate plus the spread. This kind of rate is called the *teaser rate*. The monthly mortgage payment and for the matter the investors cash flow are affected by: periodic rate caps and floors, which is the limit amount that the contract rate may increase or decrease at the reset date. The most common rate cap on annual reset loans is 200bps or 2%. There is another form of ARM, which is gaining a considerable amount of popularity, and it is called the Fixed/ Adjustable-rate mortgage. This is a hybrid of a fixed-rate mortgage and an adjustable rate mortgage. The loan is fixed for a specified period (usually 3,5,7 or 10 years) and then resets annually afterwards. Thus the fixed/ARM hybrid turns into a one-year index (Treasury) ARM after its fixed period.

The cash flow for ARMs is more complicated than that of fixed-rate, level-payment mortgages[4]. For more information on computing cash flows for ARMs see the reference in footnote

Balloon Mortgages

In a balloon mortgage the borrower is given long-term financing by the lender but at specific future dates the mortgage rate is renegotiated. Many single-family balloon originated today carry fixed rate and a 30-year amortization schedule. They typically require a balloon payment of the principal outstanding on the loan at the end of 5 or more years. Balloon mortgages are attractive to borrowers because they offer mortgages rates that are significantly lower than generic 30-year mortgages. Nowadays many balloon mortgages contract are actually hybrids that contain provisions allowing the borrower to take out a new loan from the current lender to finance the balloon payment with minimum requalification requirements. For instance for a new loan to qualify for a Fannie Mae pool, the borrower receiving the new loan to finance a balloon payment must not have been delinquent on payments at any time the 12 preceding months, must still be using be using the property as primary residence, and must have incurred no new liens on the property. The interest rate on the new loan must be no more than 500bps greater than the rate on the balloon loan.

As has being pointed out earlier, the growing complexity of lending and borrowing has led to the development of more complicated mortgage products to basically cater for specific individual needs and requirements. Most of these products are however prevalent in the secondary mortgage market. These include but not limited to “Two Step” Mortgage

[4] See Hand Book on ARMs by the Federal Reserve Board Office of Thrift Supervision for a complete discussion of ARMs.

Loans, Rate Reduction Mortgages (RRMs), Reverse Mortgages- designed basically for senior homeowners who want to convert home equity into cash, and the Growth "Alternative" Mortgages.

1.4 Mortgage Mathematics

General Mortgage Cash flow Calculations:

Monthly Payment For a Fixed-Rate Level payment mortgages the monthly payment is

$$X_i = \frac{L_0\left(\frac{r}{1200}\right)\left(1+\frac{r}{1200}\right)^N}{\left[\left(1+\frac{r}{1200}\right)^N - 1\right]}$$

Where X_i = monthly payment for month i

L_0 = Original Balance or Loan amount.

r = mortgage (coupon) rate (%)

N = original loan term in months (say 180months)

Remaining Balance The remaining balance after i months is

$$L_i = \frac{Lo\left[\left(1+\frac{r}{1200}\right)^N - \left(1+\frac{r}{1200}\right)^i\right]}{\left[\left(1+\frac{r}{1200}\right)^N - 1\right]}$$

Where L_i = the remaining balance at the end of the ith month.

Principal Payment The amount of principal paid in month i is given by

$$P_i = \frac{L_0\left(\frac{r}{1200}\right)\left(1+\frac{r}{1200}\right)^{i-1}}{\left[\left(1+\frac{r}{1200}\right) - 1\right]}$$

Where P_i = principal paid in month i

Interest Payment The amount of interest paid in month i can be represented as

$$I_i = \frac{L_0\left(\frac{r}{1200}\right)\left[\left(1+\frac{r}{1200}\right)^N - \left(1+\frac{r}{1200}\right)^{i-1}\right]}{\left[\left(1+\frac{r}{1200}\right)^N - 1\right]} = L_{i-1}\left(\frac{r}{1200}\right)$$

Where I_i = interest paid in month i

It should be noted that

$$r = S + C$$

Where S = service fee (%)

C = net coupon (%) as was described in section 1.3

Thus the servicing amount would be computed as

$$\textit{servicing} \quad \text{fee} = \left(\frac{\mathrm{S}}{\mathrm{C} + \mathrm{S}} \right) I_i$$

and the cash flow for the security holder for month I is given by

$$\textit{cashflow}_i = P_i + I_i - \textit{servicing} \ \text{fee} = P_i + \left(\frac{\mathrm{S}}{\mathrm{C} + \mathrm{S}} \right) I_i$$

CHAPTER 2

2.1 Mortgage- Backed securities

Mortgage-backed securities (MBS) are securities backed by a pool (collection) of mortgage loans. In chapter one we looked at an overview of mortgage loans and the mortgage market, which is the raw material for mortgage-backed securities. While any type of mortgage loans, residential or commercial, can be used as a collateral for a mortgage-backed security, most are backed by residential mortgages. Just as the value of any other type of security depends on the cash flow of the underlining asset, the value of mortgage-backed securities depends on the cash flow of the underlining mortgage loans. It suffice to say therefore that different types of mortgage loans comes with different cash flows and hence affect the value of the MBS differently. This chapter is intended to give an overview of the variety of mortgage-backed securities and the type of mortgage loan that characterizes them. Mortgage-backed securities include the following

2.2 Types Of Securities Backed by a Mortgage

Mortgage Passthrough securities: Passthrough securities are created when mortgages are pooled together and participation certificates in the pool are sold. Typically the mortgages backing a Passthrough security have the same loan type (fixed-rate, level payment, ARM, etc) and are similar enough with respect to maturity and loan interest rate to permit cash flow to be projected as if the pool was a single mortgage loan.

A pool may consist of several thousands of mortgages or only a few mortgages. The cash flow consists of monthly mortgage payments representing interest, scheduled principal repayment, and any prepayment. The monthly cash flows for a pass-through are less than the monthly cash flow of the underlying mortgage cash flow by an amount equal to the servicing and other fees. The other fees are fees charged by issuer or guarantor of the pass-through for guaranteeing the issue. The coupon rate of the pass-through, called the Passthrough coupon rate, is less than the mortgage rate on the underlying pool of mortgages by an amount equal to the servicing fee and guarantee fees. Not all the mortgages that are included in a pool that are securitized have the same mortgage rate and the same maturity. Consequently when describing a pass-through security, a weighted average coupon rate (WAC) obtained by weighting the mortgage rate of each mortgage loan on the pool by the amount of mortgage balance outstanding, a weighted average maturity found by weighting the remaining number of months to maturity for each mortgage loan in the pool by the amount of mortgage balance outstanding.

Mortgage originators actively pool mortgages and issue pass-throughs. The vast majority of regularly traded pass-throughs are issued and/or guaranteed by federally sponsored agencies: the Government National Mortgage Association (GNMA) or -

“Ginnie Mae”; the Federal National Mortgage Association (FNMA) or “Fannie Mae”; and the Federal Home Loan Mortgage Corporation (FHLMC), or “Freddie Mac”. A significant volume of mortgages is directly purchased, pooled, securitized by Fannie Mae and Freddie Mac.

The price of a pass-through MBS is the present value of the projected cash flows discounted at the current yield required by the market, given the specific interest rate and prepayment risk of the security in question.

Collateralized Mortgage Obligations: In 1983, a dramatic fall in mortgage rates and surging housing market caused mortgage originators to double[5]. Much of this production was sold in the capital markets; pass-through issuance increased by 58% in 1982. To accommodate this out pour in supply, financial investors designed a security that will broaden the existing MBS investor base. By the middle of 1993, the Federal Home Loan Mortgage Corporation (Freddie Mac) has issued the first CMO, a $1 billion, three class structure that offered short, intermediate, and long term securities produced from the cash flow of pool of mortgages. This instrument allowed more investors to become active in the MBS market. For instance, banks could participate in the market more efficiently by buying short-term mortgage securities to march their short-term liabilities (deposits). An investor in a mortgage pass-through security is exposed to prepayment risk. By redirecting how the cash flows of pass-through securities are paid to different bond classes CMOs provide a different exposure to prepayment risk.

The basic principle is that redirecting cash flows (interest and principal) to different bond classes, called trenches, alleviates different forms of prepayment risk.

[5] See Handbook on MBS-chapter 9 by the Mortgage Research Group, Lehman Brothers Inc.

It is never possible to eliminate prepayment risk. In order to develop realistic expectation about the performance of CMO bond, an investor must first evaluate the underlying collateral, since its performance will determine the timing and size of the cash flows reallocated by the CMO structure.

Agency and whole-loan CMOs have distinct collateral (mostly individual home mortgages, which are already pooled and securitized in a pass-through form, but whole-loan CMOs issuers create a structure directly based on the cash flow-
of a group of mortgages), credit- GNMA (Ginnie Mae). Freddie Mac and Fannie Mae are three U.S. government sponsored agencies, which guarantee the full and timely payment of all principal and interest due from pass-throughs issued under their names. GNMA securities, like U.S. Treasury securities, are backed by the full faith and credit of the U.S. government.

Striped Mortgage-Backed Security: altering the distribution of principal and interest from a pro rata distribution of pass-through security to an unequal distribution creates a striped mortgage-backed security. The most common type of striped mortgage-backed security is one in which all the interest is allocated to one class (called the interest only or IO class) and the entire principal to the other class (called the principal only or PO class). The IO receives no principal payment and the PO receives no interest payment. The IO gets all the interest payment made by the borrower and the PO gets all the principal payment made by the borrower. To illustrate how strips work in general, let us consider a \$100 face value of a newly issued or current mortgage pool carrying an interest rate of 9%. This security can be divided into the following derivatives.

The first, the *discount strip,* receives claim to $50 of the $100 face value and to $2 of every $9 in interest payments. This effectively creates a security with an interest rate of $2/$50 = 4%. Since this rate is well below the current mortgage rate, this derivative is appositely named discount strip.

The second strip, the *premium strip* receives $7 of every 9$ in interest payments and has a claim of $50 of principal, for an effective rate of 14%. Since the cash flow of $50 of each strip add up to the cash flow of the underlying mortgage, any term structure model will predict the sum of the prices of $50 of each strip will equal the price of the underlying mortgage.

CHAPTER 3

3.1 Using BDT-model to model the evolution of the short rate

The term structure of interest rates is defined as the relationship between interest rates and the maturities of the underlining securities. It is the theoretical spot rate (zero coupon) curve implied by today's Treasury securities. There are many term structures dealing with the different types of fixed-income instruments. The term structure describes the behavior of the market and the interest rate in a simple tree. Term structure consistent models is the term given to models that take into account the entire evolution of interest rates and their volatility in a way that is automatically consistent with some observed market data. In order to successfully analyze interest rate derivatives such as mortgage-backed securities, one needs a model that has a high degree of analytical tractability and can easily be calibrated. There are many such models in practice but for the purpose of this project we consider the Black, Derman, and Toy (BDT for short) model for the following reasons and assumptions:

- It is a single –factor short-rate model that matches the observed term structure of spot interest rate volatilities, as well as the term structure of interest rates.
- The model is developed algorithmically, describing the evolution of entire term structure in a discrete-time binomial lattice framework. A binomial tree is constructed for the short rate in such a way that the tree automatically returns the observed yield function and the volatility of different yields.
- The model assumes that the source of uncertainty in a term structure is randomness of the short rate.
- It also assumes that changes in the short rates are log-normally distributed, the resulting advantage being that interest rates cannot become negative.
- The BDT model approximates a continuous process by using a recombining tree.

The BDT model stipulates that the instantaneous short rate at time t is given by:

$$\mathrm{r}(t) = \mathrm{M}(t)\,\mathrm{e}^{(\sigma(t)\,\mathrm{z}(t))} \quad \text{..........................(1)}$$

Where M (t) is the median of the (lognormal) distribution for r at time t $\sigma(t)$ is the level of short rate volatility and $Z(t)$ is the level of Brownian motion. Thus $\sigma(t)Z(t)$ is a normal random variable with an expected value of zero and a variance of $\sigma(t)^2 * \Delta t$.

Therefore r (t) follows a lognormal distribution, while M (t) is a deterministic function with a degree of freedom that allows the fitting of evolution of the term structure to the observed prices of bonds. M (t) could be solved by designing a binomial tree that approximates the distribution of $\ln r(t)$, which is a normal distribution. To begin the approximation we utilize Ito's Lemma to uncover the stochastic instantaneous increments of $\ln r(t)$. The stochastic differential equation stipulating the stochastic increments of In(r) is thus:

$$d\ln r = \left(\frac{\partial}{\partial t} \ln M(t) - \frac{\partial}{\partial t} \ln \sigma(t) \left(\ln M(t) - \ln r \right) \right) dt + \sigma(t) dz \quad \text{..................2}$$

A few implications are deduced from the equation above. If sigma is constant, then

$\frac{\partial}{\partial t}\sigma(t) = 0$, And the process of dInr is a Brownian motion but with a drift that is function of M (t) only. If, on the other hand σ(t) is a decreasing function of time, then $-\frac{\partial}{\partial t}\sigma(t)$ is positive, which induces a reversion of In(r) to lnM (t).

The BDT model is actually a discreet model that approximates a continuous process described in equation (2). It does so by a recombining binomial tree. It starts by specifying the length of a period dt of each one-period binomial trees composing of the total period. $\mathrm{Inr(t+dt) - Inr(t)}$ is a random variable that takes on two values, U in state UP and V in state DOWN, each with a probability 0.5

$$\mathrm{U} = \mu(\mathrm{t}) * \mathrm{dt} + \sigma(\mathrm{t}) * \mathrm{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

And $$\mathrm{V} = \mu(\mathrm{t}) * \mathrm{dt} - \sigma(\mathrm{t}) * \mathrm{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Where $$\mu(\mathrm{t}) = \frac{\partial}{\partial \mathrm{t}}(\ln\mathrm{M(t)}) - \frac{\partial}{\partial \mathrm{t}}\ln\sigma(\mathrm{t})(\ln\mathrm{M(t)} - \ln\mathrm{r}) \,.\, \ldots\ldots\ldots\ldots (5)$$

Since $\sigma(\mathrm{t})$ is positive, the UP state is a state where $\mathrm{lnr(t+dt)} > \mathrm{lnr(t)}$

And a DOWN state is a state where $\mathrm{lnr(t+dt)} < \mathrm{lnr(t)}$.

The function $\sigma(\mathrm{t})$ specifies the volatility of short interest rate at time t, referred to as term structure of volatility and its assumed to be known. The value of $\mu(\mathrm{t})$ is constrained by the requirement that the tree will be recombining, that the no-arbitrage conditions will be satisfied, and that the observed prices will be consistent with the evolution of the term structure. Using the notation i to replace t for the discreet time and j for the number of up movement since time zero we can obtain r (i, j), the short interest rate prevailing in the market at time i if there were j up movements since time zero. Assume that at time i-1 the realization of the state of nature was j, at time i the short interest rate will be r (i, j+1) if an up movement is realized or r (i, j) if down movement is realized i.e.

$$\ln r(i, j+1) = \ln r(i-1, j) + \mu(i)\Delta t + \sigma(i)\sqrt{\Delta t} \quad \text{……………(6)}$$

Or

$$\ln r(i, j) = \ln r(i-1, j) + \mu(i)\Delta t - \sigma(i)\sqrt{\Delta t} \quad \text{………………..(7)}$$

Which implies that:

$$\frac{\ln r(i, j+1) + \ln r(i, j)}{2} = \sigma(i)\sqrt{\Delta t} \quad \text{………………………(8)}$$

The structure adopted by the binomial approximation imposes a relation on the different realization of the short rate at time i and thereby, on $\mu(\mathrm{t})$. Specifically, the rate r (i, 0) is related to r (i, 1) via the eqn (9) i.e.

$$r(i,1) = r(i,0)e^{2\sigma(t)\sqrt{\Delta t}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Since the binomial tree is required to be recombining, an up movement from (i, 0) and a down movement from (i, 1) end up at the same state namely (i+1,1). Following the same logic, the general relation

$$r(i,j) = r(i,0)e^{2j\sigma(i)\sqrt{\Delta t}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(10)$$

For every i and j=0…i, is obtained. Consequently when the function $\sigma(\mathrm{t})$ is specified, the realization of the short interest rate at time i is a function only of r (i, 0). Given the term structure of interest rates, we can solve for numerical value of r (i, 0), while ensuring the satisfaction of the no-arbitrage condition and the consistency with the observed bond prices. At time i=1 there are two possible realization of the short rate

r (1 ,0) and r (1,1) that are related to each other by

$$r(1,1) = r(1,0)e^{2\sigma(1)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Assuming a zero-coupon bond with a face value of $1, the bond maturing at time i=2 pays a dollar at that time regardless of the state of nature. Hence, if at time i=1 state one is realized, i.e., the process is at node (1,1), the price of this bond will be $e^{r_{1,1}}$. If at time 1 state 0 is realized, i.e., the process is at node (1,0), the bond maturing at time 1 will have a price of $e^{r_{1,0}}$. At time 0 the observed price of bond 1 is d(1) and that of bond 2 is d(2). To avoid arbitrage, the price of bond at time 0 should be the discounted expected value of its price at time 1. Hence for bond 1 and 2 we do obtain

$$d(2) = d(1)\left(\frac{1}{2}e^{-r_{1,1}} + \frac{1}{2}e^{-r_{1,0}}\right) \quad \text{.............(12)}$$

Substituting for r(1,1) in terms of r(1,0), based on equation (9), yields equation

$$d(2) = d(1)\left(\frac{1}{2}e^{\left(-r_{1,0}e^{(2j\sigma_i)}\right)} + \frac{1}{2}e^{-r_{1,0}}\right) \quad \text{............(13)}$$

where the only unknown is r(1,0). Solving equation (13) for r(1,0), we can recover r(1,1) in a way that the evolution of the short rate complies with the no-arbitrage conditions and consistent with the observed bond prices. Knowing r(0,1) and r(1,0) allows us to solve for u(1) and M(1).

However knowing the values of u(1) and M(1) is not needed in order to value derivatives securities based on the generated binomial tree. Once the evolution of the tem structure is determined, the tree can be used to value mortgage backed securities.

At node (2,j) the price of the bond will

$$e^{(-r_{2,j})} \quad \text{……………………………………(14)}$$

Hence, the price of it at time zero can be calculated discounting by d(i) the expected value of the price as of time i . The probability of arriving to node j at time 2 is

$$\binom{2}{j}\left(\frac{1}{2}\right)^{2} \quad \text{………………….(15)}$$

Thus, the discounted expected value of the bond is

$$\sum_{j=0}^{2}\binom{2}{j}\left(\frac{1}{2}\right)^{2} e^{(-r(i,j)} \quad \text{…. …………(16)}$$

and must satisfy

$$d_3 = d_2 \sum_{j=0}^{2}\binom{2}{j}\left(\frac{1}{2}\right)^{2} e^{(-r(2,j)} \quad \text{…………………….(17)}$$

If we substitute for $e^{(-r_{2,j})}$ in terms r (1,0) we obtain equation

$$d_3 = d_2 \sum_{j=0}^{2} \binom{2}{j} \left(\frac{1}{2}\right)^2 e^{(-r(2,0)e^{2j\sigma_2}} \qquad \text{..............(18)}$$

Which can be solved to recover the value of r (1,0). So what we are saying so far is that in general, given the current term discount factor function **d (.)**, the evolution of the of the term structure from time i to time i+1 is given by

$$d_{i+1} = d_i \sum_{j=0}^{i} \binom{2}{j} \left(\frac{1}{2}\right)^i e^{(-r(i,0)e^{2j\sigma_i}} \qquad \text{.........................(19)}$$

For i=1…N-1, where the length of a time period is dt and the number of periods in the model is N. Equation (19) stipulates the value of r (i, 0) since it is the only unknown variable in the equation. The value of r (i, j) for j=1…i, was given by equation 3 and is repeated below as

$$r_{i,j} = r_{i,0} e^{(2j\sigma_i\sqrt{\Delta t}} \qquad \text{.............................(20)}$$

It must be emphasized that many practitioners when using the BDT model, set the short-rate volatility to a constant and so only fit the model to the yield curve. In this case the stochastic differential equation and the level of short rate of equations 1 and 2 are respectively:

$$\mathrm{r(t)} = M\,(t)e^{(\sigma \mathrm{Z}(t))}$$
$$d \ln r = \sigma(t)dt + \sigma dZ$$

The process for this case follows the same as the one described above for time dependent $\sigma(t)$.

We will now illustrate the process discussed above with a numerical example utilizing an equivalent market of a flat term structure of 5 per and with annual time increments for an evolution of the short rate. We also fit a declining volatility structure, which declines from 10 per cent after one year to 1 per cent after eight years. The initial yield curve is given in a list labeled ini_yld_curve.

ini_yld_curve: = [0.05, 0.05, 0.05, 0.05,0.05,0.05,0.05,0.05,0.05,0.05,0.05].

The volatilities are also given in a list Vol, where vol_i is the volatility of the short rate at time i.

Vol: = [0.1, 0.09, 0.08, 0.07,0.06,0.05,0.04,0.03,0.02,0.00].

We will use a table (a form of an array in MAPLE) to store the values of the short rate and displayed in a spreadsheet also in MAPLE. Table 1 shows the spreadsheet representation of the short rate tree. Each column starting from 0 represent a time step and for each time step i there are 0 to time i up movements of interest rates. The lowermost value in each column indicates the rate at state zero and increases up the column to state i. The rate r[0,0] at time zero is given by the yield or interest rate spanning time interval [0,1].

r [0,0]: = 0.05

To proceed and solve for the short rate r [1,0] (at time 1) we need to solve the equation (13).

Below we ask Maple to solve the equation numerically.

$$> fsolve(d(2))/d(1) = sum('(1/2)^1 * \exp(-r[1,0] * \backslash$$
$$> \exp(j * 2 * sigma[1]))','j' = 0..1), r[1,0], 0..1);$$
$$.044$$

The obtained solution is assigned $r_{1,0}$ and used to recover $r_{1,1}$ using equation (11)

$$> r[1,1] := r[1,0] * \exp(2*1*sigma[1]);$$
$$r_{1,1} := .053$$

In a similar fashion equation (17) is evaluated numerically for $r_{2,0}$ used to uncover $r_{2,j}$

Where j = 1 to 2 using equation (20). Equation (19) and (20) would be used to generate the short rate at each time step state zero, and state j=1 to i respectively, and the result displayed as in table 1.

Table 2

Time	0	1	2	3	4	5	6	7	8	9	10
Yield	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
Vol		0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0
Price	1	0.95	0.91	0.86	0.82	0.78	0.75	0.71	0.68	0.65	0.61
ShortrateTree											
											4.9%
										5.9%	
									6.2%		4.9%
								6.4%		5.6%	
							6.6%		5.8%		4.9%
						6.5%		5.9%		5.4%	
					6.3%		5.9%		5.5%		4.9%
				6.2%		5.8%		5.4%		5.2%	
			5.9%		5.5%		5.4%		5.2%		4.9%
		5.3%		5.2%		5.1%		5.0%		5.0%	
	5.0%		4.9%		4.8%		4.9%		4.9%		4.9%
		4.4%		4.5%		4.6%		4.6%		4.8%	
			4.1%		4.2%		4.4%		4.6%		4.8%
				3.8%		4.0%		4.3%		4.6%	
					3.6%		4.0%		4.3%		4.8%
						3.6%		4.0%		4.4%	
							3.6%		4.1%		4.8%
								3.7%		4.2%	
									3.8%		4.8%
										4.1%	
											4.8%

3.2 Pricing of options on pure discount bonds within the constructed binomial tree.

Once the short-rate has been constructed and we know that rate at every time step and every state of nature is consistent with some observed market data, we can use the binomial tree to derive prices for a wide range of interest rate derivatives basically using a backward induction[6] process.

In general if $C_{i\,j}$ is the value of a contingent claim at node (i, j), then the value at this node is related to the two connecting nodes at time step i+1 according to discounted expectations adjusted for the state index j:

$$C_{i,j} = \frac{1}{2} d_{i,j} [C_{i+1,j+1} + C_{i+1,j-1}] \quad \text{..............................(21)}$$

We will now use the tree to price discount bond options to illustrate the procedure described.

We will describe in steps how to price a T maturity put option on a s-maturity discount bond (T<=s) with a strike price K. Let N and M represent the number of time steps until the maturity of the bond and the option respectively (T=Mdt, s=Ndt). We assume that the short rate tree has being constructed as far as time step N.

[6] Backward substitution ensures that every interest rate at time step i is related to the two connecting nodes at time step i+1

Step1

Let Ps [i, j] represent the value of the s-maturity bond at node (i, j). To price the derivative we set the maturity condition for the bond underlying the option. At time step N the price of the bond maturing at N is the face value of the bond. For simplicity we use a bond, which pays $1 at maturity. Thus Ps [N, j]=1 for all state j at time step N.

Step 2

We then calculate the value of the bond at every time step and every state of the world in the tree using backward induction as follows:

$$Ps_{i,j} = \frac{1}{2} d_{i,j} [Ps_{i+1,j+1} + Ps_{i+1,j-1}] \quad \forall \text{ nodes j at time step i} \quad \ldots(22)$$

For European discount bond option this second step only has to be completed as far back as time step M when the maturity condition for the bond is implemented. For American discount bond option, in order to be able to evaluate the early exercise condition when we perform backwards induction for option price, we continue applying equation (22) back to the root of the tree at

node (0, 0).

Step 3

Next we evaluate maturity condition for the options at all the nodes for time step M:

$$\mathrm{C_{M,j}} = \max\{0, Ps_{M,j} - k\} \ \forall \mathrm{j\,at\,M} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

For European options the call price can be obtained by repeatedly applying equation (21) back through to the origin of the tree.

For American option we need to allow for the possibility of early exercise in the normal way, by taking the maximum of the discounted expectation and the intrinsic value of the option at each node. The American option is given at node (i, j) is given by

$$\mathrm{C_{M,j}} = \max\{Ps_{M,j} - k, \frac{1}{2} d_{i,j}[C_{i+1,j+1} + C_{i+1,j-1}]\} \ \forall \mathrm{j\,at\,i} \qquad \ldots\ldots\ldots.(24)$$

Numerical Example:

We use the short rate tree constructed in the last section to price a six-year European call option on a pure discount bond with maturity 10 years and a strike price of $0.80. The results are summarized in table 3.

In order to price the option we first construct a tree for the 10-year pure discount bond price by setting-

$Ps_{10,j} = 1$ for j = 0 to 10 and then apply equation (22) to obtain the tree labeled *zerocouponprice* in table 3. The price at the origin of the tree is $ 0.621 and it is today's 10-year pure discount bond price.

Secondly we construct the tree for the option price resulting in the lower tree of table 3. We begin by evaluating the maturity condition for the option. For example at node (6,6) we obtain

$$C_{6,6} = \max \{Ps_{5,5} - K, 0\}$$

$$= MAX \{0.09030-0.800,0\}$$

$$= 0.0103$$

The other nodes are worked the same way. The European option price is obtained by applying equation (21), discounting backwards from maturity to the origin. For example the option price at node (2,3) is

C2, 3 = .5*09533*{0.0530 + 0.0269} = 0.0383

And the European option price today is $0.0248.

Table 3

Time	0	1	2	3	4	5	6	7	8	9	10
Yields	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%
Vol		0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
Price	1	0.9524	0.9074	0.8636	0.8224	0.7837	0.7463	0.7107	0.677	0.6447	0.6139
Shortrate Tree											
	0.05	0.0532	0.0599	0.0651	0.0706	0.0787	0.0866	0.0946	0.1048	0.1154	0.1259
		0.0436	0.049	0.0533	0.0578	0.0645	0.0709	0.0775	0.0858	0.0945	0.1031
			0.0401	0.0436	0.0473	0.0528	0.058	0.0634	0.0702	0.0773	0.0844
				0.0357	0.0387	0.0432	0.0475	0.052	0.0575	0.0633	0.0691
					0.0317	0.0354	0.0389	0.0425	0.0471	0.0518	0.0566
						0.029	0.0318	0.0348	0.0385	0.0425	0.0463
							0.0261	0.0285	0.0316	0.0348	0.0379
								0.0233	0.0258	0.0285	0.0311
									0.0212	0.0233	0.0254
										0.0191	0.0208
											0.017
Zerocouponprice											
	0.622	0.68	0.7335	0.783	0.8275	0.868	0.903	0.9335	0.9595	0.9815	1.0000
		0.6265	0.686	0.7425	0.794	0.8405	0.8825	0.9195	0.9505	0.9775	1.0000
			0.633	0.6965	0.7555	0.809	0.858	0.9015	0.9395	0.973	1.0000
				0.645	0.7115	0.773	0.83	0.8815	0.927	0.966	1.0000
					0.6625	0.733	0.7985	0.8585	0.912	0.9595	1.0000
						0.686	0.7615	0.8315	0.8945	0.9505	1.0000
							0.718	0.7995	0.8735	0.9405	1.0000
								0.7615	0.848	0.9285	1.0000
									0.8195	0.914	1.0000
										0.897	1.0000
Europeanoptionvalue											
	0.0248	0.0356	0.0485	0.0626	0.0766	0.0902	0.103				
		0.0166	0.0259	0.0383	0.053	0.0679	0.0825				
			0.009	0.016	0.0269	0.0422	0.058				
				0.0032	0.0067	0.0142	0.03				
					0	0	0				
						0	0				
							0				

American type option could be priced in a similar manner using equation (24) taking into consideration early exercise conditions.

Thus far we have used our knowledge of the term structure of interest rates today to model future interest rates using the equal probability specification of Black, Derman and Toy in a binomial lattice framework. We have also used the generated tree to value options on bonds. The process can be used to value several other interest rate derivatives such as Swap ions, Barrier options and Captions.

In the next section we will look at a more a different, more flexible way of valuing interest rate derivatives with the BDT-binomial tree framework using Monte Carlo Simulation processes.

3.3 Using Monte Carlo simulation to price pure discount bonds options.

The idea of any simulation is to estimate one or more expectation of the form $E\phi(X)$. (This may not be obvious at first sight, but even a cumulative distribution function is a collection of expectations[7]). Thus we will regard the problem of this section as evaluating a frequently complex and high dimensional integral.

A more unswerving way to price interest rate derivatives are by modeling the underlying asset's source of uncertainty, in our case interest rates, directly. One way of constructing a no-arbitrage model for interest rates is in terms of the process followed by the instantaneous short rate, r. It has been shown from previous sections that the process for the short rate in a risk-neutral world assumption can be used to resolve the current term structure of interest rates and vice versa. For example, arbitrage pricing tells us that the s-maturity bond prices are given by

$$Ps(t,s) = \hat{E}_t\left[\exp\left(-\int_t^s r(\tau)d\tau\right)\right] \qquad \text{......................}(25)$$

[7] See Stochastic Simulation by B Ripley

Where $\hat{E}_t$ denotes the expectations (with information set at t) in a risk-neutral world, -

With $r(\tau)$ denoting the path of short interest rate from time t to s. Time t interest rate derivative prices, C (t) are determined in the same way as:

$$C(t) = \hat{E}_t\left[\exp\left(-\int_t^s r(\tau)d\tau\right)C(T)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

Where C (T) is the pay-off from a derivative when it is exercised at time T. For example, for the European discount bond put option described in section 2 we have

$$C(t) = \hat{E}_t\left[\exp\left(-\int_t^s r(\tau)d\tau\right)\max(Ps(T,s) - K,0)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

Both equation (26) and (27) are expectations and we calculate them by simulating many paths $r(\tau)$ in the interest rate tree and taken average of the resulting values

The price of a European call option on a pure discount bond (equation (27)) can be rewriting as

$$C(t) = \hat{E}_t\left[\max\left(Ps(T,s)Y(T,s) - KPs(t,T)Y(T,T)\right)\right] \ldots\ldots\ldots(28)$$

Where

$$Y(T,s) = \exp\left(-\int_t^T r(\tau)\,d\tau\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

We can implement this equation using Monte Carlo simulations with

j = 1,…,M simulations

$$C(t) = \frac{1}{M}\sum_{j=1}^{M}\left[\max\left(Ps(T,s)Y_j(T,s) - KPs(t,T)Y_j(t,T)\right)\right] \ldots\ldots\ldots\ldots\ldots(30)$$

Where

$$Y_j = \left(\prod_{i=0}^{N} w(t_j)\frac{1}{\left(1 + dt * r(i,k)\right)} \ \text{k = 0 to i}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

Where w (t_j) is a function that describes the path [8] that interest rate take to reach state j at time t from time zero. W (t_j) can be modeled in different ways but for simplicity we model it with the generation of uniform random numbers between 0 and 1. At every node in the binomial tree if rates go up

[8] Random walk-a process that will ensure that all possible path that interest rate can thread is taken care of.

we assign w (t_j) a value of 1 and 0 otherwise. The simulation works by generating many different future interest rate paths.

The simulation is normalized so that the average simulated price of the pure discount bond equals today's actual price.

To illustrate this we use bond specification in section 3.2 where we priced a six-year European call option on a pure discount bond with maturity 10 years and a strike price of \$0.80 using analytical methods.

What we do here is to generate random numbers, which takes values 0 or 1 with equal probability of 0.5. we assume 1 if interest rates go up, and down if it is 0. Table 4 shows how this is done. The zeros and ones in blue print denote the upward and downward movement of interest rates and a simulation represent a unique path that rates can thread to get to the expiration of the option. Every path leads to a particular pay-off at expiration equation (29) is solved basically by finding the product of all the discount factors along a particular path. The set of numbers labeled pathwise discount rates represents the discount factors corresponding to the rates on a particular path. The product of these discount factor is multiplied by the pay-off of the option at expiration corresponding to the node on that path to determine the price of the option on that for that path.

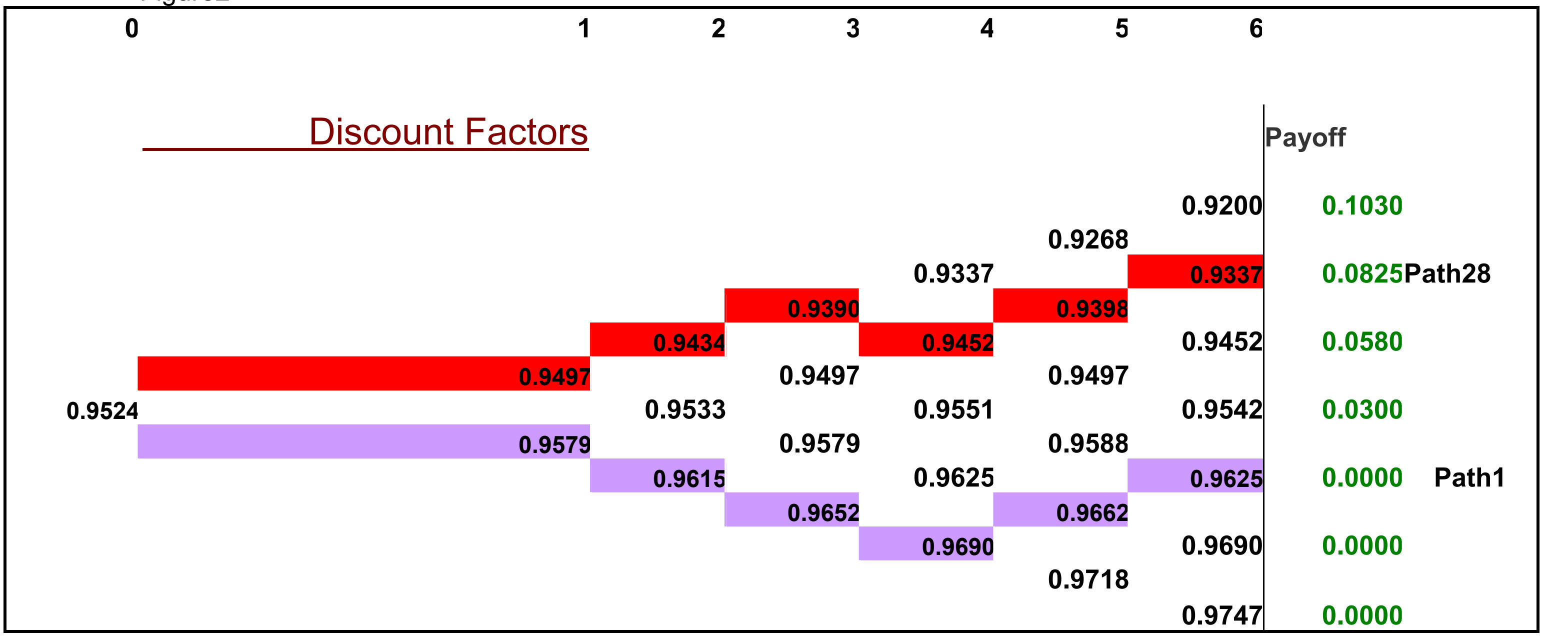
The simulation is done for many times and the corresponding option price is obtained for all paths under consideration. The arithmetic average of all the option prices for all the paths represent the overall price of the option with is $0.0233 in this case.

Table 4

	1	2	3	4	5	6	0	1	2	3	4	5	6	Payoff	Price/si	Price
Simulations							Discount Factor along path									0.0233
1	0	0	0	0	1	1	0.9524	0.9579	0.9615	0.9652	0.9690	0.9662	0.9625	0.0000	0.0000	
2	1	1	1	1	0	0	0.9524	0.9497	0.9434	0.9390	0.9337	0.9398	0.9452	0.0015	0.0010	
3	0	1	0	0	1	0	0.9524	0.9579	0.9533	0.9579	0.9625	0.9588	0.9625	0.0000	0.0000	
4	1	0	0	1	0	1	0.9524	0.9497	0.9533	0.9579	0.9551	0.9588	0.9542	0.0000	0.0000	
5	0	1	1	0	1	1	0.9524	0.9579	0.9533	0.9497	0.9551	0.9497	0.9452	0.0015	0.0011	
6	1	1	0	1	0	0	0.9524	0.9497	0.9434	0.9497	0.9452	0.9497	0.9542	0.0000	0.0000	
7	1	0	0	0	1	0	0.9524	0.9497	0.9533	0.9579	0.9625	0.9588	0.9625	0.0000	0.0000	
8	0	1	1	1	0	1	0.9524	0.9579	0.9533	0.9497	0.9452	0.9497	0.9452	0.0015	0.0011	
9	1	1	0	0	0	1	0.9524	0.9497	0.9434	0.9497	0.9551	0.9588	0.9542	0.0000	0.0000	
10	0	0	0	1	1	0	0.9524	0.9579	0.9615	0.9652	0.9625	0.9588	0.9625	0.0000	0.0000	
11	1	1	1	0	0	0	0.9524	0.9497	0.9434	0.9390	0.9452	0.9497	0.9542	0.0000	0.0000	
12	0	1	0	1	1	1	0.9524	0.9579	0.9533	0.9579	0.9551	0.9497	0.9452	0.0015	0.0011	
13	1	0	0	0	0	1	0.9524	0.9497	0.9533	0.9579	0.9625	0.9662	0.9625	0.0000	0.0000	
14	1	0	1	1	1	0	0.9524	0.9497	0.9533	0.9497	0.9452	0.9398	0.9452	0.0015	0.0010	
15	0	1	0	0	0	0	0.9524	0.9579	0.9533	0.9579	0.9625	0.9662	0.9690	0.0000	0.0000	
16	1	0	0	1	1	1	0.9524	0.9497	0.9533	0.9579	0.9551	0.9497	0.9452	0.0015	0.0011	
17	0	0	1	0	0	1	0.9524	0.9579	0.9615	0.9579	0.9625	0.9662	0.9625	0.0000	0.0000	
18	1	1	0	1	0	0	0.9524	0.9497	0.9434	0.9497	0.9452	0.9497	0.9542	0.0000	0.0000	
19	0	0	1	0	1	0	0.9524	0.9579	0.9615	0.9579	0.9625	0.9588	0.9625	0.0000	0.0000	
20	1	0	1	0	0	1	0.9524	0.9497	0.9533	0.9497	0.9551	0.9588	0.9542	0.0000	0.0000	
21	1	1	0	1	1	1	0.9524	0.9497	0.9434	0.9497	0.9452	0.9398	0.9337	0.0385	0.0259	
22	0	0	1	0	0	0	0.9524	0.9579	0.9615	0.9579	0.9625	0.9662	0.9690	0.0000	0.0000	
23	1	0	1	1	1	0	0.9524	0.9497	0.9533	0.9497	0.9452	0.9398	0.9452	0.0015	0.0010	
24	0	1	0	0	0	1	0.9524	0.9579	0.9533	0.9579	0.9625	0.9662	0.9625	0.0000	0.0000	
25	1	0	1	1	1	1	0.9524	0.9497	0.9533	0.9497	0.9452	0.9398	0.9337	0.0385	0.0261	
26	0	0	1	0	1	0	0.9524	0.9579	0.9615	0.9579	0.9625	0.9588	0.9625	0.0000	0.0000	
27	1	1	0	1	0	0	0.9524	0.9497	0.9434	0.9497	0.9452	0.9497	0.9542	0.0000	0.0000	
28	1	1	1	0	1	1	0.9524	0.9497	0.9434	0.9390	0.9452	0.9398	0.9337	0.0385	0.0256	
29	0	0	1	1	0	1	0.9524	0.9579	0.9615	0.9579	0.9551	0.9588	0.9542	0.0000	0.0000	
30	1	1	0	0	1	0	0.9524	0.9497	0.9434	0.9497	0.9551	0.9497	0.9542	0.0000	0.0000	

Figure 2 shows path 1 and 28 in the simulation.

Figure2



CHAPTER 4

4.1 Valuing Mortgage-Backed Securities

In order to value a mortgage-backed security, it is necessary to project its cash flows. The difficult is that the cash flows are unknown because of prepayment. The only way to project cash flows is to make some assumption about the prepayment rate over the life of the underlying mortgage pool. The prepayment rate is sometimes referred to as the speed (that is how fast a pool can be prepaid). Two conventions have been used as a benchmark for prepayment rates-conditional prepayment rate and the public securities association benchmark.

Conditional Prepayment Rate (CPR): One convention for projecting prepayments and the cash flows of a MBS assumes that some fraction of the remaining principal in a pool of mortgages is prepaid each month for the remaining term of the mortgage. CPR is based on the characteristic of the underlying mortgage pool.

The Public Securities Association (PSA) prepayment benchmark is expressed as a monthly series of CPR's. The PSA benchmark assumes that prepayment rates are low for newly originated mortgages and then will speed up as the mortgages become seasoned.

4.2 Modeling Prepayments due to refinancing

A prepayment model is a statistical model that is used to forecast prepayments. It begins by modeling the statistical relationship among the factors that are expected to affect prepayment. The factors that affect prepayment behavior are: (1) prevailing mortgage rate, (2) characteristics of the underlying mortgage pool, (3) seasonal factors and (4) General economic factors.

In section 3.1 three we looked at how to model interest rates that apply in a short time period and in sections 3.2 and 3.3 we saw how to price an interest rate derivative based on the constructed binomial tree of short interest rates. Particularly in section 3.3 we saw how to value a bond option using Monte Carlo simulation. In this section we will use the idea of generating interest rate paths to model a very important factor affecting prepayments, which is the at which people refinance in a pool of mortgages.

The single most important factor affecting prepayment because of refinancing is the current level of **mortgage rates** relative to the borrowers contract rate. The more the contract rate exceeds the prevailing mortgage rate the greater the incentive to refinance the mortgage loan. For refinance to make economic sense the interest saving must be greater than the cost associated with refinancing the mortgage. These cost include legal expenses, origination fees, title insurance and the value of the time associated with obtaining the mortgage loan.

Historical patterns of prepayment and economic theories suggest that, it is not only the level of mortgage rates that affects prepayment behavior, but also the path that mortgage rates take to get to the current level.

If the mortgage rate follows a particular path, those who can benefit from refinancing will more than likely take advantage of this opportunity when the mortgage rate. Anytime interest rates drop people who can refinance will likely do it. When interest rates keep on dropping, then those who can benefit by taken advantage of the refinancing opportunity will have done so already when rates declined in previous periods. This prepayment behavior is referred to as *refinancing burnout*. The expected prepayment behavior when mortgage rates follow two unique paths is different. Burnout is related to the path of mortgage rates. This is the primary reason why the binomial tree model (which uses backward) is not used. In order words prepayment or refinancing behavior is path dependent.

In this project we considered how to model refinancing rates based on the path that interest rates take to reach a particular level. **Refinancing rate** represents the fraction of the mortgages in a pool that will be completely repaid in the current period.

The factors that go into modeling refinancing are:

- Current interest rates
- Trend of interest rates
- Interest rates in previous periods (burnout effect)
- (Other economic factors).

Since different firms and commercial vendors model prepayments due to refinancing based on their historic experience with refinancing, we will provide a very simple prototype refinancing function that could be altered to suit the situation a specific firm.

Notation

$I(i)$ = interest rate in period i (from interest rate tree)

M = Contract mortgage rate

R(i) = refinancing rate in period i

Definition of function

$$R(i)=\begin{cases} 0 \text{ if } \; I(i)>M-275bps \\ 0.05\left(M-275bps-I(i)\right) \quad \text{if} \quad \mathrm{I(i)<M\text{-}275bps< I(i\text{-}1)} \\ \mathrm{R(i\text{-}1)}+0.04\left(M-275bps-I(i)\right) \quad \text{if} \;\; \mathrm{I(i)<I(i\text{-}1)<M\text{-}275bps} \\ 0 \;\; \text{if} \quad \mathrm{I(i\text{-}1)<I(i)} \end{cases}$$

What this means is that, we are considering refinancing behavior of mortgagors parallel to the movements of interest rates. If the prevailing interest rate in the first year is higher than contract mortgage rate minus some basis points of it, which is basically to account for the cost involved in refinancing, then it does not make any economic sense to refinance, and so the refinance rate will be zero. If by the second year interest rates drop then, those who can benefit from refinancing will do so.

In this case the fraction of the pool that will be refinanced will be some percentage of the difference between the mortgage rate and the prevailing interest rate. Also if rates continue to drop for a second conservative time then, a large portion of the pool is being completely repaid. Eventually refinancing activities will lease for a particular pool, which has experienced prepayments in earlier stages.

4.3 Valuation methodology

The cash flow of a mortgage-backed security is interest rate path dependent. This means that the cash flow received in one period is determined not only by the current interest rate level, but also by the path interest rates took to get to the current level. Each of the various mortgage instruments listed in section 1.3 has is own cash flow pattern and prepayment behavior. Conceptually, valuation of the securities using Monte Carlo simulation is simple. It involves generating a set of cash flows based on simulated future mortgage refinancing rates, which in turn imply simulated prepayment rates. The simulation works by generating many scenarios of future interest rate paths. In each month of the scenario a monthly interest rate and a mortgage refinancing rates are generated. The monthly interest rates are used to discount the projected cash flows for each scenario. The mortgage refinancing rates is needed in order to determine the cash flows because it represents the opportunity cost the mortgagor is facing at that time. If the refinancing rates are higher relative to the mortgagors original coupon rate (contract rate), the mortgagor will have less incentive to refinance or it could even be a disincentive. On the other had if the refinancing rate is lower than the mortgagors original coupon then the mortgagor can decide to refinance if it makes economic sense. To make this point more concrete let us consider a newly issued MBS with a maturity of 360 months. Table 5 shows three simulated interest rate path scenarios. Each scenario consists of 360 1-month future interest rates (we showed only 6 terms out of it). For comparison purpose we have also showed in table5 the cash flow in a case where there is no prepayment. It should be noted that, when no prepayment is assumed for the entire mortgage life the cash flow would be different for a particular month when there is prepayment but a zero rate of refinancing. For instance on the fourth month in simulation three there was an O% refinancing rate and the cash flow is entirely different from the cash flow without prepayment. This can be explained by the fact that there were some refinancing activities in the first three months so the outstanding principal amount has reduced by the fraction, which has already being refinanced. The value of the MBS is the average of all the cash flows for the total number of simulations done.

Table 5

SIMULATION ONE						
Without Prepayment	1	2	3	4	5	6
Interest Rate	4.00%	5.22%	6.44%	7.48%	8.53%	6.40%
Discount Factor	0.9615	0.9506	0.9398	0.9302	0.9217	0.9398
Discounted Cash Flow	989.0110	929.4725	864.5954	796.8975	748.9243	700.6187
With Prepayment						
Refinancing Rate	0.2308%	0.1763%	0.1223%	0.0743%	0.0318%	0.1223%
Total Cash Flow	1051.0542	994.2779	938.7996	889.9849	847.1800	936.7301
Discounted Cash flow	1010.5886	898.4471	789.1035	689.4984	616.8248	638.0348
SIMULATION TWO						
Without Prepayment						
Interest Rate	4.00%	6.37%	9.23%	12.09%	14.94%	9.17%
Discount Factor	0.9615	0.9398	0.9158	0.8921	0.8703	0.9158
Discounted Cash Flow	989.0110	905.7363	808.0073	703.2088	643.9986	593.5735
With Prepayment						
Refinancing Rate	0.2308%	0.1223%	0.0023%	0.0000%	0.0000%	0.0023%
Total Cash Flow	1051.0542	940.4335	819.8346	817.5453	817.6065	819.9969
Discounted Cash flow	1010.5886	828.0910	644.0057	558.9131	511.8909	473.1893
SIMULATION THREE						
Without Prepayment						
Interest Rate	4.00%	5.22%	7.71%	8.78%	9.81%	7.21%
Discount Factor	0.9615	0.9506	0.9285	0.9191	0.9107	0.9328
Discounted Cash Flow	989.0110	918.2967	844.0065	768.6367	716.9843	670.7389
With Prepayment						
Refinancing Rate	0.2308%	0.1763%	0.0658%	0.0188%	0.0000%	0.0873%
Total Cash Flow	1051.0542	994.2779	882.5782	835.3146	816.4917	903.3125
Discounted Cash flow	1010.5886	887.6443	724.1810	624.1937	569.1276	589.0330

Implementation

The valuation methodology outlined in the previous sections is implemented as a computer program written in MAPLE.

The inputs of the program are the spot interest rates and their volatilities covering the entire time until the maturity of the MBS and the specification of the mortgages in the underlying pool.

The program first builds the Black-Derman-Toy interest rate tree. Then it generates random paths in the tree and evaluates both refinancing rates and the resulting cash flows along each simulated path. The security is valued by averaging the discounted cash flows along a user specified number of interest rate paths.

The refinancing rate is a function of current and past interest rates along the path. The user can change this function as it is implemented as a subroutine. This makes it possible to experiment with various functional dependences between refinancing and the interest rates.

The MAPLE program communicates with the user through a spreadsheet interface

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