

Financial Mathematics Project

A Directed Research Project

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Abstract

This project describes the underlying principles of Modern Portfolio Theory (MPT), the Capital Asset Pricing Model (CAPM), and multi-factor models in detail. It also explores the process of constructing optimal portfolios using Modern Portfolio Theory, as well as estimates the expected return and covariance matrix of assets using the CAPM and multi-factor models. Finally, the project applies these models in real markets to analyze our portfolios and compare their performances.

Acknowledgements

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In addition, I have special thanks to Professor Marcel Blais who introduced us to the concept of Modern Portfolio Theory, the CAPM and multi-factor model and guided us throughout this project. His encouragement and motivation helped us since we chose the stocks to build the portfolio for analysis. His time allotment to us despite his busy schedule and project instruction helped us make huge improvements in our project.

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1. INTRODUCTION

Modern Portfolio Theory was developed by Markowitz ^[1] in the 1950s through the early 1970s. It is one of the earliest and most fundamental theories in the field of mathematical finance and is extensively used in practice in the financial industry. It attempts to maximize the portfolio return for a given level of risk and suggests that trying to pick a few great stocks is not an efficient investment strategy. In this project we will introduce the concept of portfolio optimization and the power of diversification in risk reduction.

Built on the work of Markowitz, the Capital Asset Pricing Model (CAPM) was later introduced independently by William Sharpe, John Lintner, Jack Treynor and Jan Mossin ^[2]. It is based on the assumptions similar to those underlying Modern Portfolio Theory and is a very powerful and simple tool to use and reference in practice. Beta is perhaps the most important concept introduced by the CAPM and is broadly referenced in the media every day. It measures the sensitivity of the return of stocks with respect to the return of the broad market. With beta we can calculate the theoretically required rate of return of assets given the return of the market. Although CAPM provides us with this insight, it is too simple to be effective in predicting asset returns in the complex real market because of its constraining assumptions and limitations.

Since the real markets are so complex, a reliable and appropriate explanation is that there are many factors which affect the return of assets. Compared with CAPM, which has only one factor and is a poor predictor of asset returns, the multi-factor models are introduced to try to address some of the limitations of the CAPM. They are based on the same principles on which the CAPM is based but expand the number of risk factors, such as macroeconomic, fundamental and statistical factors.

In this project, we will examine the risk and return characteristics of the individual assets and estimate the returns and covariance matrix of assets using the CAPM and multi-factor models. Finally, we will create the optimal portfolio based on the estimated returns and covariance matrix in Matlab ^[3], form our real market positions in Interactive Brokers ^[4] and analyze their performances.

An asset has many characteristics. The most important characteristics are risk, return and correlations among assets. The following gives the definition of return and risk of the asset.

- I. Return: return is defined as the sum of price change and cash inflow of an asset during a period divided by the previous price.

$$R = \frac{P_t - P_{t-1} + C_t}{P_{t-1}} = \text{Capital gain} + \text{Dividend yield}$$

Where P is price, C is cash inflow, t is time.

- II. Risk: risk is defined as the variance of the return, a measure of the volatility of returns. Although this definition is convenient and simple in use, we must note the drawback of the definition. Since we only care about the downside of the return, the variance of the return may not fully represent the real risk.

2. Modern Portfolio Theory

2.1 Assumptions of Modern Portfolio Theory

In this section we examine the assumptions underlying Modern Portfolio Theory. As with most mathematical models, Modern Portfolio Theory simplifies the complex real world and starts from a simple model that has many assumptions. Some of these assumptions are constraining and cannot be relaxed easily without a much deeper analysis, while some other assumptions can be removed without much influence on the conclusion of the theory. Here are the assumptions of Modern Portfolio Theory ^[12]:

- Investors based their decisions on expected return and risk, as measured by the mean and variance of the returns on various assets.
- All investors have the same time horizon. In other words, they are concerned only with the utility of their terminal wealth, and not with the state of their portfolio beforehand, and this terminal time is the same for all investors.
- All investors are in agreement as to the parameters necessary, and their values, in the investment decision making process, namely, the means, variances and correlations of returns on various investments.
- Financial assets are arbitrarily fungible.

Next we will start from a simple portfolio that consists of only one risky asset and one risk free asset and move to the more complex portfolios to see how we can construct the efficient frontier, a collection of all possible portfolios that have the best return given a certain level of risk.

2.2 Portfolio Risk

2.2.1 Portfolio of one risk-free asset and one risky asset

Let w be the weight of the risky asset, $E(R_r)$ be its expected return, R_f be the risk free return and $E(R_p)$ be the expected portfolio return. When we have only one risky asset, our portfolio return is:

$$E(R_p) = wE(R_r) + (1-w)R_f \quad (1)$$

The variance of the return is:

$$\sigma_p^2 = w^2\sigma_r^2 + (1-w)^2(0)^2 \quad (2)$$

$$\sigma_p = w\sigma_r \quad (3)$$

By substitution for w , we can express $E(R_p)$ in terms of σ_p :

$$E(R_p) = R_f + \frac{E(R_r) - R_f}{\sigma_r} * \sigma_p \quad (4)$$

Figure 1 is the plot of equation (4).

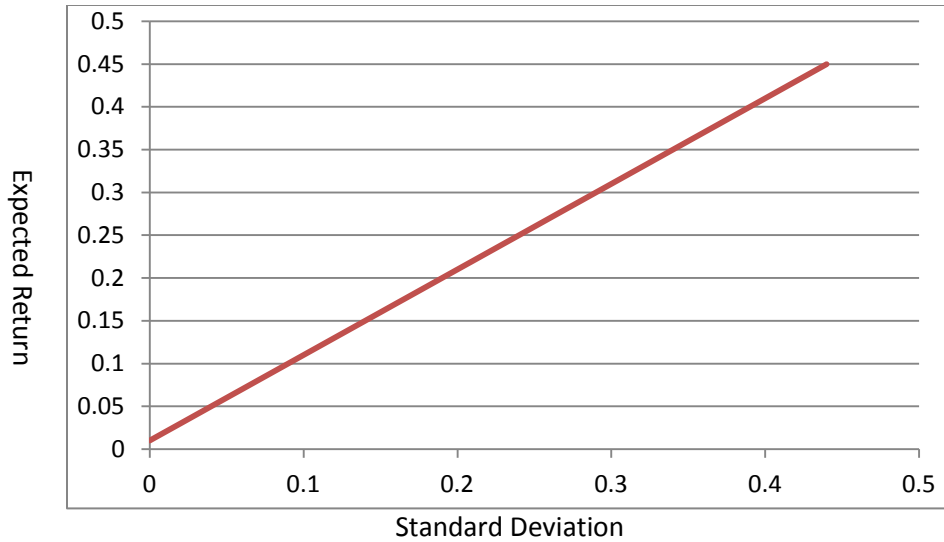


Figure 1

2.2.2 Portfolio of Two Risky Assets

Let w be the weight of one risky asset, R_1 be its return and R_2 be the return of another asset. When we have only two risky assets, the portfolio return is:

$$R_p = wR_1 + (1-w)R_2$$

Let ρ_{12} be the correlation between the returns of two assets, the variance of the portfolio return is:

$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2$$

The graph below is generated using Microsoft Excel and Microsoft Paint. It shows that the risk-return relationship varies as the correlation changes. As the correlation decreases, the risk becomes smaller given the same return.

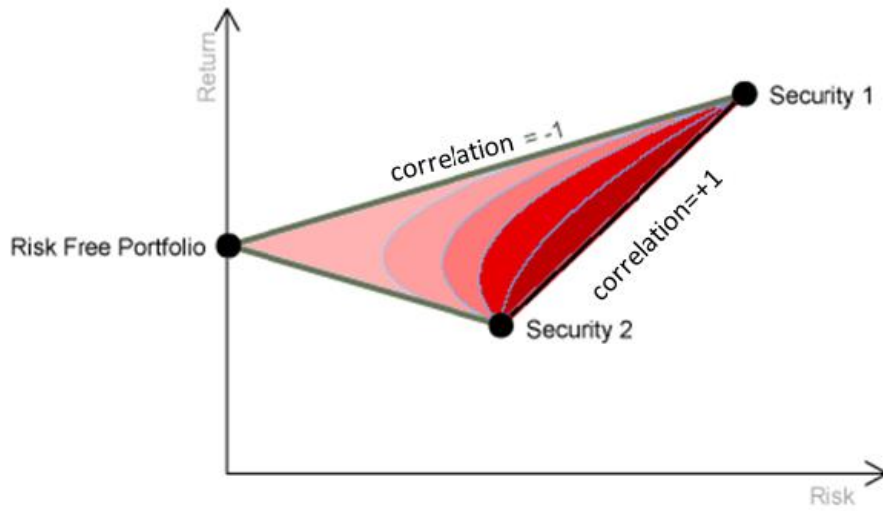


Figure 2

2.2.3 Portfolio of Many Risky Assets

As we have discussed in the previous section, correlation plays a vital role in reducing the risk of a portfolio with two risky assets. The smaller the correlation, the greater the risk reduction we have. The analysis above can be extended to a portfolio with many risky assets (N). The multi-assets portfolio return and variance are:

$$E(R_p) = \sum_1^N w_i E(R_i)$$

$$\sigma_p^2 = (\sum_1^N w_i^2 \sigma_i^2 + \sum_{i,j=1, i \neq j}^N w_i w_j \text{Cov}(i,j)), \sum_1^N w_i = 1$$

To examine the risk and return characteristics of a portfolio with many risky assets and the roles correlation play in reducing the risk of such a portfolio, assume that the portfolio has equal weights ($1/N$) for all N assets. Let $\bar{\sigma}^2$ and $\overline{\text{Cov}}$ be the average variance and average

covariance. Given equal weights, average variance and covariance, we can rewrite the portfolio variance as below ^[5]:

$$\sigma_p^2 = (\sum_1^N w_i^2 \sigma_i^2 + \sum_{i,j=1, i \neq j}^N w_i w_j \text{Cov}(i,j)) \quad (6)$$

$$\sigma_p^2 = \frac{\bar{\sigma}^2}{N} + \frac{N-1}{N} \overline{\text{Cov}} \quad (7)$$

The equation (7) indicates that as N becomes larger, the first term $\frac{\bar{\sigma}^2}{N}$ becomes smaller. This means that the individual risk of each asset is very small in the total portfolio risk. However, the second term $\frac{N-1}{N} \overline{\text{Cov}}$ is close to the average covariance as N increases. So we can see that for portfolios with a large number of assets, correlations among the assets determine the total risk of the portfolio.

2.3 Efficient Frontier and Optimal Portfolios

2.3.1 Efficient Frontier

Here we describe the efficient frontier, explain how we choose the optimal portfolio based on risk and return analysis and mix the risk free asset with the tangency portfolio to create the efficient portfolio.

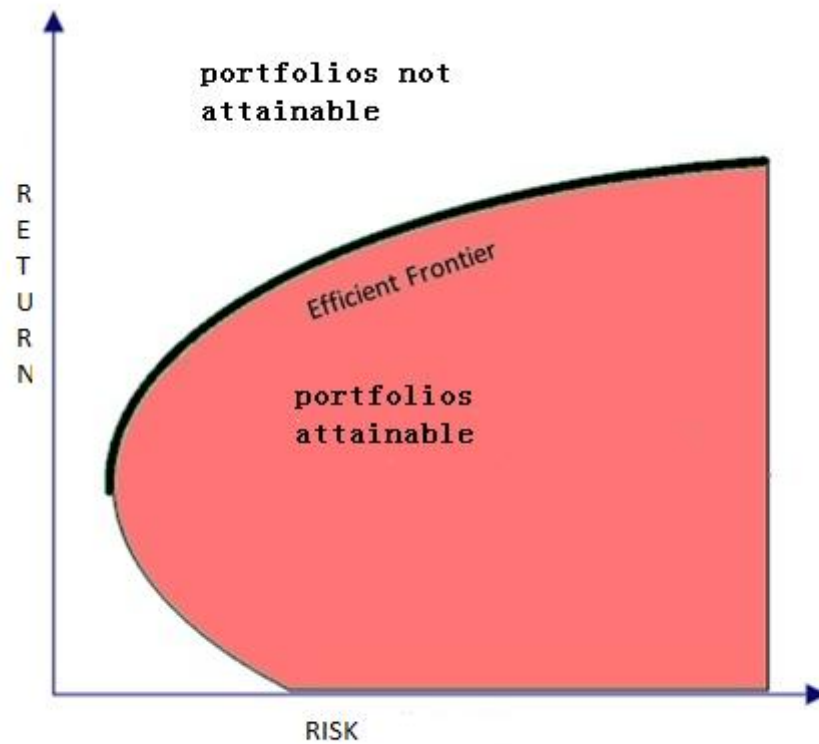


Figure 3

In figure 3, the red area includes all of the attainable portfolios we can choose. Each portfolio has its corresponding risk and return. Any rational investor would pursue the portfolio that has the highest return and lowest risk. For a portfolio given a constant return, an investor would prefer lower risk to higher risk. Also for a portfolio given a constant risk, an investor would prefer higher return to lower return. Based on this perspective, the portfolios in the upper left area from figure 3 are preferred by the investors. The black line, called the efficient frontier, represents the best portfolios in the attainable portfolios.

2.3.2 Efficient Portfolio Mathematics

Assume that we have N risky assets and that the return on the i th asset is R_i and has expected value u_i . Define

$$\mathbf{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix} \text{ and } \mathbf{E}(\mathbf{R}) = \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}.$$

Let Ω_{ij} be the covariance between R_i and R_j , $\sigma_{ij} = \sqrt{\Omega_{ij}}$ be the standard deviation and $\mathbf{\Omega}$ be the covariance matrix of \mathbf{R} . In addition, let $\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}$ be a vector of portfolio weights and $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ be a column of N ones. We assume that $w_1 + \dots + w_N = \mathbf{1}^T \mathbf{w} = 1$. The expected return on a portfolio with weights \mathbf{w} is:

$$\sum_{i=1}^N w_i u_i = \mathbf{w}^T \mathbf{u}$$

For $N \geq 3$, there will be an infinite number of portfolios achieving the target expected return u_p . The portfolio with the smallest variance is called the efficient portfolio and we aim to find this efficient portfolio. The variance of the return on the portfolio with weights \mathbf{w} is:

$$\sum_{i=1}^N \sum_{j=1}^N w_i w_j \Omega_{ij} = \mathbf{w}^T \mathbf{\Omega} \mathbf{w}$$

Thus, given a target return u_p , the efficient portfolio that minimizes the portfolio variance is subject to

$$\mathbf{w}^T \mathbf{u} = u_p \text{ and } \mathbf{1}^T \mathbf{w} = 1$$

The target expected return, u_p , can be varied over some range of values, e.g.,

$$\min_{i=1, \dots, N} u_i \leq u_p \leq \max_{i=1, \dots, N} u_i$$

As u_p varies over this range, we get a locus \mathbf{w}_{u_p} of efficient portfolios called the efficient frontier. We can illustrate the efficient frontier by the following algorithm ^[6].

I. Vary u_p along the grid. For each value of u_p on the grid, compute σ_{u_p} by :

a) Compute $\mathbf{w}_{u_p} = \mathbf{g} + \mu_p \mathbf{h}$;

$$\text{where } \mathbf{g} = \frac{B}{D} \boldsymbol{\Omega}^{-1} \mathbf{1} - \frac{A}{D} \boldsymbol{\Omega}^{-1} \mathbf{u}, \quad \mathbf{h} = \frac{C}{D} \boldsymbol{\Omega}^{-1} \mathbf{u} - \frac{A}{D} \boldsymbol{\Omega}^{-1} \mathbf{1}$$

$$A = \mathbf{1}^T \boldsymbol{\Omega}^{-1} \boldsymbol{\mu}, \quad B = \boldsymbol{\mu}^T \boldsymbol{\Omega}^{-1} \boldsymbol{\mu}$$

$$C = \mathbf{1}^T \boldsymbol{\Omega}^{-1} \mathbf{1}, \quad D = BC - A^2$$

b) Then compute $\sigma_{u_p} = \sqrt{\mathbf{w}_{u_p}^T \boldsymbol{\Omega} \mathbf{w}_{u_p}}$.

II. Plot the values (u_p, σ_{u_p}) . The values (u_p, σ_{u_p}) with $u_p \geq u_{min}$ are the efficient frontier, where u_{min} is the expected return of minimum variance portfolio.

2.3.3 Efficient Portfolio

Now suppose that we have a risk-free asset and we want to mix the risk-free asset with some portfolios on the efficient frontier to create the efficient portfolio. The mixture is also called the capital allocation line. We can derive a tangent line connecting the risk free return and the efficient frontier. This tangent line is called the efficient portfolio. The slope of the efficient portfolio is called the Sharpe ratio. Each efficient portfolio has two properties ^[6]:

- It has a higher expected return than any other portfolio with the same (or smaller) risk, and
- It has a smaller risk than any other portfolio with the same (or smaller) expected return.

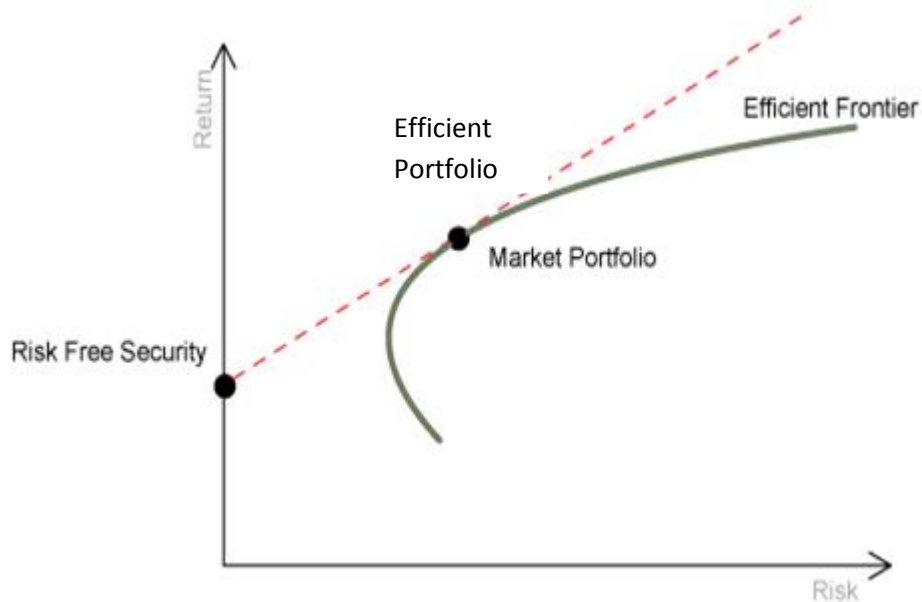


Figure 4

Thus we can only reduce the risk of an efficient portfolio by accepting a smaller return and we can only increase the expected return of an efficient portfolio by accepting higher risk.

2.4 Application of Portfolio Theory

2.4.1 Stock Selection and Data Estimation

We chose 15 stocks trading in the United States and used the mean of their historical daily returns ^[7] as the expected daily return and historical covariance matrix of returns as the expected covariance matrix. The time length of data is two years. The tickers of the stocks are: **AAPL, AMZN, CHL, COKE, DELL, EDU, GOOG, GS, JOBS, MCD, MS, MSFT, SNP, WMT and YUM**. The risk-free asset is a debt security with no default risk, no inflation risk, no liquidity risk and no interest rate risk. In this project we choose **3-month US Treasury bills rate** as a proxy of the risk-free return. The initial portfolio weights of the stocks are shown in the graph below.

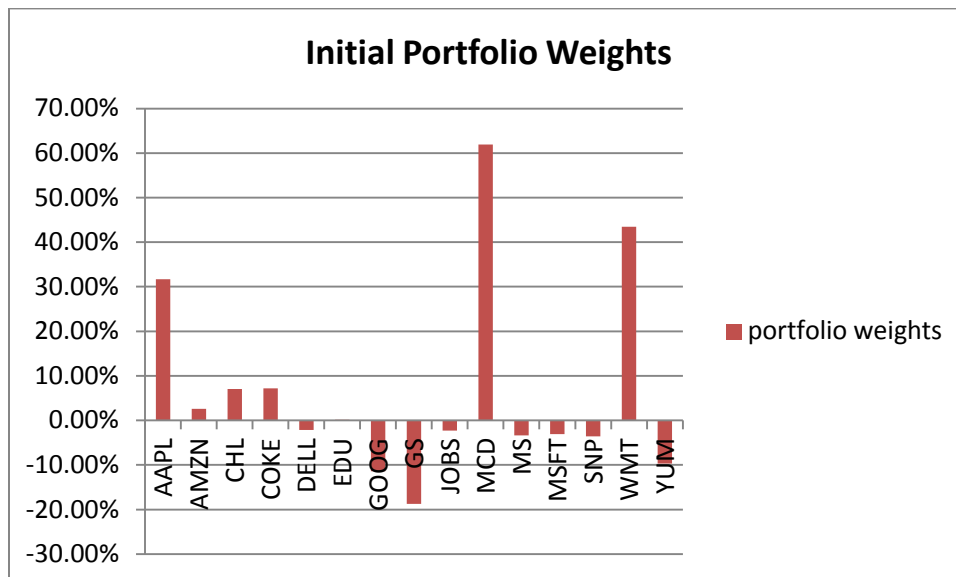


Figure 5

We formed the position using \$500,000 as initial capital in our simulated account in Interactive Brokers ^[4] on November 3, 2011 and we rebalanced the portfolio once every week to ensure we were investing in the efficient portfolio at that time.

2.4.2 Performance Report

Date	Company	Weights	Value	Num of Shares	Change of Size	Purchase Price	Buying Time
2/11/ 2011	AAPL	31.68%	\$158,400	399	0	\$397.09	2:54:52 PM
	AMZN	2.61%	\$13,050	60	0	\$215.01	2:55:55 PM
	CHL	7.08%	\$35,400	723	0	\$48.93	2:57:12 PM
	COKE	7.21%	\$36,050	644	0	\$55.99	2:58:32 PM
	DELL	-2.11%	-\$10,550	-688	0	\$15.32	2:59:50 PM
	EDU	0.18%	\$900.00	31	0	\$29.26	3:01:28 PM
	GOOG	-11.42%	-\$57,100	-98	0	\$583.57	3:02:42 PM
	GS	-18.71%	-\$93,550	-885	0	\$105.72	3:03:13 PM
	JOBS	-2.32%	-\$11,600	-254	0	\$45.36	3:03:45 PM
	MCD	61.93%	\$309,650	3355	0	\$92.32	3:04:53 PM
	MS	-3.34%	-\$16,700	-1006	0	\$26.02	3:16:36 PM
	MSFT	-3.04%	-\$15,200	-584	0	\$95.95	3:11:53 PM
	SNP	-3.56%	-\$17,800	-186	0	\$56.90	3:12:29 PM
	WMT	43.48%	\$217,400	3820	0	\$53.00	3:13:02 PM
	YUM	-9.67%	-\$48,350	-912	0	\$16.58	3:13:33 PM
Total	100.00%	\$500,000					

Table 1

Date	Company	Weights	Value	Num of Shares	Change of Size	Purchase Price	Buying Time
7/11/ 2011	AAPL	29.63%	\$150,846	377	-22	\$401.75	9:30:25 AM
	AMZN	2.28%	\$11,621	54	-6	\$219.01	9:30:27 AM
	CHL	4.34%	\$22,089	454	-269	\$48.85	9:30:45 AM
	COKE	7.29%	\$37,097	660	16	\$56.92	9:30:39 AM
	DELL	-1.78%	-\$9,041	-581	107	\$15.67	9:30:28 AM
	EDU	-0.67%	-\$3,415	-119	-150	\$28.66	9:30:40 AM
	GOOG	-9.79%	-\$49,815	-82	16	\$609.50	9:30:21 AM
	GS	-20.63%	-	-995	-110	\$106.61	9:30:34 AM
	JOBS	-2.32%	-\$11,801	-255	-1	\$46.31	9:30:40 AM
	MCD	62.38%	\$317,558	3356	1	\$95.07	9:31:49 AM
	MS	-3.11%	-\$15,816	-935	71	\$17.21	9:30:36 AM
	MSFT	-3.02%	-\$15,391	-574	10	\$27.00	9:30:07 AM
	SNP	1.26%	\$6,426	61	247	\$105.60	9:31:46 AM
	WMT	43.32%	\$220,544	3806	-14	\$58.12	9:30:38 AM
	YUM	-9.20%	-\$46,837	-857	55	\$54.96	9:30:39 AM
Total	100.00%	\$509,067					

Table 2

Date	Company	Weights	Value	Num of Shares	Change of Size	Purchase Price	Buying Time
14/11/2011	AAPL	25.35%	\$129,638	337	-40	\$384.10	9:30:12 AM
	AMZN	2.40%	\$12,299	57	3	\$214.75	9:30:28 AM
	CHL	7.29%	\$37,279	757	303	\$49.05	9:30:57 AM
	COKE	6.09%	\$31,171	572	-88	\$54.25	9:30:57 AM
	DELL	-1.98%	-\$10,115	-659	-78	\$15.29	9:30:12 AM
	EDU	-1.53%	-\$7,841	-289	-170	\$27.10	9:30:57 AM
	GOOG	-7.74%	-\$39,565	-65	17	\$608.50	9:30:12 AM
	GS	-22.29%	-	-1121	-126	\$100.08	9:30:16 AM
	JOBS	-2.64%	-\$13,498	-308	-53	\$43.45	9:30:58 AM
	MCD	59.56%	\$304,626	3215	-141	\$94.40	9:31:00 AM
	MS	-3.28%	-\$16,768	-1025	-90	\$16.15	9:30:57 AM
	MSFT	-1.92%	-\$9,845	-366	208	\$26.89	9:30:02 AM
	SNP	1.71%	\$8,747	82	21	\$106.84	9:30:58 AM
	WMT	45.29%	\$231,636	3913	107	\$58.73	9:30:20 AM
	YUM	-6.32%	-\$32,337	-579	278	\$55.60	9:30:15 AM
Total	100.00%	\$511,440					

Table 3

Date	Company	Weights	Value	Num of Shares	Change of Size	Purchase Price	Buying Time
21/11/2011	AAPL	17.34%	\$88,788	237	-100	370	9:30:17 AM
	AMZN	-0.03%	-\$154	-1	-58	191.96	9:30:14 AM
	CHL	17.29%	\$88,532	1817	1060	48.65	9:30:28 AM
	COKE	7.54%	\$38,608	695	123	55.55	9:30:28 AM
	DELL	-3.42%	-\$17,512	-1175	-516	14.72	9:30:15 AM
	EDU	-2.65%	-\$13,569	-596	-307	23.08	9:30:40 AM
	GOOG	-2.29%	-\$11,726	-20	45	584.75	9:30:16 AM
	GS	-9.04%	-\$46,289	-504	617	90.243	9:30:29 AM
	JOBS	-2.53%	-\$12,955	-308	0	N/A	N/A
	MCD	49.64%	\$254,178	2741	-474	91.99	9:30:27 AM
	MS	-8.89%	-\$45,521	-3203	-2178	13.76	9:30:50 AM
	MSFT	2.06%	\$10,548	417	783	25.21	9:30:17 AM
	SNP	-2.68%	-\$13,723	-135	-217	100.98	9:30:27 AM
	WMT	42.28%	\$216,492	3783	-130	56.91	9:30:48 AM
	YUM	-4.63%	-\$23,708	-440	139	53.23	9:30:18 AM
Total	100.00%	\$512,043					

Table 4

Date	Company	Weights	Value	Num of Shares	Change of Size	Purchase Price	Buying Time
28/11/2011	AAPL	14.78%	\$75,853	209	-28	375	9:30:17
	AMZN	-1.18%	-\$6,056	-33	-32	194.67	9:30:16
	CHL	18.86%	\$96,792	2039	222	48.16	9:30:54
	COKE	5.81%	\$29,818	578	-117	53.03	9:30:52
	DELL	-4.69%	-\$24,070	-1693	-518	14.96	9:30:09
	EDU	-1.76%	-\$9,033	-385	211	23.587	9:30:53
	GOOG	-1.84%	-\$9,443	-17	3	588.45	9:30:13
	GS	-2.35%	-\$12,060	-136	368	90.222	9:30:24
	JOBS	-2.71%	-\$13,908	-344	-36	44.48	9:30:55
	MCD	47.46%	\$243,570	2645	-96	93.32	9:30:24
	MS	-11.39%	-\$58,455	-4408	-1205	13.544	9:30:54
	MSFT	3.35%	\$17,193	708	291	24.843	9:30:08
	SNP	-2.96%	-\$15,191	-149	-14	102.94	9:31:16
	WMT	43.30%	\$222,220	3906	123	57.39	9:30:16
	YUM	-4.67%	-\$23,967	-455	-15	54.56	9:30:55
Total	100.00%	\$513,211					

Table 5

Date	Company	Weights	Value	Num of Shares	Change of Size	Purchase Price	Buying Time
5/12/2011	AAPL	18.84%	\$97,776	251	42	393.55	9:30:11
	AMZN	-1.13%	-\$5,864	-30	3	198.98	9:30:05
	CHL	14.78%	\$76,705	1569	-470	48.93	9:30:18
	COKE	5.88%	\$30,516	557	-21	55.01	9:30:18
	DELL	-3.57%	-\$18,528	-1180	513	15.972	9:30:17
	EDU	-1.92%	-\$9,964	-400	-15	25.14	9:30:18
	GOOG	-1.77%	-\$9,186	-15	2	626.88	9:30:05
	GS	-7.95%	-\$41,259	-424	-288	99.516	9:31:13
	JOBS	-2.84%	-\$14,739	-326	18	46.24	9:30:16
	MCD	53.73%	\$278,848	2914	269	96.49	9:31:11
	MS	-8.42%	-\$43,698	-2816	1592	16.07	9:31:15
	MSFT	-0.72%	-\$3,737	-148	-856	25.732	9:30:16
	SNP	-1.56%	-\$8,096	-77	72	106.73	9:32:40
	WMT	42.00%	\$217,971	3752	-154	58.2	9:30:43
YUM	-5.35%	-\$27,765	-494	-39	56.7	9:30:11	
Total	100.00%	\$518,979					

Table 6

Figure 6 describes the weights of each stock every week in our portfolio.

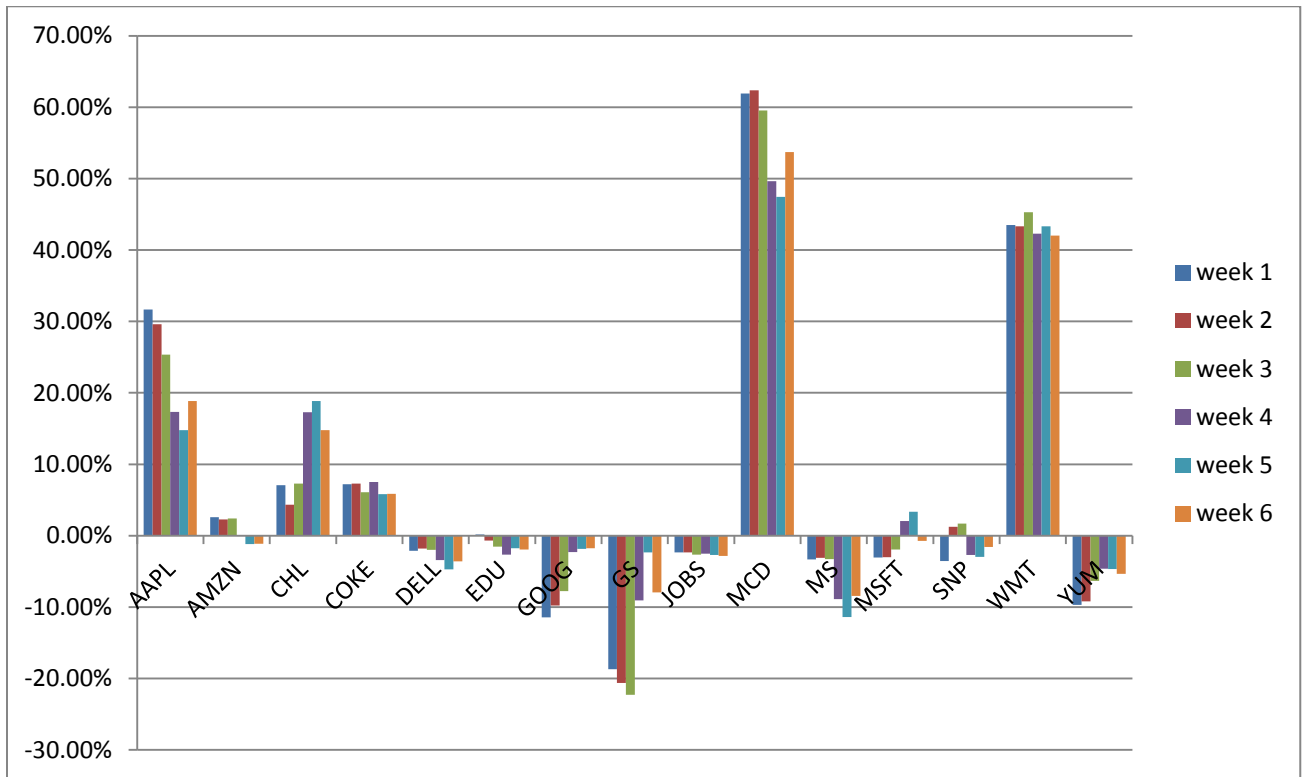


Figure 6

We included the daily returns of each stock in our portfolio in the week prior to our rebalancing date to calculate a new expected return, covariance matrix and tangency portfolio. In Figure 6 we can see the change of weights in our portfolio. We conclude that if a stock performs better than others in the week, the weight of this stock would increase. Otherwise if a stock performs poorer than others, its weight would decrease.

3. The Capital Asset Pricing Model (CAPM)

3.1 Assumptions of the CAPM

The CAPM is developed upon the work of Modern Portfolio Theory. As we have discussed in Modern Portfolio Theory, CAPM simplifies the complex real world and has many similar assumptions to that of Modern Portfolio Theory. Once this basic relationship is established, we can relax some of the assumptions to perform deeper analysis. Moreover, relaxing most of the assumptions will only have a minor impact on the conclusion of the model.

Assumptions of the CAPM ^[5]:

- Investors are risk-averse, utility-maximizing, rational individuals.
- Markets are frictionless, including no transaction costs and no taxes.
- Investors plan for the same single holding period.
- Investors have homogeneous expectations or beliefs.
- All investments are infinitely divisible.
- Investors are price takers.

3.2 The Capital Market Line (CML)

3.2.1 What is the “Market”

The theoretical market in the capital asset pricing model includes all of the risky assets in the world, such as stocks, bonds and commodities, but not all of these risky assets can be traded and the prices of many assets that have low liquidity cannot be seen directly in the market. Moreover, there is no market index that can represent the prices of all risky assets.

In this project we choose the S&P 500 as a proxy for the market because it is practical and convenient to do so. It is commonly used by analysts as a benchmark for market performance throughout the United States. It contains 500 of the largest stocks that are domiciled in the United States.

3.2.2 The Capital Market Line

We have described the capital allocation line in Modern Portfolio Theory. A capital allocation line is a tangent line that connects the risk free return and the efficient frontier. The capital market line is a special case of the capital allocation line, where the risky portfolio is the market portfolio, i.e. the S&P 500.

The expected return on the market portfolio is expressed as $E(R_m)$. The capital market line is shown in figure 7, where the standard deviation (σ_p), or total risk, is on the x-axis and expected market return, $E(R_p)$, is on the y-axis.



Figure 7

3.3 Betas and Security Market Line

3.3.1 Systematic Risk and Nonsystematic Risk

The risk of an asset has two components: systematic risk and nonsystematic risk. Systematic risk is also called non-diversifiable risk and is the risk that influences the entire market. In contrast, nonsystematic risk is the risk that belongs to a single company or industry and is also called diversifiable risk.

Systematic risk is risk that cannot be removed or diversified. It is inherent in the market and affects every asset in the market. There are many systematic risks in the market, such as the inflation rate, the economic growth rate, the unemployment rate and interest rate. The return of an asset should be impacted by all of these systematic risks and thus one factor is far from enough to model asset returns.

Nonsystematic risk is risk that is limited to the company itself. Every company has its unique characteristic. These unique characteristics are not shared by the broad market and thus have no impact on other companies.

Later we will show that the sum of systematic risk and nonsystematic risk equals the total risk of the security or portfolio ^[5]:

$$\text{Total risk} = \text{Systematics risk} + \text{Nonsystematic risk}$$

3.3.2 The Security Characteristic Line

A return-generating model is a model that can estimate the expected return of a security given certain risk factors that may influence the return of the assets. It is expressed in the following manner: ^[5]

$$E(R_i) - R_f = \sum_{j=1}^k \beta_{ij} E(F_j)$$

With the introduction of return-generating models, we can see the composition of total risk clearly. In CAPM, the security characteristic line is a regression model ^[2]:

$$R_{j,t} = \mu_{f,t} + \beta_j * (R_{m,t} - \mu_{f,t}) + \varepsilon_{j,t}$$

The characteristic line implies that the total risk of *i*th asset is:

$$\sigma_i^2 = \beta_i * \sigma_m^2 + \sigma_{\varepsilon,t}^2$$

For $i \neq j$,

$$\sigma_{j,j'}^2 = \beta_j \beta_{j'}' \sigma_m^2$$

As we have discussed, $\beta_j * \sigma_m^2$ is the systematic component of risk and $\sigma_{\epsilon,j}^2$ is the unsystematic component of risk^[6].

3.3.3 Calculation and Interpretation of Beta

First we define that the securities are indexed by i and $Cov(R_i, R_m)$ is the covariance between the returns on the i th security and the market portfolio.

Also, we define

$$\beta_i = \frac{Cov(R_i, R_m)}{Var(R_m)}$$

We know from linear regression that β_i is the best linear predictor of the i th security's return choosing the return of the market portfolio as the predictor variable. Beta captures a security's systematic risk and measures the sensitivity of the security's return with respect to the market return.

A positive beta shows that the security has the same trend with the market, whereas a negative beta shows that the security has the opposite trend with the market. By definition, the beta for the market portfolio is 1; i.e., $\beta_M = 1$. Therefore^[6],

$$\beta_i > 1 \rightarrow \text{“aggressive”}$$

$$\beta_i = 1 \rightarrow \text{“average risk”}$$

$$\beta_i < 1 \rightarrow \text{“not aggressive”}$$

Figure 8 shows the betas of stocks from our portfolio.

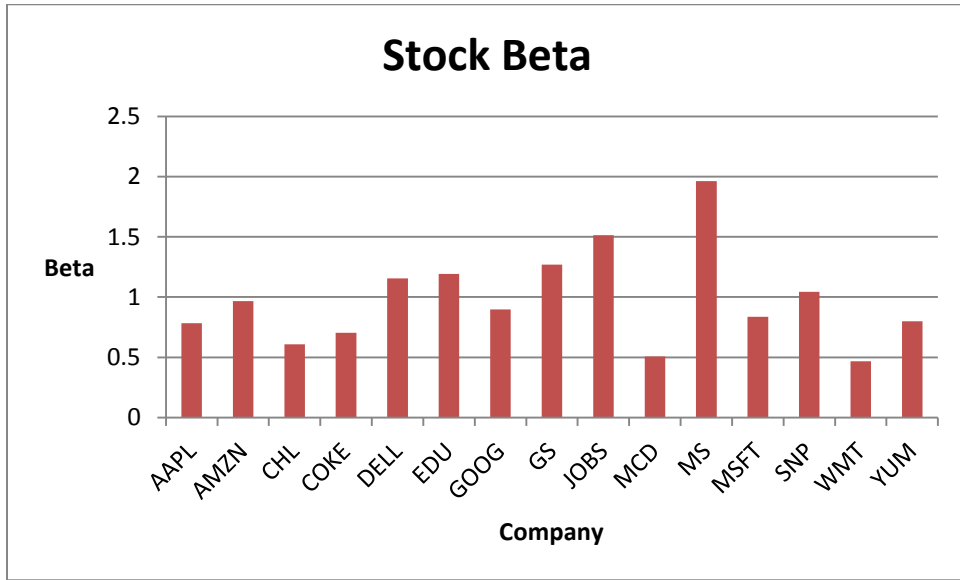


Figure 8

We can see that the stocks from our portfolio are not very aggressive. From the analysis above we can also calculate the total risk of each stock.

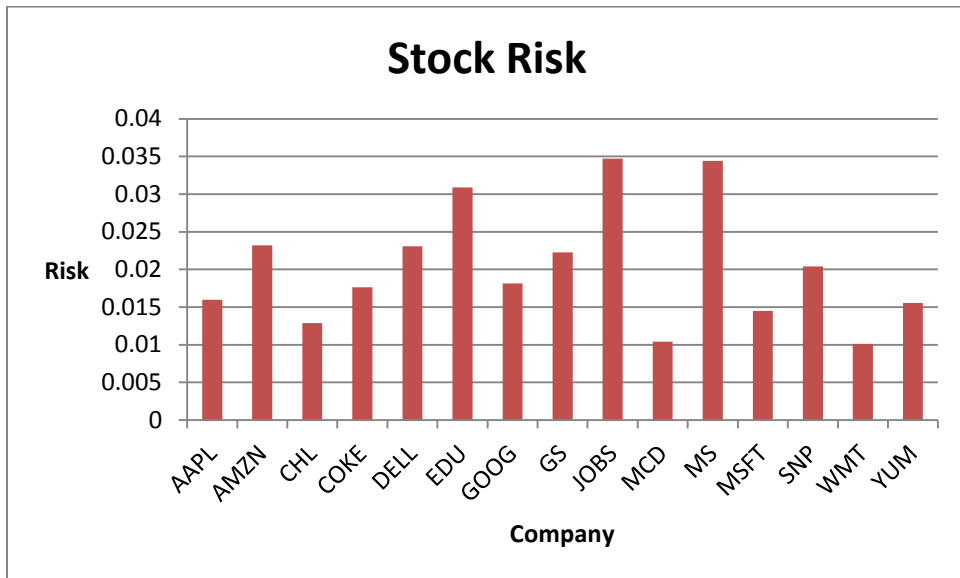


Figure 9

3.3.4 Security Market Line (SML)

The Security Market Line shows the relationship between the excess return on an asset and the slope of its regression on the market portfolio. The SML differs from the CML in that the SML applies to all assets while the CML applies only to efficient portfolio.^[6] Let μ_j be the expected return on the j th security and $\mu_j - \mu_f$ be the risk premium for that security. Using CAPM, it can be shown that:

$$\mu_j = \mu_f + \beta_j * (\mu_m - \mu_f)$$

Figure 10 is a graphical representation of the security market line.

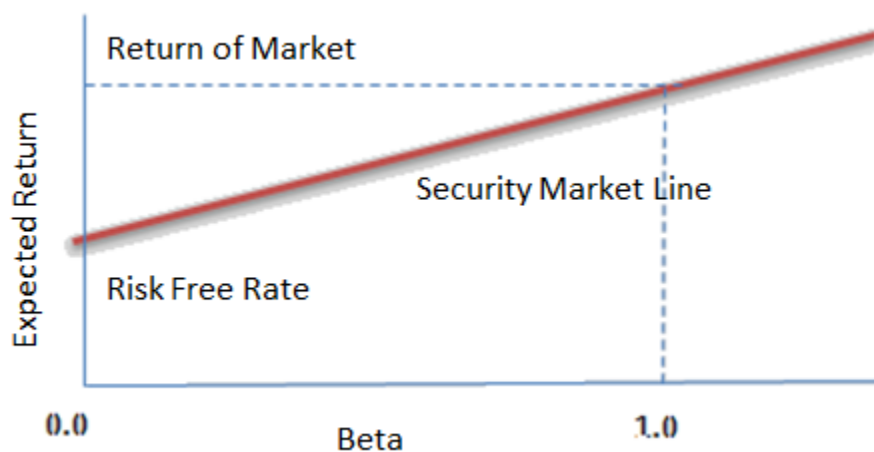


Figure 10

3.3.5 Portfolio Beta and Diversification

As we have seen, the risk of an asset has two components: systematic risk and unique risk. The systematic risk cannot be diversified away but the unique risk can be diversified away. Suppose that there are N assets with returns $R_{1,t}, \dots, R_{N,t}$ for holding period t . If we form a portfolio with weights w_1, \dots, w_N then the return of the portfolio is:

$$R_{p,t} = w_1 R_{1,t} + \dots + w_N R_{N,t}$$

Let $R_{M,t}$ be the return on the market portfolio. According to the security characteristic line, the portfolio return can also be written as:

$$R_{p,t} = \mu_{f,t} + \left(\sum_{i=1}^N \beta_i w_i \right) * (R_{m,t} - \mu_{f,t}) + \sum_{i=1}^N w_i \varepsilon_{i,t}$$

Therefore, the portfolio beta is:

$$\beta_P = \sum_{j=1}^N w_j \beta_j$$

The “epsilon” (risk) of the portfolio is:

$$\varepsilon_{P,t} = \sum_{j=1}^N w_j \varepsilon_{j,t}$$

We now assume that $\varepsilon_{1,t}, \dots, \varepsilon_{N,t}$ are uncorrelated. Therefore we have:

$$\sigma_{\varepsilon,P}^2 = \sum_{j=1}^N w_j^2 \sigma_{\varepsilon,j}^2$$

The risk of our portfolio is due to two components: one is portfolio beta; the other is its unique risk. Our portfolio beta is 0.276, which means that it is not very aggressive. If we assume that the unique risks of the stocks are uncorrelated, then the unique risk of our portfolio is:

$$\sigma_{\varepsilon,P}^2 = \sum_{i=1}^N w_i^2 \sigma_{\varepsilon,i}^2 = 5.5 * 10^{-9}$$

Below is the “epsilon” of our portfolio for the 2 year time length of our portfolio:

Risk
0.00828229245013019
0.000424289366833158
-0.00246796529272042
-0.00221775069686653
-0.00366640783381596
0.0113146151246076
0.000266707688003285
.....
-0.00107509111685723
-0.00466611682557540
0.0108648183209402
-0.00115726669429455
-9.89519048337289e-05
-0.00498106031954456
-0.00277579834182683
-0.00319741990365105
0.000807286219249118
-0.00107509111685723

Table 7

3.4 Applications of the CAPM

3.4.1 Estimate of Expected Return and Covariance Matrix

The capital asset pricing model provides a clear relationship between risk and return. It is important not only in theory but also in practice. In practice, when given a certain level of risk, we could use the CAPM and the SML to give indication of the market return. However, the actual return may differ from the expected return significantly.

After we have estimated beta and σ_{ε}^2 for each asset in our portfolio, we can use the actual market data to model the returns and covariance matrix of all assets by the formulas ^[6]:

$$\mu_j = \mu_f + \beta_j * (\mu_m - \mu_f)$$

$$\sigma_j^2 = \beta_j + \sigma_m^2 + \sigma_{\varepsilon,j}^2$$

$$\sigma_{j,j'}^2 = \beta_j \beta_{j'} \sigma_m^2$$

Note here, all these estimates are effective only when the CAPM assumption is valid. Generally we assume that the betas, σ_ε^2 , σ_m^2 and μ_m of the market are independent of t so that these parameters can be estimated by assuming stationarity of the time series of return ^[5]. Then we compare them to the actual returns to see if CAPM works well.

Company	Actual Returns	CAPM Returns	Difference
AAPL	-0.00655	-0.00031	-0.00624
AMZN	-0.03609	-0.00038	-0.03571
CHL	0.00102	-0.00024	0.00126
COKE	0.01404	-0.00028	0.01432
DELL	-0.00134	-0.00046	-0.00089
EDU	-0.08478	-0.00047	-0.08431
GOOG	-0.00997	-0.00035	-0.00962
GS	-0.00477	-0.00050	-0.00427
JOBS	0.00023	-0.00060	0.00083
MCD	0.00487	-0.00020	0.00507
MS	0.00566	-0.00078	0.00644
MSFT	-0.00940	-0.00033	-0.00907
SNP	0.00602	-0.00041	0.00644
WMT	0.00881	-0.00018	0.00899
YUN	0.00130	-0.00031	0.00161

Table 8

We can see that the differences between actual returns and CAPM returns are very large for some stocks. Since the return of the stock is normally distributed by assumption, the phenomenon that many actual returns are beyond the two standard deviations from theoretical returns could be explained by the ineffectiveness of CAPM in predicting the return.

4. The Multi-Factor Model

4.1 Descriptions of Factors

A multi-factor model is expressed as follows ^[5]:

$$E(R_i) - R_f = \sum_{j=1}^k \beta_{ij} E(F_j) = \beta_{i1} [E(R_m) - R_f] + \sum_{j=2}^k \beta_{ij} E(F_j)$$

In this model, we have k factors, $E(F_1), E(F_2), \dots, E(F_k)$. The left part of the equation is the excess return and the right part of the equation provides the risk factors that would generate the return required to assume the risk.

In this project we will use a total of five factors to estimate the expected return. Here we give an in-depth description of all factors, including reasons for our choice and sources of data.

<i>Factors</i>	<i>Definition</i>	<i>Reason to choose</i>	<i>Source of data</i>
Market Risk Premium (MRP)	The excess return over risk-free rate on the market portfolio.	The sole factor in the CAPM model, every stock is influenced more or less by the broad market.	S&P 500 as market portfolio and 3-month Treasury bills as risk-free rate ^[10] .
Small Minus Large (SMB)^[8]	The difference in returns between small-cap stocks and large-cap stocks.	The small capitalization stocks tend to perform better than large capitalization stocks.	Kenneth R. French Data Library - Fama/French Factors ^[9] .
High Minus Low (HML)^[8]	The difference in returns between high-book-to-market stocks and low-book-to-market stocks.	The high-book-to-market stocks tend to perform better than low-book-to-market stocks.	Kenneth R. French Data Library - Fama/French Factors
Momentum (UMD)^[11]	The difference in returns of the prior year's winners and losers.	If a stock has performed poorly (well) for months leading up to the end of a period, the stock tend to keep this trend in the next period. So the winner (loser) tends to be the winner (loser) of next period.	Kenneth R. French Data Library - Momentum Factors
Interest Rates Spread (IRP)	The difference of the yields between long-term AAA corporate bonds and Treasury bonds.	The interest rates spread has important influence on company's operation since bond is the major financing tools for company and borrowing costs have direct impact on company's profit.	BofA Merrill Lynch US Corporate AAA Option-Adjusted Spread ^[10] .

Table 9

4.2 Model Descriptions

With these five factors, we will implement a total of seven models to analyze our portfolios and compare the different performance of each model. Below is the summary of these seven factor models.

Model Name	Factors in the Model	Model Equation
Fama French (FF)	MRP, SMB, HML	$E(R_{j,t}) - R_f = \beta_{0,j} + \beta_{1,j}MRP_t + \beta_{2,j}SML_t + \beta_{3,j}HML_t$
CAPMI	MRP, IRP	$E(R_{j,t}) - R_f = \beta_{0,j} + \beta_{1,j}MRP_t + \beta_{2,j}IRS_t$
CAPMM	MRP, UMD	$E(R_{j,t}) - R_f = \beta_{0,j} + \beta_{1,j}MRP_t + \beta_{2,j}UMD_t$
Fama French IRS (FFI)	MRP, SMB, HML, IRP	$E(R_{j,t}) - R_f = \beta_{0,j} + \beta_{1,j}MRP_t + \beta_{2,j}SML_t + \beta_{3,j}HML_t + \beta_{4,j}IRS_t$
Fama French M (FFM)	MRP, SMB, HML, UMD	$E(R_{j,t}) - R_f = \beta_{0,j} + \beta_{1,j}MRP_t + \beta_{2,j}SML_t + \beta_{3,j}HML_t + \beta_{4,j}UMD_t$
Fama French and MIRS (FFMI)	MRP, SMB, HML, UMD, IRP	$E(R_{j,t}) - R_f = \beta_{0,j} + \beta_{1,j}MRP_t + \beta_{2,j}SML_t + \beta_{3,j}HML_t + \beta_{4,j}IRS_t + \beta_{5,j}UMD_t$
CAPMMI	MRP, UMD, IRP	$E(R_{j,t}) - R_f = \beta_{0,j} + \beta_{1,j}MRP_t + \beta_{2,j}UMD_t + \beta_{3,j}IRS_t$

Table 10

4.3 Application of the Multi-Factor Model

4.3.1 Estimate of the Expected Return and the Covariance Matrix

We chose the same 15 stocks as we did in CAPM and Modern Portfolio Theory. For the parameters estimation, we used one year time length from Jan 03, 2010 to Dec 31, 2010 to estimate the coefficients of each stock. Then we used the actual return of each factor on Jan 4, 2011 to estimate the expected return of each stock and the corresponding covariance matrix to form our optimal tangency portfolios as of Jan 4, 2011 with a 10 month holding period. The following tables are estimated coefficients, expected returns and variances calculated from each model in Matlab.

Fama French	β_0	β_{MRP}	β_{SML}	β_{HML}	Expected Return	Actual Return	Expected Variance	Actual Variance
AAPL	0.0009	1.18	-0.0803	0.5821	0.0105	0.0217	0.000287	0.000283
AMZN	0.0004	1.2596	0.3274	-0.8403	0.0101	0.0234	0.000429	0.000593
CHL	0	0.7704	-0.1781	-0.1032	0.0072	0.0069	0.000161	0.000185
COKE	-0.0003	0.4918	0.9162	0.2423	0.0113	0.0247	0.000281	0.000346
DELL	-0.0011	1.1569	0.4223	-0.549	0.0101	0.0103	0.000387	0.000585
EDU	0.001	0.6823	0.3987	0.2448	0.0125	0.0023	0.000556	0.001091
GOOG	-0.0008	1.2147	-0.1812	-0.683	0.0072	0.0175	0.000312	0.000373
GS	0.0002	0.4972	-0.4862	1.6209	0.0154	0.0291	0.000377	0.000498
JOBS	0.0035	0.9106	0.6669	-0.8909	0.0105	0.0388	0.001076	0.001312
MCD	0.0005	0.7078	-0.0953	-0.4762	0.0047	-0.0021	9.29E-05	0.000119
MS	-0.0004	0.8375	-0.498	1.7604	0.0197	0.0374	0.000441	0.001279
MSFT	-0.0009	1.0836	-0.2185	-0.6041	0.0060	0.0026	0.000193	0.000221
SNP	0.0001	1.0193	-0.3787	0.1794	0.0113	0.0203	0.000298	0.000413
WMT	-0.0002	0.6178	-0.2404	-0.6559	0.0009	0.0117	7.77E-05	0.00011
YUM	0.0008	1.029	0.0893	-0.2981	0.0108	0.0008	0.000204	0.00027

Table 11

Fama French M	β_0	β_{MRP}	β_{SML}	β_{HML}	β_{UMD}	Expected Return	Actual Return	Expected Variance	Actual Variance
AAPL	0.0009	0.9217	-0.1175	-0.7640	0.0103	0.0055	0.0217	0.000287	0.000283
AMZN	0.0004	1.1111	0.2137	-0.9449	0.0059	0.0073	0.0234	0.000429	0.000593
CHL	0.0000	0.7687	-0.1793	-0.1044	0.0001	0.0071	0.0069	0.000162	0.000185
COKE	-0.0003	0.4899	0.9148	0.2410	0.0001	0.0113	0.0247	0.000282	0.000346
DELL	-0.0011	1.1760	0.4370	-0.5355	-0.0008	0.0104	0.0103	0.000387	0.000585
EDU	0.0010	0.4596	0.2282	0.0880	0.0089	0.0082	0.0023	0.000558	0.001091
GOOG	-0.0008	1.1916	-0.1989	-0.6992	0.0009	0.0067	0.0175	0.000313	0.000373
GS	0.0002	0.5729	-0.4282	1.6741	-0.0030	0.0169	0.0291	0.000378	0.000498
JOBS	0.0035	0.9217	0.6754	-0.8831	-0.0004	0.0107	0.0388	0.001080	0.001312
MCD	0.0005	0.6710	-0.1235	-0.5021	0.0015	0.0039	-0.0021	0.000093	0.000119
MS	-0.0004	0.9333	-0.4246	1.8278	-0.0038	0.0215	0.0374	0.000442	0.001279
MSFT	-0.0009	1.1453	-0.1713	-0.5607	-0.0025	0.0072	0.0026	0.000193	0.000221
SNP	0.0001	0.9687	-0.4174	0.1439	0.0020	0.0103	0.0203	0.000298	0.000413
WMT	-0.0002	0.6705	-0.2000	-0.6188	-0.0021	0.0019	0.0117	0.000078	0.00011
YUM	0.0008	0.9857	0.0561	-0.3286	0.0017	0.0099	0.0008	0.000204	0.00027

Table 12

Fama French IRS	β_0	β_{MRP}	β_{SML}	β_{HML}	β_{IRS}	Expected Return	Actual Return	Expected Variance	Actual Variance
AAPL	0.0092	1.1915	0.0625	-0.5954	-0.0118	0.0111	0.0217	0.000288	0.000283
AMZN	0.0032	1.2635	0.3213	-0.8449	-0.0041	0.0103	0.0234	0.00043	0.000593
CHL	-0.0046	0.7641	-0.1682	-0.0958	0.0066	0.0069	0.0069	0.000162	0.000185
COKE	-0.0157	0.4707	0.9489	0.2667	0.0218	0.0103	0.0247	0.000282	0.000346
DELL	-0.0017	1.1561	0.4235	-0.5481	0.0008	0.0100	0.0103	0.000387	0.000585
EDU	-0.0040	0.6755	0.4094	0.2528	0.0071	0.0121	0.0023	0.000558	0.001091
GOOG	-0.0012	1.2142	-0.1804	-0.6823	0.0006	0.0072	0.0175	0.000313	0.000373
GS	-0.0162	0.4748	-0.4513	1.6470	0.0232	0.0143	0.0291	0.000378	0.000498
JOBS	-0.0078	0.8950	0.6911	-0.8728	0.0161	0.0097	0.0388	0.001079	0.001312

MCD	-0.0004	0.7066	-0.0934	-0.4748	0.0013	0.0046	-0.0021	9.31E-05	0.000119
MS	-0.0033	0.8334	-0.4917	1.7651	0.0042	0.0195	0.0374	0.000442	0.001279
MSFT	0.0086	1.0968	-0.2389	-0.6194	-0.0136	0.0067	0.0026	0.000193	0.000221
SNP	-0.0007	1.0182	-0.3770	0.1807	0.0011	0.0112	0.0203	0.000298	0.000413
WMT	0.0008	0.6192	-0.2427	-0.6576	-0.0015	0.0010	0.0117	7.79E-05	0.00011
YUM	0.0079	1.0388	0.0741	-0.3095	-0.0101	0.0113	0.0008	0.000204	0.00027

Table 13

Fama French MIRS	β_0	β_{MRP}	β_{SML}	β_{HML}	β_{IRS}	β_{UMD}	Expected Return	Actual Return	Expected Variance	Actual Variance
AAPL	0.0120	0.9336	-0.1437	-0.7840	0.0104	-0.0157	0.0062	0.0217	0.000288	0.000283
AMZN	0.0048	1.1159	0.2032	-0.9529	0.0060	-0.0063	0.0075	0.0234	0.00043	0.000593
CHL	-0.0046	0.7638	-0.1684	-0.0960	0.0000	0.0066	0.0069	0.0069	0.000162	0.000185
COKE	-0.0157	0.4734	0.9511	0.2687	-0.0001	0.0218	0.0103	0.0247	0.000283	0.000346
DELL	-0.0019	1.1752	0.4388	-0.5341	-0.0008	0.0011	0.0104	0.0103	0.000388	0.000585
EDU	-0.0016	0.4567	0.2345	0.0928	0.0088	0.0038	0.0080	0.0023	0.000559	0.001091
GOOG	-0.0010	1.1915	-0.1986	-0.6990	0.0009	0.0002	0.0067	0.0175	0.000314	0.000373
GS	-0.0170	0.5544	-0.3876	1.7052	-0.0032	0.0244	0.0158	0.0291	0.000379	0.000498
JOBS	-0.0080	0.9094	0.7026	-0.8623	-0.0006	0.0163	0.0100	0.0388	0.001083	0.001312
MCD	0.0000	0.6705	-0.1223	-0.5012	0.0015	0.0007	0.0039	-0.0021	9.33E-05	0.000119
MS	-0.0043	0.9290	-0.4153	1.8350	-0.0039	0.0056	0.0213	0.0374	0.000442	0.001279
MSFT	0.0080	1.1549	-0.1925	-0.5769	-0.0023	-0.0127	0.0078	0.0026	0.000194	0.000221
SNP	-0.0002	0.9685	-0.4168	0.1444	0.0020	0.0004	0.0103	0.0203	0.000299	0.000413
WMT	0.0003	0.6711	-0.2012	-0.6197	-0.0021	-0.0007	0.0019	0.0117	7.81E-05	0.00011
YUM	0.0084	0.9939	0.0381	-0.3423	0.0018	-0.0108	0.0104	0.0008	0.000204	0.00027

Table 14

CAPMM	β_0	β_{MRP}	β_{UMD}	Expected Return	Actual Return	Expected Variance	Actual Variance
AAPL	0.0011	0.6957	0.0093	0.0092	0.0217	0.000287	0.000283
AMZN	0.0007	0.8828	0.0054	0.0109	0.0234	0.000428	0.000593
CHL	0.0000	0.7146	-0.0004	0.0081	0.0069	0.000161	0.000185
COKE	-0.0001	0.6865	0.0024	0.0078	0.0247	0.000281	0.000346
DELL	-0.0008	1.0918	-0.0003	0.0117	0.0103	0.000386	0.000585
EDU	0.0011	0.5163	0.0095	0.0071	0.0023	0.000554	0.001091
GOOG	-0.0007	0.9717	-0.0002	0.0104	0.0175	0.000312	0.000373
GS	-0.0005	0.9703	-0.0023	0.0105	0.0291	0.000377	0.000498
JOBS	0.0040	0.7763	0.0002	0.0129	0.0388	0.001072	0.001312
MCD	0.0006	0.5159	0.0007	0.0065	-0.0021	9.27E-05	0.000119
MS	-0.0011	1.3733	-0.0029	0.0145	0.0374	0.000441	0.001279
MSFT	-0.0008	0.9673	-0.0034	0.0102	0.0026	0.000193	0.000221
SNP	-0.0001	0.9486	0.0012	0.0108	0.0203	0.000297	0.000413
WMT	-0.0001	0.4725	-0.0032	0.0052	0.0117	7.75E-05	0.00011
YUM	0.0009	0.9037	0.0015	0.0112	0.0008	0.000203	0.00027

Table 15

CAPMIR S	β_0	β_{MRP}	β_{IRS}	Expected Return	Actual Return	Expected Variance	Actual Variance
AAPL	0.0086	1.0207	-0.0106	0.0133	0.0217	0.000287	0.000283
AMZN	0.0037	1.0729	-0.0043	0.0132	0.0234	0.000428	0.000593
CHL	-0.0058	0.6969	0.0082	0.0075	0.0069	0.000161	0.000185
COKE	-0.0094	0.7646	0.0133	0.0079	0.0247	0.000281	0.000346
DELL	0.0000	1.0803	-0.0011	0.0116	0.0103	0.000386	0.000585
EDU	-0.0010	0.8450	0.0030	0.0106	0.0023	0.000554	0.001091
GOOG	-0.0035	0.9624	0.0040	0.0101	0.0175	0.000312	0.000373
GS	-0.0161	0.8853	0.0222	0.0084	0.0291	0.000377	0.000498
JOBS	-0.0051	0.7780	0.0129	0.0122	0.0388	0.001072	0.001312

MCD	-0.0018	0.5385	0.0034	0.0066	-0.0021	9.27E-05	0.000119
MS	-0.0033	1.2716	0.0032	0.0132	0.0374	0.000441	0.001279
MSFT	0.0061	0.8515	-0.0099	0.0094	0.0026	0.000193	0.000221
SNP	-0.0027	0.9904	0.0037	0.0110	0.0203	0.000297	0.000413
WMT	-0.0018	0.3612	0.0024	0.0039	0.0117	7.75E-05	0.00011
YUM	0.0078	0.9593	-0.0098	0.0124	0.0008	0.000203	0.00027

Table 16

CAPMI RS	β_0	β_{MRP}	β_{IRS}	β_{UMD}	Expected Return	Actual Return	Expected Variance	Actual Variance
AAPL	0.0096	0.6977	-0.0121	0.0093	0.0099	0.0217	0.000287	0.000283
AMZN	0.0043	0.8837	-0.0051	0.0054	0.0112	0.0234	0.000429	0.000593
CHL	-0.0059	0.7131	0.0083	-0.0005	0.0076	0.0069	0.000161	0.000185
COKE	-0.0092	0.6843	0.0129	0.0023	0.0071	0.0247	0.000282	0.000346
DELL	0.0000	1.0919	-0.0010	-0.0003	0.0117	0.0103	0.000387	0.000585
EDU	0.0000	0.5160	0.0015	0.0095	0.0071	0.0023	0.000556	0.001091
GOOG	-0.0035	0.9710	0.0040	-0.0002	0.0102	0.0175	0.000312	0.000373
GS	-0.0164	0.9664	0.0226	-0.0023	0.0093	0.0291	0.000378	0.000498
JOBS	-0.0051	0.7741	0.0129	0.0001	0.0121	0.0388	0.001076	0.001312
MCD	-0.0017	0.5153	0.0033	0.0007	0.0063	-0.0021	9.29E-05	0.000119
MS	-0.0036	1.3726	0.0037	-0.0029	0.0143	0.0374	0.000442	0.001279
MSFT	0.0058	0.9688	-0.0093	-0.0034	0.0107	0.0026	0.000193	0.000221
SNP	-0.0025	0.9480	0.0035	0.0012	0.0106	0.0203	0.000298	0.000413
WMT	-0.0021	0.4720	0.0029	-0.0032	0.0050	0.0117	7.78E-05	0.00011
YUM	0.0080	0.9055	-0.0101	0.0015	0.0118	0.0008	0.000204	0.00027

Table 17

4.3.2 Performance Comparison

In the previous section we estimated the expected returns and covariance matrix for each model, now we can use these data to establish the optimal tangency portfolios as of January 3, 2011 with a 10 month holding period and compare their performances.

Optimal Portfolio Weights								
Stocks	Actual Return	CAPMI	CAPM M	FFI	FFM	FFMI	FF	CAPM MI
AAPL	0.2287	20.82%	-6.51%	25.79%	15.71%	21.45%	23.90%	-2.23%
AMZN	0.1797	8.21%	0.51%	8.54%	-0.92%	0.99%	8.77%	1.34%
CHL	-0.0008	1.49%	9.34%	0.46%	-0.82%	-1.60%	2.47%	5.12%
COKE	0.0064	-0.90%	-0.18%	5.25%	10.90%	-1.96%	11.70%	-3.69%
DELL	0.1914	1.20%	2.55%	5.06%	-3.75%	-1.22%	5.39%	2.78%
EDU	0.1403	4.20%	-2.60%	0.01%	-13.26%	-7.88%	0.25%	-2.99%
GOOG	-0.0070	-0.21%	2.78%	8.47%	11.55%	2.81%	12.36%	1.64%
GS	-0.3253	-3.93%	4.76%	-16.60%	-20.57%	-24.32%	-9.84%	0.40%
JOBS	-0.0326	4.64%	4.77%	6.19%	8.31%	5.18%	6.92%	4.19%
MCD	0.2453	19.76%	18.40%	-0.04%	9.91%	12.54%	-1.99%	16.19%
MS	-0.3099	-0.70%	6.23%	11.38%	11.91%	4.88%	14.32%	5.37%
MSFT	-0.0176	5.83%	14.19%	-1.30%	-16.45%	3.48%	-9.16%	20.47%
SNP	0.0318	4.21%	3.88%	22.26%	33.51%	17.98%	26.33%	2.68%
WMT	0.0694	1.06%	22.81%	-0.14%	36.23%	41.23%	-11.23%	22.23%
YUM	0.1253	34.32%	19.08%	24.68%	17.75%	26.44%	19.82%	26.51%
Total Return	N/A	0.1765	0.0343	0.1385	0.1210	0.1971	0.0885	0.0644

Table 18

We can see from the table 18 that the FFMI model, the model with all of five factors, has the best performance. The CAPMM model, the model with least factors, has the worst performance. So we can make a simple conjecture that the models with more factors tend to perform better than the models with less factors. Also this may indicate that it is more appropriate to model the asset return using more factors since the real market is so complicated that there are many factors that influence asset return. However, we must also be careful about the problem of overfitting, which can cause big differences in results even if there is only a minor change in the raw data.

5. CONCLUSION

In this project, we discussed Modern Portfolio Theory, Capital Asset Pricing Model and Multi-Factor model in detail. These models simplify a complex investment environment and allow investors to understand the relationship between risk and return. Although they afford us this insight, their assumptions sometimes can be constraining and unrealistic. More importantly, they are ineffective in modeling asset returns in real markets. However, these works lay down the foundation of financial econometrics that can be used to model the real markets and give us a direction for further analysis.

6. REFERENCES

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7. APPENDIX

Matlab Code:

Beta of stocks

```
ReturnMatrix=proprice(1:end-1,:)./proprice(2:end,:)-1;
ExcessRetMat=ReturnMatrix-0.000004;
Beta=zeros(1,15);
Xsqr=sum(ExcessRetMat(:,16).^2);
for ii=1:15
Beta(ii)=sum(ExcessRetMat(:,16).*ExcessRetMat(:,ii))/Xsqr;
end
```

Total risk

```
residual=zeros(265,15);
for ii=1:15
    for jj=1:265
        residual(jj,ii)=ExcessRetMat(jj,ii)-ExcessRetMat(jj,16)*Beta(ii);
    end
end
RsdSigma=zeros(1,15);
for zz=1:15
    RsdSigma(zz)=sum(residual(:,zz).^2)/264;
end
MktRisk=var(ExcessRetMat(:,16)+0.000004);
TotalRisk=Beta.^2*MktRisk+RsdSigma';
```

```
Risk=sqrt(TotalRisk);
```

Portfolio Beta

```
portfoliobeta=sum(weight.*Beta)
```

The epsilon for the portfolio

```
portfolio_epsilon=zeros(265,1);  
for ii=1:265  
    portfolio_epsilon(ii)=sum(weight'.*ExcessRetMat(ii,1:15));  
end
```

Factors Model Betas Estimation

```
function [beta,expReturns,CovMatrix] = final(Y,X,newfactor)
```

```
[m n]=size(X);
```

```
beta=zeros(15,n+1);
```

```
regX=ones(m,n+1);
```

```
regX(:,2:end)=X;
```

```
for ii=1:15
```

```
    beta(ii,:)=regress(Y(:,ii),regX);
```

```
end
```

```
expReturns=newfactor*beta';
```

```
mse=zeros(15,15);
```

```
for jj=1:15
```

```
    g=regstats(Y(:,jj),X,'linear','mse');
```

```
    mse(jj,jj)=g.mse;
```

```
end
```

```
SigmaF=cov(X);
```

```
CovMatrix=beta(:,2:end)*SigmaF*beta(:,2:end)'+mse;
```

```
end
```

Portfolio Optimization

```
function [weights,Std,imin] = optimalportfolio2(expReturns,CovMatrix,rf)

n=length(expReturns);

maxReturn=max(expReturns);

minReturn=min(expReturns);

number=50;

ReturnInt=linspace(minReturn,maxReturn,number);

weights=zeros(n,number);

Aeq=[ones(1,n);expReturns];

Std=zeros(number,1);

for ii=1:number

    weights(:,ii)=quadprog(CovMatrix,zeros(n,1), "", "", Aeq,[1;ReturnInt(ii)]);

    Std(ii)=sqrt(weights(:,ii)*CovMatrix*weights(:,ii));

end

imin=find(Std==min(Std));

Ieff=(ReturnInt >= ReturnInt(imin));

sharperatio=(ReturnInt-rf)'./Std;

Itangency=find(sharperatio==max(sharperatio));

plot(Std(Ieff),ReturnInt(Ieff),'r',Std(Itangency),ReturnInt(Itangency),'*',Std(imin),ReturnInt(imin),'o',0,rf,'x');

line([0 Std(Itangency)],[rf,ReturnInt(Itangency)]);

fsize=16;

xlabel('standard deviation of return','fontsize',fsize);

ylabel('expected return','fontsize',fsize);

grid;

weights(:,Itangency)

end
```

