On the Decay Rate of Singular Values of Integral Operators

by

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Contents

1 Introduction

Integral operators are ubiquitous in all areas of pure and applied mathematics, as well as in modeling in population biology, wave propagation theory, mechanical engineering, and image compression and deblurring. The simplest argument explaining the omnipresence of integral operators is that an ordinary differential equation is equivalent to an integral equation, and this provides the basis for the classic proof of the Picard-Lindelof theorem for existence and uniqueness of solutions. Regarding partial differential equations, integral operators appear as fundamental solutions [3]. Integral operators are also instrumental in applications such as image compression and deblurring [9, 6] and other more general inverse problems [10].

The decay rate of the singular values of integral operators is crucial to building computational inversions. Indeed, this decay rate is intimately related to the dilation parameter in Tykhonov regularization and truncations of singular vector expansions [10, 11, 12]. This decay rate is intimately related to the regularity properties of the integration kernel. In dimension one, this relation is well understood and can be analyzed using relatively elementary integral operator theory tools [5, 7, 2]. In this thesis, we revisit the convergence rate proofs given in [7] (their work is also based on [8]). We found that some arguments in [7, 8] are too succinct and hard to grasp for a graduate student or just someone with limited familiarity in this field. We provide additional explanations and a few more lemmas to make these arguments more accessible. In addition, we explore how these arguments could be extended to higher dimensions. We explain why this is a non-trivial endeavor; it will be the subject of future work.

2 Fundamental functional analysis results involved in this thesis

We state and prove a few functional analysis results that we will use to prove our main result. The first two propositions pertain to the separability of classic functional spaces. Next results show how given a separable Hilbert space, some continuous linear operators can be defined using a Hilbert basis. We also show that in some cases, simple criteria ensuring compactness can be found. Finally we state three results on singular values of a compact operator on a separable Hilbert space. The first result expresses the n-th singular value as the infimum over operator norms on restrictions to subspaces of codimension $n-1$. The other two results state general estimates on singular values that will be essential in the rest of this thesis.

Proposition 2.1. *Let E be a separable metric space and $F \subset E$. Then F is also separable.*

Proof. [Adapted from 1, p. 73] Let $\{u_n\}$ be a countable dense subset of E and $\{r_m\}$ a sequence of positive numbers such that $r_m \to 0$. Choose $a_{n,m} \in B(u_n, r_m) \cap F$

whenever this intersection is non-empty. Clearly, the set $\{a_{n,m}\}$ is countable. Fix $\epsilon > 0$ and let $x \in F$. Then there exist $n, m \geq 1$ such that $d(x, u_n) < r_m < \frac{\epsilon}{2}$ in E. Notice that $B(u_n, r_m) \cap F$ is non-empty, since x is in this intersection. Therefore $a_{n,m}$ exists, and $d(x, a_{n,m}) \leq d(x, u_n) + d(u_n, a_{n,m}) < 2r_m < \epsilon$. Thus the set $\{a_{n,m}\}$ is dense in F. □

Theorem 2.1. *Let Ω be an open set in $\mathbb{R}^d$, $1 \leq d < \infty$. $L^2(\Omega)$ is separable.*

Proof. [Adapted from 1, pp. 98–99] Let

$$\mathcal{R} = \{(a_1, b_1) \times \cdots \times (a_d, b_d) \mid a_i, b_i \in \mathbb{Q}, 1 \leq i \leq d\}.$$

Set

$$\mathcal{E} = \left\{ \sum_{i=1}^{n} \alpha_i 1_{R_i} \mid n \in \mathbb{N}, \alpha_i \in \mathbb{Q}, R_i \in \mathcal{R}, 1 \leq i \leq n \right\}.$$

Notice that $\mathcal{R}$ is countable, and thus $\mathcal{E}$ is also countable.

We claim that $\mathcal{E}$ is dense in $L^2(\mathbb{R}^d)$. Let $f \in L^2(\mathbb{R}^d)$ and $\epsilon > 0$. We know that $C_C(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, so there exists $f_1 \in C_C(\mathbb{R}^d)$ such that $\|f - f_1\|_2 < \frac{\epsilon}{2}$. Choose $R \in \mathcal{R}$ such that $\operatorname{supp}(f_1) \subset R$. Since $f_1 \in C_C(\mathbb{R}^d)$, f_1 is uniformly continuous, that is, $\forall \epsilon_1 > 0, \exists \delta > 0$ such that

$$\forall x, y \in \mathbb{R}^d, \quad \|x - y\|_2 < \delta \implies |f_1(x) - f_1(y)| < \epsilon_1.$$

We can divide R into finitely many smaller intervals $R_i \in \mathcal{R}$ with $\|x - y\|_2 < \delta$ for all $x, y \in R_i$ and choose rational $\alpha_i \in (\inf_{x \in R_i} f_1(x), \inf_{x \in R_i} f_1(x) + \epsilon_1)$. Define

$$f_2 = \sum_{i=1}^{n} \alpha_i 1_{R_i}.$$

Clearly $f_2 \in \mathcal{E}$, and $\|f_1 - f_2\|_\infty < \epsilon_1$.

As all norms are equivalent, $\|f_1 - f_2\|_2 \leq c\|f_1 - f_2\|_\infty$ for some constant $c > 0$. If we choose $\epsilon_1 = \frac{\epsilon}{2c}$, then $\|f_1 - f_2\|_2 \leq c\|f_1 - f_2\|_\infty < c\epsilon_1 = \frac{\epsilon}{2}$ Finally, we have

$$\|f - f_2\|_2 \leq \|f - f_1\|_2 + \|f_1 - f_2\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\mathcal{E}$ is dense in $L^2(\mathbb{R}^d)$, and thus $L^2(\mathbb{R}^d)$ is separable.

For a subset Ω of $\mathbb{R}^d$, the extension by 0 outside Ω defines an isometry from $L^2(\Omega)$ to a subset of $L^2(\mathbb{R}^d)$. As $L^2(\mathbb{R}^d)$ is separable, so is this subset, and therefore $L^2(\Omega)$ is separable. □

Proposition 2.2 (Bessel's inequality)**.** *Let H be a Hilbert space, and let $\{e_k\}$ be an orthonormal sequence in H. Then for any $x \in H$, $\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2$.*

Proof. For any $n \in \mathbb{N}$, we have that

$$\begin{aligned} 0 \leq \left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \right\|^2 &= \|x\|^2 - 2\sum_{k=1}^{n} \langle x, \langle x, e_k \rangle e_k \rangle + \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \\ &= \|x\|^2 - 2\sum_{k=1}^{n} |\langle x, e_k \rangle|^2 + \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \\ &= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2. \end{aligned}$$

Therefore

$$\sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \leq \|x\|^2,$$

and as this holds for all $n \in \mathbb{N}$, the statement is proved. □

Theorem 2.2. *Consider the sequence $\{v_j\}$ in H. Define $A : V \to H$ by setting $Ae_j = v_j$, where $\{e_j\}$ is an orthonormal basis for V and $\overline{V} = H$. Assume that the v_j are pairwise orthogonal, that is, $\langle v_i, v_j \rangle = 0$ for $i \neq j$. A is continuous if and only if $\|v_j\|$ is bounded. In this case, extend A to H. A is compact if and only if* $\lim_{j\to\infty} \|v_j\| = 0$.

Proof. Suppose $\|v_j\|$ is bounded. Then there exists $c > 0$ such that $\|v_j\| < c$ for all $j \geq 1$. Then for all $x, y \in V$,

$$\|Ax - Ay\| = \left(\sum_{j=1}^{\infty} (x_j - y_j)^2 \|v_j\|^2 \right)^{1/2} \leq c \left(\sum_{j=1}^{\infty} (x_j - y_j)^2 \right)^{1/2} = c\|x - y\|.$$

Thus given $\epsilon > 0$,

$$\|x - y\| < \epsilon/c \implies \|Ax - Ay\| < \epsilon.$$

Hence A is uniformly continuous.

Suppose instead that $\|v_j\|$ is not bounded. Fix $x_0 \in V$ and let $\epsilon = 1/2$. Given $\delta > 0$, we can choose $c > 0$ such that $c > 1/\delta$. Since $\|v_j\|$ is not bounded, we have that $\|v_k\| > c$ for some $k \geq 1$. Choose $x = x_0 + \frac{\delta}{2} e_k$. Then

$$\|x - x_0\| = \frac{\delta}{2}\|e_k\| = \frac{\delta}{2} < \delta$$

and

$$\|Ax - Ax_0\| = \frac{\delta}{2}\|v_k\| > \frac{\delta}{2} c > \frac{1}{2} = \epsilon.$$

We can construct such an x in this way for any $\delta > 0$. Hence A is not continuous.

If A is continuous, then we can extend A to H. Suppose that $\lim_{j\to\infty} \|v_j\| = 0$. Then given $\epsilon > 0$, there exists an integer N such that $\|v_j\| < \epsilon$ for all $j > N$. Define

$A_k : V \to H$ by $A_k e_j = v_j$ for $1 \leq j \leq k$ and $A_k e_j = 0$ for $j \geq k+1$. Each A_k is continuous, and so can be extended to H. Additionally, $\dim \operatorname{Im} A_k = k$, so each A_k is of finite rank. For $k > N$, we have that for all $x \in H$,

$$\|Ax - A_k x\| = \left(\sum_{j=k+1}^{\infty} x_j^2 \|v_j\|^2 \right)^{1/2} < \epsilon \left(\sum_{j=k+1}^{\infty} x_j^2 \right)^{1/2} \leq \epsilon \|x\|.$$

Thus $\|A - A_k\| < \epsilon$ for $k > N$, so $\|A - A_k\| \to 0$. As each A_k is of finite rank, this shows that A is compact.

Conversely, suppose A is compact. Note that the sequence $\{e_j\}$ converges weakly to zero since for all x in H, $\lim_{j\to\infty} \langle x, e_j \rangle = 0$, as $\sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2$ converges. Since A is compact, the sequence Ae_j converges strongly to zero. □

Theorem 2.3. *Consider the sequence $\{v_j\}$ in H. Define $A : V \to H$ by setting $Ae_j = v_j$, where $\{e_j\}$ is an orthonormal basis for V and $\overline{V} = H$. If $\sum_{j=1}^{\infty} \|v_j\|^2 < \infty$, then A is continuous and compact.*

Proof. Let $c^2 = \sum_{j=1}^{\infty} \|v_j\|^2$. For all $x, y \in V$,

$$\|Ax - Ay\| \leq \sum_{j=1}^{\infty} (x_j - y_j) \|v_j\| \leq \left(\sum_{j=1}^{\infty} (x_j - y_j)^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} \|v_j\|^2 \right)^{1/2} = \|x - y\| c.$$

Thus given $\epsilon > 0$,

$$\|x - y\| < \epsilon / c \implies \|Ax - Ay\| < \epsilon.$$

Hence A is uniformly continuous. Define $A_k : V \to H$ by $A_k e_j = v_j$ for $1 \leq j \leq k$ and $A_k e_j = 0$ for $j \geq k+1$. Then $\dim \operatorname{Im} A_k \leq k$, so each A_k is of finite rank. Fix $\epsilon > 0$. Since $\sum_{j=1}^{\infty} \|v_j\|^2 < \infty$, there exists an integer N such that for all $k > N$, $\sum_{j=k+1}^{\infty} \|v_j\|^2 < \epsilon$. Thus for $k > N$, we have that for all $x \in H$,

$$\|Ax - A_k x\| \leq \sum_{j=k+1}^{\infty} x_j \|v_j\| \leq \left(\sum_{j=k+1}^{\infty} x_j^2 \right)^{1/2} \left(\sum_{j=k+1}^{\infty} \|v_j\|^2 \right)^{1/2} < \|x\| \epsilon.$$

Thus $\|A - A_k\| < \epsilon$ for $k > N$, so $\|A - A_k\| \to 0$. As each A_k is of finite rank, this shows that A is compact. □

Theorem 2.4. *Let a_k be a sequence such that $\sum_{k=1}^{\infty} a_k^2$ diverges. There exists an operator $A : V \to H$, where $\overline{V} = H$, such that $\|Ae_k\| = a_k$ for $k \geq 1$ and A is not continuous.*

Proof. Define $A : V \to H$ by setting $Ae_k = a_k e_1$. Clearly $\|Ae_k\| = a_k$. Set $s_n = \left(\sum_{k=1}^{n} a_k^2\right)^{-1/2}$ and $u_n = s_n \sum_{k=1}^{n} a_k e_k$. Notice that $\|u_n\| = 1$ for each $n \geq 1$. But

$$\|Au_n\| = \left\| s_n \sum_{k=1}^{n} a_k A e_k \right\| = \left\| s_n \sum_{k=1}^{n} a_k^2 e_1 \right\| = \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2}$$

Thus $\|Au_n\| \to \infty$ as $n \to \infty$, so A is not continuous. □

Theorem 2.5. *Suppose T is a compact operator on $L^2(\Omega)$. The singular values of T have the characterization that*

$$s_n(T) = \inf\{\|T|_H\| : H \text{ is a vector subspace, } \operatorname{codim} H \leq n-1\},$$

where $T|_H$ denotes the restriction of T to H. The infimum is attained when H is the vector subspace formed by $f \in L^2(\Omega)$ which are orthogonal to the first $n-1$ eigenfunctions $\phi_1, \dots, \phi_{n-1}$ of T^*T.

Proof. Let $\{\phi_k\}, \{\lambda_k\}$ be a basic system of eigenvectors and eigenvalues of T^*T, and let H be a subspace of codimension $\leq n-1$. Then $H^\perp$ is a subspace of dimension $\leq n-1$, so there exists $f_0 \in \operatorname{span}\{\phi_1, \dots, \phi_n\}$ such that $\|f_0\| = 1$ and $f_0 \perp H^\perp$, that is, $f_0 \in H$. Write $f_0 = \sum_{k=1}^n a_k \phi_k$. Therefore

$$\begin{aligned}\|T|_H\|^2 \geq \|Tf_0\|^2 = \langle Tf_0, Tf_0\rangle = \langle T^*Tf_0, f_0\rangle = \sum_{k=1}^n \lambda_k a_k^2 \geq \lambda_n \sum_{k=1}^n a_k^2 \\ = \lambda_n \|f_0\|^2 = \lambda_n = s_n^2(T),\end{aligned}$$

so we have that

$$\inf\{\|T|_H\| : H \text{ is a vector subspace, } \operatorname{codim} H \leq n-1\} \geq s_n(T).$$

Suppose $H = \operatorname{span}\{\phi_1, \dots, \phi_{n-1}\}^\perp$ and let $f \in H$ with $\|f\| = 1$. We can write $f = \sum_{k=n}^\infty a_k \phi_k$. Therefore

$$\|Tf\|^2 = \langle Tf, Tf\rangle = \langle T^*Tf, f\rangle = \sum_{k=n}^\infty \lambda_k a_k^2 \leq \lambda_n \sum_{k=n}^\infty a_k^2 = \lambda_n \|f\|^2 = \lambda_n.$$

But $\phi_n \in H$ and

$$\|T\phi_n\|^2 = \langle T\phi_n, T\phi_n\rangle = \langle T^*T\phi_n, \phi_n\rangle = \lambda_n \|\phi_n\|^2 = \lambda_n,$$

so we have that

$$\|T|_H\|^2 = \max_{\substack{f \in H \\ \|f\|=1}} \|Tf\|^2 = \lambda_n = s_n^2(T).$$

Hence

$$s_n(T) = \inf\{\|T|_H\| : H \text{ is a vector subspace, } \operatorname{codim} H \leq n-1\}.$$

□

Corollary 2.1. *Suppose T is a compact operator on a Hilbert space H. If $r \geq 1$ and G is a vector subspace of codimension $\leq r$, then*

$$s_{n+r}(T) \leq s_n(T|_G). \tag{2.1}$$

Proof. Since $\operatorname{codim} G = m \le r$, there exist $f_1, \ldots, f_m \in H$ such that

$$G = \{f \in H : \langle f, f_j \rangle = 0,\ 1 \le j \le m\}.$$

Let $\{\psi_k\}$ be a basic system of eigenvectors of $T|_G$ and

$$G_1 = \{g \in G : \langle g, \psi_k \rangle = 0,\ 1 \le k \le n-1\}.$$

Then

$$s_n(T|_G) = \max_{\substack{\|g\|=1 \\ g \in G_1}} \|Tg\|.$$

But G_1 is the vector subspace orthogonal to the $n + m - 1$ functions $f_1, \ldots, f_m$, $\psi_1, \ldots, \psi_{n-1}$, so $\operatorname{codim} G_1 \le n + m - 1 \le n + r - 1$, and thus

$$s_{n+r}(T) \le \max_{\substack{\|g\|=1 \\ g \in G_1}} \|Tg\| = s_n(T|_G).$$

□

Proposition 2.3. *If $A : H \to H$ is a compact operator and $B, C : H \to H$ are bounded linear operators, then for $n \ge 1$,*

$$s_n(BAC) \le \|B\|\|C\|s_n(A)$$

Proof. From the characterization of the singular values in Theorem 2.5, we have that

$$s_n(BA) = \min_{\dim H = n-1} \max_{\substack{\|x\|=1 \\ x \perp H}} \|BAx\| \le \|B\| \min_{\dim H = n-1} \max_{\substack{\|x\|=1 \\ x \perp H}} \|Ax\| = \|B\|s_n(A).$$

Since an operator and its adjoint have the same norm and the same singular values, it follows that

$$\begin{aligned} s_n(BAC) = s_n((BAC)^*) = s_n(C^*(BA)^*) \le \|C^*\|s_n((BA)^*) &= \|C\|s_n(BA) \\ &\le \|B\|\|C\|s_n(A). \end{aligned}$$

□

Proposition 2.4. *Suppose T is a compact operator on $L^2(\Omega)$. If K is positive definite Hermitian, then*

$$\lambda_{m+n-1}(T^*KT) = s_{m+n-1}(T^*KT) \le s_m(K)s_n^2(T) = \lambda_m(K)\lambda_n(T^*T). \tag{2.2}$$

Proof. Since K is positive definite Hermitian, we can write $T^*KT = (K^{1/2}T)^*(K^{1/2}T)$, where $K^{1/2}$ denotes the positive square root of K. So we have that

$$\lambda_{m+n-1}(T^*KT) = s_{m+n-1}(T^*KT) = s_{m+n-1}((K^{1/2}T)^*(K^{1/2}T)) = s_{m+n-1}^2(K^{1/2}T).$$

Let $\phi_1, \dots, \phi_{n-1}$ be the first $n-1$ eigenvectors of T^*KT, and let $H = \operatorname{span}\{\phi_1, \dots, \phi_{n-1}\}^\perp$. By Theorem 2.5, Corollary 2.1, and Proposition 2.3,

$$s_{m+n-1}(K^{1/2}T) \le s_m(K^{1/2}T|_H) \le s_m(K^{1/2})\|T|_H\| = s_m(K^{1/2})s_n(T).$$

Hence

$$s_{m+n-1}(T^*KT) = s^2_{m+n-1}(K^{1/2}T) \le s^2_m(K^{1/2})s^2_n(T) = s_m(K)s^2_n(T).$$

□

3 Application to integral operator theory, Mercer's theorem, and regular integration kernels

We first show that integral operators are compact. It follows easily that if λ_n is the sequence of singular values of such an operator then $\sum_{n=1}^{\infty} \lambda_n^2$ converges due to Proposition 5.1. In case of continuous kernels over compact sets, Mercer's theorem asserts that even $\sum_{n=1}^{\infty} \lambda_n$ converges. Next we show that additional smoothness of integration kernels implies a Mercer type expansion for derivatives. The proof is adapted from [8] with a crucial additional argument that is lacking in Kadota's paper.

Proposition 3.1. *Let Ω be a compact subset of $\mathbb{R}^d$, and fix $n > 0$. Then Ω has a finite cover $\{S_i\}_{1\le i\le m}$ such that the S_i are pairwise disjoint and* $\operatorname{diam} S_i < 1/n$ *for each $1 \le i \le m$.*

Proof. Set

$$C = \{B(x, 1/2n) \mid x \in \Omega\}.$$

Clearly C is a cover of Ω. But Ω is compact, so there exists a finite subcover $A \subset C$ of Ω. Set $m = |A|$, and label the elements of A as $S_1, \dots, S_m$. For $1 \le i \le m$, set $S_i' = S_i \setminus \cup_{j=1}^{i-1} S_j$. The S_i' are pairwise disjoint and form a cover of Ω. Furthermore, since each $S_i' \subset B\left(x_i, \frac{1}{2n}\right)$ for some $x_i \in \Omega$, $\operatorname{diam} S_i < 1/n$. □

Theorem 3.1. *Let Ω be a subset of $\mathbb{R}^d$ and $k \in L^2(\Omega \times \Omega)$. We write $k(x,t)$, where $x, t \in \Omega$. The integral operator K defined by*

$$Kf(x) = \int_\Omega k(x,t)f(t)dt$$

is a compact operator on $L^2(\Omega)$.

Proof. Assume $k \in C_C(\Omega \times \Omega)$. As k has compact support, $k(x,t)$ is zero outside some compact set $S \subset \Omega$. For any given $n \in \mathbb{Z}^+$, since S is compact, it has a finite cover $\{S_{i,n}\}_{1\le i\le a_n}$ such that the $S_{i,n}$ are pairwise disjoint and $\operatorname{diam} S_{i,n} < 1/n$ for each $1 \leq i \leq a_n$. Choose a representative point $t_{i,n}$ from each $S_{i,n} \cap \Omega$ if this set is non-empty, and let K_n be the sequence of integral operators defined by

$$K_n f(x) = \sum_{i=1}^{a_n} k(x, t_{i,n}) \int_{S_{i,n}} f(t)dt.$$

Since $\dim \operatorname{Im} K_n \leq a_n$, each K_n is of finite rank, and since $k \in C_C(\Omega \times \Omega)$, k is uniformly continuous. Fix $\epsilon > 0$. Then there exists $\delta > 0$ such that for all $x, s, t \in \Omega$,

$$\|s - t\| < \delta \implies |k(x,s) - k(x,t)| < \epsilon.$$

Hence for $n > 1/\delta$, for each $1 \leq i \leq a_n$,

$$|k(x,t) - k(x, t_{i,n})| < \epsilon \quad \text{for all} \quad t \in S_{i,n}.$$

Therefore

$$\begin{aligned}
\|Kf - K_n f\|^2 &= \int_\Omega \left| \int_\Omega k(x,t)f(t)dt - \sum_{i=1}^{a_n} k(x,t_{i,n}) \int_{S_{i,n}} f(t)dt \right|^2 dx \\
&= \int_\Omega \left| \int_S k(x,t)f(t)dt - \sum_{i=1}^{a_n} k(x,t_{i,n}) \int_{S_{i,n}} f(t)dt \right|^2 dx \\
&= \int_\Omega \left| \sum_{i=1}^{a_n} \int_{S_{i,n}} [k(x,t) - k(x,t_{i,n})] f(t)dt \right|^2 dx \\
&\leq \int_\Omega \left(\sum_{i=1}^{a_n} \int_{S_{i,n}} |k(x,t) - k(x,t_{i,n})||f(t)|dt \right)^2 dx \\
&< \epsilon^2 \int_\Omega \left(\int_S |f(t)| \right)^2 dx \\
&\leq \epsilon^2 \int_\Omega \int_S |f(t)|^2 dt\, dx \\
&= \epsilon^2 \int_S \int_\Omega |f(t)|^2 dx\, dt \\
&= \epsilon^2 \|f\|^2.
\end{aligned}$$

Thus $\|K - K_n\| < \epsilon$. This shows that $\|K - K_n\| \to 0$, so K is compact.

Now let $k \in L^2(\Omega \times \Omega)$. Then there exists a sequence of functions $k_n \in C_C(\Omega \times \Omega)$ such that $\|k - k_n\|_{L^2(\Omega\times\Omega)} \to 0$. Define the sequence of integral operators K_n' by

$$K_n' f(x) = \int_\Omega k_n(x,t) f(t)dt.$$

We know that each K_n' is compact. Fix $\epsilon > 0$. Then for n large enough,

$$\begin{aligned}
\|Kf - K_n'f\|^2 &= \int_\Omega \left| \int_\Omega [k(x,t) - k_n(x,t)] f(t) dt \right|^2 dx \\
&\leq \int_\Omega \left(\int_\Omega |k(x,t) - k_n(x,t)||f(t)| dt \right)^2 dx \\
&\leq \int_\Omega \left(\int_\Omega |k(x,t) - k_n(x,t)|^2 dt \right) \|f\|^2 dx \\
&= \|f\|^2 \int_\Omega \int_\Omega |k(x,t) - k_n(x,t)|^2 dx\, dt \\
&= \|f\|^2 \|k - k_n\|^2_{L^2(\Omega\times\Omega)} \\
&< \|f\|^2 \epsilon^2.
\end{aligned}$$

Hence $\|K - K_n'\| \to 0$, so K is compact. □

Theorem 3.2. *Let K be an integral operator with kernel function $k \in L^2(\Omega^2)$. Then K^*K is also an integral operator with kernel function $l(x,t) = \int_\Omega k(s,x)k(s,t)ds$, and $l \in L^2(\Omega^2)$*

Proof. Let K' be the integral operator defined by

$$K'f(x) = \int_\Omega k(t,x) f(t) dt.$$

Notice that for all $f, g \in L^2(\Omega)$,

$$\begin{aligned}
\langle Kf, g \rangle &= \int_\Omega \left(\int_\Omega k(x,t) f(t) dt \right) g(x) dx \\
&= \int_\Omega f(t) \left(\int_\Omega k(x,t) g(x) dx \right) dt \\
&= \langle f, K'g \rangle.
\end{aligned}$$

Hence $K' = K^*$. Therefore

$$\begin{aligned}
K^*Kf(x) &= \int_\Omega k(s,x) \left(\int_\Omega k(s,t) f(t) dt \right) ds \\
&= \int_\Omega \left(\int_\Omega k(s,x) k(s,t) ds \right) f(t) dt \\
&= \int_\Omega l(x,t) f(t) dt
\end{aligned}$$

Moreover,

$$\begin{aligned}\int_\Omega \int_\Omega |l(x,y)|^2 dx\, dy &= \int_\Omega \int_\Omega \left| \int_\Omega k(t,x)k(t,y)dt \right|^2 dx\, dy \\ &\le \int_\Omega \int_\Omega \left[\left(\int_\Omega |k(t,x)|^2 dt \right)^{1/2} \left(\int_\Omega |k(t,y)|^2 dt \right)^{1/2} \right]^2 dx\, dy \\ &= \left(\int_\Omega \int_\Omega |k(t,x)|^2 dt\, dx \right) \left(\int_\Omega \int_\Omega |k(t,y)|^2 dt\, dy \right) < \infty.\end{aligned}$$

Thus $l \in L^2(\Omega^2)$ □

Theorem 3.3 (Hilbert-Schmidt theorem)**.** *Let k be a Lebesgue measurable function on $[a,b] \times [a,b]$ such that $\overline{k(t,s)} = k(s,t)$ a.e. and $\sup_{t \in [a,b]} \int_a^b |k(t,s)|^2 ds < \infty$. Let $\{\phi_n\}, \{\lambda_n\}$ be a basic system of eigenvectors and eigenvalues of K, where K is the integral operator with kernel function k. Then for all $f \in L^2([a,b])$,*

$$\int_a^b k(t,s) f(s) ds = \sum_{k=1}^{\infty} \lambda_k \langle f, \phi_k \rangle \phi_k(t) \ \ a.e.$$

The series converges absolutely and uniformly on $[a,b]$.

Proof. [Adapted from 5, pp. 132–133] By the Cauchy-Schwarz inequality,

$$\sum_{j=m}^{n} |\lambda_j \langle f, \phi_j \rangle \phi_j(t)| \le \left(\sum_{j=m}^{n} |\lambda_j \phi_j(t)|^2 \right)^{1/2} \left(\sum_{j=m}^{n} |\langle f, \phi_j \rangle|^2 \right)^{1/2}.$$

We have that

$$\lambda_j \phi_j(t) = K\phi_j(t) = \int_a^b k(t,s)\phi_j(s) ds = \langle k_t, \phi_j \rangle,$$

where $k_t(s) = k(t,s)$. Since $k_t \in L^2([a,b])$, it follows from Bessel's inequality that

$$\begin{aligned}\sum_{j=1}^{\infty} |\lambda_j \phi_j(t)|^2 = \sum_{j=1}^{\infty} |\langle k_t, \phi_j \rangle|^2 \le \|k_t\|^2 = \int_a^b |k(t,s)|^2 ds \le \sup_{t \in [a,b]} \int_a^b |k(t,s)|^2 ds \\ = C^2 < \infty.\end{aligned}$$

Let $\epsilon > 0$. Since $\sum_{j=1}^{\infty} |\langle f, \phi_j \rangle|^2 \le \|f\|^2$, there exists $N \in \mathbb{N}$ such that for all $n > m > N$,

$$\sum_{j=m}^{n} |\langle f, \phi_j \rangle|^2 \le \epsilon^2.$$

Putting together the previous three inequalities, we obtain that for all $n > m > N$ and $t \in [a,b]$,

$$\sum_{j=m}^{n} |\lambda_j \langle f, \phi_j \rangle \phi_j(t)| \leq C\epsilon.$$

Hence $\sum_{j=1}^{\infty} \lambda_j \langle f, \phi_j \rangle \phi_j(t)$ converges absolutely and uniformly on $[a,b]$. Since this series also converges to $Kf(t)$, it follows that $Kf(t)$ is the limit of the series for almost every t. □

Lemma 3.1. *If k is continuous on $[a,b] \times [a,b]$ and $\int_a^b \int_a^b k(t,s) f(s) \overline{f(t)} ds\, dt \geq 0$ for all $f \in L^2([a,b])$, then $k(t,t) \geq 0$ for all $t \in [a,b]$.*

Proof. [Adapted from 5, pp. 134–135] The function $k(t,t)$ is real valued. Hence $\overline{k(t,t)} = k(t,t)$. Suppose $k(t_0,t_0) < 0$ for some $t_0 \in [a,b]$. It follows from the continuity of k that $\operatorname{Re} k(t,s) < 0$ for all (t,s) in some square $[c,d] \times [c,d]$ containing (t_0,t_0). Let $g(s) = 1_{[c,d]}(s)$. Then

$$0 \leq \int_a^b \int_a^b k(t,s) g(s) \overline{g(t)} ds\, dt = \operatorname{Re} \int_c^d \int_c^d k(t,s) ds\, dt < 0$$

which is a contradiction. Hence $k(t,t) \geq 0$ for all $t \in [a,b]$. □

Lemma 3.2. *If k is a continuous complex-valued function on $[a,b] \times [a,b]$, then for any $\phi \in L^2([a,b])$, $h(t) = \int_a^b k(t,s)\phi(s) ds$ is continuous on $[a,b]$.*

Proof. [Adapted from 5, p. 135] Fix $t_0 \in [a,b]$ and $\epsilon > 0$. As k is uniformly continuous on $[a,b] \times [a,b]$, there exists $\delta > 0$ such that for all $t \in [a,b]$,

$$|t - t_0| < \delta \implies |k(t,s) - k(t_0,s)| < \epsilon \quad \forall s \in [a,b].$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} |h(t) - h(t_0)| \leq \int_a^b |k(t,s) - k(t_0,s)||\phi(s)| ds &\leq \|\phi\| \left(\int_a^b |k(t,s) - k(t_0,s)|^2 ds \right)^{1/2} \\ &\leq \|\phi\| \epsilon (b-a)^{1/2}. \end{aligned}$$

Hence h is continuous on $[a,b]$. □

Proposition 3.2 (Cantor's intersection theorem)**.** *Let $\{C_n\}$ be a sequence of non-empty compact, closed sets such that $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. Then $\cap_{n=1}^{\infty} C_n \neq \emptyset$.*

Proof. Choose a point $x_i \in C_i$ for $i = 1, 2, \ldots$. Then $x_i \in C_i \subset C_1$ for each i. Since C_1 is compact, there exists a subsequence x_{i_j} converging to a point $x \in C_1$. Notice that for any n, there exists an integer N for which $i_N > n$, so the subsequence $\{x_{i_j}\}_{j=N}^{\infty}$ is in C_n. Since C_n is also compact, it follows that $x \in C_n$. Since $x \in C_n$ for all n, we have that $x \in \cap_{n=1}^{\infty} C_n$. □

Theorem 3.4 (Dini's theorem)**.** *Let $\{f_n\}$ be a sequence of real-valued continuous functions on $[a,b]$. Suppose $f_1(t) \leq f_2(t) \leq \cdots$ for all $t \in [a,b]$ and $f(t) = \lim_{n\to\infty} f_n(t)$ is continuous on $[a,b]$. Then $\{f_n\}$ converges uniformly to f on $[a,b]$.*

Proof. [Adapted from 5, pp. 135–136] Given $\epsilon > 0$, let $F_n = \{t : f(t) - f_n(t) \geq \epsilon\}$ for $n \in \mathbb{N}$. Clearly $F_{n+1} \subset F_n$.

Let $t_0 \in \overline{F_n}$. Arguing by contradiction, suppose $t_0 \notin F_n$, that is, $f(t_0) - f_n(t_0) = c < \epsilon$. Define $\epsilon_1 = \epsilon - c$. As $f - f_n$ is continuous at t_0, there exists $\delta > 0$ such that for all $t \in (t_0 - \delta, t_0 + \delta)$,

$$|f(t) - f_n(t) - c| = |(f(t) - f_n(t)) - (f(t_0) - f_n(t_0))| < \epsilon_1,$$

But $(t_0 - \delta, t_0 + \delta) \cap F_n \neq \emptyset$, that is, there exists $t_1 \in F_n$ such that

$$|f(t_1) - f_n(t_1) - c| = |(f(t_1) - f_n(t_1)) - (f(t_0) - f_n(t_0))| < \epsilon_1,$$

so

$$\epsilon_1 = \epsilon - c \leq |f(t_1) - f_n(t_1)| - c = |f(t_1) - f_n(t_1) - c| < \epsilon_1.$$

This is the desired contradiction. Thus $t_0 \in F_n$, and hence F_n is a closed set.

Since f_n converges pointwise to f, it follows that $\cap_{n=1}^{\infty} F_n = \emptyset$. Suppose $F_n \neq \emptyset$ for all $n \in \mathbb{N}$. Then since each F_n is closed, $\cap_{n=1}^{\infty} F_n \neq \emptyset$ by Cantor's intersection theorem. This is a contradiction, so there must exist $N \in \mathbb{N}$ such that $F_n = \cap_{n=1}^{N} F_n = \emptyset$. Thus for all $n \geq N$ and $t \in [a,b]$,

$$|f(t) - f_n(t)| = f(t) - f_n(t) \leq f(t) - f_N(t) < \epsilon,$$

so $\{f_n\}$ converges uniformly to f on $[a,b]$. □

Theorem 3.5 (Mercer's theorem)**.** *Let k be continuous on $[a,b] \times [a,b]$. Suppose that for all $f \in L^2([a,b])$, $\int_a^b \int_a^b k(t,s) f(s) \overline{f(t)} ds\, dt \geq 0$. If $\{\phi_n\}, \{\lambda_n\}$ is a basic system of eigenvectors and eigenvalues of the integral operator with kernel function k, then for all $(t,s) \in [a,b] \times [a,b]$,*

$$k(t,s) = \sum_{j=1}^{\infty} \lambda_j \phi_j(t) \overline{\phi_j(s)}.$$

The series converges absolutely and uniformly on $[a,b] \times [a,b]$.

Proof. [Adapted from 5, pp. 136–138] Let K be the integral operator with kernel function k. It follows from the assumptions that K is compact and positive and $\lambda_j = \langle K\phi_j, \phi_j \rangle \geq 0$. Let

$$k_n(t,s) = k(t,s) - \sum_{j=1}^{n} \lambda_j \phi_j(t) \overline{\phi_j(s)}.$$

Since each ϕ_j is an eigenvector of K, it follows from Lemma 3.2 that ϕ_j is continuous, implying that k_n is continuous. Also, we can verify that for all $f \in L^2([a,b])$,

$$\int_a^b \left(\int_a^b k_n(t,s) f(s) ds \right) \overline{f(t)} dt = \int_a^b \left(\int_a^b k(t,s) f(s) ds - \sum_{j=1}^{n} \lambda_j \langle f, \phi_j \rangle \phi_j(t) \right) \overline{f(t)} dt$$

$$= \langle Kf, f \rangle - \sum_{j=1}^{n} \lambda_j |\langle f, \phi_j \rangle|^2 = \sum_{j=n+1}^{\infty} \lambda_j |\langle f, \phi_j \rangle|^2 \geq 0,$$

so by Lemma 3.1 we have that for each $t \in [a,b]$,

$$0 \leq k_n(t,t) = k(t,t) - \sum_{j=1}^{n} \lambda_j |\phi_j(t)|^2.$$

As n is arbitrary, it follows that

$$\sum_{j=1}^{\infty} \lambda_j |\phi_j(t)|^2 \leq k(t,t) \leq \max_{s \in [a,b]} |k(s,s)| = C^2. \tag{3.1}$$

Applying the Cauchy-Schwarz inequality to the sequences $\{\sqrt{\lambda_j}\phi_j(t)\}$ and $\{\sqrt{\lambda_j}\phi_j(s)\}$ yields

$$\sum_{j=m}^{n} \lambda_j |\phi_j(t) \overline{\phi_j(s)}| \leq \left(\sum_{j=m}^{n} \lambda_j |\phi_j(t)|^2 \right)^{1/2} \left(\sum_{j=m}^{n} \lambda_j |\phi_j(s)|^2 \right)^{1/2}. \tag{3.2}$$

Fix $t \in [a,b]$ and $\epsilon > 0$. There exists an integer $N(t)$ such that for $n > m > N(t)$,

$$\sum_{j=m}^{n} \lambda_j |\phi_j(t)|^2 < \epsilon^2. \tag{3.3}$$

From (3.1), (3.2), and (3.3), we have that for $n > m > N(t)$,

$$\sum_{j=m}^{n} \lambda_j |\phi_j(t) \overline{\phi_j(s)}| \leq C\epsilon.$$

Therefore $\sum_{j=1}^{\infty} \lambda_j \phi_j(t) \overline{\phi_j(s)}$ converges absolutely and uniformly in s for each t. Let

$$\tilde{k}(t,s) = \sum_{j=1}^{\infty} \lambda_j \phi_j(t) \overline{\phi_j}(s).$$

For $f \in L^2([a,b])$ and fixed $t \in [a,b]$, the uniform convergence of the series in s and the continuity of each ϕ_j imply that $\tilde{k}(t,s)$ is continuous as a function of s, and

$$\int_a^b [k(t,s) - \tilde{k}(t,s)] f(s) ds = Kf(t) - \sum_{j=1}^{\infty} \lambda_j \langle f, \phi_j \rangle \phi_j(t).$$

If $f \in \ker K = (\operatorname{Im} K)^{\perp}$, then $Kf = 0$ and $\langle f, \phi_j \rangle = \frac{1}{\lambda_j}\langle f, K\phi_j \rangle = 0$, so $\int_a^b [k(t,s) - \tilde{k}(t,s)]f(s)ds = 0$. If $f = \phi_i$ for some i, then

$$(Kf)(t) - \sum_{j=1}^{\infty} \lambda_j \langle f, \phi_j \rangle \phi_j(t) = \lambda_i \phi_i(t) - \lambda_i \phi_i(t) = 0,$$

so again $\int_a^b [k(t,s) - \tilde{k}(t,s)]f(s)ds = 0$. Thus for each t, $k(t,s) - \tilde{k}(t,s)$ is orthogonal to $L^2([a,b])$. Hence $\tilde{k}(t,s) = k(t,s)$ for every t and almost every s. But $k(t,s)$ and $\tilde{k}(t,s)$ are continuous, so

$$k(t,s) = \tilde{k}(t,s) = \sum_{j=1}^{\infty} \lambda_j \phi_j(t) \overline{\phi_j(s)} \quad \forall (t,s) \in [a,b] \times [a,b].$$

In particular,

$$k(t,t) = \sum_{j=1}^{\infty} \lambda_j |\phi_j(t)|^2.$$

The partial sums of this series form an increasing sequence of continuous functions which converges pointwise to $k(t,t)$, which is also continuous. By Dini's theorem, this series converges uniformly to $k(t,t)$. Thus given $\epsilon > 0$, there exists an integer N such that for $n > m > N$,

$$\sum_{j=m}^{n} \lambda_j |\phi_j(t)|^2 < \epsilon \quad \forall t \in [a,b].$$

This implies that for all $n > m > N$ and $(t,s) \in [a,b] \times [a,b]$,

$$\sum_{j=m}^{n} \lambda_j |\phi_j(t) \overline{\phi_j(s)}| < \epsilon$$

Hence $\sum_{j=1}^{\infty} \lambda_j \phi_j(t) \overline{\phi_j(s)}$ converges absolutely and uniformly on $[a,b] \times [a,b]$. □

Theorem 3.6 (Trace formula for integral operators)**.** *Let k be continuous on $[a,b] \times [a,b]$. Suppose that for all $f \in L^2([a,b])$,*

$$\int_a^b \int_a^b k(t,s) f(s) \overline{f(t)} ds\, dt \geq 0.$$

If K is the integral operator with kernel function k and $\{\lambda_j\}$ is the basic system of eigenvalues of K, then

$$\sum_{j=1}^{\infty} \lambda_j = \int_a^b k(t,t) dt.$$

Proof. [5, p. 139] Let $\{\phi_j\}$ be a basic system of eigenvectors of K corresponding to $\{\lambda_j\}$. By Mercer's theorem, the series

$$k(t,t) = \sum_{j=1}^{\infty} \lambda_j |\phi_j(t)|^2$$

converges uniformly on $[a,b]$. Hence

$$\int_a^b k(t,t)dt = \sum_{j=1}^{\infty} \lambda_j \|\phi_j\|^2 = \sum_{j=1}^{\infty} \lambda_j.$$

□

Theorem 3.7. *Let $K(x,t)$ be a real, symmetric, continuous, non-negative definite kernel on $[0,1]^2$. Let $\{\phi_j\}, \{\lambda_j\}$ be a basic system of eigenvectors and eigenvalues of the integral operator generated by K. If*

$$K_r(x,t) = \frac{\partial^{2r}}{\partial x^r \partial t^r} K(x,t)$$

exists and is continuous on $[0,1]^2$, then $\phi_j^{(r)}$ exists and is continuous on $[0,1]$ for $j \geq 1$ and

$$K_r(x,t) = \sum_{k=1}^{\infty} \lambda_j \phi_j^{(r)}(x) \phi_j^{(r)}(t)$$

uniformly on $[0,1]^2$.

Proof. [Adapted from 8] As ϕ_j is an eigenvector of the integral operator generated by K, we have that

$$\phi_j(x) = \frac{1}{\lambda_j} \int_0^1 K(x,t)\phi_j(t)dt$$

for $j \geq 1$. Since K_r exists and is continuous on $[0,1]^2$, we can differentiate both sides r times to obtain

$$\phi_j^{(r)}(x) = \frac{1}{\lambda_j} \int_0^1 \frac{\partial^r}{\partial x^r} K(x,t)\phi_j(t)dt.$$

Hence $\phi_j^{(r)}$ exists and is continuous on $[0,1]$. Define

$$R_{r,k}(x,t) = K_r(x,t) - \sum_{j=1}^{k} \lambda_j \phi_j^{(r)}(x) \phi_j^{(r)}(t).$$

Suppose $R_{1,k}(x_0, x_0) < 0$ for some $x_0 \in [0, 1]$. Then, since $R_{1,k}$ is continuous, there exists $\delta > 0$ such that $R_{1,k}(x, y) < 0$ for all $x, y \in [x_0 - \delta, x_0 + \delta]$. Write $x_0^- = x_0 - \delta$ and $x_0^+ = x_0 + \delta$. Then

$$\begin{aligned}
0 &> \int_{x_0^-}^{x_0^+} \int_{x_0^-}^{x_0^+} R_{1,k}(x,t)dx\,dt \\
&= \int_{x_0^-}^{x_0^+} \int_{x_0^-}^{x_0^+} \left[K_1(x,t) - \sum_{j=1}^{k} \lambda_j \phi_j'(x)\phi_j'(t) \right] dx\,dt \\
&= K(x_0^+, x_0^+) - K(x_0^+, x_0^-) - K(x_0^-, x_0^+) + K(x_0^-, x_0^-) \\
&\quad - \sum_{j=1}^{k} \lambda_j \int_{x_0^-}^{x_0^+} \phi_j'(x)dx \int_{x_0^-}^{x_0^+} \phi_j'(t)dt \\
&= \sum_{j=1}^{\infty} \lambda_j \left[\phi_j(x_0^+)\phi_j(x_0^+) - \phi_j(x_0^+)\phi_j(x_0^-) - \phi_j(x_0^-)\phi_j(x_0^+) + \phi_j(x_0^-)\phi_j(x_0^-) \right] \\
&\quad - \sum_{j=1}^{k} \lambda_j \int_{x_0^-}^{x_0^+} \phi_j'(x)dx \int_{x_0^-}^{x_0^+} \phi_j'(t)dt \\
&= \sum_{k+1}^{\infty} \lambda_j \int_{x_0^-}^{x_0^+} \phi_j'(x)dx \int_{x_0^-}^{x_0^+} \phi_j'(t)dt > 0.
\end{aligned}$$

This is a contradiction. Thus $R_{1,k}(x, x) \geq 0$ for all $x \in [0, 1]$. Hence

$$R_{1,k}(x,x) = K_1(x,x) - \sum_{j=1}^{k} \lambda_j |\phi_j'|^2 \geq 0,$$

and therefore

$$K_1(x,x) \geq \sum_{j=1}^{k} \lambda_j |\phi_j'(x)|^2$$

for all $x \in [0, 1]$, for all $k \geq 1$. Since the partial sums of this series form a non-decreasing sequence which is bounded above by $K_1(x, x)$, the sum

$$\sum_{j=1}^{\infty} \lambda_j |\phi_j'(x)|^2$$

converges. Fix $t \in [0, 1]$ and define $M = \max_{x \in [0,1]} K_1(x, x)$. Then, given $\epsilon > 0$, there exists an integer N such that for all $n > m > N$,

$$\sum_{j=m}^{n} \lambda_j |\phi_j'(t)|^2 < \epsilon.$$

By the Cauchy-Schwarz inequality, for $n > m > N$,

$$\left|\sum_{j=m}^{n} \lambda_j \phi_j'(x)\phi_j'(t)\right|^2 \leq \left(\sum_{j=m}^{n} \lambda_j |\phi_j'(x)|^2\right)\left(\sum_{j=m}^{n} \lambda_j |\phi_j'(t)|^2\right) \leq M \sum_{j=m}^{n} \lambda_j |\phi_j'(t)|^2 < M\epsilon.$$

Therefore

$$K_1'(x,t) = \sum_{j=1}^{\infty} \lambda_j \phi_j'(x)\phi_j'(t)$$

converges absolutely and uniformly in x for every fixed t. Similarly, it converges absolutely and uniformly in t for every fixed x.

Note that K_1 and K_1' are both measurable. Additionally,

$$\begin{aligned}
&\int_0^t \int_0^x [K_1(u,v) - K_1'(u,v)] du\, dv \\
&= \int_0^t \int_0^x K_1(u,v) du\, dv - \sum_{j=1}^{\infty} \lambda_j \int_0^x \phi_j'(u) du \int_0^t \phi_j'(v) dv \\
&= \int_0^t \int_0^x K_1(u,v) du\, dv - [K(x,t) - K(x,0) - K(0,t) + K(0,0)] = 0
\end{aligned}$$

Hence $K_1(x,t) = K_1'(x,t)$ a.e. Thus for fixed x, $K_1(x,t) = K_1'(x,t)$ for almost every t. But for any fixed x, both K_1 and K_1' are continuous in t, so this equality holds for every t. Thus for every t, $K_1(x,t) = K_1'(x,t)$ for almost every x. But for any fixed t, K_1 and K_1' are also continuous in x, so the equality holds for every x and t.

We now have that

$$K_1(x,x) = \sum_{j=1}^{\infty} \lambda_j |\phi_j'(x)|^2.$$

The partial sums of this series form a non-decreasing sequence of continuous functions converging to another continuous function. Therefore by Dini's theorem, this convergence is uniform. In particular, given $\epsilon > 0$, there exists an integer N such that for all $n > m > N$, for all t,

$$\sum_{j=m}^{n} \lambda_k |\phi_j'(t)|^2 < \epsilon$$

Hence by the Cauchy-Schwarz inequality,

$$\left|\sum_{j=m}^{n} \lambda_j \phi_j'(x)\phi_j'(t)\right|^2 \leq \left(\sum_{j=m}^{n} \lambda_j |\phi_j'(x)|^2\right)\left(\sum_{j=m}^{n} \lambda_j |\phi_j'(t)|^2\right) \leq M \sum_{j=m}^{n} \lambda_j |\phi_j'(t)|^2 < M\epsilon,$$

and thus the series converges uniformly in both x and t simultaneously.

Replacing ϕ_j, K, ϕ_j', K_1, K_1^*, and $R_{1,j}$ in the above proof by $\phi_j^{(s)}$, K_s, $\phi_j^{(s+1)}$, K_{s+1}, K_{s+1}^*, and $R_{s+1,j}$, respectively, where $s + 1 \leq r$, we establish that the result holds for $s + 1$ if it holds for s. Therefore, by induction, the result holds for r. $\square$

4 Singular Values of Integral Operators

The proof of the following proposition is adapted from [7]. However, we provide more details at two crucial stages of the proof. Specifically, we show the full calculations for the characterization of J^* and for finding the eigenvalues of J^*J.

Proposition 4.1. *Let J be the operator on $L^2([0,1])$ defined by*

$$Jf(x) = \int_x^1 f(t)dt.$$

Then J^* is characterized by

$$J^*f(x) = \int_0^x f(t)dt$$

and the singular values of J are given by

$$s_n(J) = \frac{2}{(2n-1)\pi}, \quad n \geq 1. \tag{4.1}$$

Proof. Let J' be the integral operator defined by

$$J'f(x) = \int_0^x f(t)dt.$$

Notice that for all $f, g \in L^2([0,1])$,

$$\begin{aligned}
\langle Jf, g\rangle &= \int_0^1 \left(\int_t^1 f(s)ds\right) g(t)dt \\
&= \left[\int_t^1 f(s)ds \int_0^t g(s)ds\right]_{t=0}^1 + \int_0^1 f(t)\left(\int_0^t g(s)ds\right)dt \\
&= \int_0^1 f(t)\left(\int_0^t g(s)ds\right)dt = \langle f, J'g\rangle.
\end{aligned}$$

Hence $J' = J^*$. Let ϕ_n, $n \geq 1$ be the eigenfunctions of J^*J. Then

$$\phi_n(x) = \frac{1}{s_n(J)^2} J^*J\phi_n(x) = \frac{1}{s_n(J)^2}\int_0^x \left(\int_t^1 \phi_n(s)ds\right)dt. \tag{4.2}$$

Notice that $\int_1^t \phi_n(s)ds$ is continuous in t, so $\phi_n(x) = \frac{1}{s_n(J)^2}\int_0^x \left(\int_1^t \phi_n(s)ds\right)dt$ is in $C^1[0,1]$. But if $\phi_n \in C^k[0,1]$, then $\int_1^t \phi_n(s)ds$ is in $C^{k+1}[0,1]$, and it follows that $\phi_n \in C^{k+2}[0,1]$. Hence, by induction, $\phi_n \in C^\infty[0,1]$. In particular, $\phi_n \in C^2([0,1])$, and so from (4.2) we obtain

$$\phi_n''(x) + \frac{1}{s_n(J)^2}\phi_n(x) = 0, \quad \phi_n(0) = \phi_n'(1) = 0.$$

Solving this ODE yields

$$\phi_n(x) = c_1 \sin\left(\frac{x}{s_n(J)}\right).$$

We cannot have $c = 0$, since then $\phi_n(x) \equiv 0$. Therefore to satisfy $\phi_n'(1) = 0$, we must have that

$$\cos\left(\frac{1}{s_n(J)}\right) = 0 \implies s_n(J) = \frac{2}{(2n-1)\pi}.$$

□

The main result of this section is a theorem from [7]. The proof uses the result of Proposition 4.2, and in Ha's paper, they refer to a proof of this proposition from [4, p. 122]. We believe that Gohberg's proof of that proposition is unnecessarily complicated and hard to follow. We present here a simple proof that only uses elementary results on series.

Lemma 4.1. *Let b_k be a sequence in $\mathbb{N}$ such that $b_k < b_{k+1}$ for all $k \geq 1$. Then*

$$\sum_{k=1}^{\infty} \left(1 - \left(\frac{b_k}{b_{k+1}}\right)^p\right)$$

diverges.

Proof. Arguing by contradiction, assume

$$\sum_{k=1}^{\infty} \left(1 - \left(\frac{b_k}{b_{k+1}}\right)^p\right)$$

converges. Then

$$\lim_{k\to\infty} \left(1 - \left(\frac{b_k}{b_{k+1}}\right)^p\right) = 0.$$

Set $a_k = 1 - \left(\frac{b_k}{b_{k+1}}\right)^p$. Then $\lim_{k\to\infty} a_k = 0$, so

$$\lim_{k\to\infty} \frac{-\ln(1-a_k)}{a_k} = 1$$

Since $a_k > 0$ and $-\ln(1-a_k) > 0$ for all k and $\sum_{k=1}^{\infty} a_k$ converges, $\sum_{k=1}^{\infty}[-\ln(1-a_k)]$ converges by the limit comparison test. Also,

$$-\ln(1-a_k) = \ln\left(\left(\frac{b_{k+1}}{b_k}\right)^p\right) = p\ln(b_{k+1}) - p\ln(b_k),$$

so

$$\sum_{k=1}^{n}[-\ln(1-a_k)] = \sum_{k=1}^{n} \left(p\ln(b_{k+1}) - p\ln(b_k)\right) = p\ln(b_{n+1}) - p\ln(b_1).$$

But $b_n \geq n$, so

$$\sum_{k=1}^{\infty}[-\ln(1-a_k)] = \lim_{n\to\infty}\sum_{k=1}^{n} -\ln(1-a_k) = \lim_{n\to\infty} p\ln(b_{n+1}) - p\ln(b_1)$$
$$\geq \lim_{k\to\infty} p\ln(n) - p\ln(b_1)$$

diverges, a contradiction. □

Proposition 4.2. *Let $p > 0$, and let $a_n > 0$ be a decreasing sequence such that*

$$\sum_{n=1}^{\infty} n^p a_n$$

converges. Then

$$\lim_{n\to\infty} n^{p+1} a_n = 0.$$

Proof. Arguing by contradiction, suppose

$$\lim_{n\to\infty} n^{p+1} a_n \neq 0.$$

Then there exists $\epsilon > 0$ and a subsequence a_{n_k} such that

$$n_k^{p+1} a_{n_k} \geq \epsilon \implies a_{n_k} \geq \frac{\epsilon}{n_k^{p+1}}$$

for all $k \geq 1$. Therefore

$$\sum_{j=n_k+1}^{n_{k+1}} j^p a_j \geq a_{n_{k+1}} \sum_{j=n_k+1}^{n_{k+1}} j^p \geq \frac{\epsilon}{n_{k+1}^{p+1}} \sum_{j=n_k+1}^{n_{k+1}} j^p \geq \frac{\epsilon}{n_{k+1}^{p+1}} \int_{n_k}^{n_{k+1}} x^p dx$$
$$= \frac{\epsilon}{p+1}\left(1 - \left(\frac{n_k}{n_{k+1}}\right)^{p+1}\right).$$

But then

$$\sum_{j=1}^{\infty} j^p a_j \geq \sum_{j=n_1}^{\infty} j^p a_j \geq \frac{\epsilon}{p+1}\sum_{k=1}^{\infty}\left(1 - \left(\frac{n_k}{n_{k+1}}\right)^{p+1}\right)$$

diverges by the previous lemma, a contradiction. □

Theorem 4.1. *If $K(x,t)$ is positive definite Hermitian and the symmetric derivative*

$$K_r(x,t) = \frac{\partial^{2r}}{\partial x^r \partial t^r} K(x,t)$$

exists and is continuous on $[0,1]^2$, then

$$\sum_{n=1}^{\infty} n^{2r} \lambda_n(K) < \infty.$$

Consequently,

$$\lim_{n\to\infty} n^{2r+1} \lambda_n(K) = 0.$$

Proof. [Adapted from 7] Define the operator J by

$$Jf(x) = \int_x^1 f(t)dt.$$

Let H_1 be the vector subspace formed by $f \in L^2([0,1])$ which are orthogonal to the constant function $e(t) \equiv 1$ and the function $K(t,0)$. Then H_1 is of codimension ≤ 2. If $f \in H_1$, then

$$\int_0^1 f(t)dt = 0 \quad \text{and} \quad \int_0^1 K(0,t)f(t)dt = 0,$$

and so we have

$$\begin{aligned}
Kf(x) &= \int_0^1 K(x,t)f(t)dt \\
&= \int_0^x \frac{\partial}{\partial y}\left[\int_0^1 K(y,t)f(t)dt\right]dy + \int_0^1 K(0,t)f(t)dt \\
&= \int_0^x \left(\left[\frac{\partial}{\partial y}K(y,t)\int_1^t f(s)ds\right]_{t=0}^1 - \int_0^1 K_1(y,t)\left(\int_1^t f(s)ds\right)dt\right)dy \\
&= \int_0^x \left[\int_0^1 K_1(y,t)\left(\int_t^1 f(s)ds\right)dt\right]dy = J^*K_1Jf(x).
\end{aligned} \tag{4.3}$$

Let G be the vector subspace formed by $g \in L^2([0,1])$ which are orthogonal to the functions $e(t)$ and $K(t,1)$. Then for $g \in G$,

$$\int_0^1 g(t)dt = 0 \quad \text{and} \quad \int_0^1 K(1,t)g(t)dt = 0,$$

and so we have

$$\begin{aligned}
Kg(x) &= \int_0^1 K(x,t)g(t)dt \\
&= \int_1^x \frac{\partial}{\partial y}\left[\int_0^1 K(y,t)g(t)dt\right]dy + \int_0^1 K(1,t)g(t)dt \\
&= \int_1^x \left(\left[\frac{\partial}{\partial y}K(y,t)\int_0^t g(s)ds\right]_{t=0}^1 - \int_0^1 K_1(y,t)\left(\int_0^t g(s)ds\right)dt\right)dy \\
&= \int_x^1 \left[\int_0^1 K_1(y,t)\left(\int_0^t g(s)ds\right)dt\right]dy = JK_1J^*g(x).
\end{aligned} \tag{4.4}$$

For $r = 2$, let H_2 be the vector subspace formed by $f \in H_1$ which are orthogonal to the functions $J^*e(t)$ and $J^*K_1(t,1)$. Then H_2 is of codimension ≤ 4. If $f \in H_2$, then in addition to the above, f also satisfies

$$\int_0^1 Jf(t)dt = 0 \quad \text{and} \quad \int_0^1 K_1(1,t)Jf(t)dt = 0.$$

Applying (4.4) with K_1 and $Jf(x)$ in place of K and $g(x)$, respectively yields

$$K_1 Jf(x) = JK_2 J^* Jf(x).$$

Substituting this into (4.3), we have

$$Kf(x) = J^* JK_2 J^* Jf(x).$$

For $r \geq 3$, we can continue to iterate. Let T_0 be the identity operator and for $1 \leq j \leq r$, let

$$T_j = \begin{cases} J(J^*J)^{(j-1)/2} & \text{if } j \text{ is odd} \\ (J^*J)^{j/2} & \text{if } j \text{ is even} \end{cases}$$

Let H_r be the vector subspace formed by $f \in L^2([0,1])$ which are orthogonal to the $2r$ functions $T_j^* e(t)$ and $T_j^* K_j(t, a_j)$ for $0 \leq j \leq r-1$, where $a_j = 0$ if j is even and $a_j = 1$ if j is odd. Then H_r is of codimension $\leq 2r$ and for $f \in H_r$,

$$Kf(x) = T_r^* K_r T_r f(x).$$

Since $T_r^* T_r = (J^*J)^r$ and is positive definite hermitian, $\lambda_n(T_r^* T_r) = [\lambda_n(J^*J)]^r$. By (2.1), (2.2), and (4.1) for $n \geq 2r+1$,

$$\begin{aligned} \lambda_{2n}(K) \leq \lambda_{2n-1}(K) &\leq \lambda_{2n-2r-1}(T_r^* K_r T_r) \\ &\leq \lambda_{n-2r}(T_r^* T_r)\lambda_n(K_r) = [\lambda_{n-2r}(J^*J)]^r \lambda_n(K_r) \\ &\leq \frac{4^r}{(2n-2r-1)^{2r}\pi^{2r}} \lambda_n(K_r) \leq \frac{1}{n^{2r}} \lambda_n(K_r). \end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned} \sum_{n=2r+1}^{\infty} n^{2r} \lambda_n(K) &= \sum_{n=r+1}^{\infty} (2n)^{2r} \lambda_{2n}(K) + \sum_{n=r+1}^{\infty} (2n-1)^{2r} \lambda_{2n-1}(K) \\ &\leq 2^{2r} \sum_{n=2r+1}^{\infty} \lambda_n(K_r) \leq \infty, \end{aligned}$$

and thus

$$\sum_{n=1}^{\infty} n^{2r} \lambda_n(K) < \infty.$$

Consequently, by Proposition 4.2,

$$\lim_{n \to \infty} n^{2r+1} \lambda_n(K) = 0.$$

□

Theorem 4.2. *If $K \in C^p([0,1]^2)$ is positive definite Hermitian, then*

$$\lambda_n(K) = o\left(\frac{1}{n^{p+1}}\right)$$

as $n \to \infty$.

Proof. [Adapted from 7] If p is even, then $p = 2r$ for some integer $r \geq 1$, so we have from the previous theorem that

$$\lim_{n\to\infty} n^{p+1}\lambda_n(K) = 0.$$

If p is odd, set $r = (p-1)/2$. Since $K_r \in C^1([0,1]^2)$ is positive definite Hermitian,

$$\lim_{n\to\infty} n^2\lambda_n(K_r) = 0,$$

so from (4.5),

$$0 \leq \lim_{n\to\infty} n^{p+1}\lambda_n(K) = \lim_{n\to\infty} (2n)^{2r+2}\lambda_{2n}(K) \leq 2^{2r+2} \lim_{n\to\infty} n^2\lambda_n(K_r) = 0.$$

□

4.1 MATLAB Code

The following MATLAB code uses three different methods to approximate the first n singular values of the integral operator with kernel function $k(x,t) = (x-t)^p \cdot \mathbf{1}_{x>t}$. We expect

$$s_n \sim Cn^\alpha \implies \ln s_n \sim \ln C + \alpha \ln n$$

If $p = 1.5$, k is C^1 regular, so according to Theorem 4.1 we expect that $\alpha < -2$. In the plots below, we take $n = 100$ and perform a linear regression to approximate the decay rate α using the first 80 computed singular values, as the last several values are subject to high numerical error.

```
close all

n = 100; % dimension of subspace
p = 1;   % number of continuous derivatives

[A1,s1] = singular(p,n);
[A2,s2] = singular2(p,n);
[A3,s3] = singularFFT(p,n);

log_singular = log([s1,s2,s3]);

X = [ones(80,1), log(1:80)'];
```

```
beta = X\log_singular(1:80,:);
alpha = beta(2,:);

% plot the approximated singular values of K
plot(log(1:80), log_singular(1:80, :))
legend("Explicitly Computed Integral \alpha \approx " +
   alpha(1), ...
    "Numerical Integration \alpha \approx " + alpha(2),
       ...
    "FFT \alpha \approx " + alpha(3))
title("Approximated Singular Values for p = " + p)
xlabel('ln(n)')
ylabel('ln(s_n(K))')

% Using explicitly computed integral with kernel function
% k(x,t) = (x-t)^p * (x > t) and f_j(x) = n * ((j-1)/n <=
    x < j/n)
function[A,s] = singular(p,n)
    pow = p + 2;
    denom = pow .* (p+1) .* n.^(p+1);

    A = zeros(n);
    for j = 1:n
        % <Kf_j, f_j>
        A(j,j) = 1./denom;

        % <Kf_j, f_k> for k > j. If k < j, this is 0
        for k = j+1:n
            numer = (k-j+1).^pow - 2.*(k-j).^pow + (k-j
               -1).^pow;
            A(j,k) = numer./denom;
        end
    end

    s = svd(A);
end

% Using numerical integration
function[A,s] = singular2(p,n)
    % Numerically approximates the matrix A, where A_jk =
        <Kf_j, f_k>
    % and f_j(x) = n * ((j-1)/n <= x < j/n)
```

```
    % kernel function
    K = @(x,t) (x-t).^p .* (x > t);

    A = zeros(n);
    for j = 1:n
        for k = 1:n
            A(j,k) = n .* integral2(K, (k-1)./n, k./n, (j
               -1)./n, j./n);
        end
    end

    s = svd(A);
end

% Using FFT
function[A,s] = singularFFT(p,n)
    % Numerically approximates the matrix A, where A_jk =
        <Kf_j, f_k>
    % and f_j(x) = exp(2pi*ijx)

    % kernel function
    K = @(x,t) (x-t).^p .* (x > t);

    K_eval = zeros(n);
    for j = 1:n
        for k = 1:n
            % K(x,t) evaluated at a grid ofdiscrete
               points (j/n, k/n)
            K_eval(j,k) = K(j./n, k./n);
        end
    end

    % Kf_eval(j,k) is an approximation of Kf_j(k/n)
    Kf_eval = ifft(K_eval, n).';

    A = ifft(Kf_eval, n);

    s = svd(A);
end
```

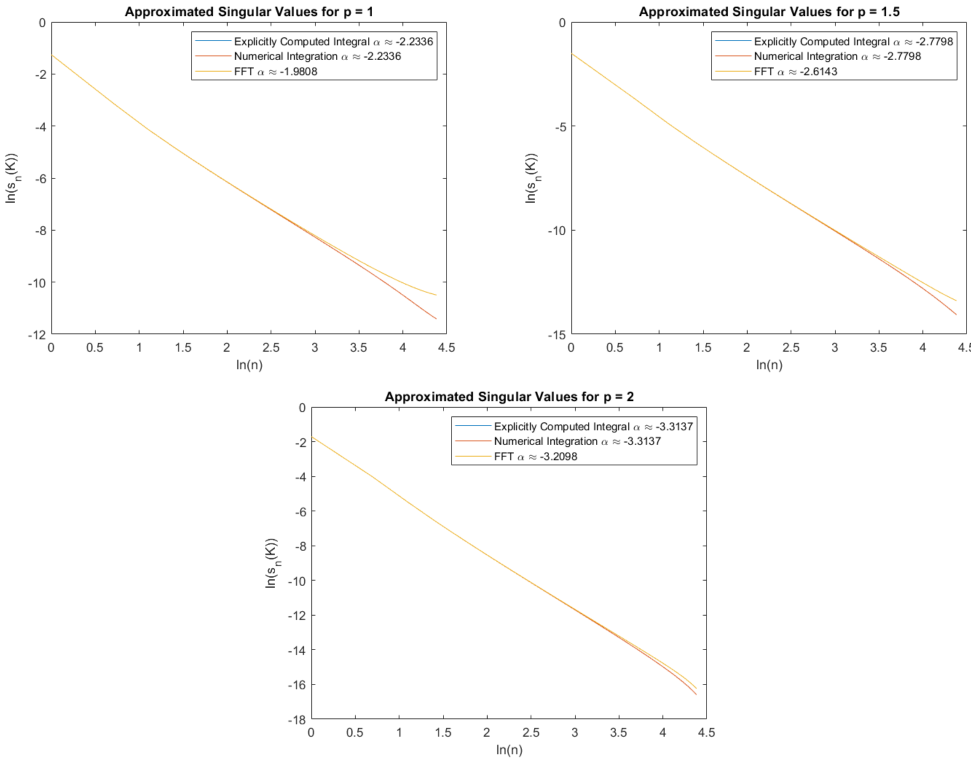

Figure 1: Approximations of the first 100 singular values for the integral operator K with kernel function $k(x,t) = (x-t)^p \cdot \mathbf{1}_{x>t}$, computed for several different values of p.

5 The Two-Dimensional Case

Proposition 5.1. *If ϕ_n, $n \geq 1$ form a Hilbert basis for $L^2([0,1])$, then $\phi_{m,n}(x,y) = \phi_n(x)\phi_m(y)$, $m,n \geq 1$ form a Hilbert basis for $L^2([0,1]^2)$.*

Proof. It is simple to show that the functions $\{\phi_{m,n}\}_{m,n\geq 1}$ are pairwise orthogonal:

$$\begin{aligned}\int_0^1\int_0^1 \phi_{m,n}(x,y)\phi_{k,l}(x,y)dx\,dy &= \int_0^1\int_0^1 \phi_m(x)\phi_n(y)\phi_k(x)\phi_l(y)dx\,dy \\ &= \left(\int_0^1 \phi_m(x)\phi_k(x)dx\right)\left(\int_0^1 \phi_n(y)\phi_l(y)dy\right) \\ &= \begin{cases} 1, & \text{if } m = k \text{ and } n = l \\ 0, & \text{otherwise}\end{cases}.\end{aligned}$$

Let $f \in L^2([0,1]^2)$. Fix $x \in [0,1]$. For almost all x, the function $y \to f(x,y)$ is in $L^2([0,1])$, and thus

$$f(x,y) = \sum_{n=1}^{\infty}\left(\int_0^1 f(x,y)\phi_n(y)dy\right)\phi_n(y).$$

It follows that

$$\int_0^1 f(x,y)^2 dy = \sum_{n=1}^{\infty} \left| \int_0^1 f(x,y)\phi_n(y)dy \right|^2$$

for almost all $x \in [0,1]$. Denote

$$g_m(x) = \int_0^1 f(x,y)\phi_m(y)dy.$$

Then clearly

$$[g_m(x)]^2 \le \sum_{n=1}^{\infty} \left| \int_0^1 f(x,y)\phi_n(y)dy \right|^2 = \int_0^1 f(x,y)^2 dy,$$

so we have that

$$\int_0^1 [g_m(x)]^2 dx \le \int_0^1 \int_0^1 f(x,y)^2 dx\, dy,$$

and therefore $g_m \in L^2([0,1])$. Thus

$$\int_0^1 [g_m(x)]^2 dx = \sum_{n=1}^{\infty} \left| \int_0^1 g_m(x)\phi_n(x)dx \right|^2.$$

Hence

$$\begin{aligned}
\langle f, f \rangle &= \int_0^1 \int_0^1 f(x,y)^2 dy\, dx = \int_0^1 \sum_{m=1}^{\infty} \left| \int_0^1 f(x,y)\phi(y)dy \right|^2 dx \\
&= \int_0^1 \sum_{m=1}^{\infty} [g_m(x)]^2 dx = \sum_{m=1}^{\infty} \int_0^1 [g_m(x)]^2 dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^1 g_m(x)\phi_n(x)dx \right|^2 \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^1 \int_0^1 f(x,y)\phi_m(y)\phi_n(x)dy\, dx \right|^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle f, \phi_{m,n} \rangle|^2.
\end{aligned}$$

Suppose $\langle f, \phi_{m,n} \rangle = 0$ for all $\phi_{m,n}$, $m, n \ge 1$. Then

$$\langle f, f \rangle = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle f, \phi_{m,n} \rangle|^2 = 0 \implies f = 0.$$

Thus $\text{span}\{\phi_{m,n}\}_{m,n \ge 1}$ is dense in $L^2([0,1]^2)$. □

Proposition 5.2. *Let J be the integral operator on $L^2([0,1]^2)$ defined by*

$$Jf(x,y) = \int_y^1 \int_x^1 f(s,t)ds\, dt.$$

Then J^* is characterized by

$$J^*f(x,y) = \int_0^y \int_0^x f(s,t)ds\,dt.$$

and the singular values of J are given by

$$s_{m,n}(J) = \frac{4}{(2m-1)(2n-1)\pi^2}, \quad m,n \geq 1.$$

Proof. Let J' be the integral operator defined by

$$J'f(x,y) = \int_0^y \int_0^x f(s,t)ds\,dt.$$

Then for all $f, g \in L^2([0,1]^2)$, we have that

$$\begin{aligned}
&\langle Jf, g\rangle = \int_0^1 \int_0^1 \left(\int_y^1 \int_x^1 f(s,t)ds\,dt \right) g(x,y)dx\,dy \\
&= \int_0^1 \int_y^1 \left(\left[\int_x^1 f(s,t)ds \int_0^x g(s,y)ds \right]_{x=0}^1 + \int_0^1 f(x,t) \left(\int_0^x g(s,y)ds \right) dx \right) dt\,dy \\
&= \int_0^1 \int_0^1 \int_y^1 \int_0^x f(x,t)\, g(s,y)\, ds\, dt\, dy\, dx \\
&= \int_0^1 \int_0^x \left(\left[\int_y^1 f(x,t)dt \int_0^y g(s,t)dt \right]_{y=0}^1 + \int_0^1 f(x,y) \left(\int_0^y g(s,t)dt \right) dy \right) ds\,dx \\
&= \int_0^1 \int_0^1 f(x,y) \left(\int_0^y \int_0^x g(s,t)ds\,dt \right) dx\,dy = \langle f, J'g\rangle.
\end{aligned}$$

Hence $J' = J^*$. Let ϕ be an eigenfunction of J^*J with corresponding eigenvalue λ^2, so that λ is a singular value of J. Then

$$\phi(x,y) = \frac{1}{\lambda^2} J^*J\phi(x,y) = \frac{1}{\lambda^2} \int_0^y \int_0^x \left(\int_t^1 \int_s^1 \phi(u,v)du\,dv \right) ds\,dt. \tag{5.1}$$

Notice that $\int_1^t \int_1^s \phi(u,v)du\,dv$ is continuous in s and t, so we have that $\phi(x,y) = \frac{1}{\lambda^2} \int_0^y \int_0^x \left(\int_t^1 \int_s^1 \phi(u,v)du\,dv \right) ds\,dt$ is in $C^1([0,1]^2)$. But if $\phi \in C^k([0,1]^2)$, then $\int_1^t \int_1^s \phi(u,v)du\,dv$ is in $C^{k+1}([0,1]^2)$, and it follows that $\phi \in C^{k+2}(0,1]^2)$. Hence, by induction, $\phi \in C^\infty([0,1]^2)$. In particular, $\phi \in C^4([0,1]^2)$, and so from (5.1) we obtain

$$\frac{\partial^4}{\partial x^2 \partial y^2}\phi(x,y) - \frac{1}{\lambda^2}\phi(x,y) = 0 \tag{5.2}$$

with

$$\phi(0,y) = \phi(x,0) = \frac{\partial}{\partial x}\phi(1,y) = \frac{\partial}{\partial y}\phi(x,1) = 0.$$

Assume ϕ is of the form $\phi(x,y) = f(x)g(y)$ for some functions $f, g \in L^2([0,1])$. Then (5.2) becomes

$$f''(x)g''(y) - \frac{1}{\lambda^2} f(x)g(y) = 0$$

with

$$f(0) = f'(1) = g(0) = g'(1) = 0.$$

Thus we have that

$$f''(x) - \frac{g(y)}{\lambda^2 g''(y)} f(x) = 0$$

for all $x, y \in [0,1]$. Hence it must be that $-g(y)/g''(y) = c$ for some constant $c \in \mathbb{R}$ for all $y \in [0,1]$. We now have

$$f''(x) + \frac{c}{\lambda^2} f(x) = 0$$

and

$$g''(y) + \frac{1}{c} g(y) = 0.$$

Solving these ODEs yields

$$f(x) = c_1 \sin\left(\frac{x\sqrt{c}}{\lambda}\right) \quad \text{and} \quad g(y) = c_2 \sin\left(\frac{y}{\sqrt{c}}\right).$$

Moreover, from the conditions $f'(1) = 0$ and $g'(1) = 0$ we find that

$$\frac{\lambda}{\sqrt{c}} = \frac{2}{(2n-1)\pi} \quad \text{and} \quad \sqrt{c} = \frac{2}{(2m-1)\pi},$$

for some $m, n \in \mathbb{Z}^+$. Thus we have that

$$s_{m,n}(J) = \lambda = \frac{4}{(2m-1)(2n-1)\pi^2}$$

is a singular value of J with corresponding singular function

$$\phi_{m,n}(x,y) = \sin\left(\frac{(2n-1)\pi x}{2}\right) \sin\left(\frac{(2m-1)\pi y}{2}\right).$$

By Proposition 5.1, the functions $\{\phi_{m,n}\}_{m,n\geq 1}$ form a Hilbert basis for $L^2([0,1]^2)$, and therefore these are the only singular functions, and thus the only singular values, of J. □

Proposition 5.3. *Let J be the integral operator on $L^2([0,1]^2)$ defined by*

$$Jf(x,y) = \int_y^1 \int_x^1 f(s,t) ds\, dt$$

and let $s_n(J)$, $n \geq 1$ be the singular values of J in decreasing order. Define $f : [1,\infty) \to [1,\infty)$ by $f(x) = x + x \ln x$. Then

$$s_n(J) \leq \frac{4}{\pi^2 f^{-1}(x)}.$$

Proof. By Proposition 5.2, the singular values of J are given by

$$s'_{j,k}(J) = \frac{4}{\pi^2(2j-1)(2k-1)},$$

so we have that

$$s'_{j,k}(J) \geq \frac{4}{\pi^2 m} \implies (2j-1)(2k-1) \leq m \implies jk \leq m \implies k \leq \frac{m}{j}.$$

It follows that

$$\left|\left\{(j,k) : s'_{j,k}(J) \geq \frac{4}{\pi^2 m}\right\}\right| = \sum_{j=1}^{m} \left|\left\{k : s'_{j,k}(J) \geq \frac{4}{\pi^2 m}\right\}\right| \leq \sum_{j=1}^{m} \frac{m}{j} \leq m + \int_1^m \frac{m}{x} dx$$
$$= m + m \ln m,$$

that is, the number of singular values of J which are greater than or equal to $\frac{4}{\pi^2 m}$ is at most $f(m) = m + m \ln m$. Therefore if $n > f(m)$, then $s_n(J) < \frac{4}{\pi^2 m}$. Since $s_n(J) = \frac{4}{\pi^2 l}$ for some integer $l \geq 1$, it follows that $s_n(J) \leq \frac{4}{\pi^2(m+1)}$. Notice that f is a strictly increasing function, and is therefore invertible. Moreover, its inverse is also strictly increasing. Hence if $f(m) < n \leq f(m+1)$, then

$$s_n(J) \leq \frac{4}{\pi^2(m+1)} = \frac{4}{\pi^2 f^{-1}(f(m+1))} \leq \frac{4}{\pi^2 f^{-1}(n)}.$$

As this inequality holds for all m, we obtain the desired result that

$$s_n(J) \leq \frac{4}{\pi^2 f^{-1}(n)} \quad \text{for all} \quad n \geq 1.$$

□

6 Conclusion

Although we found the eigenfunctions of J in the two-dimensional case and an upper bound on its eigenvalues, we were unable to generalize calculations (4.3) and (4.4). In the one-dimensional case, we considered a subspace of finite codimension, but the higher dimensional case would require subspaces that do not have finite codimension. All in all, we believe that there is no straightforward generalization of Ha's arguments [7], unless the integration kernel $k(x, y)$ is compactly supported in the y variable. In that case, integration by parts are more easily manipulated since no boundary terms appear. In future work, we will examine the case of general smooth kernels k over general compact domains Ω. We believe that using the Dirichlet and Neumann eigenvalues for the Laplacian in combination with Weyl's Theorem for their decay rate will be relevant.

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