Tiling of Prime and Composite Kirchhoff Graphs

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Abstract

A Kirchhoff graph is a vector graph with orthogonal cycles and vertex cuts. We present an algorithm that constructs all the Kirchhoff graphs up to a fixed edge multiplicity. We explore the tiling of prime Kirchhoff graphs. Specifically, we show the existence of countably infinitely many prime Kirchhoff graphs given a set of initial fundamental Kirchhoff graphs. We also explore the minimal multiplicity for which nontrivial Kirchhoff graphs exist.

1 Introduction

Kirchhoff graphs are originally motivated by the study of electrochemical reaction networks; they are circuit diagrams for these networks ([1], [2]). A Kirchhoff graph is a connected vector graph whose cycles are orthogonal to its vertex cuts. More specifically, given a matrix R, a Kirchhoff graph has properties that all of its cycles form a basis for $\mathrm{Null}(R)$, and all of its vertex cuts lie in $\mathrm{Row}(R)$.

Many properties of Kirchhoff graphs have been explored in the past, for example, Kirchhoff graph uniformity ([3],[4]). Fehribach & McDonald [5] showed how to construct rank-two, nullity-two Kirchhoff graphs. In a previous MQP [6], Gietzmann-Sanders presented an algorithm for construction Kirchhoff graphs inside a given frame (bounding box) . This MQP aims to explore the construction and the tiling of Kirchhoff graphs in order to study how Kirchhoff graphs interact with one another.

Section 2 gives a brief background on Kirchhoff graphs. Section 3 presents an algorithm that constructs all Kirchhoff graphs up to a fixed edge multiplicity given R. Section 4 explores the tiling of prime Kirchhoff graphs. Specifically, it explores the number of possible prime Kirchhoff graphs given an initial fundamental set of Kirchhoff graphs.

2 Kirchhoff Graphs

Kirchhoff graphs are graphs whose edges are vectors (or whose edges are assigned vectors) that satisfy an orthogonality condition between its cycles and its vertex cuts. Consider a set $S := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_n\}$ of vectors in a vector space $\mathcal{V}$ over $\mathbb{Q}$. For simplicity, suppose that no vector in S is a scalar multiple of another vector is S. Suppose there is a k where $1 < k < n$ so that $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_k\}$ is the basis for $\mathrm{Span}(S)$. Then for $[\boldsymbol{s}_1,\ \boldsymbol{s}_2,\ \cdots\ \boldsymbol{s}_n]$, a row vector of vectors, there is a coefficient matrix C' such that $[\boldsymbol{s}_1,\ \boldsymbol{s}_2,\ \cdots\ \boldsymbol{s}_n]\cdot[C'/-I_k] = 0$ where $[C'/-I_k]$ is a block matrix with C' over I_k. Note that if the entries of R are not integers, we can write R in the form of $[qI_k|C]$ where $q \in \mathbb{Z}^+$ is the least common multiple of denominators of the fractional entries in R, so the entries of R are integers, and C is the block matrix with entries multiplied by q. Then $N := [C/-qI_k]$ is the null matrix for S, and $R := [qI_k|C]$ is the row matrix for S. Specifically, the columns of N is a basis for $\mathrm{Null}(R)$, and the columns of R can be used to represent S. This means that any matrix A that is row equivalent to R has the same row space and null space as R.

Definition 2.1. A **vector graph $\boldsymbol{G}$** is a pair $\boldsymbol{G} = (V, S)$, where V is a set of vertices, and S is a set of edge vectors as discussed above. There is a cycle in the graph only when the corresponding vectors add to zero in the vector space.

Definition 2.2. For a vertex v in a vector graph $\boldsymbol{G}$, the **vertex cut** of v, denoted $\boldsymbol{\lambda}(v) = \{\lambda_1, \cdots, \lambda_n\}$, has entries correspond to the vectors $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_n$. For each i, entry λ_i is the net number of $\mathbf{s}_i$ that exit vertex v. $\lambda_i \in \mathbb{Z}$ is negative if a vector is entering v, positive if a vector is leaving v, and zero if it is not involved. Each vertex is situated in $\mathbb{Z}^k$ and is associated with a coordinate $(x_1, \cdots, x_k)$, where $k = \dim(\text{Row}(R))$.

Definition 2.3. A **cycle** C in a vector graph $\boldsymbol{G}$ is an alternating sequence of vertices and edges that starts and ends with the same vertex in which no vertex appears twice except for the first and the last vertex. Cycles in a vector graph corresponds to linear combinations of the edge vectors $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_n$ that add to the zero vector.

Definition 2.4. The **cycle vector** of a cycle C, denoted $\boldsymbol{\chi}(C) = \{\chi_1, \cdots, \chi_n\}$, has entries correspond to vectors $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_n$. For each i, entry χ_i is the net number of times $\boldsymbol{s}_i$ appears in the cycle. Add 1 to the i-th component each time C traverses an $\mathbf{s}_i$ in the forward direction, and subtract 1 for each $\mathbf{s}_i$ in the backward direction.

This leads to the definition of a Kirchhoff graph.

Definition 2.5. Let A be any matrix that is row equivalent to R. A vector graph $\boldsymbol{G}$ is a **Kirchhoff graph** for A if and only if the following conditions are satisfied:

1. For $u_j \in \mathbb{Z}$, $\mathbf{u} = [u_1, u_2, \cdots, u_n]^T \in \text{Null}(A)$ if and only if there is a basis of cycles in $\boldsymbol{G}$ where, for each j, $1 \le j \le n$, the j-th directed edge appears with multiplicity $|u_j|$.
2. For a given vertex of $\boldsymbol{G}$, if the j-th edge exists with multiplicity $v_j \in \mathbb{Z}$, then $\boldsymbol{\lambda}(v) \in \text{Row}(A)$.

Note that because of the fundamental theorem of linear algebra, Row(A) and Null(A) are orthogonal compliments:

$$\text{Row}(A) \perp \text{Null}(A), \quad \text{Row}^T(A) \oplus \ \text{Null}(A) = \mathbb{Q}^n$$

Additionally, A has a Kirchhoff graph if and only if R has a Kirchhoff graph, since they are row-equivalent to each other.

Definition 2.6. Let $\boldsymbol{G}$ be a Kirchhoff graph with edge vectors $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_n$. $\boldsymbol{G}$ is **uniform** if each $\boldsymbol{s}_i$ occurs the same number of times in $\boldsymbol{G}$.

Definition 2.7. A vector graph $\boldsymbol{G}$ is **vector 2-connected** if and only if for any pair of vector edges $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$, there exists a cycle c such that the cycle vector $\boldsymbol{\chi}(c)$ is nonzero with respect to both $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$.

Theorem 2.8. ***(Reese, Fehribach, Paffenroth, Servatius, 2019 [4])*** *Every vector 2-connected Kirchhoff graph is uniform.*

The authors use a linear algebraic equivalent to Kirchhoff graphs to prove this result. All Kirchhoff graphs mentioned in this MQP will assumed to be vector 2-connected, hence uniform.

Definition 2.9. Given a 2-connected Kirchhoff graph $\boldsymbol{G}$, the **multiplicity** $m(\boldsymbol{G})$ is the number of times each edge appears in $\boldsymbol{G}$. When it is unambiguous, we simply write m.

Definition 2.10. A **null Kirchhoff graph** is a Kirchhoff graph with only one vertex and no edges.

Example 2.11. Suppose we have vectors $\boldsymbol{s}_1 = \sin^2 x$, $\boldsymbol{s}_2 = \cos^2 x$, $\boldsymbol{s}_3 = 1$, $\boldsymbol{s}_4 = 1 + \sin^2 x$ in $L^2[0, \pi]$. Then the null matrix is

$$N = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which means

$$\begin{aligned} \boldsymbol{s}_1 + \boldsymbol{s}_2 &= \boldsymbol{s}_3 \\ 2\boldsymbol{s}_1 + \boldsymbol{s}_2 &= \boldsymbol{s}_4 \end{aligned}$$

are satisfied by the null matrix. The row matrix then is

$$R = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

where each column represents an edge vector. The vector graph in Figure 1 is a Kirchhoff graph for R and any graph that is row equivalent to R.

The vertex cut for each vertex in $\boldsymbol{G}$ lies in $\mathrm{Row}(R)$. For example, the vertex $v_0 = (0, 0)$ has one copy of $\boldsymbol{s}_1$, zero copy of $\boldsymbol{s}_2$, one copy of $\boldsymbol{s}_3$, and two copies of $\boldsymbol{s}_4$ coming out of it, so

$$\boldsymbol{\lambda}(v) = \begin{bmatrix} 1 & 0 & 1 & 2 \end{bmatrix} \in \mathrm{Row}(R).$$

All cycles of $\boldsymbol{G}$ also lie in $\mathrm{Null}(R)$. For example, the cycle $C = (0, 0) - (1, 0) - (1, 1) - (0, 0)$ has cycle vector

$$\boldsymbol{\chi}(C) = \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \in \mathrm{Null}(R).$$

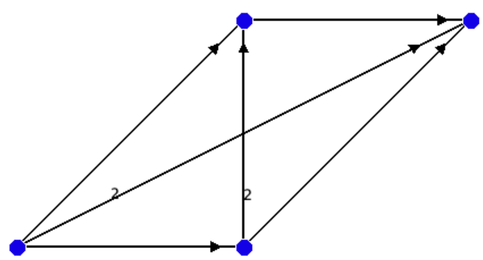


Figure 1: Kirchhoff graph for R. Hash marks indicate multiplicity and vertex coordinates are given by ordered pairs.

Definition 2.12. Let $\boldsymbol{G}$ be a Kirchhoff graph with vertices $\{v_1, \cdots, v_n\}$ situated in $\mathbb{Z}^k$. Let $p_v = (x_1, \cdots, x_k)$ be Cartesian coordinate of vertex v. Let $|p_v| = \sum_{i=1}^{k} x_i$. Then the **anchor vertex** v_ϵ is the vertex with minimal $|p_v|$. If for two vertices v_1, v_2, $|p_{v_1}| = |p_{v_2}|$, then the anchor vertex is the vertex with smallest x_1, then x_2 etc. We then place the anchor vertex of $\boldsymbol{G}$ at $(0, \cdots, 0)$ and place other vertices correspondingly.

Definition 2.13. Given two Kirchhoff graphs $\boldsymbol{K}_1, \boldsymbol{K}_2$, and a coordinate $\boldsymbol{x} \in \mathbb{Z}^n$, we define addition $(\boldsymbol{K}_1 + \boldsymbol{K}_2, \boldsymbol{x})$ as drawing the anchor vertex of $\boldsymbol{K}_1$ at $(0, 0)$ and the anchor vertex of $\boldsymbol{K}_2$ at coordinate $\boldsymbol{x}$. And $V(\boldsymbol{K}_1 + \boldsymbol{K}_2) = V(\boldsymbol{K}_1) \cup V(\boldsymbol{K}_2)$, $E(\boldsymbol{K}_1 + \boldsymbol{K}_2) = E(K) \cup E(K)$, with $m(\boldsymbol{K}_1 + \boldsymbol{K}_2) = m(\boldsymbol{K}_1) + m(\boldsymbol{K}_2)$. When it is unambiguous, we write $\boldsymbol{K}_1 + \boldsymbol{K}_2$ as a shorthand.

Example 2.14. Given

$$R = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 \end{bmatrix},$$

and two Kirchhoff graphs $\boldsymbol{K}_1, \boldsymbol{K}_2$ (Figures 2 and 3) associated with it,

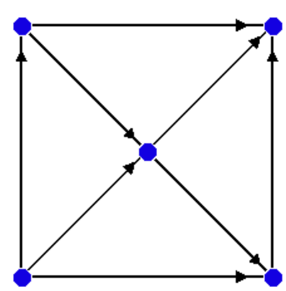


Figure 2: Kirchhoff graph $\boldsymbol{K}_1$. Notice that it has vertex cuts in the row space and the cycles for a basis for the null space.

The addition $(\boldsymbol{K}_1 + \boldsymbol{K}_2, (1, 1))$ is shown in Figures 4.

Definition 2.15. Given a Kirchhoff graph $\boldsymbol{G}$, its **chiral** graph is achieved by rotating $\boldsymbol{G}$ 180 degrees and reversing edge vectors.

Definition 2.16. A Kirchhoff graph is a **self-chiral** if its chiral is itself. In other words, it is invariant under the chiral action.

Definition 2.17. Given n Kirchhoff graphs $\boldsymbol{K}_1, \cdots, \boldsymbol{K}_n$, the set generated from the graphs is defined as $\langle \boldsymbol{K}_1, \cdots, \boldsymbol{K}_n \rangle = \{a_1 \boldsymbol{K}_1 + \cdots + a_n \boldsymbol{K}_n : a_i \in \mathbb{Z}\}$, where $a_i \boldsymbol{K}_i$ is the set of all graphs that can be made by tiling or subtracting $\boldsymbol{K}_i$

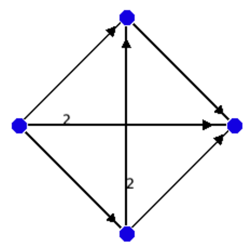


Figure 3: Kirchhoff graph $\boldsymbol{K}_2$.

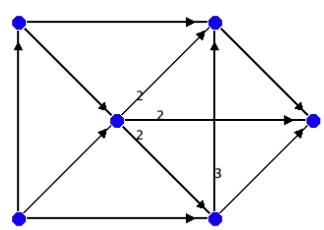


Figure 4: Addition example for $(\boldsymbol{K}_1 + \boldsymbol{K}_2, (1, 1))$. When it is unambiguous, we can just write $\boldsymbol{K}_1 + \boldsymbol{K}_2$.

together, at any coordinate, and a_i is negative only when there exists a_i copies of $\boldsymbol{K}_i$ in the tiled Kirchhoff graph that can be taken out.

Definition 2.18. If $\mathbb{S}'$ is a set of Kirchhoff graphs and $\mathbb{S}$ is a subset of $\mathbb{S}'$ such that $\mathbb{S}' = \langle \mathbb{S} \rangle$, then $\mathbb{S}$ is called a **generating set** of $\mathbb{S}'$.

Definition 2.19. A Kirchhoff graph $\boldsymbol{G}$ is **prime** if and only if $\boldsymbol{G}$ has no nontrivial sub-graph decomposition. In other words, $\boldsymbol{G}$ cannot be written as $\boldsymbol{G}_1 + \boldsymbol{G}_2$ where both $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are nontrivial Kirchhoff graphs. A Kirchhoff graph is **composite** otherwise.

Definition 2.20. A **fundamental set** for S is a minimal generating set with respect to multiplicity and cardinality.

3 Finding Kirchhoff Graphs via an Algorithm

Given a matrix $R = [qI|C]$ with entries in $\mathbb{Z}$ and its associated set of edges vectors S, we are interested in whether a Kirchhoff graph exists for R. If so, we want to find all Kirchhoff graphs associated to R. One way to do so is through an exhaustive search algorithm. Specifically, given R and an edge vector multiplicity $m_{\max}$, we want to find all Kirchhoff graphs with multiplicity up to $m_{\max}$ that associate with R.

For this algorithm, we use Theorem 2.8 to assume that the Kirchhoff graphs we are looking for are uniform, i.e., all edges have the same multiplicity. Doing so allows us to have exactly m copies of each edge vector and stop the algorithm when the multiplicity exceeds $m_{\max}$.

The code is implemented in java and can be found at `https://github.com/Jessica-Wang-Math/Kirchhoff.git`

3.1 Structure of the Algorithm

Below is a short description of the backtracking exhaustive search algorithm for a given matrix R and multiplicity $m_{\max}$.

1. Find all possible vertex cuts with entries between $-m_{\max}$ and $m_{\max}$ by finding all linear combinations of the row vectors of R. Let Λ be the set of all possible vertex cuts with an arbitrary order. Initialize $\mathbb{T}$ to be an empty list which we will add vertices into. This will serve as our "to-do" list.

2. Construct an anchor vertex, assign the first vertex cut in Λ to the anchor vertex. Add the set of edges according to the vertex cut. If doing so results in vertices to have coordinates (x_1, x_2) where $x_1 < x_2$, then we abandon

this vertex cut and remove all the vertices that were constructed. Add all vertices neighboring to the anchor vertex to $\mathbb{T}$.

3. Go to the next vertex in the graph (according to the order in $\mathbb{T}$), assign an appropriate vertex cut to it. Delete this vertex in $\mathbb{T}$ and add its neighboring vertices to $\mathbb{T}$. If the current vertex cut is not in Λ or doing so results in having $m(\boldsymbol{G})$ greater than $m_{\max}$, then we abandon this vertex and goes back to the previous vertex, and assign the next vertex cut in Λ to it.

4. We repeat step 3 until either

   - we find a graph with all vertices assigned to a vertex cut in S, which means we have likely found a Kirchhoff graph, or
   - we have exhausted all cuts in S, which means there is no Kirchhoff graph with multiplicity n, $n < m_{\max}$ associated to R.

5. If a Kirchhoff graph is found, we add it to a list of graphs, and continue the process with step 2 to find the next possible graph until $\mathbb{T}$ is empty.

3.2 Acceleration of the Algorithm

In addition to developing and implementing the algorithm, there are also a few steps we took in order to speed up the graph-finding process.

For step 1 in Section 3.1, we want to find all possible vertex cuts $\boldsymbol{c}$ that are in Row(R). Originally, we generated all linear combinations of the row vectors of R that have entries between $-m_{\max}$ and $m_{\max}$. However, this was computationally costly and the algorithm consistently spent hours on finding the vertex cuts even for relatively small $m_{\max}$. To improve this, we can start with constructing the vertex cut instead. We generate all possible permutations of $\boldsymbol{c}$ with entries from $-m_{\max}$ to $m_{\max}$, then check if it is in Row(R). To do so, We can construct the augmented matrix B=$\left[R^T \,|\, \boldsymbol{c}\right]$. Then $\boldsymbol{c}$ is in Row(R) if and only if B is consistent. This would result in only checking $(2m_{\max}+1)^n$ vertex cuts, where n is the number of edge vectors in R.

Another concern is that the algorithm would waste time by generating duplicate Kirchhoff graphs. To solve this, we can consider only the Kirchhoff graphs whose anchor vertex lies above or on the line $x_2 = x_1$ (as described in step 2 above.) If assigning the anchor vertex (which is situated at the origin) a vertex cut would result in having vertices with coordinate (x_1, x_2) with $x_1 < x_2$, then we deem this vertex cut invalid for the null vertex. This improves the efficiency by cutting down the time spent on finding duplicate graphs.

3.3 Examples of Kirchhoff Graphs found by the Algorithm

Given $R = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 3 & 1 \end{bmatrix}$ and an upper bound multiplicity $m_{\max} = 6$, the algorithm finds 16 non-trivial Kirchhoff graphs, as shown in Figure 5. Notice that $m_{\max} = 6$ is the smallest multiplicity for any Kirchhoff graphs associated to this R, meaning that all 16 are prime Kirchhoff graphs. Interestingly, the first 8 Kirchhoff graphs are self-chirals, and the rest show up in pairs in which one is the chiral of the other.

For another example, let us consider $R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$, $m_{\max} = 6$ that was shown in Example 2.11. The algorithm finds 4 prime Kirchhoff graphs, as shown in Figure 6. Two form a chiral pair, and two are self-chirals.

4 Tiling of Kirchhoff Graphs

Definition 4.1. The **frame size** of a Kirchhoff graph $\boldsymbol{G}$ situated in Cartesian space $\mathbb{Z}^n$ is an n-tuple $(x_1, \cdots, x_n)$ with x_i being the maximal length of any s_j in that direction.

Proposition 4.2. *Given a Kirchhoff graph $\boldsymbol{G}$ for matrix R, there exists an algorithm to determine if $\boldsymbol{G}$ is prime.*

Proof. To check if a Kirchhoff graph is prime, we start by removing any edge vector. We then remove additional edge vectors associated with this vertex until this vertex has a vertex cut in Row(R), which may result in a null vertex. After that, we move on to a neighboring vertex and continue the same process. If doing so results in every edge being removed, then $\boldsymbol{G}$ is prime. If such process terminates and a nontrivial Kirchhoff graph is left, then $\boldsymbol{G}$ is composite, and is the sum of the remaining Kirchhoff graph and the Kirchhoff graph formed from the removed edges. $\square$

Proposition 4.3. *For the two fundamental Kirchhoff graphs $\boldsymbol{F}_1, \boldsymbol{F}_2$ (see Figures 2 and 3) for matrix*

$$R = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 \end{bmatrix},$$

there exists a set of prime Kirchhoff graphs $\mathcal{A} \subseteq \langle \boldsymbol{F}_1, \boldsymbol{F}_2 \rangle$ of infinite cardinality.

Proof. Graphics of this proof can be found in Figure 7. First, we write the edges of $\boldsymbol{F}_2$ as a multiset $M = E(\boldsymbol{F}_2) = \{\boldsymbol{s}_1, \boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_3, \boldsymbol{s}_4, \boldsymbol{s}_4\}$ where s_i is the i-th column of R and the multiplicity of each s_i is the the multiplicity of $\boldsymbol{F}_2$. Now, we will generate a prime Kirchhoff graph from $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ by tiling $\boldsymbol{F}_1$ together in a way that forms a copy of $\boldsymbol{F}_2$ inside the resulted composite graph, always placing each $\boldsymbol{F}_1$ so the edges not included in constructing $\boldsymbol{F}_2$ lie to the

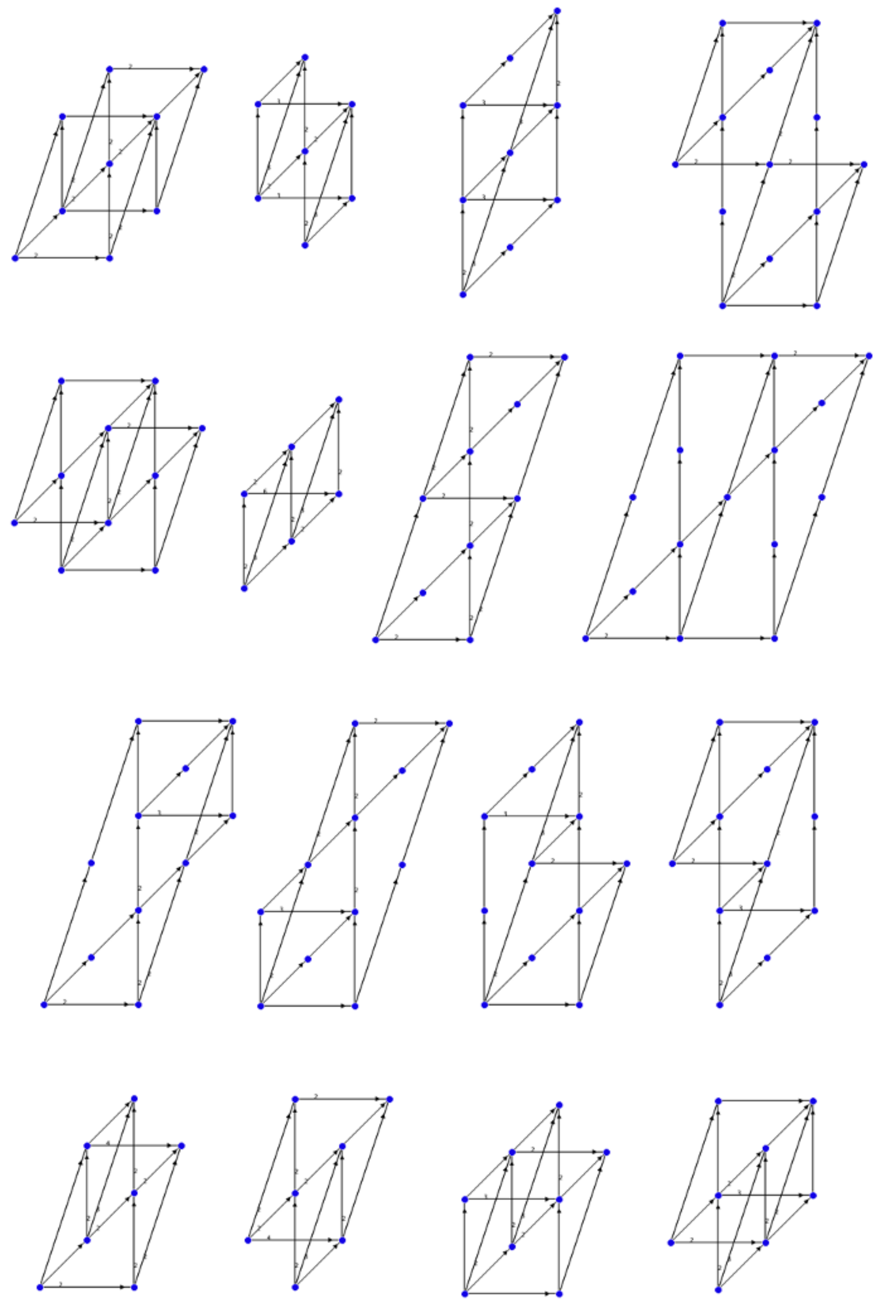


Figure 5: 16 non-trivial prime Kirchhoff graphs for matrix R

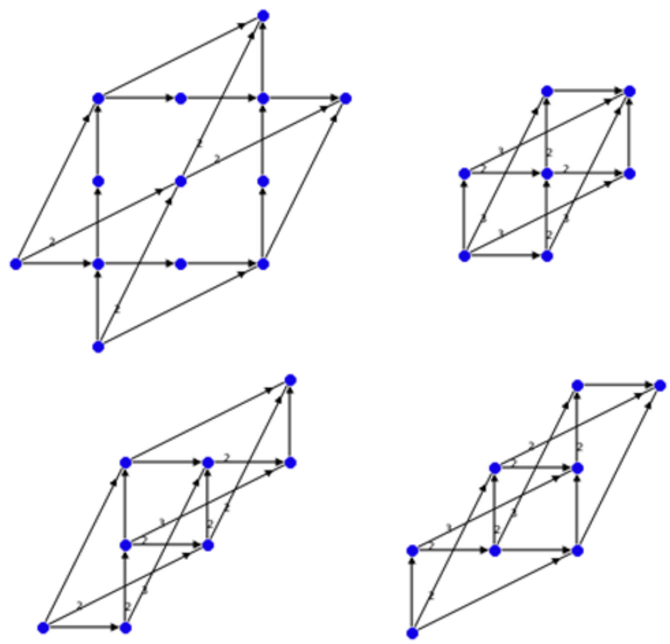


Figure 6: 4 prime Kirchhoff graphs for matrix R

outside of the partial construction. One way to do so is by creating a composite graph $\boldsymbol{C}_1 = (\boldsymbol{F}_1 + \boldsymbol{F}_1, (1,1))$, which covers five out of eight edges of $\boldsymbol{F}_2$. Remove the five edges from M to get $M_1 = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$. To include the second s_1, we can create $\boldsymbol{C}_2 = (\boldsymbol{C}_1 + \boldsymbol{F}_1, (1,-1))$, since we want to enclose $\boldsymbol{F}_2$ in the graph. Now, s_1 and s_3 have been completed, so we only have $M_2 = \{s_2\}$ to tile. We will create $\boldsymbol{C}_3 = (\boldsymbol{C}_2 + \boldsymbol{F}_1, (2,0))$, which now has a copy of $\boldsymbol{F}_2$ inside. We can remove it to result in our desired graph $\boldsymbol{P}_1 = (\boldsymbol{C}_3 - \boldsymbol{F}_2, (0,1))$.

Note that $\boldsymbol{P}_1$ has no copies of $\boldsymbol{F}_1$ in it because every $\boldsymbol{F}_1$ has contributed at least one edge in the creating of $\boldsymbol{F}_2$. P has no copies of $\boldsymbol{F}_2$ in it since only one was created and it was taken out. We can see P is prime if we apply algorithm 4.2 to it. Starting from the anchor vertex, have to remove edges in order for the vertex cut to remain in the row space, which consequently will remove all edges in P.

From P, we can continue expanding the graph by tiling more $\boldsymbol{F}_1$ to create an enclosed $\boldsymbol{F}_2$ to take out. One way to do so to create $\boldsymbol{C}^1 = ((\boldsymbol{P} + \boldsymbol{F}_1, (2,2)) + \boldsymbol{F}_1, (3,1))$. Then we can take out a copy of $\boldsymbol{F}_2$ inside to create the prime graph $\boldsymbol{P}_2 = (\boldsymbol{C}^1 - \boldsymbol{F}_2, (2,2))$. Notice that $\boldsymbol{P}_2$ is prime by the same argument using algorithm 4.2.

We can keep expanding the graph by tiling $\boldsymbol{C}^{i+1} = ((\boldsymbol{C}^i + \boldsymbol{F}_1, (i,i)), \boldsymbol{F}_1, (i+1, i-1))$, then obtain a prime graph by subtracting $\boldsymbol{P}_{i+1} = (\boldsymbol{C}^i - \boldsymbol{F}_2, (i,i))$. $\boldsymbol{P}_{i+1}$ is prime since it has the same pattern of vertex cuts as $\boldsymbol{P}_2$, i.e., the outside vertices have vertex cuts $(1,1,1,0), (-1,-1,-1,0), (1,-1,1,0), (-1,1,0,-1)$, and the inside vertices are null vertices.

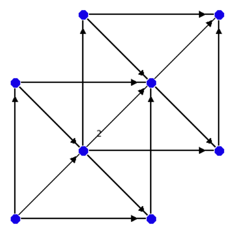
The set of graph $\mathcal{A} = \{\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots\}$ are unique from each other since they have distinct frame sizes and all graphs are connected graphs. Hence $\mathcal{A}$ must have infinite cardinality. □

Note that this kind of construction is possible in many other cases, especially when the Kirchhoff graphs are symmetric. In many cases, it is possible to expand a given fundamental graph and obtain prime graphs of bigger multiplicity. It seems possible to construct countably infinitely many prime Kirchhoff graphs for any edge set S.

5 Future Work

This project has motivated some further questions, such as:

- Given a matrix R, what is the smallest m such that nontrivial Kirchhoff graphs exist?
- Given a matrix R and multiplicity m, how many prime Kirchhoff graphs are there?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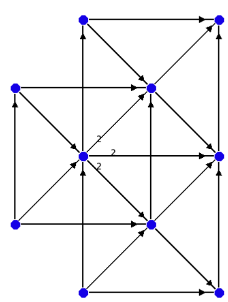
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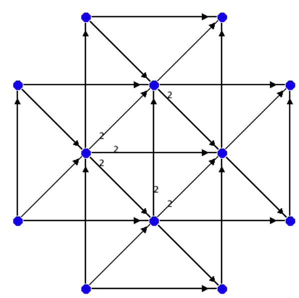
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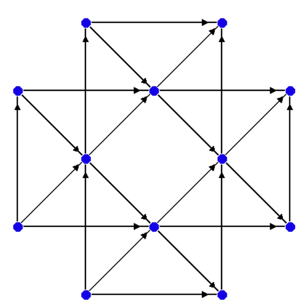

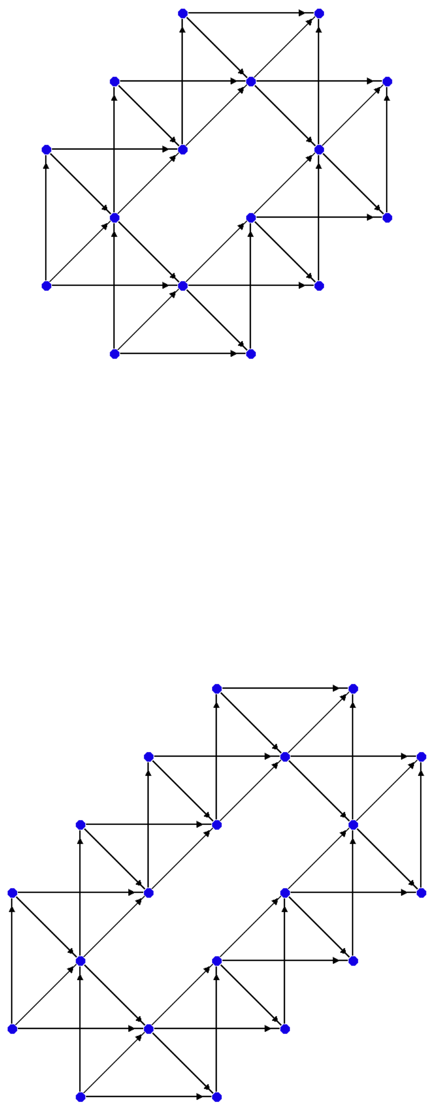

Figure 7: The expansion method mentioned in Proposition 4.3

- Given a matrix R, does there exist a large fundamental graph $\boldsymbol{F}^* \notin \langle \boldsymbol{F}_1, \boldsymbol{F}_2 \rangle$?
- Is there a condition for the existence of chiral graphs given a matrix?
- What algebraic structure does a family of Kirchhoff graphs form?

These questions will better help us to understand Kirchhoff graphs and their tilings. We hope to explore them in the future.

References

[1] Joseph D. Fehribach. Vector-space methods and kirchhoff graphs for reaction networks. *SIAM Journal on Applied Mathematics*, 70(2):543–562, 2009.

[2] Joseph D. Fehribach. Matrices and their kirchhoff graphs. *Ars Math. Contemp.*, 9:125–144, 2015.

[3] Tyler M. Reese, Joseph D. Fehribach, Randy C. Paffenroth, and Brigitte Servatius. Matrices over finite fields and their kirchhoff graphs. *Linear Algebra and its Applications*, 547:128–147, 2018.

[4] Tyler M. Reese, Joseph D. Fehribach, Randy C. Paffenroth, and Brigitte Servatius. Uniform kirchhoff graphs. *Linear Algebra and its Applications*, 566:1–16, 2019.

[5] Joseph D. Fehribach, Judi J. McDonald. Matrices and kirchhoff graphs, a rank-two, nullity-two construction. *Congressus Numerantium*, 230:199–207, 2018.

[6] Marcel C. Gietzmann-Sanders, Joseph D. Fehribach. An algorithm for kirchhoff graph construction. *Major Qualifying Project*, 2016.