Comparison Between Confidence Intervals of Multiple Linear Regression Model with or without Constraints

by

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Abstract

Regression analysis is one of the most applied statistical techniques. The statistical inference of a linear regression model with a monotone constraint had been discussed in early analysis. A natural question arises when it comes to the difference between the cases of with and without the constraint. Although the comparison between confidence intervals of linear regression models with and without restriction for one predictor variable had been considered, this discussion for multiple regression is required.

In this thesis, I discuss the comparison of the confidence intervals between a multiple linear regression model with and without constraints.

Keywords: Least favorable distribution, Chi-bar-square distribution, Likelihood ratio test, Confidence interval.

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Chapter 1

Introduction

Regression analysis has been applied in a large number of areas in statistics. First we consider the standard linear regression model

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{1.1}$$

where $\boldsymbol{Y}$ is an $(n \times 1)$ vector, $\boldsymbol{X}$ is an $(n \times p)$ fixed or random matrix of rank p, $\boldsymbol{\beta}$ is a $(p \times 1)$ vector of unknown parameters, and ϵ is an $(n \times 1)$ multivariate normal vector of errors with mean zero and covariance matrix $\sigma^2\boldsymbol{I}$. In the usual unrestricted case, $\boldsymbol{\beta}$ is simply assumed to lie in $\mathbb{R}^p$. Suppose that $\boldsymbol{R}$ is a $(k \times p)$ matrix of constants with rank k, where $k \leq p$. For a given $(k \times 1)$ vector $\boldsymbol{r}$, testing involves

$$\boldsymbol{R\beta}{=}\boldsymbol{r} \quad \text{against} \quad \boldsymbol{R\beta}{\geq}\boldsymbol{r}, \quad \boldsymbol{R\beta}{\neq}\boldsymbol{r},$$

there seems much more needs to be done. (Mukerjee and Tu,1995) discussed the inference for the mean of the response variable when p=2. In this thesis we consider three dimensional case in the same format. When it comes to higher dimensional case, the inference becomes much more complicated.

(Peiris and Bhattacharya, 2016) have obtained point estimators and confidence intervals for model parameters as well as mean response variable by inverting several tests in early analysis. By using least favorable distribution, we calculated critical values of those tests and now we can try to compare confidence intervals of linear regression models with and without restriction in high dimensional case.

1.1 First Order Model With Two Variables

Consider the standard linear regression model with two predictor variables,

$$\boldsymbol{Y_i} = \beta_0 + \beta_1 \boldsymbol{X_{1i}} + \beta_2 \boldsymbol{X_{2i}} + \epsilon_i, \tag{1.2}$$

where ϵ_i are iid N(0, σ^2).

Let $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ be the unrestricted maximum likelihood estimators of β_0, β_1 and β_2 respectively. Now consider the constraints,

$$\beta_1 \geq 0 \quad and \quad \beta_2 \geq 0. \tag{1.3}$$

The following Lemma shows that the restricted MLEs of β_i are functions of corresponding unrestricted MLEs.

***Lemma* 1.0.1**. *Restricted MLEs of* β_0, β_1, β_2 *under* (1.2) *are given by,*

$$\beta_0^* = \hat{\beta}_0, \quad \beta_1^* = max\{\hat{\beta}_1, 0\}, \quad \beta_2^* = max\{\hat{\beta}_2, 0\}.$$

Proof. This follows using the constraint (1.3) and monotonicity of the likelihood in β_1 and β_2.

1.2 Inference for β_0, β_1 and β_2

Let,

$$S_{X_1}^2 = \Sigma X_{1i}^2, S_{X_2}^2 = \Sigma X_{2i}^2 \quad \text{and} \quad S^2 = \Sigma(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})^2/\nu$$

where ν= n-3. We assume that the entries of matrix X satisfy,

$$\Sigma X_{1i} = 0, \Sigma X_{2i} = 0 \quad \text{and} \quad \Sigma X_{1i} X_{2i} = 0. \tag{1.4}$$

The following well known result shows that sampling distribution of unrestricted MLEs.

***Lemma* 1.1.1**. *Let $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ be the unrestricted MLEs of β_0, β_1 and β_2 respectively, and S^2 be as defined above. Then $\{\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, S^2\}$ are mutually independent. Further, $\hat{\beta}_0 \sim$ N(β_0, σ^2/n), $\hat{\beta}_1 \sim$ N(β_1, σ^2/S_{X1}^2), $\hat{\beta}_2 \sim$ N(β_2, σ^2/S_{X2}^2) and $\nu S^2/\sigma^2 \sim {\chi_\nu}^2$.*

Proof. It is known that (Kutner, et.al, 2005),

$$COV(\hat{\beta}) = \sigma^2 \begin{pmatrix} n & \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i2} \\ \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i1}^2 & \sum_{i=1}^n x_{i1}x_{i2} \\ \sum_{i=1}^n x_{i2} & \sum_{i=1}^n x_{i1}x_{i2} & \sum_{i=1}^n x_{i2}^2 \end{pmatrix} \quad \text{where} \;\; \hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)'$$

then using (1.3) cov($\hat{\beta}_0$,$\hat{\beta}_1$) $= \Sigma x_{i1} = 0$, cov($\hat{\beta}_0$,$\hat{\beta}_2$) $= \Sigma x_{i2} = 0$, and cov($\hat{\beta}_1$,$\hat{\beta}_2$) $= \Sigma x_{i1} x_{i2} = 0$. Let,

$$\boldsymbol{Y} = (y_1, y_2, \cdots, y_n)' \quad and \quad \boldsymbol{X} = \begin{pmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} \end{pmatrix}.$$

Since,

$$\begin{aligned} COV(\hat{\beta}, Y - X\hat{\beta}) &= COV((X^{'}X)^{-1}X^{'}Y, Y - X(X^{'}X)^{-1}X^{-1}X^{'}Y) \\ &= COV((X^{'}X)^{-1}X^{'}Y, (I_n - X(X^{'}X)^{-1}X^{-1}X^{'})Y) \\ &= (X^{'}X)^{-1}(\sigma^2 I)(I_n - X(X^{'}X)^{-1}X^{-1}X^{'}) \\ &= \sigma^2((X^{'}X)^{-1}X^{'} - X^{'}X)^{-1}X^{'}X(X^{'}X)^{-1}X^{-1}X^{'}) = 0, \end{aligned}$$

so that $\hat{\beta}$ and $Y - X\hat{\beta}$ are independent, thus $\hat{\beta}$ and $S^2 = \left\| Y - X\hat{\beta} \right\|^2 / \nu$ are independent. Therefore $\{\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, S^2\}$ are mutually independent. Following the properties of multivariate normal distribution, $\hat{\beta}_0 \sim \mathrm{N}(\beta_0, \sigma^2/\mathrm{n})$, $\hat{\beta}_1 \sim \mathrm{N}(\beta_1, \sigma^2/S_{X1}^2)$, $\hat{\beta}_2 \sim \mathrm{N}(\beta_2, \sigma^2/S_{X2}^2)$. Further $y_i - \beta_0 - \beta_1 X_{1i} - \beta_2 X_{2i} = \epsilon_i \sim \mathrm{N}(0, \sigma^2)$, so $\nu S^2/\sigma^2 \sim {\chi_\nu}^2$ where $\nu = n - 3$.

Chapter 2

Inference of mean response $\boldsymbol{E(Y)}$

We consider inferences about the mean function E(Y) = $\beta_0+\beta_1 x_{01}+\beta_2 x_{02}$ at predictor variable values (x_{01}, x_{02}). Here we have four possible cases based on the signs of x_{01} and x_{02}. First we consider the case with $x_{01} > 0$, $x_{02} > 0$.

2.1 Test for $\beta_0+\beta_1 x_{01}+\beta_2 x_{02}$ (when $x_{01} > 0$, $x_{02} > 0$)

We consider the hypotheses,

$$G_0 : \beta_0 + \beta_1 x_{01} + \beta_2 x_{02} \leq l, \quad \beta_1 \geq 0, \beta_2 \geq 0, \quad G_1 : \beta_1 \geq 0, \beta_2 \geq 0, \tag{2.1}$$

and test G_0 vs $G_1 - G_0$ for some l $\in \mathbb{R}$. Using the transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\gamma}$, where $\gamma_0 = \sqrt{n}\beta_0$, $\gamma_1 = S_{x_1}\beta_1$, $\gamma_2 = S_{x_2}\beta_2$. The constraint (2.1) becomes $\frac{\gamma_0}{\sqrt{n}} + \frac{\gamma_1 x_{01}}{S_{x_1}} + \frac{\gamma_2 x_{02}}{S_{x_2}} \leq$ l, or, $\gamma_2 \leq b_1 - c_1\gamma_0 - d_1\gamma_1$, where $b_1 = \frac{lS_{x_2}}{x_{02}}$, $c_1 = \frac{S_{x_2}}{x_{02}\sqrt{n}}$ and $d_1 = \frac{x_{01}S_{x_2}}{x_{02}S_{x_1}}$. Then, the hypotheses(2.1) becomes:

$$G_{01} : 0 \leq \gamma_2 \leq b_1 - c_1\gamma_0 - d_1\gamma_1, \quad 0 \leq \gamma_1, \quad G_{11} : \gamma_1 \geq 0, \gamma_2 \geq 0, \tag{2.2}$$

and test G_{01} vs $G_{11} - G_{01}$. Let $\boldsymbol{K}$ be the closed convex cone bounded by the hyperplanes $\{c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = 0, \gamma_1 \geq 0, \gamma_2 \geq 0\}$,$\{\gamma_2 = 0, 0 \leq \gamma_1 \leq \frac{-c_1\gamma_0}{d_1}, \gamma_0 \leq 0\}$, and $\{\gamma_1 = 0, 0 \leq \gamma_2 \leq -c_1\gamma_0, \gamma_0 \leq 0\}$ and let $\boldsymbol{L}$=$(\frac{b_1}{c_1}, 0, 0)$, then $\boldsymbol{G_{01}}$ is the shifted cone $\boldsymbol{K}$+$\boldsymbol{L}$. The faces of $\boldsymbol{G_{01}}$ are $\{c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1, \gamma_1 \geq 0, \gamma_2 \geq 0\}$, $\{\gamma_2 = 0, c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \leq b_1, \gamma_0 \leq b_1/c_1\}$, and $\{\gamma_1 = 0, c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \leq b_1, \gamma_0 \leq b_1/c_1\}$(see Figure 2.1 below).

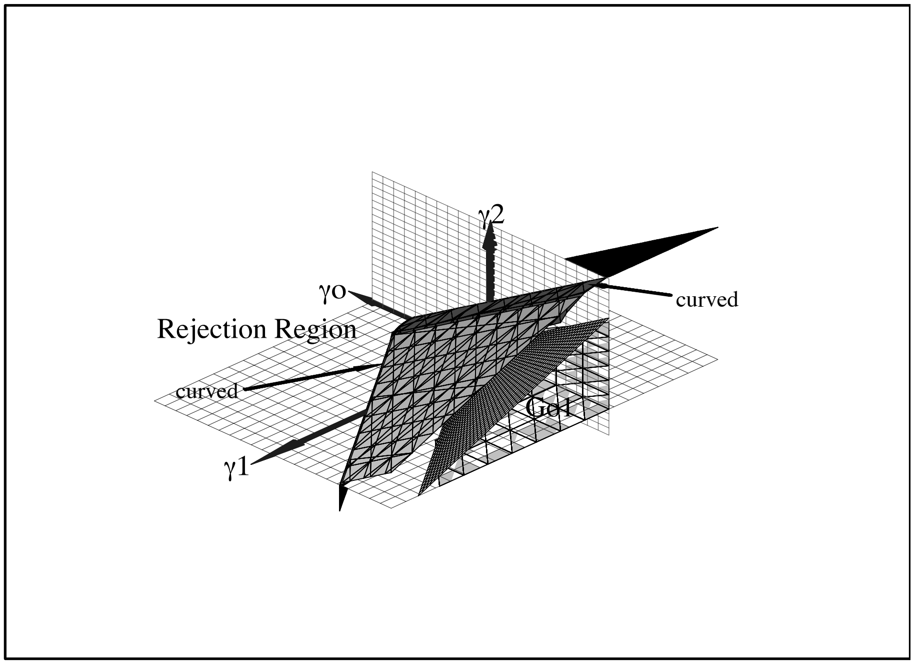

Figure 2.1: The region G_{01} and the rejection region

Let $\hat{\boldsymbol{\gamma}}$ denotes the MLE of $\boldsymbol{\gamma}$ under $\boldsymbol{G_{01}}$ and $\boldsymbol{\gamma}^*$ denotes the MLE of $\boldsymbol{\gamma}$ under $\boldsymbol{G_{11}}$. For testing $\boldsymbol{G_{01}}$ versus $\boldsymbol{G_{11}} - \boldsymbol{G_{01}}$, the LRT rejects $\boldsymbol{G_{01}}$ for large values of the test statistic,

$$\bar{\chi}^2_{01} \equiv -2log\Lambda = (\|\hat{\gamma} - \bar{\gamma}\|^2 - \|\hat{\gamma} - \gamma^*\|^2)/\sigma^2 = \|\bar{\gamma} - \gamma^*\|^2 / \sigma^2, \qquad (2.3)$$

where Λ is the appropriate LRT statistic. Now we move on to investigate the rejection region of LRT in (2.3).

We divide the $\mathbb{R}^3$ space into thirteen disjoint polyhedral cone regions and calculate the test statistic $\hat{\chi}^2_{01}$ in (2.3) for each region. First consider when $\hat{\gamma} \in \{(\gamma_0, \gamma_1, \gamma_2) : \gamma_1 < 0, \gamma_2 < 0\} = \boldsymbol{S_1} \uplus \boldsymbol{S_2}$, where $\uplus$ means disjoint union, $\boldsymbol{S_1} = \{\gamma_0 < b_1/c_1, \gamma_1 < 0, \gamma_2 < 0\}$ and $\boldsymbol{S_2} = \{\gamma_0 \geq b_1/c_1, \gamma_1 < 0, \gamma_2 < 0\}$.

Let $\{\hat{\gamma} : \|\gamma^* - \hat{\gamma}\| > C_\alpha \sigma\}$ be the rejection region for level α test for some critical value C_α. From (2.3), when $\hat{\gamma} \in \boldsymbol{S_1}$, $\|\gamma^* - \hat{\gamma}\|$=$\|(\hat{\gamma_0}, 0, 0) - (\hat{\gamma_0}, 0, 0)\|$=0. When $\hat{\gamma} \in \boldsymbol{S_2}$, $\|\gamma^* - \hat{\gamma}\|$=$\|(\hat{\gamma_0}, 0, 0) - (b_1/c_1, 0, 0)\|$=$\hat{\gamma_0} - b_1/c_1 \geq C_\alpha \sigma$ and hence the boundary of the rejection region in $\boldsymbol{S_2}$ is $\hat{\gamma_0} = b_1/c_1 + C_\alpha \sigma$.

Consider when $\hat{\gamma} \in \{(\gamma_0, \gamma_1, \gamma_2) : \gamma_1 < 0, \gamma_2 \geq 0\} = \boldsymbol{S_3} \uplus \boldsymbol{S_4} \uplus \boldsymbol{S_5}$, where $\boldsymbol{S_3} = \{\gamma_1 < 0, 0 \leq \gamma_2 < b_1 - c_1\gamma_0\}$, $\boldsymbol{S_4} = \{\gamma_1 < 0, \gamma_2 \geq \max\{b_1 - c_1\gamma_0, \gamma_0/c_1 - b_1/c_1^2\}\}$ and $\boldsymbol{S_5} = \{\gamma_1 < 0, 0 \leq \gamma_2 < \gamma_0/c_1 - b_1/c_1^2\}$, where the line $c_1\gamma_0 + \gamma_2 = b_1$ (ML in Figure 2.2) is the intersects of the plane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ and the $\gamma_0\gamma_2$ plane. The line $\gamma_2 = \gamma_0/c_1 - b_1/c_1^2$ (NL in Figure 2.2) is orthogonal to the line ML. These two hyperplanes divide the space $\hat{\gamma} \in \{(\gamma_0, \gamma_1, \gamma_2) : \gamma_1 < 0, \gamma_2 \geq 0\}$ into $\boldsymbol{S_3}, \boldsymbol{S_4}, \boldsymbol{S_5}$. Now when $\hat{\gamma} \in \boldsymbol{S_3}$, $\|\gamma^* - \hat{\gamma}\| = \|(\hat{\gamma_0}, 0, \hat{\gamma_2}) - (\hat{\gamma_0}, 0, \hat{\gamma_2})\| = 0$. When $\hat{\gamma} \in \boldsymbol{S_4}$, $\|\gamma^* - \hat{\gamma}\|^2 = \|(\hat{\gamma_0}, 0, \hat{\gamma_2}) - ((\hat{\gamma_0}, 0, \hat{\gamma_2}) \cdot \boldsymbol{u})\boldsymbol{u}\|^2 \geq C_\alpha^2 \sigma^2$, where $\boldsymbol{u}$ is a unit vector along the line $c_1\gamma_0 + \gamma_2 = b_1$ on the $\gamma_0\gamma_2$ plane, which means the boundary plane is parallel and has $C_\alpha \sigma$ distance to the hyperplane $c_1\gamma_0 + \gamma_2 = b_1$. So the boundary of the rejection region is $c_1\gamma_0 + \gamma_2 = b_1 + \sqrt{1 + c_1^2} C_\alpha \sigma$. When $\hat{\gamma} \in \boldsymbol{S_5}$, the rejection region $\|\gamma^* - \hat{\gamma}\|^2 = \|(\hat{\gamma_0}, 0, \hat{\gamma_2}) - (b_1/c_1, 0, 0)\| = (\hat{\gamma_0} - b_1/c_1)^2 + \hat{\gamma_2}^2 \geq C_\alpha^2 \sigma^2$, which is a partly cylindrical region with axis $\gamma_0 = b_1/c_1$, $\gamma_2 = 0$ with radius $C_\alpha \sigma$.

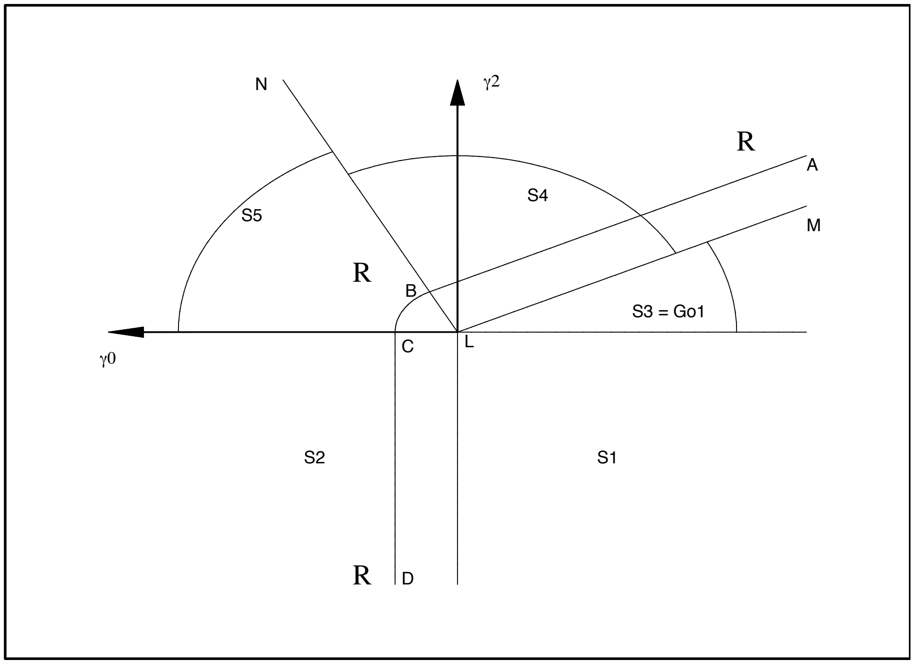

Figure 2.2: Two dimensional views of the rejection region when $\gamma_1 = 0$

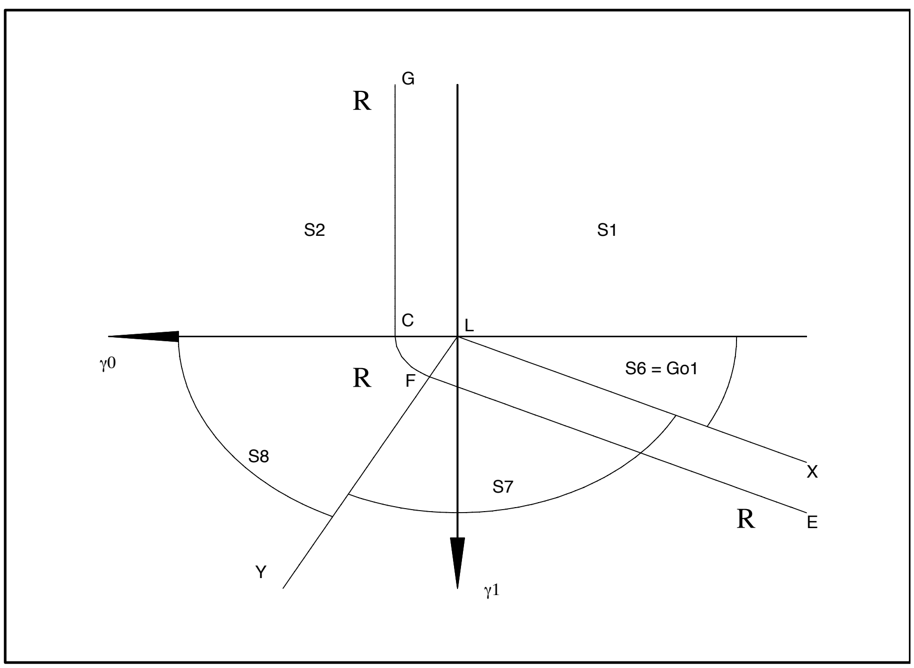

Figure 2.3: Two dimensional views of the rejection region when $\gamma_2 = 0$

We continue to find the rejection region in these cases. When $\hat{\gamma} \in \boldsymbol{S_6}$, $\|\gamma^* - \hat{\gamma}\| =$

$\|(\hat{\gamma_0}, \hat{\gamma_1}, 0) - (\hat{\gamma_0}, \hat{\gamma_1}, 0)\| = 0$. If $\hat{\gamma} \in \boldsymbol{S_7}$, $\|\gamma^* - \hat{\gamma}\|^2 = \|(\hat{\gamma_0}, \hat{\gamma_1}, 0) - ((\hat{\gamma_0}, \hat{\gamma_1}, 0) \cdot \boldsymbol{v})\boldsymbol{v}\|^2 \geq C_\alpha^2 \sigma^2$, where $\boldsymbol{v}$ is a unit vector along the line $c_1\gamma_0 + d_1\gamma_1 = b_1$ on the $\gamma_0\gamma_1$ plane, which means the boundary plane is parallel and has $C_\alpha\sigma$ distance to the hyperplane $c_1\gamma_0 + d_1\gamma_1 = b_1$. So the boundary of the rejection region is $c_1\gamma_0 + d_1\gamma_1 = b_1 + \sqrt{c_1^2 + d_1^2} C_\alpha \sigma$. When $\hat{\gamma} \in \boldsymbol{S_8}$, the rejection region $\|\gamma^* - \hat{\gamma}\|^2 = \|(\hat{\gamma_0}, \hat{\gamma_1}, 0) - (b_1/c_1, 0, 0)\| = (\hat{\gamma_0} - b_1/c_1)^2 + \hat{\gamma_1}^2 \geq C_\alpha^2 \sigma^2$, which is a partly cylindrical region with axis $\gamma_0 = b_1/c_1$, $\gamma_1 = 0$ with radius $C_\alpha\sigma$.

Furthermore, we consider the hyperplane that is orthogonal to the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ and contains the line $c_1\gamma_0 + \gamma_2 = b_1$, given by $c_1 d_1 \gamma_0 - (1 + c_1^2)\gamma_1 + d_1\gamma_2 = b_1 d_1$. Also consider the hyperplane that is orthogonal to the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ and contains the line $c_1\gamma_0 + d_1\gamma_1 = b_1$, given by $c_1\gamma_0 + d_1\gamma_1 - (c_1^2 + d_1^2)\gamma_2 = b_1$.

These two hyperplanes divide the space $\hat{\gamma} \in \{(\gamma_0, \gamma_1, \gamma_2) : \gamma_1 \geq 0, \gamma_2 \geq 0\}$ into $\boldsymbol{S_9}, \boldsymbol{S_{10}}, \boldsymbol{S_{11}}, \boldsymbol{S_{12}}, \boldsymbol{S_{13}}$, where $\boldsymbol{S_9} = \boldsymbol{G_{01}} = \{c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \leq b_1, 0 \leq \gamma_1, 0 \leq \gamma_2\}$, $\boldsymbol{S_{10}} = \{0 \leq \gamma_1 \leq \frac{c_1 d_1}{1+c_1^2}\gamma_0 + \frac{d_1}{1+c_1^2}\gamma_2 - \frac{b_1 d_1}{1+c_1^2}, \gamma_2 \geq \frac{1}{c_1}\gamma_0 - \frac{b_1}{c_1^2}\}$, where $\gamma_2 = \frac{1}{c_1}\gamma_0 - \frac{b_1}{c_1^2}$. $\boldsymbol{S_{11}} = \{\gamma_1 \geq \frac{d_1}{c_1}\gamma_0 - \frac{b_1 d_1}{c_1^2}, 0 \leq \gamma_2 \leq \frac{c_1}{c_1^2+d_1^2}\gamma_0 + \frac{d_1}{c_1^2+d_1^2}\gamma_1 - \frac{b_1}{c_1^2+d_1^2}\}$, $\boldsymbol{S_{10}} = \{0 \leq \gamma_1 \leq \frac{d_1}{c_1}\gamma_0 - \frac{b_1 d_1}{c_1^2}, 0 \leq \gamma_2 \geq \frac{1}{c_1}\gamma_0 - \frac{b_1}{c_1^2}\}$ and $\boldsymbol{S_{12}} = \{\gamma_1 \geq 0, \gamma_2 \geq 0\} - \boldsymbol{S_9} \uplus \boldsymbol{S_{10}} \uplus \boldsymbol{S_{11}} \uplus \boldsymbol{S_{13}}$.

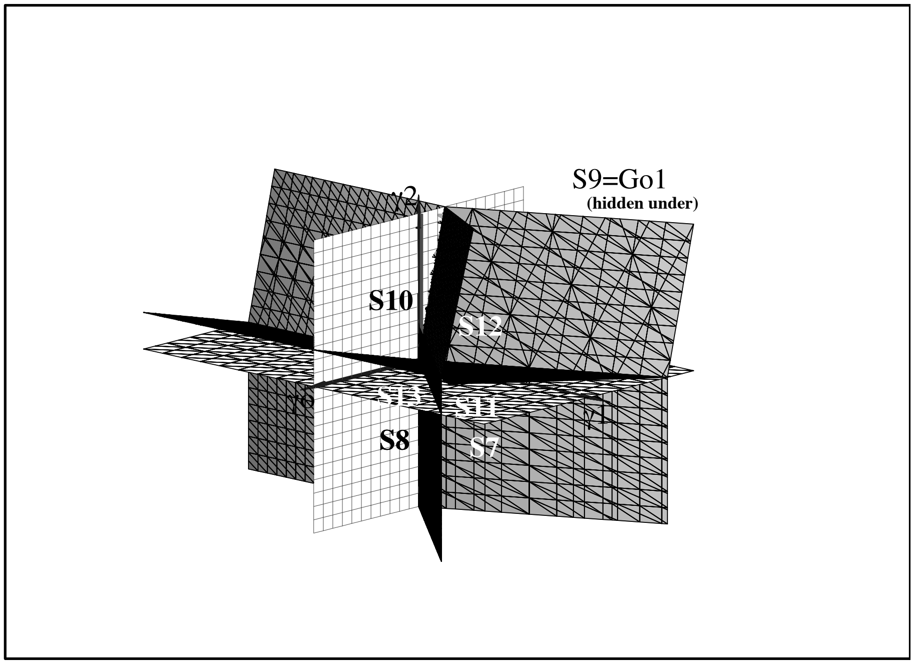

Figure 2.4: Regions $S_9 - S_{13}$

When $\hat{\gamma} \in \boldsymbol{S_9}$, $\|\gamma^* - \hat{\gamma}\| = \|\hat{\gamma} - \hat{\gamma}\| = 0$. If $\hat{\gamma} \in \boldsymbol{S_{10}}$, $\|\gamma^* - \hat{\gamma}\|^2 = \|\hat{\gamma} - (\hat{\gamma} \cdot \boldsymbol{u})\boldsymbol{u}\|^2 \geq C_\alpha^2 \sigma^2$, where $\boldsymbol{u}$ is a unit vector along the line $c_1\gamma_0 + \gamma_2 = b_1$. Thus the boundary of the rejection region is the a part of a cylinder whose axis is the line $c_1\gamma_0 + \gamma_2 = b_1, \gamma_1 = 0$ and the radius is $C_\alpha \sigma$. Let ω_2 be the angle between $c_1\gamma_0 + \gamma_2 = b_1$ and γ_0 axis, then $\tan \omega_2 = c_1$ We find the equation of the boundary of the rejection region in $\boldsymbol{S_{10}}$ by rotating this cylinder by an angle $\theta_1 = \frac{\pi}{2} - \omega_2$. Then using the rotation matrix, we get

$$\begin{bmatrix} \gamma_0 - \frac{b_1}{c_1} \\ \gamma_1 \\ \gamma_2 \end{bmatrix} \to \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \gamma_0 - \frac{b_1}{c_1} \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} (\gamma_0 - \frac{b_1}{c_1})\cos\theta_1 + \gamma_2 \sin\theta_1 \\ \gamma_1 \\ -(\gamma_0 - \frac{b_1}{c_1})\sin\theta_1 + \gamma_2\cos\theta_1 \end{bmatrix}. \tag{2.4}$$

Then the equation of the rotated cylinder is $\gamma_1^2 + ((\gamma_0 - \frac{b_1}{c_1})cos\theta_1 + \gamma_2 sin\theta_1)^2 = C_\alpha^2\sigma^2$. Since $\tan \omega_2 = c_1$, thus $\sin\theta_1 = \sin(\frac{\pi}{2} - \omega_2) = \frac{1}{\sqrt{1+c_1^2}}$ and $\cos\theta_1 = \cos(\frac{\pi}{2} - \omega_2) =$

$\frac{c_1}{\sqrt{1+c_1^2}}$, so the equation is $\gamma_1^2 + (\frac{1}{\sqrt{1+c_1^2}}\gamma_2 + \frac{c_1}{\sqrt{1+c_1^2}}(\gamma_0 - \frac{b_1}{c_1}))^2 = C_\alpha^2\sigma^2$, which is the boundary of the rejection region in $\boldsymbol{S_{10}}$.

When $\hat{\gamma} \in \boldsymbol{S_{11}}$, $\|\gamma^* - \hat{\gamma}\|^2 = \|\hat{\gamma} - (\hat{\gamma}\cdot\boldsymbol{v})\boldsymbol{v}\|^2 \geq C_\alpha^2\sigma^2$, where $\boldsymbol{v}$ is a unit vector along the line $c_1\gamma_0 + d_1\gamma_1 = b_1, \gamma_2 = 0$. Therefore the boundary of the rejection region is a part of a cylinder whose axis is the line $c_1\gamma_0 + d_1\gamma_1 = b_1, \gamma_2 = 0$ and the radius is $C_\alpha\sigma$. By using the similar technique rotating the cylinder in $\boldsymbol{S_{10}}$, the boundary of the rejection region in $\boldsymbol{S_{11}}$ is $\gamma_2^2 + (\frac{d_1}{\sqrt{c_1^2+d_1^2}}\gamma_1 + \frac{c_1}{\sqrt{c_1^2+d_1^2}}(\gamma_0 - \frac{b_1}{c_1}))^2 = C_\alpha^2\sigma^2$.

When $\hat{\gamma} \in \boldsymbol{S_{12}}$, $\|\gamma^* - \hat{\gamma}\|^2 = \|\hat{\gamma} - (\hat{\gamma}\cdot\boldsymbol{w})\boldsymbol{w}\|^2 \geq C_\alpha^2\sigma^2$, where $\boldsymbol{w}$ is a unit vector along the vector $(c_1, d_1, 1)$ which is orthogonal to the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$. This gives the hyperplane which is parallel and has $C_\alpha\sigma$ distance to the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ given by $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1 + \sqrt{1 + c_1^2 + d_1^2}C_\alpha\sigma$.

When $\hat{\gamma} \in \boldsymbol{S_{13}}$, $\|\gamma^* - \hat{\gamma}\|^2 = (\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1{}^2 + \hat{\gamma}_2{}^2 \geq C_\alpha^2\sigma^2$, thus the boundary of the rejection region in $\boldsymbol{S_{13}}$ is $(\gamma_0 - \frac{b_1}{c_1})^2 + \gamma_1^2 + \gamma_2^2 = C_\alpha^2\sigma^2$, which is part of a sphere with radius $C_\alpha\sigma$ and center L.

From the discussion above, we get the rejection region which is the union of the following nine disjoint regions,

1. $\{\hat{\gamma}_0 \geq \frac{b_1}{c_1} + C_\alpha\sigma, \hat{\gamma}_1 \leq 0, \hat{\gamma}_2 \leq 0\}$,
2. $\{(\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_2{}^2 \geq C_\alpha^2\sigma^2, \hat{\gamma}_1 < 0, 0 \leq \hat{\gamma}_2 < \frac{1}{c_1}\hat{\gamma}_0 - \frac{b_1}{c_1^2}\}$,
3. $\{(\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1{}^2 \geq C_\alpha^2\sigma^2, 0 \leq \hat{\gamma}_1 < \frac{d_1}{c_1}\hat{\gamma}_0 - \frac{b_1 d_1}{c_1^2}, \hat{\gamma}_2 < 0\}$,
4. $\{(\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1{}^2 + \hat{\gamma}_2{}^2 \geq C_\alpha^2\sigma^2, 0 \leq \hat{\gamma}_1 < \frac{d_1}{c_1}\hat{\gamma}_0 - \frac{b_1 d_1}{c_1^2}, 0 \leq \hat{\gamma}_2 < \frac{1}{c_1}\hat{\gamma}_0 - \frac{b_1}{c_1^2}\}$,
5. $\{c_1\hat{\gamma}_0 + \hat{\gamma}_2 \geq b_1 + \sqrt{1 + c_1^2}C_\alpha\sigma, \hat{\gamma}_1 < 0, \hat{\gamma}_2 \geq \frac{1}{c_1}\hat{\gamma}_0 - \frac{b_1}{c_1^2}\}$,
6. $\{c_1\hat{\gamma}_0 + d_1\hat{\gamma}_1 - b_1 \geq \sqrt{c_1^2 + d_1^2}C_\alpha\sigma, \hat{\gamma}_1 \geq \frac{d_1}{c_1}\hat{\gamma}_0 - \frac{b_1 d_1}{c_1^2}, \hat{\gamma}_2 < 0\}$,
7. $\{\hat{\gamma}_1{}^2 + (\frac{1}{\sqrt{1+c_1^2}}\hat{\gamma}_2 + \frac{c_1}{\sqrt{1+c_1^2}}(\hat{\gamma}_0 - \frac{b_1}{c_1}))^2 \geq C_\alpha^2\sigma^2,$
   $0 \leq \hat{\gamma}_1 \leq \frac{c_1 d_1}{1+c_1^2}\hat{\gamma}_0 + \frac{d_1}{1+c_1^2}\hat{\gamma}_2 - \frac{b_1 d_1}{1+c_1^2}, \hat{\gamma}_2 \geq \frac{1}{c_1}\hat{\gamma}_0 - \frac{b_1}{c_1^2}\}$,
8. $\{\hat{\gamma}_2{}^2 + (\frac{d_1}{\sqrt{c_1^2+d_1^2}}\hat{\gamma}_1 + \frac{c_1}{\sqrt{c_1^2+d_1^2}}(\hat{\gamma}_0 - \frac{b_1}{c_1}))^2 \geq C_\alpha^2\sigma^2,$
   $\hat{\gamma}_1 \geq \frac{d_1}{c_1}\hat{\gamma}_0 - \frac{b_1 d_1}{c_1^2}, 0 \leq \hat{\gamma}_2 \leq \frac{c_1}{c_1^2+d_1^2}\hat{\gamma}_0 + \frac{d_1}{c_1^2+d_1^2}\hat{\gamma}_1 - \frac{b_1}{c_1^2+d_1^2}\}$,

9. $\{c_1\hat{\gamma}_0 + d_1\hat{\gamma}_1 + \hat{\gamma}_2 - b_1 \geq C_\alpha\sigma\sqrt{1+c_1^2+d_1^2}$,

$\hat{\gamma}_1 \geq \max\{0, \frac{c_1 d_1}{1+c_1^2}\hat{\gamma}_0 + \frac{d_1}{1+c_1^2} - \frac{b_1 d_1}{1+c_1^2}\}, \hat{\gamma}_2 \geq \max\{0, \frac{c_1}{c_1^2+d_1^2}\hat{\gamma}_0 + \frac{d_1}{c_1^2+d_1^2}\hat{\gamma}_1 - \frac{b_1}{c_1^2+d_1^2}\}\}$,

where $C_\alpha = C_\alpha(\omega_1, \omega_2)$, ω_1 is the angle between $c_1\gamma_0 + \gamma_2 = b_1$ and $\gamma_2 = 0$ on the $\gamma_0\gamma_2$-plane, and ω_2 is the angle between $c_1\gamma_0 + d_1\gamma_1 = b_1$ and $\gamma_1 = 0$ on the $\gamma_0\gamma_1$-plane.

To find C_α, we need to find the least favorable distribution of $\bar{\chi}^2_{01}$ in (2.3),

$$Pr(LRT \leq t) = \sum_{i=1}^{13} Pr(LRT \leq t|\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_i})Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_i}). \tag{2.5}$$

It is shown in (Peiris and Bhattacharya, 2016), the least favorable null value of $\bar{\chi}^2_{01}$ is attained at $\boldsymbol{L} = (\frac{b_1}{c_1}, 0, 0)$. When $\hat{\boldsymbol{\gamma}} = \boldsymbol{L}$, $\hat{\gamma} \sim \boldsymbol{N}_3(\boldsymbol{L}, \sigma^2\boldsymbol{I})$, the length and the direction of the $\hat{\boldsymbol{\gamma}}$ are independent. Then for each region $\boldsymbol{S_i}$, $Pr(LRT \leq t|\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_i}) = Pr(LRT \leq t)$. When $i = 1, 3, 6, 9, LRT = 0$. When $i = 2, LRT = (\hat{\gamma}_0 - \frac{b_1}{c_1})^2/\sigma^2$, which is the squared length of the first coordinate, therefore LRT has a χ^2_1 distribution. When $i = 5, 8$, $LRT = ((\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_2{}^2)/\sigma^2$ and $LRT = ((\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1{}^2)/\sigma^2$ respectively, which are both the summation of two squared lengths. These two are both distributed as χ^2_2 distribution. When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_{13}}$, $LRT = ((\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1{}^2 + \hat{\gamma}_2{}^2)/\sigma^2$, which is obviously distributed as a χ^2_3 distribution.

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_4}$, we consider a new orthogonal coordinate system. New γ_0 and γ_2 axis becomes the line $\gamma_0 - c_1\gamma_2 = \frac{b_1}{c_1}$ and $c_1\gamma_0 + \gamma_2 = b_1$ on $\gamma_1 = 0$ hyperplane. Then $LRT = \|\gamma^* - \hat{\gamma}\|^2 = \|(\hat{\gamma}_0, 0, \hat{\gamma}_2) - ((\hat{\gamma}_0, 0, \hat{\gamma}_2) \cdot \boldsymbol{u})\boldsymbol{u}\|^2$ is the squared length of one coordinate only, which is distributed as χ^2_1 distribution. Similarly when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_7}$, we consider a new orthogonal coordinate system with axis along the line $d_1\gamma_0 - c_1\gamma_1 = \frac{b_1 d_1}{c_1}$ and $c_1\gamma_0 + d_1\gamma_1 = b_1, \gamma_2 = 0$ as new γ_0 and γ_1 axis respectively. Thus the LRT given $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_7}$ is also distributed as χ^2_1 distribution.

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_{10}}$, it is obvious that $\gamma^* = \hat{\gamma}$ and $\bar{\gamma}$ is the projection of $\hat{\gamma}$ (also

γ^*) onto the line $c_1\gamma_0 + \gamma_2 = b_1, \gamma_1 = 0$.(i.e. $\bar{\gamma} = \Pi(\hat{\gamma}|\boldsymbol{G_{01}})$). Thus $LRT = \left\|\hat{\gamma} - \Pi(\hat{\gamma}|\boldsymbol{G_{01}})\right\|^2/\sigma^2 = \left\|\Pi(\hat{\gamma}|\boldsymbol{G^*_{01}})\right\|^2/\sigma^2 \sim \chi^2_{3-1}$ (Silvapulle and Sen, 2005). Therefore the LRT has χ^2_2 distribution when $\hat{\gamma} \in \boldsymbol{S_{10}}$. When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_{11}}$, $\gamma^* = \hat{\gamma}$ and $\bar{\gamma}$ is the projection of $\hat{\gamma}$ (also γ^*) onto the line $c_1\gamma_0 + d_1\gamma_1 = b_1, \gamma_2 = 0$.(i.e. $\bar{\gamma} = \Pi(\hat{\gamma}|\boldsymbol{G_{01}})$). So similarly $LRT = \left\|\Pi(\hat{\gamma}|\boldsymbol{G^*_{01}})\right\|^2/\sigma^2 \sim \chi^2_{3-1}$ (Silvapulle and Sen, 2005). Thus the LRT given $\hat{\gamma} \in \boldsymbol{S_{11}}$ also has χ^2_2 distribution.

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_{12}}$, $LRT = \left\|\hat{\gamma} - \Pi(\hat{\gamma}|\boldsymbol{G_{01}})\right\|^2/\sigma^2 = \left\|\Pi(\hat{\gamma}|\boldsymbol{G^*_{01}})\right\|^2/\sigma^2$. The $\Pi(\hat{\gamma}|\boldsymbol{G_{01}})$ is the projection onto the face of $\boldsymbol{G_{01}}$. Thus $\hat{\gamma} - \Pi(\hat{\gamma}|\boldsymbol{G_{01}})$ is the projection onto the line $(\frac{b_1}{c_1}, 0, 0) + u(c_1, d_1, 1)$, which is orthogonal to the face of G_{01} and hence $\left\|\Pi(\hat{\gamma}|\boldsymbol{G^*_{01}})\right\|^2/\sigma^2 \sim \chi^2_1$ (Silvapulle and Sen, 2005). Therefore the LRT has χ^2_1 distribution when $\hat{\gamma} \in \boldsymbol{S_{12}}$.

We get the probabilities $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_i})$ by using the lemma 2 in (Peiris and Bhattacharya, 2016) which gives us $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}) = (4\pi)^{-1}(\theta_1 + \theta_2 + \theta_3 - \pi)$, where $\theta_1, \theta_2, \theta_3$ are the angles between the faces of S. Thus we can show that $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_1}) = (4\pi)^{-1}(\pi/2 + \pi/2 + \pi/2 - \pi) = 1/8$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_2}) = 1/8$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_3}) = (4\pi)^{-1}(\cos^{-1}\frac{1}{\sqrt{1+c_1^2}})$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_4}) = 1/8$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_5}) = (4\pi)^{-1}(\pi/2 - \cos^{-1}\frac{1}{\sqrt{1+c_1^2}})$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_6}) = (4\pi)^{-1}(\cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}})$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_7}) = 1/8$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_8}) = (4\pi)^{-1}(\pi/2 - \cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}})$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_9}) = (4\pi)^{-1}(\cos^{-1}\frac{1}{\sqrt{1+c_1^2+d_1^2}} + \cos^{-1}\frac{d_1}{\sqrt{1+c_1^2+d_1^2}} - \pi/2)$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_{10}}) = (4\pi)^{-1}(\cos^{-1}\frac{\sqrt{1+c_1^2}}{\sqrt{1+c_1^2+d_1^2}})$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_{11}}) = (4\pi)^{-1}(\cos^{-1}\frac{\sqrt{c_1^2+d_1^2}}{\sqrt{1+c_1^2+d_1^2}})$, $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_{12}}) = (4\pi)^{-1}(\cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}})$ and $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_{13}}) = 1 - \sum_{i=1}^{12} Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S_i})$.

Since the least favorable null value of $\bar{\chi}^2_{01}$ is attained at $\hat{\boldsymbol{\gamma}} = \boldsymbol{L} = (\frac{b_1}{c_1}, 0, 0)$, the least favorable null distribution of LRT is

$$Pr(LRT \leq t|\hat{\gamma} = \boldsymbol{L}) = \sum_{i=0}^{3} \omega_i Pr(\chi^2_i \leq t), \tag{2.6}$$

and we can classify the probabilities above and get our weights $\omega_0, \omega_1, \omega_2, \omega_3$, where,

$\omega_0 = (4\pi)^{-1}(\cos^{-1}\frac{1}{\sqrt{1+c_1^2}} + \cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}} + \cos^{-1}\frac{1}{\sqrt{1+c_1^2+d_1^2}} + \cos^{-1}\frac{d_1}{\sqrt{1+c_1^2+d_1^2}}),$

$\omega_1 = (4\pi)^{-1}(\frac{3\pi}{2} + \cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}}),$

$\omega_2 = (4\pi)^{-1}(\pi + \cos^{-1}\frac{\sqrt{1+c_1^2}}{\sqrt{1+c_1^2+d_1^2}} + \cos^{-1}\frac{\sqrt{c_1^2+d_1^2}}{\sqrt{1+c_1^2+d_1^2}} - \cos^{-1}\frac{1}{\sqrt{1+c_1^2}} - \cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}}),$

$\omega_3 = (4\pi)^{-1}(\frac{3\pi}{2} - \cos^{-1}\frac{\sqrt{1+c_1^2}}{\sqrt{1+c_1^2+d_1^2}} - \cos^{-1}\frac{\sqrt{c_1^2+d_1^2}}{\sqrt{1+c_1^2+d_1^2}} - \cos^{-1}\frac{1}{\sqrt{1+c_1^2+d_1^2}}$
$- \cos^{-1}\frac{d_1}{\sqrt{1+c_1^2+d_1^2}} - \cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}}).$

Since $\omega_0+\omega_2 = \omega_1+\omega_3 = \frac{1}{2}$ hold, we can rewrite $\omega_3 = (4\pi)^{-1}(\frac{\pi}{2} - \cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}})$.

The lower bound of the confidence interval for the mean response is obtained by inverting the acceptance region of hypotheses (2.2). We rewrite the rejection region in terms of β, since $b_1 = \frac{lS_{x_2}}{x_{02}}$, $c_1 = \frac{S_{x_2}}{x_{02}\sqrt{n}}$, $d_1 = \frac{x_{01}S_{x_2}}{x_{02}S_{x_1}}$, $\hat{\gamma_0} = \sqrt{n}\hat{\beta}_0$, $\hat{\gamma_1} = S_{x_1}\hat{\beta}_1$, $\hat{\gamma_2} = S_{x_2}\hat{\beta}_2$, then the rejection region can be written as

1. $\{\hat{\beta}_0 \geq l + C_\alpha\sigma\frac{1}{\sqrt{n}}, \hat{\beta}_1 \leq 0, \hat{\beta}_2 \leq 0\}$,
2. $\{n(\hat{\beta}_0 - l)^2 + S_{x_2}^2\hat{\beta}_2^{\,2} \geq C_\alpha^2\sigma^2, \hat{\beta}_1 < 0, 0 \leq \hat{\beta}_2 < \frac{nx_{02}}{S_{x_2}^2}(\hat{\beta}_0 - l)\}$,
3. $\{n(\hat{\beta}_0 - l)^2 + S_{x_1}^2\hat{\beta}_1^{\,2} \geq C_\alpha^2\sigma^2, 0 \leq \hat{\beta}_1 < \frac{nx_{01}}{S_{x_1}^2}(\hat{\beta}_0 - l), \hat{\beta}_2 < 0\}$,
4. $\{n(\hat{\beta}_0 - l)^2 + S_{x_1}^2\hat{\beta}_1^{\,2} + S_{x_2}^2\hat{\beta}_2^{\,2} \geq C_\alpha^2\sigma^2,$
   $0 \leq \hat{\beta}_1 < \frac{nx_{01}}{S_{x_1}^2}(\hat{\beta}_0 - l), 0 \leq \hat{\beta}_2 < \frac{nx_{02}}{S_{x_2}^2}(\hat{\beta}_0 - l)\}$,
5. $\{\hat{\beta}_0 + \hat{\beta}_2 x_{02} \geq l + \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}C_\alpha\sigma, \hat{\beta}_1 < 0, \hat{\beta}_2 \geq \frac{nx_{02}}{S_{x_2}^2}(\hat{\beta}_0 - l)\}$,
6. $\{\hat{\beta}_0 + \hat{\beta}_1 x_{01} \geq l + \sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}C_\alpha\sigma, \hat{\beta}_1 \geq \frac{nx_{01}}{S_{x_1}^2}(\hat{\beta}_0 - l), \hat{\beta}_2 < 0\}$,
7. $\{S_{x_1}^2\hat{\beta}_1^{\,2} + (\hat{\beta}_0 + \hat{\beta}_2 x_{02} - l)^2\frac{1}{\frac{1}{n}+\frac{x_{02}^2}{S_{x_2}^2}} \geq C_\alpha^2\sigma^2,$
   $0 \leq \hat{\beta}_1 < \frac{x_{01}}{S_{x_1}^2}\frac{1}{(\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n})}(\hat{\beta}_0 + \hat{\beta}_2 x_{02} - l), \hat{\beta}_2 \geq \frac{nx_{02}}{S_{x_2}^2}(\hat{\beta}_0 - l)\}$,
8. $\{S_{x_2}^2\hat{\beta}_2^{\,2} + (\hat{\beta}_0 + \hat{\beta}_1 x_{01} - l)^2\frac{1}{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}} \geq C_\alpha^2\sigma^2,$
   $\hat{\beta}_1 \geq \frac{nx_{01}}{S_{x_1}^2}(\hat{\beta}_0 - l), 0 \leq \hat{\beta}_2 < \frac{x_{02}}{S_{x_2}^2}\frac{1}{(\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n})}(\hat{\beta}_0 + \hat{\beta}_1 x_{01} - l)\}$,
9. $\{\hat{\beta}_0 + \hat{\beta}_1 x_{01} + \hat{\beta}_2 x_{02} \geq l + C_\alpha\sigma\sqrt{\frac{1}{n} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{x_{02}^2}{S_{x_2}^2}},$

$\hat{\beta}_1 \geq \max\{0, \frac{x_{01}}{S_{x_1}^2}\frac{1}{\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n}}(\hat{\beta}_0+\hat{\beta}_2 x_{02}-l)\}, \hat{\beta}_2 \geq \max\{0, \frac{x_{02}}{S_{x_2}^2}\frac{1}{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}}(\hat{\beta}_0+\hat{\beta}_1 x_{01}-l)\}\}$, Thus the lower bound L of the confidence interval for mean response is, $L =$

1. $\hat{\beta}_0 - C_{\alpha/2}\sigma\frac{1}{\sqrt{n}}, \quad if\ \hat{\beta}_1 \le 0, \hat{\beta}_2 \le 0,$
2. $\hat{\beta}_0 - \sqrt{\frac{C_{\alpha/2}^2\sigma^2 - S_{x_2}^2\hat{\beta}_2{}^2}{n}}, \quad if\ \hat{\beta}_1 < 0, 0 \le \hat{\beta}_2 < C_{\alpha/2}\sigma\sqrt{\frac{nx_{02}^2}{S_{x_2}^4 + nx_{02}^2 S_{x_2}^2}},$
3. $\hat{\beta}_0 - \sqrt{\frac{C_{\alpha/2}^2\sigma^2 - S_{x_1}^2\hat{\beta}_1{}^2}{n}}, \quad if\ 0 \le \hat{\beta}_1 < C_{\alpha/2}\sigma\sqrt{\frac{nx_{01}^2}{S_{x_1}^4 + nx_{01}^2 S_{x_1}^2}}, \hat{\beta}_2 < 0,$
4. $\hat{\beta}_0 - \sqrt{\frac{C_{\alpha/2}^2\sigma^2 - S_{x_1}^2\hat{\beta}_1{}^2 - S_{x_2}^2\hat{\beta}_2{}^2}{n}}, \quad if\ 0 \le \hat{\beta}_1 < C_{\alpha/2}\sigma\sqrt{\frac{(1-\frac{S_{x_2}^2\hat{\beta}_2{}^2}{C_{\alpha/2}^2\sigma^2})nx_{01}^2}{S_{x_1}^4 + nx_{01}^2 S_{x_1}^2}},$

   $0 \le \hat{\beta}_2 < C_{\alpha/2}\sigma\sqrt{\frac{(1-\frac{S_{x_1}^2\hat{\beta}_1{}^2}{C_{\alpha/2}^2\sigma^2})nx_{02}^2}{S_{x_2}^4 + nx_{02}^2 S_{x_2}^2}},$
5. $\hat{\beta}_0 + \hat{\beta}_2 x_{02} - C_{\alpha/2}\sigma\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}, \quad if\ \hat{\beta}_1 < 0, \hat{\beta}_2 \ge C_{\alpha/2}\sigma\sqrt{\frac{nx_{02}^2}{S_{x_2}^4 + nx_{02}^2 S_{x_2}^2}},$
6. $\hat{\beta}_0 + \hat{\beta}_1 x_{01} - C_{\alpha/2}\sigma\sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}, \quad if\ \hat{\beta}_1 \ge C_{\alpha/2}\sigma\sqrt{\frac{nx_{01}^2}{S_{x_1}^4 + nx_{01}^2 S_{x_1}^2}}, \hat{\beta}_2 < 0,$
7. $\hat{\beta}_0 + \hat{\beta}_2 x_{02} - \sqrt{(C_{\alpha/2}^2\sigma^2 - S_{x_1}^2\hat{\beta}_1^2)(\frac{1}{n} + \frac{x_{02}^2}{S_{x_2}^2})},$

   $if\ 0 \le \hat{\beta}_1 < C_{\alpha/2}\sigma/\sqrt{S_{x_1}^2 + \frac{S_{x_1}^4}{x_{01}^2}(\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})}, \hat{\beta}_2 > C_{\alpha/2}\sigma\sqrt{\frac{(1-\frac{S_{x_1}^2\hat{\beta}_1{}^2}{C_{\alpha/2}^2\sigma^2})nx_{02}^2}{S_{x_2}^4 + nx_{02}^2 S_{x_2}^2}},$
8. $\hat{\beta}_0 + \hat{\beta}_1 x_{01} - \sqrt{(C_{\alpha/2}^2\sigma^2 - S_{x_2}^2\hat{\beta}_2^2)(\frac{1}{n} + \frac{x_{01}^2}{S_{x_1}^2})},$

   $if\ \hat{\beta}_1 > C_{\alpha/2}\sigma\sqrt{\frac{(1-\frac{S_{x_2}^2\hat{\beta}_2{}^2}{C_{\alpha/2}^2\sigma^2})nx_{01}^2}{S_{x_1}^4 + nx_{01}^2 S_{x_1}^2}}, 0 \le \hat{\beta}_2 < C_{\alpha/2}\sigma/\sqrt{S_{x_2}^2 + \frac{S_{x_2}^4}{x_{02}^2}(\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n})},$
9. $\hat{\beta}_0 + \hat{\beta}_1 x_{01} + \hat{\beta}_2 x_{02} - C_{\alpha/2}\sigma\sqrt{\frac{1}{n} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{x_{02}^2}{S_{x_2}^2}},$

   $if\ \hat{\beta}_1 > C_{\alpha/2}\sigma/\sqrt{S_{x_1}^2 + \frac{S_{x_1}^4}{x_{01}^2}(\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})}, \hat{\beta}_2 > C_{\alpha/2}\sigma/\sqrt{S_{x_2}^2 + \frac{S_{x_2}^4}{x_{02}^2}(\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n})},$

2.2 Test (2.1) in opposite direction(when $x_{01} > 0$, $x_{02} > 0$)

We consider the hypotheses,

$$H_0 : \beta_0 + \beta_1 x_{01} + \beta_2 x_{02} \geq u, \quad \beta_1 \geq 0, \beta_2 \geq 0, \quad H_1 : \beta_1 \geq 0, \beta_2 \geq 0, \tag{2.7}$$

for some u $\in \mathbb{R}$ and test $\boldsymbol{H_0}$ against $\boldsymbol{H_1 - H_0}$. Using the transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\gamma}$, the constraint in (2.7) becomes $\gamma_2 \geq b_1' - c_1\gamma_0 - d_1\gamma_1$, where $b_1' = \frac{uS_{x_2}}{x_{02}}$, $c_1 = \frac{S_{x_2}}{x_{02}\sqrt{n}}$ and $d_1 = \frac{x_{01}S_{x_2}}{x_{02}S_{x_1}}$. The hypotheses in terms of $\boldsymbol{\gamma}$ can be written as,

$$H_{01} : \gamma_2 \geq b_1' - c_1\gamma_0 - d_1\gamma_1, \quad \gamma_1 \geq 0, \gamma_2 \geq 0, \quad H_{11} : \gamma_1 \geq 0, \gamma_2 \geq 0, \tag{2.8}$$

and test H_{01} against $H_{11} - H_{01}$. The faces of $\boldsymbol{H_{01}}$ are $\{c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1', \gamma_1 \geq 0, \gamma_2 \geq 0\}$, $\{\gamma_2 = 0, c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \geq b_1', \gamma_1 \geq 0\}$, and $\{\gamma_1 = 0, c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \geq b_1', \gamma_2 \geq 0\}$.

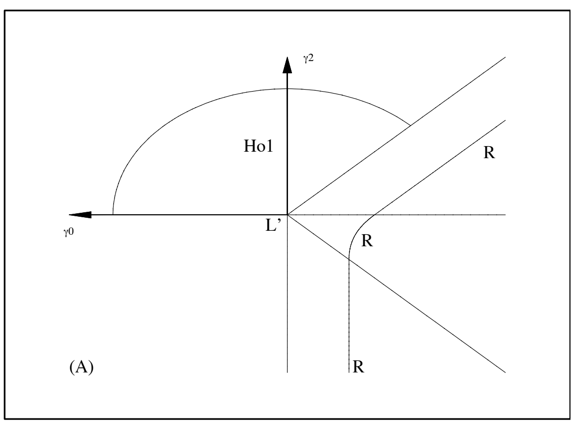

Figure 2.5: Two dimensional views of the rejection region of the LRT (2.9) when $\gamma_1 = 0$

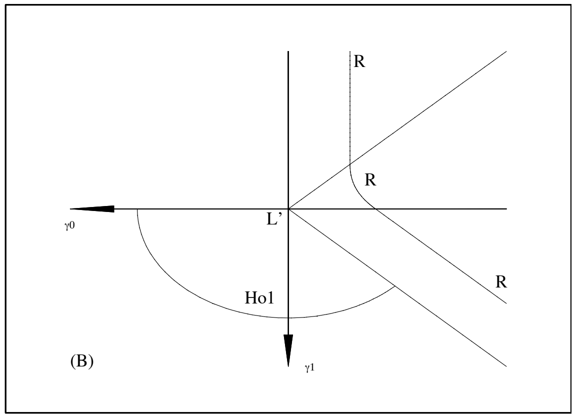

Figure 2.6: Two dimensional views of the rejection region of the LRT (2.9) when $\gamma_2 = 0$

Again, the LRT rejects $\boldsymbol{H_{01}}$ for large values of the test statistics and

$$\bar{\chi}^2_{02} \equiv -2log\Lambda = (\|\hat{\gamma} - \bar{\gamma}\|^2 - \|\hat{\gamma} - \gamma^*\|^2)/\sigma^2, \tag{2.9}$$

Let $\{\|\hat{\gamma} - \bar{\gamma}\|^2 - \|\hat{\gamma} - \gamma^*\|^2 > D^2_\alpha \sigma^2\}$ be the rejection region. We obtain the critical value D_α by investigating the least favorable distribution of LRT. It is shown that in (Peiris and Bhattacharya, 2016), the least favorable null value of LRT is attained at infinity and

$\sup_{\boldsymbol{\gamma} \in \boldsymbol{H_{01}}} Pr_{\boldsymbol{\gamma}}\{\hat{\boldsymbol{\gamma}} : \|\hat{\gamma} - \bar{\gamma}\|^2 - \|\hat{\gamma} - \gamma^*\|^2 \geq D^2_\alpha \sigma^2\} = \lim_{t\to\infty, s\to\infty} Pr_(b'_1/c_1 - s - c_1 t, c_1 t, c_1 s)$ $\{\bar{\chi}^2_{02} > D^2_\alpha\}$, when it is attained, the critical value is $D^2_\alpha = \chi^2_{1,\alpha}$, $D_\alpha = Z_\alpha$.

According to discussion in (Peiris and Bhattacharya, 2016), the power of the test of LRT is quite low at the vertex of the null region. We consider a new test ignoring the restrictions $\gamma_1 \geq 0, \gamma_2 \geq 0$. Now the hypotheses

$$M_{01} : \gamma_2 \geq b'_1 - c_1\gamma_0 - d_1\gamma_1, \quad M_{11} : \gamma_2 < b'_1 - c_1\gamma_0 - d_1\gamma_1, \tag{2.10}$$

The rejection region is $\{\hat{\boldsymbol{\gamma}} : \frac{c_1\hat{\gamma}_0 + d_1\hat{\gamma}_1 + \hat{\gamma}_2 - b'_1}{\sqrt{1 + c_1^2 + d_1^2}\sigma} < -Z_\alpha \sigma\}$, which contains the rejection region of the restricted case so this test is more powerful than the restricted case but this test also creates a philosophical dilemma that in some of the rejection region of this case (but not the restricted case). It is possible to reject H_{01} though γ^* is in H_{01}. To solve it, we need to construct a modified rejection region. Following the argument in (Mukerjee and Tu, 1995) we propose the following as the modified LRT for hypothesis(2.8).

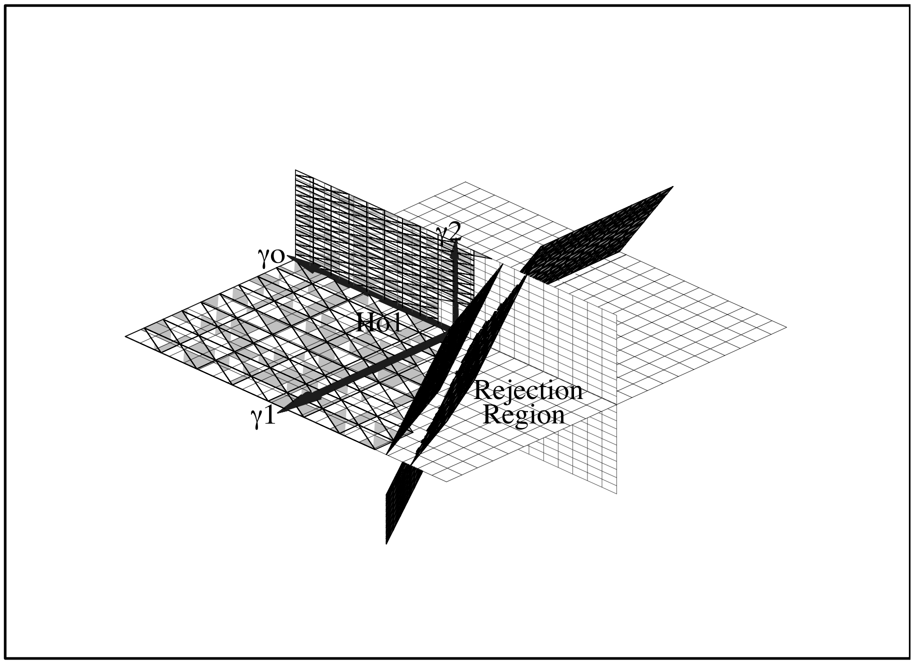

Figure 2.7: Rejection region of modified LRT of test (2.9)

Here we remove the region that cause dilemma and the rejection region is

1. $c_1\hat{\gamma_0} + \hat{\gamma_2} \le b_1', \quad if\ \hat{\gamma_1} \le \frac{-\sqrt{1+c_1^2+d_1^2}Z_\alpha\sigma}{d_1}, \hat{\gamma_2} \ge 0,$
2. $c_1\hat{\gamma_0} + d_1\hat{\gamma_1} \le b_1', \quad if\ \hat{\gamma_1} \ge 0, \hat{\gamma_2} \le -\sqrt{1+c_1^2+d_1^2}Z_\alpha\sigma,$
3. $\hat{\gamma_0} < \frac{b_1'}{c_1}, \quad if\ \hat{\gamma_1} \le 0, \hat{\gamma_2} \le \min\{0, -d_1\hat{\gamma_1} - \sqrt{1+c_1^2+d_1^2}Z_\alpha\sigma\}$
4. $\hat{\gamma_0} + d_1\hat{\gamma_1} + \hat{\gamma_2} \le b_1' - \sqrt{1+c_1^2+d_1^2}Z_\alpha\sigma$, otherwise.

The β form rejection region,

1. $\{\hat{\beta}_0 + \hat{\beta}_2 X_{02} \le u, \quad \hat{\beta}_1 < -\frac{\sqrt{\frac{1}{n}+\frac{X_{01}^2}{S_{X_1}^2}+\frac{X_{02}^2}{S_{X_2}^2}}}{X_{01}} Z_\alpha \sigma, \quad \hat{\beta}_2 \ge 0\},$
2. $\{\hat{\beta}_0 + \hat{\beta}_1 X_{01} \le u, \quad \hat{\beta}_1 \ge 0, \quad \hat{\beta}_2 < -\frac{\sqrt{\frac{1}{n}+\frac{X_{01}^2}{S_{X_1}^2}+\frac{X_{02}^2}{S_{X_2}^2}}}{X_{02}} Z_\alpha \sigma\},$
3. $\{\hat{\beta}_0 \le u, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < \min\{0, -\frac{\sqrt{\frac{1}{n}+\frac{X_{01}^2}{S_{X_1}^2}+\frac{X_{02}^2}{S_{X_2}^2}}}{X_{02}} Z_\alpha \sigma\}\},$
4. $\{\hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} \le u - \sqrt{\frac{1}{n}+\frac{X_{01}^2}{S_{X_1}^2}+\frac{X_{02}^2}{S_{X_2}^2}} Z_\alpha \sigma, \quad otherwise\}.$

Thus we define the upper bound of the confidence interval by inverting the acceptance region of hypotheses (2.7). So the upper bound U of the confidence interval can be obtained, $U =$

1. $\hat{\beta}_0 + \hat{\beta}_2 x_{02}, \quad if\ \hat{\beta}_1 < -\frac{\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}}}{x_{01}} Z_{\alpha/2} \sigma, \quad \hat{\beta}_2 \ge 0,$
2. $\hat{\beta}_0 + \hat{\beta}_1 x_{01}, \quad if\ \hat{\beta}_1 \ge 0, \quad \hat{\beta}_2 < -\frac{\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}}}{x_{02}} Z_{\alpha/2} \sigma,$
3. $\hat{\beta}_0, \quad if\ \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < \min\{0, -\frac{\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}}}{x_{02}} Z_{\alpha/2} \sigma\},$
4. $\hat{\beta}_0 + \hat{\beta}_1 x_{01} + \hat{\beta}_2 x_{02} + \sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}} Z_{\alpha/2} \sigma$, otherwise

Now we get both the lower bound and the upper bound of the restricted confidence interval of the both positive case.

Then we consider the both negative case, when $x_{01} < 0$ and $x_{02} < 0$, we have G_{02} to compare with the G_{01} in the both positive case and the rejection region of the modified LRT is shown below, which is mirror image of the figure of the both positive case. Same thing happens when we consider the test in opposite direction and all the formulas can be obtained using the symmetric property.

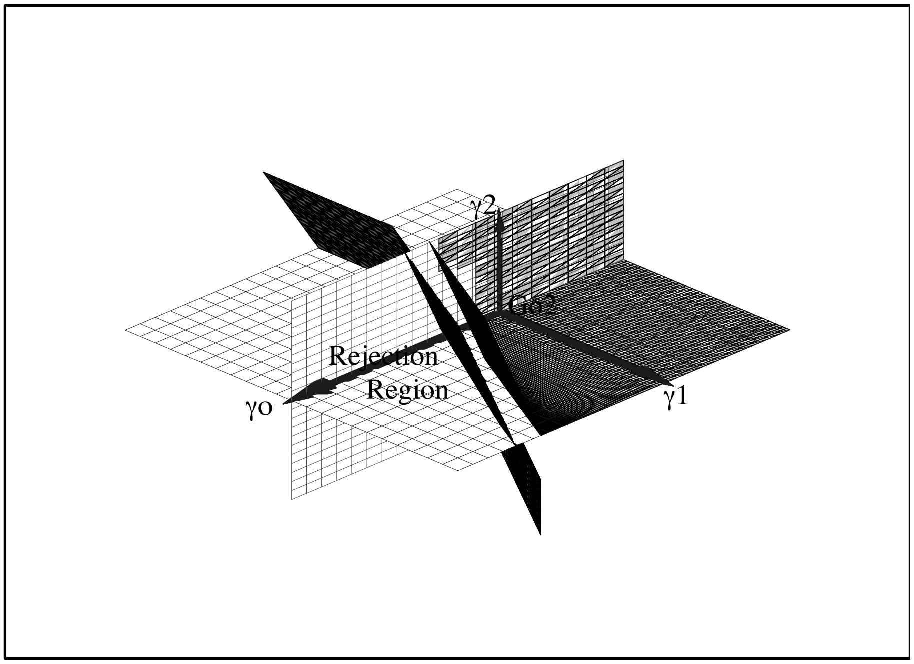

Figure 2.8: Rejection region of the both negative case

2.3 Inference for mixed signs case

We now consider hypothesis when $x_{01} > 0, x_{02} < 0$. Let $b_2 = \frac{lS_{x_2}}{x_{02}}, c_2 = \frac{S_{x_2}}{x_{02}\sqrt{n}}, d_2 = \frac{x_{01}S_{X_2}}{x_{02}S_{x_1}}$. Now note that c_2, d_2 are both negative. The hypotheses are,

$$G_{03} : \gamma_2 \geq b_2 - c_2\gamma_0 - d_2\gamma_1, \quad \gamma_1 \geq 0, \gamma_2 \geq 0, \quad G_{13} : \gamma_1 \geq 0, \gamma_2 \geq 0. \tag{2.11}$$

The faces of $\boldsymbol{G_{03}}$ are$\{c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2, \gamma_1 \geq 0, \gamma_2 \geq 0\}$,$\{\gamma_1 = 0, c_2\gamma_0 + d_2\gamma_1 + \gamma_2 \leq b_2, \gamma_2 \geq 0\}$ and $\{\gamma_2 = 0, c_2\gamma_0 + d_2\gamma_1 + \gamma_2 \geq b_2, \gamma_1 \geq 0\}$. The LRT rejects $\boldsymbol{G_{03}}$ for large values of the test statistics and

$$\bar{\chi}^2_{03} \equiv -2log\Lambda = (\|\hat{\gamma} - \bar{\gamma}\|^2 - \|\hat{\gamma} - \gamma^*\|^2)/\sigma^2, \tag{2.12}$$

Again, We consider the rejection region $\{\|\hat{\gamma}-\bar{\gamma}\|^2 - \|\hat{\gamma}-\gamma^*\|^2 > E_\alpha^2\sigma^2\}$. According to (Peiris and Bhattacharya, 2016), the null least favorable distribution of LRT is attained at $\lim_{\gamma_0\to\infty}(\gamma_0, 0, b_2 - c_2\gamma_0)$. Thus,

$$\sup_{\boldsymbol{\gamma}\in \boldsymbol{G_{03}}} Pr_{\boldsymbol{\gamma}}\{\hat{\boldsymbol{\gamma}} : \|\hat{\gamma}-\bar{\gamma}\|^2 - \|\hat{\gamma}-\gamma^*\|^2 \geq E_\alpha^2\sigma^2\} = \lim_{\gamma_0\to\infty} Pr_(\gamma_0, 0, b_2 - c_2\gamma_0)\{\bar{\chi}_{03}^2 > E_\alpha^2\}.$$

Further we can derive the null least favorable distribution of LRT, which is

$$\sup_{\boldsymbol{\gamma}\in \boldsymbol{G_{03}}} Pr(LRT \geq c) = (\frac{1}{4}+\frac{\theta_1}{2\pi})P(\chi_0^2 \geq c)+\frac{1}{2}P(\chi_1^2 \geq c)+(\frac{1}{4}-\frac{\theta_1}{2\pi})P(\chi_2^2 \geq c) \quad (2.13)$$

where θ_1 is the angle between the hyperplanes $c_2\gamma_0+d_2\gamma_1+\gamma_2 = b_2$ and $\gamma_1 = 0$. Since the least favorable is attained at infinity. The power will be low near the vertex of G_{03}. Therefore, we consider a more powerful test that ignores the restriction $\gamma_2 \geq 0$ that is,

$$M_{02} : \gamma_2 \geq b_2 - c_2\gamma_0 - d_2\gamma_1, \quad \gamma_1 \geq 0 \quad and \quad M_{12} : \gamma_1 \geq 0. \quad (2.14)$$

LRT rejects M_{02} for large values of

$$\bar{\chi}_{04}^2 = (\|\hat{\gamma}-\bar{\bar{\gamma}}\|^2 - \|\hat{\gamma}-\gamma^{**}\|^2)/\sigma^2, \quad (2.15)$$

where $\bar{\bar{\gamma}}$ is the MLE under M_{02} and γ^{**} is the MLE under M_{12}. Notice that $\bar{\chi}_{04}^2 = \|\gamma^{**}-\bar{\bar{\gamma}}\|^2/\sigma^2$. In this case, we can divide the space into five disjoint regions and calculate $\bar{\chi}_{04}^2$ for each region. Then we combine them like the previous case.

We use the hyperplane $c_2\gamma_0 + \gamma_2 = b_2$ to divide $\gamma_1 < 0$ to get S_1 and S_2, where $S_1 = \{\gamma : \gamma_1 < 0, c_2\gamma_0 + \gamma_2 \geq b_2\}$ and $S_2 = \{\gamma : \gamma_1 < 0, c_2\gamma_0 + \gamma_2 < b_2\}$, $S_5 = M_{02}$. Now let $c_2\gamma_0 + \gamma_2 = b_2$, $\gamma_1 = 0$ be the center axis. When $\hat{\gamma} \in S_1$, $\bar{\chi}^2_{04} = \|\gamma^{**} - \bar{\bar{\gamma}}\|^2 / \sigma^2 = 0$. When $\hat{\gamma} \in S_2$, for $\bar{\bar{\gamma}}$, we need to project $\hat{\gamma}$ onto the center axis. Then, $\|\gamma^{**} - \bar{\bar{\gamma}}\|^2 = \|(\hat{\gamma_0}, 0, \hat{\gamma_2}) - ((\hat{\gamma_0}, 0, \hat{\gamma_2}) \cdot u)u\|^2$, where u is an unit vector along the center axis. Therefore the rejection region will be $\|(\hat{\gamma_0}, 0, \hat{\gamma_2}) - ((\hat{\gamma_0}, 0, \hat{\gamma_2}) \cdot u)u\|^2 \geq F_\alpha^2\sigma^2$, which gives a parallel hyperplane to the hyperplane $c_2\gamma_0 + \gamma_2 = b_2$ with distance $F_\alpha\sigma$. The boundary of the rejection region will be $c_2\gamma_0 + \gamma_2 = b_2 - \sqrt{1+c_2^2}F_\alpha\sigma$.

We then use the hyperplane $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$ as well as the hyperplane that is orthogonal to $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$ and contains the center axis (which is $c_2d_2\gamma_0 - (1+c_2^2)\gamma_1 + d_2\gamma_2 = b_2d_2$) to divide the $\gamma_1 > 0$ region into three disjoint regions S_3, S_4, S_5.

$S_3 = \{\boldsymbol{\gamma} : 0 \leq \gamma_1 < \frac{c_2d_2}{1+c_2^2}\gamma_0 + \frac{d_2}{1+c_2^2}\gamma_2 - \frac{b_2d_2}{1+c_2^2}\}$, $S_4 = \{\boldsymbol{\gamma} : \gamma_1 \geq \frac{c_2d_2}{1+c_2^2}\gamma_0 + \frac{d_2}{1+c_2^2}\gamma_2 - \frac{b_2d_2}{1+c_2^2}, \gamma_1 \geq \frac{b_2}{d_2} - \frac{c_2}{d_2}\gamma_0 - \frac{1}{d_2}\gamma_2\}$, $S_5 = \{\boldsymbol{\gamma} : 0 \leq \gamma_1 < \frac{b_2}{d_2} - \frac{c_2}{d_2}\gamma_0 - \frac{1}{d_2}\gamma_2\}$.

When $\hat{\gamma} \in S_3$, then $\|\gamma^{**} - \bar{\bar{\gamma}}\|^2 = \|\hat{\gamma} - (\hat{\gamma} \cdot u)u\|^2$, where u is a unit vector along the center axis. Let $\|\gamma^{**} - \bar{\bar{\gamma}}\|^2 \geq F_\alpha^2\sigma^2$ and this gives a part of cylinder with radius $F_\alpha\sigma$ and its axis is center axis. Thus, by using the technique that rotates cylinder mentioned before, we can get the boundary of the rejection region in S_3 is $\gamma_1^2 + (\frac{1}{\sqrt{1+c_2^2}}\gamma_2 + \frac{c_2}{\sqrt{1+c_2^2}}(\gamma_0 - \frac{b_2}{c_2}))^2 = F_\alpha^2\sigma^2$. When $\hat{\gamma} \in S_4$, then γ^{**} will be equal to $\hat{\gamma}$ and $\bar{\bar{\gamma}}$ will be the projection of $\hat{\gamma}$ onto the hyperplane $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$, thus $\|\gamma^{**} - \bar{\bar{\gamma}}\|^2 = \|\hat{\gamma} - (\hat{\gamma} \cdot w)w\|^2$, where w is the unit vector that is orthogonal to the hyperplane $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$. Then the boundary of the rejection region will be the hyperplane which is parallel and has $F_\alpha\sigma$ distance to the hyperplane $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$, that is $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2 - \sqrt{1+c_2^2+d_2^2}F_\alpha\sigma$. When $\hat{\gamma} \in S_5$, γ^{**} will be equal to $\hat{\gamma}$ and $\bar{\bar{\gamma}}$ will be equal to $\hat{\gamma}$, thus $\|\gamma^{**} - \bar{\bar{\gamma}}\|^2 = 0$.

From the analysis above, we can get the rejection region,

$c_2\hat{\gamma}_0 + \hat{\gamma}_2 \leq b_2 - \sqrt{1+c_2^2}F_\alpha\sigma, \quad \hat{\gamma}_1 < 0,$

$\hat{\gamma}_1{}^2 + (\frac{1}{\sqrt{1+c_2^2}}\hat{\gamma}_2 + \frac{c_2}{\sqrt{1+c_2^2}}(\hat{\gamma}_0 - \frac{b_2}{c_2}))^2 \geq F_\alpha^2\sigma^2, \quad 0 \leq \hat{\gamma}_1 \leq \frac{c_2 d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2 d_2}{1+c_2^2},$

$c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 + \hat{\gamma}_2 \leq b_2 - F_\alpha\sigma\sqrt{1+c_2^2+d_2^2}, \quad \hat{\gamma}_1 \geq \max\{0, \frac{c_2 d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2 d_2}{1+c_2^2}\}.$

Again, test (2.14) is a more powerful test than (2.12) but also creates a philosophical dilemma when $\hat{\gamma}$ is in some regions. Thus a modified rejection region is needed.

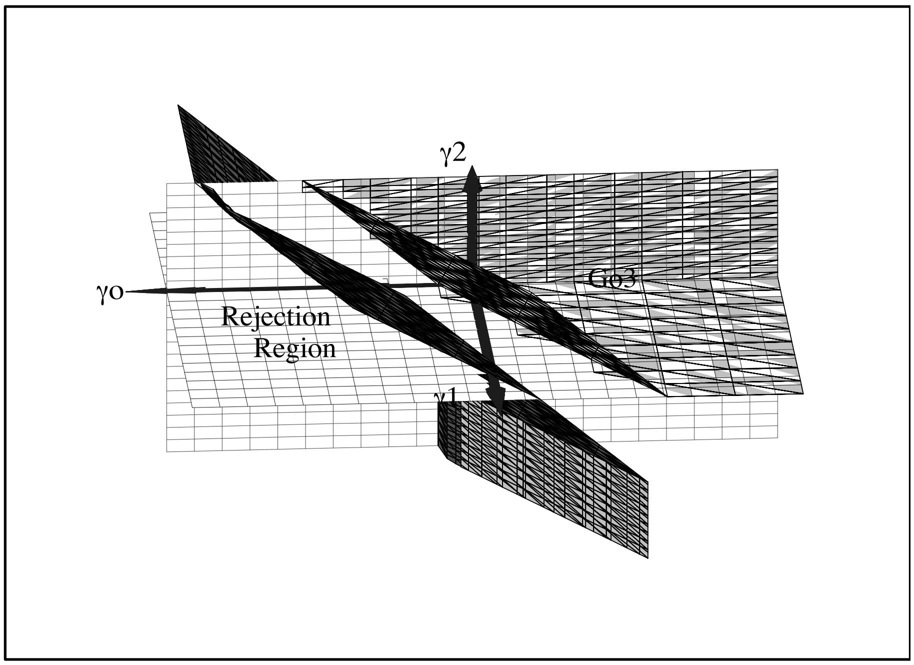

Figure 2.9: Modified rejection region

From (Peiris and Bhattacharya, 2016), we reject G_{03} when

1. $\hat{\gamma}_0 > \frac{b_2}{c_2}, \quad \hat{\gamma}_1 < 0, \hat{\gamma}_2 < -\sqrt{1+c_2^2}E_\alpha\sigma,$

2. $c_2^2(\hat{\gamma}_0 - (b_2/c_2 + \sqrt{1+c_2^2}/c_2 E_\alpha\sigma))^2 + (1+c_2^2)\hat{\gamma}_1{}^2 \geq (1+c_2^2)E_\alpha^2\sigma^2,$

$0 \leq \hat{\gamma}_1 \leq \frac{c_2 d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2 d_2}{1+c_2^2}, \hat{\gamma}_2 < -\sqrt{1+c_2^2}E_\alpha\sigma,$

3. $c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 \leq b_2 - (\sqrt{1+c_2^2+d_2^2} - \sqrt{1+c_2^2})E_\alpha\sigma,$

$\hat{\gamma}_1 \geq \max\{0, \frac{c_2 d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2 d_2}{1+c_2^2}\}, \hat{\gamma}_2 < -\sqrt{1+c_2^2}E_\alpha\sigma,$

4. $c_2\hat{\gamma}_0 + \hat{\gamma}_2 \leq b_2 - \sqrt{1+c_2^2}E_\alpha\sigma, \quad \hat{\gamma}_1 < 0, \hat{\gamma}_2 \geq -\sqrt{1+c_2^2}E_\alpha\sigma,$

5. $\hat{\gamma}_1{}^2 + (\frac{1}{\sqrt{1+c_2^2}}\hat{\gamma}_2 + \frac{c_2}{\sqrt{1+c_2^2}}(\hat{\gamma}_0 - b_2/c_2))^2 \geq E_\alpha^2\sigma^2,$

$0 < \hat{\gamma}_1 \leq \frac{c_2d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2d_2}{1+c_2^2}, \hat{\gamma}_2 \geq -\sqrt{1+c_2^2}E_\alpha\sigma,$

6. $c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 + \hat{\gamma}_2 \leq b_2 - \sqrt{1+c_2^2+d_2^2}E_\alpha\sigma,$

$\hat{\gamma}_1 \geq \max\{0, \frac{c_2d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2d_2}{1+c_2^2}\}, \hat{\gamma}_2 \geq -\sqrt{1+c_2^2}E_\alpha\sigma,$

In terms of the original variables, we reject G_{03} when

1. $\hat{\beta}_0 > l, \hat{\beta}_1 < 0, \hat{\beta}_2 < \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma,$

2. $(\hat{\beta}_0 - l - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma)^2 + (\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})S_{x_1}^2\hat{\beta}_1{}^2 \geq (\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})E_\alpha^2\sigma^2,$

$0 \leq \hat{\beta}_1 < (\hat{\beta}_0 - l + x_{02}\hat{\beta}_2)\frac{x_{01}}{S_{x_1}^2}\frac{1}{\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n}}, \hat{\beta}_2 < \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma,$

3. $\hat{\beta}_0 + x_{01}\hat{\beta}_1 \geq l + (\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}})E_\alpha\sigma,$

$\hat{\beta}_1 \geq (\hat{\beta}_0 - l + x_{02}\hat{\beta}_2)\frac{x_{01}}{S_{x_1}^2}\frac{1}{\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n}}, \hat{\beta}_2 < \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma,$

4. $\hat{\beta}_0 + x_{02}\hat{\beta}_2 \geq l + \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma, \hat{\beta}_1 < 0, \hat{\beta}_2 \geq \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma,$

5. $(\hat{\beta}_0 - l + x_{02}\hat{\beta}_2)^2 + (\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})S_{x_1}^2\hat{\beta}_1{}^2 \geq (\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})E_\alpha^2\sigma^2,$

$0 \leq \hat{\beta}_1 < (\hat{\beta}_0 - l + x_{02}\hat{\beta}_2)\frac{x_{01}}{S_{x_1}^2}\frac{1}{\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n}}, \hat{\beta}_2 \geq \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma,$

6. $\hat{\beta}_0 + x_{01}\hat{\beta}_1 + x_{02}\hat{\beta}_2 \geq l + \sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma,$

$\hat{\beta}_1 \geq (\hat{\beta}_0 - l + x_{02}\hat{\beta}_2)\frac{x_{01}}{S_{x_1}^2}\frac{1}{\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n}}, \hat{\beta}_2 \geq \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_\alpha\sigma,$

Thus the lower bound L of the confidence interval can be obtained, $L =$

1. $\hat{\beta}_0, \quad$ if $\quad \hat{\beta}_1 < 0, \hat{\beta}_2 < \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$

2. $\hat{\beta}_0 - \sqrt{(\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})(E_{\alpha/2}^2\sigma^2 - S_{x_1}^2\hat{\beta}_1{}^2)} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$

if $\quad 0 \leq \hat{\beta}_1 < \frac{x_{02}\hat{\beta}_2 + (\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}})E_{\alpha/2}\sigma}{\frac{S_{x_1}^2}{x_{01}}(\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n})}, \hat{\beta}_2 < \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$

3. $\hat{\beta}_0 + x_{01}\hat{\beta}_1 - (\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}})E_{\alpha/2}\sigma,$

if $\hat{\beta}_1 \geq \frac{x_{02}\hat{\beta}_2 + (\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}})E_{\alpha/2}\sigma}{\frac{S_{x_1}^2}{x_{01}}(\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n})}, \hat{\beta}_2 < \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$

4. $\hat{\beta}_0 + x_{02}\hat{\beta}_2 - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$ if $\hat{\beta}_1 < 0, \hat{\beta}_2 \geq \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$

5. $\hat{\beta}_0 + \hat{\beta}_2 x_{02} - \sqrt{(\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})(E_{\alpha/2}^2\sigma^2 - S_{x_1}^2\hat{\beta}_1^{\,2})},$

if $0 \leq \hat{\beta}_1 < \frac{E_{\alpha/2}\sigma}{\sqrt{S_{x_1}^2 + \frac{S_{x_1}^4}{x_{01}^2}(\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})}}, \hat{\beta}_2 \geq \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$

6. $\hat{\beta}_0 + x_{01}\hat{\beta}_1 + x_{02}\hat{\beta}_2 - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$

if $\hat{\beta}_1 \geq \frac{E_{\alpha/2}\sigma}{\sqrt{S_{x_1}^2 + \frac{S_{x_1}^4}{x_{01}^2}(\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n})}}, \hat{\beta}_2 \geq \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}\sigma,$

2.4 Test in opposite direction of mixed signs case

Again, we follow the similar steps in section 2.2. First consider the hypothesis, let $b_2' = \frac{uS_{x_2}}{x_{02}}$.

$$H_{03} : 0 \leq \gamma_2 < b_2' - c_2\gamma_0 - d_2\gamma_1, \quad \gamma_1 \geq 0 \quad \text{and} \quad H_{13} : \gamma_1 \geq 0, \gamma_2 \geq 0 \tag{2.16}$$

Next, state the test statistics. Since the least favorable null value is attained at infinity thus by ignoring the restriction $\gamma_1 \geq 0$, we can get a more powerful test and then we modify the test because of philosophical dilemma arises.

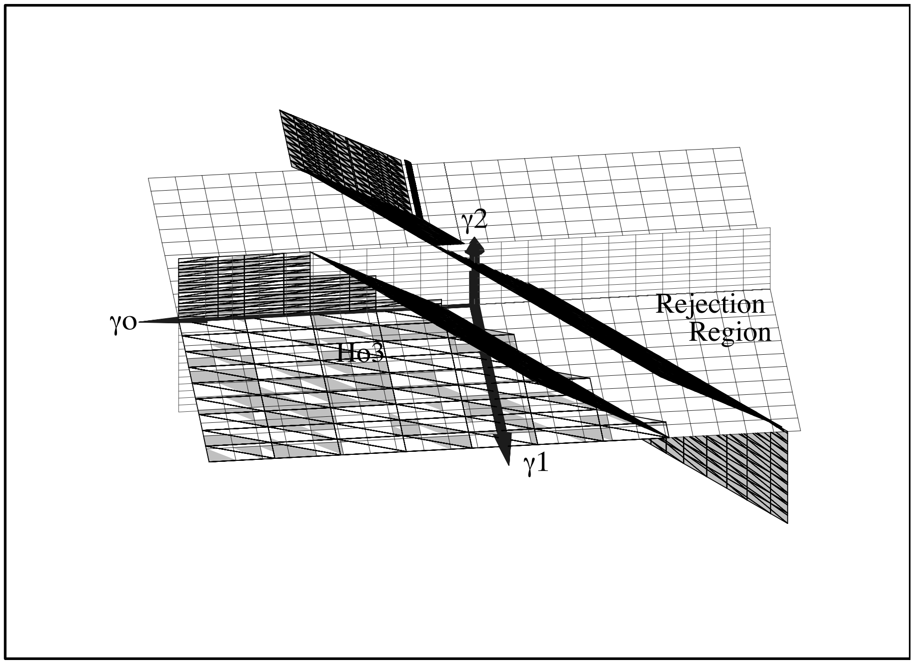

Figure 2.10: Rejection region of modified LRT of test (2.16)

From (Peiris and Bhattacharya, 2016), we reject H_{03} at level α when

1. $\hat{\gamma}_0 > \frac{b_2'}{c_2},\ \hat{\gamma}_1 < \frac{1}{d_2}\sqrt{c_2^2+d_2^2}K_\alpha\sigma, \hat{\gamma}_2 < 0,$
2. $(c_2\hat{\gamma}_0 - (b_2' - \sqrt{c_2^2+d_2^2}K_\alpha\sigma))^2 + (c_2^2+d_2^2)\hat{\gamma}_2{}^2 \geq (c_2^2+d_2^2)K_\alpha^2\sigma^2,\ \hat{\gamma}_1 < \frac{1}{d_2}\sqrt{c_2^2+d_2^2}K_\alpha\sigma,$
   $0 \leq \hat{\gamma}_2 < \frac{c_2}{c_2^2+d_2^2}\hat{\gamma}_0 + \frac{d_2}{c_2^2+d_2^2}\hat{\gamma}_1 - \frac{b_2'}{c_2^2+d_2^2},$
3. $c_2\hat{\gamma}_0 + \hat{\gamma}_2 \geq b_2' + (\sqrt{1+c_2^2+d_2^2} - \sqrt{c_2^2+d_2^2})K_\alpha\sigma,\ \hat{\gamma}_1 < \frac{1}{d_2}\sqrt{c_2^2+d_2^2}K_\alpha\sigma,$
   $\hat{\gamma}_2 \geq \frac{c_2}{c_2^2+d_2^2}\hat{\gamma}_0 + \frac{d_2}{c_2^2+d_2^2}\hat{\gamma}_1 - \frac{b_2'}{c_2^2+d_2^2},$
4. $c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 \geq b_2' + \sqrt{c_2^2+d_2^2}K_\alpha\sigma,\ \hat{\gamma}_1 \leq \frac{1}{d_2}\sqrt{c_2^2+d_2^2}K_\alpha\sigma, \hat{\gamma}_2 < 0,$
5. $\hat{\gamma}_2{}^2 + (\frac{d_2}{\sqrt{c_2^2+d_2^2}}\hat{\gamma}_1 + \frac{c_2}{\sqrt{c_2^2+d_2^2}}(\hat{\gamma}_0 - \frac{b_2'}{c_2}))^2 \geq K_\alpha^2\sigma^2, \hat{\gamma}_1 \geq \frac{1}{d_2}\sqrt{c_2^2+d_2^2}K_\alpha\sigma,$
   $0 \leq \hat{\gamma}_2 < \frac{c_2}{c_2^2+d_2^2}\hat{\gamma}_0 + \frac{d_2}{c_2^2+d_2^2}\hat{\gamma}_1 - \frac{b_2'}{c_2^2+d_2^2},$
6. $c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 + \hat{\gamma}_2 \geq b_2' + \sqrt{1+c_2^2+d_2^2}K_\alpha\sigma,\ \hat{\gamma}_1 \geq \frac{1}{d_2}\sqrt{c_2^2+d_2^2}K_\alpha\sigma,$
   $\hat{\gamma}_2 \geq \frac{c_2}{c_2^2+d_2^2}\hat{\gamma}_0 + \frac{d_2}{c_2^2+d_2^2}\hat{\gamma}_1 - \frac{b_2'}{c_2^2+d_2^2},$

In terms of the original variables, we reject H_{03} when

1. $\hat{\beta}_0 \leq u,\ \hat{\beta}_1 < \frac{1}{x_{01}}\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}K_\alpha\sigma, \hat{\beta}_2 < 0$

2. $(\hat{\beta}_0 - u - \sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}K_\alpha\sigma)^2 + (\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2})^2 S_{x_2}^2\hat{\beta}_2{}^2 \geq (\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2})K_\alpha^2\sigma^2,$

   $\hat{\beta}_1 < \frac{1}{x_{01}}\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}K_\alpha\sigma, 0 \leq \hat{\beta}_2 < (\hat{\beta}_0 + x_{01}\hat{\beta}_1 - u)\frac{(-x_{02})}{S_{x_2}^2}\frac{1}{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}$

3. $\hat{\beta}_0 + x_{02}\hat{\beta}_2 \leq u - (\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}} - \sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}})K_\alpha\sigma,$

   $\hat{\beta}_1 < \frac{1}{x_{01}}\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}K_\alpha\sigma, \hat{\beta}_2 \geq (\hat{\beta}_0 + x_{01}\hat{\beta}_1 - u)\frac{(-x_{02})}{S_{x_2}^2}\frac{1}{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}$

4. $\hat{\beta}_0 + x_{01} \leq u - \sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}K_\alpha\sigma, \hat{\beta}_1 \geq \frac{1}{x_{01}}\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}K_\alpha\sigma, \hat{\beta}_2 < 0$

5. $S_{x_2}^2\hat{\beta}_2{}^2 + \frac{1}{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}(\hat{\beta}_0 + x_{01}\hat{\beta}_1 - u)^2 \geq K_\alpha^2\sigma^2,$

   $\hat{\beta}_1 \geq \frac{1}{x_{01}}\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}K_\alpha\sigma, 0 \leq \hat{\beta}_2 < (\hat{\beta}_0 + x_{01}\hat{\beta}_1 - u)\frac{(-x_{02})}{S_{x_2}^2}\frac{1}{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}$

6. $\hat{\beta}_0 + x_{01}\hat{\beta}_1 + x_{02}\hat{\beta}_2 \leq u - \sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}}K_\alpha\sigma,$

   $\hat{\beta}_1 \geq \frac{1}{x_{01}}\sqrt{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}K_\alpha\sigma, \hat{\beta}_2 \geq (\hat{\beta}_0 + x_{01}\hat{\beta}_1 - u)\frac{(-x_{02})}{S_{x_2}^2}\frac{1}{\frac{1}{n}+\frac{x_{01}^2}{S_{x_1}^2}}$

Thus the upper bound U of the confidence interval can be obtained, $U =$

1. $\hat{\beta}_0$, if $\hat{\beta}_1 < \frac{1}{x_{01}}\sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}}K_{\alpha/2}\sigma, \hat{\beta}_2 < 0,$

2. $\hat{\beta}_0 - \sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}}K_{\alpha/2}\sigma + \sqrt{(\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n})(K_{\alpha/2}^2\sigma^2 - S_{x_2}^2\hat{\beta}_2{}^2)},$

   if $\hat{\beta}_1 < \frac{1}{x_{01}}\sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}}K_{\alpha/2}\sigma, 0 \leq \hat{\beta}_2 < \frac{(\sqrt{\frac{x_{02}^2}{S_{x_2}^2}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}} - \sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}})K_{\alpha/2}\sigma + x_{01}\hat{\beta}_1}{\frac{S_{x_2}^2}{x_{02}}(\frac{x_{02}^2}{S_{x_2}^2}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n})},$

3. $\hat{\beta}_0 + x_{02}\hat{\beta}_2 - (\sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n}} - \sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}})K_{\alpha/2}\sigma,$

   if $\hat{\beta}_1 < \frac{1}{x_{01}}\sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}}K_{\alpha/2}\sigma, \hat{\beta}_2 \geq \frac{(\sqrt{\frac{x_{02}^2}{S_{x_2}^2}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}} - \sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}})K_{\alpha/2}\sigma + x_{01}\hat{\beta}_1}{\frac{S_{x_2}^2}{x_{02}}(\frac{x_{02}^2}{S_{x_2}^2}+\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n})},$

4. $\hat{\beta}_0 + x_{01}\hat{\beta}_1 - \sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}}K_{\alpha/2}\sigma,$ if $\hat{\beta}_1 \geq \frac{1}{x_{01}}\sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n}}K_{\alpha/2}\sigma, \hat{\beta}_2 < 0,$

5. $\hat{\beta}_0 + x_{01}\hat{\beta}_1 + \sqrt{(\frac{x_{01}^2}{S_{x_1}^2}+\frac{1}{n})(K_{\alpha/2}^2\sigma^2 - S_{x_2}^2\hat{\beta}_2^2)},$

if $\quad \hat{\beta}_1 \geq \frac{1}{x_{01}}\sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}K_{\alpha/2}\sigma, 0 \leq \hat{\beta}_2 < \frac{1}{\sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n}}}\frac{(-x_{02})}{S_{x_2}^2}K_{\alpha/2}\sigma,$

6. $\hat{\beta}_0 + x_{01}\hat{\beta}_1 + x_{02}\hat{\beta}_2 - \sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}K_{\alpha/2}\sigma,$

if $\quad \hat{\beta}_1 \geq \frac{1}{x_{01}}\sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}K_{\alpha/2}\sigma, \hat{\beta}_2 \geq \frac{1}{\sqrt{\frac{x_{01}^2}{S_{x_1}^2}+\frac{x_{02}^2}{S_{x_2}^2}+\frac{1}{n}}}\frac{(-x_{02})}{S_{x_2}^2}K_{\alpha/2}\sigma,$

Now we get both the lower bound and the upper bound of the restricted confidence interval of the mixed signs case.

Chapter 3

Comparison Between Confidence Intervals with and without Restrictions

As discussed in (Peiris and Bhattacharya, 2016), the length of the intervals strictly depend on the values of x_{01} and x_{02}. So a comparison of restricted and unrestricted confidence intervals is needed to identify which method works better for a given data set. In this chapter we compare the restricted intervals we obtained in previous chapters with confidence intervals for the unrestricted general linear regression model.

3.1 Some properties of the critical values

Before we compare the confidence intervals, we discuss some properties of the critical values we introduce in previous chapters.

Lemma 1 *If* $\alpha/2 \in (0, \alpha_0/2]$, *then*

$$z_{\alpha/2} \leq c_{\alpha/2} \leq 2z_{\alpha/2}, \tag{3.1}$$

where $c_{\alpha/2}$ *is the solution to the equation*

$$\alpha/2 = \frac{1}{2}P\{\chi_1^2 > c_{\alpha/2}^2\} + (\frac{1}{4} - \frac{\omega}{2\pi})P\{\chi_2^2 > c_{\alpha/2}^2\}, \tag{3.2}$$

where $\alpha_0/2 = P\{z > \frac{\sqrt{2\pi}}{4}\} = 0.2643$ *and* z *is the* $N(0,1)$ *random variable.*

Proof We have

$$z_{\alpha/2} \leq c_{\alpha/2},$$

because we get $c_{\alpha/2}$ from the least favorable distribution. Now we are going to prove that $2z_{\alpha/2} - c_{\alpha/2} \geq 0$ for all $\alpha/2$ in $(0, \alpha_0/2]$. Here this proof is given only for the case $x_{01} > 0$ and $x_{02} < 0$. The proof is similar for the other mixed case. Let x and y be the solutions to the equations

$$\frac{\alpha}{2} = \frac{1}{2}P\{\chi_1^2 > x^2\} + \frac{1}{4}P\{\chi_2^2 > x^2\}, \tag{3.3}$$

and

$$\frac{\alpha}{2} = P\{z > y\} \tag{3.4}$$

respectively. Then $x \geq c_{\alpha/2}$ because $\frac{\alpha}{2} = \frac{1}{2}P\{\chi_1^2 > c_{\alpha/2}^2\} + (\frac{1}{4} - \frac{\omega}{2\pi})P\{\chi_2^2 > c_{\alpha/2}^2\}$ and $y = z_{\alpha/2}$. We are going to prove a stronger argument $2y - x \geq 0$, so that we have $2z_{\alpha/2} - c_{\alpha/2} \geq 0$. Let (3.3) and (3.4) equal, since $\frac{1}{2}P\{\chi_1^2 > x^2\} = P\{z > x\}$, then

$$\frac{1}{\sqrt{2\pi}} \int_y^x e^{-\frac{t^2}{2}} dt = \frac{1}{4} e^{-\frac{x^2}{2}}, \tag{3.5}$$

Thus there exists a t^* such that $\frac{1}{\sqrt{2\pi}} \int_y^x e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^{*2}}{2}} (x - y) = \frac{1}{4} e^{-\frac{x^2}{2}}$, where

$y \leq t^* \leq x$.

Hence

$$x - y = \frac{\sqrt{2\pi}}{4} e^{-\frac{(x^2 - t^{*2})}{2}} \leq \frac{\sqrt{2\pi}}{4}.$$

Therefore, $2y - x \geq 2y - (y + \frac{\sqrt{2\pi}}{4}) \geq 0$ when

$$y \geq \frac{\sqrt{2\pi}}{4} \approx 0.626657$$

We notice that $\frac{\alpha}{2} = P\{z > y\}$ and $\alpha_0/2 = P\{z > \frac{\sqrt{2\pi}}{4}\} = 0.2643$. Thus when $\frac{\alpha}{2} < \frac{\alpha_0}{2} = 0.2643$, $y \geq \frac{\sqrt{2\pi}}{4}$. Therefore, we have $2z_{\alpha/2} - c_{\alpha/2} \geq 0$ when $\frac{\alpha}{2} < \frac{\alpha_0}{2} = 0.2643$.

Lemma 2 *The solution $x(\alpha)$ of*

$$\alpha/2 = \frac{1}{2} P\{F_{1,n-2} > x^2\} + \frac{1}{4} P\{F_{2,n-2} > \frac{x^2}{2}\} \tag{3.6}$$

is a decreasing function of α for $x > 0$.

Proof To prove $x(\alpha)$ is decreasing, we take the derivative respect to α.

$$\frac{1}{2} = \frac{d}{d\alpha}(\frac{1}{2}\int_{x^2}^{\infty} f_{1,n-2}(t)dt + \frac{1}{4}\int_{x^2/2}^{\infty} f_{2,n-2}(t)dt) = -x(f_{1,n-2}(x^2) + \frac{1}{4} f_{2,n-2}(x^2/2))\frac{dx}{d\alpha}$$

Therefore, we obtain

$$\frac{dx}{d\alpha} = -\frac{2}{x[4f_{1,n-2}(x^2) + f_{2,n-2}(x^2/2)]} < 0$$

where f is the pdf of the F-distribution. Therefore, $x(\alpha)$ is a decreasing function of α.

Lemma 3 *The function $g(z) = z - 2\tan\frac{A}{\sqrt{1+z^2}} - z^2 \tan\frac{A}{\sqrt{1+z^2}}$ is a strictly increasing* *function of z for $z > 0$, where*

$$A = A_n = \frac{\sqrt{\pi}\Gamma(\frac{n-2}{2})}{4\Gamma(\frac{n-1}{2})}, n = 3, 4, \cdots$$

Proof Consider the derivative $g'(z)$,

$$g'(z) = 1 + \frac{2Az[1 + \tan^2(\frac{A}{\sqrt{1+z^2}})]}{(1 + z^2)^{3/2}} - 2z\tan(\frac{A}{\sqrt{1 + z^2}}) + \frac{Az^3[1 + tan^2(\frac{A}{\sqrt{1+z^2}})]}{(1 + z^2)^{3/2}} \quad (3.7)$$

now we are going to prove $g'(z)$ is positive on both $[0, 1]$ and $[1, \infty)$.

First consider when $z \in [0, 1]$. Since $A = A_n = \frac{\sqrt{\pi}\Gamma(\frac{n-2}{2})}{4\Gamma(\frac{n-1}{2})}, n = 3, 4, \cdots$, thus when $n = 3$, A attains its maximum value $\pi/4$. For $0 < A \leq \pi/4, \sqrt{2}/2 \leq \cos A < 1$, we have

$$\tan\frac{A}{\sqrt{1 + z^2}} = \frac{\sin\frac{A}{\sqrt{1+z^2}}}{\cos\frac{A}{\sqrt{1+z^2}}} \leq \frac{\frac{A}{\sqrt{1+z^2}}}{\cos\frac{A}{\sqrt{1+z^2}}} \leq \frac{\frac{A}{\sqrt{1+z^2}}}{\cos A} \quad (3.8)$$

Thus,

$g'(z) \geq g_1(z) = 1 + \frac{2Az}{(1+z^2)^{3/2}} + \frac{Az^3}{(1+z^2)^{3/2}} - \frac{2}{\cos A}\frac{Az}{\sqrt{1+z^2}}$.

Now we are going to prove $g_1'(z) < 0$ and minimum value of $g_1(z) = g_1(1) > 0$, then we can get $g'(z) > 0$.

$g_1'(z) = -A\frac{2(\frac{1}{\cos A}-1)+(1+\frac{2}{\cos A})z^2}{(1+z^2)^{5/2}} < 0$ and $g_1(1) = 1 - \frac{A}{2\sqrt{2}}(\frac{4}{\cos A} - 3) \geq 1 - \frac{\pi}{8\sqrt{2}}(4\sqrt{2} - 3) > 0$. Therefore, when $z \in [0, 1]$, $g'(z)$ is positive.

Next, we consider the case when $z \in [1, \infty)$. To prove this, we consider the function

$$g_2(z) = 2z\tan\frac{A}{\sqrt{1 + z^2}} \quad (3.9)$$

Note that $\lim_{z\to\infty} g_2(z) = 2A$. To see this, we investigating $\lim_{z\to\infty} g_2(z) = \lim_{z\to\infty} \frac{2z\sin\frac{A}{\sqrt{1+z^2}}}{\cos\frac{A}{\sqrt{1+z^2}}} = \lim_{z\to\infty} \frac{2\sin\frac{A}{\sqrt{1+z^2}}}{\frac{1}{z}}$, by using the *L'Hopital's rule*, $\lim_{z\to\infty} \frac{2\sin\frac{A}{\sqrt{1+z^2}}}{\frac{1}{z}} = \lim_{z\to\infty} \frac{2\cos\frac{A}{\sqrt{1+z^2}}}{-\frac{1}{z^2}}(-z)A(1+$

$z^2)^{-3/2} = \lim_{z\to\infty} 2z^3A(1+z^2)^{-3/2} = 2A$.

Furthermore, $g_2'(z) = 2\tan\frac{A}{\sqrt{1+z^2}} - 2A\frac{z^2}{(1+z^2)^{3/2}} - 2Az^2\frac{\tan^2(\frac{A}{\sqrt{1+z^2}})}{(1+z^2)^{3/2}}$. Since $0 < A \le \frac{\pi}{4}, \sqrt{2}A < \frac{5\pi}{12}$ and $\cos\frac{5\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, thus we have $\cos^2\frac{A}{\sqrt{2}} = \frac{1+\cos\frac{2A}{\sqrt{2}}}{2} \ge 5/8$. Thus, $\tan^2\frac{A}{\sqrt{1+z^2}} \le \frac{\frac{A^2}{1+z^2}}{\frac{5}{8}} = \frac{\frac{8}{5}A^2}{1+z^2}$.

Furthermore, the Taylor expansion of $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5\cdots$, thus $\tan x \ge x + \frac{1}{3}x^3$ for $0 < x \le \frac{\pi}{4}$ and we have, $2\tan\frac{A}{\sqrt{1+z^2}} - 2A\frac{z^2}{(1+z^2)^{3/2}} \ge 2(\frac{A}{\sqrt{1+z^2}} + \frac{1}{3}\frac{A^3}{(1+z^2)^{3/2}}) - 2A\frac{z^2}{(1+z^2)^{3/2}} = \frac{2A+\frac{2}{3}A^3}{(1+z^2)^{3/2}}$.

Hence, we have

$$g_2'(z) \ge \frac{2A + \frac{2}{3}A^3 - 2Az^2\frac{8A^2}{5(1+z^2)}}{(1+z^2)^{3/2}} \ge \frac{2A + \frac{2}{3}A^3 - \frac{16}{5}A^3}{(1+z^2)^{3/2}} > 0$$

Therefore, $g_2(z)$ is increasing and $2A \ge 2z\tan\frac{A}{\sqrt{1+z^2}}$. Thus, when $z \to \infty$, $g'(z)$ attains lower bound that is, $g'(z) \ge (1-2A) + A\frac{2z+z^3}{(1+z^2)^{3/2}}$.

Now we only need to prove the lower bound is positive for $z \ge 1$, which is true when we take the derivative. Therefore, $g(z)$ is a increasing function for $z > 0$.

Lemma 4 *If* $\alpha/2 \in (0, 0.218]$, *then*

$$t_{n-2,\alpha/2} \le c_{n-2,\alpha/2,w} \le 2t_{n-2,\alpha/2}, \tag{3.10}$$

where $c_{n-2,\alpha/2,w}$ *in (3.11) is the solution to the equation*

$$\alpha/2 = \frac{1}{2}P\{F_{1,n-2} > c_{n-2,\alpha/2,w}\} + (\frac{1}{4} - \frac{w}{2\pi})P\{F_{2,n-2} > c^2_{n-2,\alpha/2,w}/2\}. \tag{3.11}$$

Proof We have

$$t_{n-2,\alpha/2} \le c_{n-2,\alpha/2,w},$$

We now prove

$$2t_{\alpha/2} - c_{\alpha/2} \geq 0, \tag{3.12}$$

Again, we follow the steps in the proof of lemma 1, let $x(\alpha)$ and $y(\alpha)$ be the solutions to

$$\alpha/2 = \frac{1}{2}P\{F_{1,n-2} > x^2\} + \frac{1}{4}P\{F_{2,n-2} > \frac{x^2}{2}\}, \tag{3.13}$$

and

$$\alpha/2 = P\{T_{n-2} > y\}. \tag{3.14}$$

We have,

$$P\{F_{2,n-2} > \frac{x^2}{2}\} = \int_{x^2/2}^{\infty} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{2}{2})\Gamma(\frac{n-2}{2})}(\frac{2}{n-2})\frac{1}{(1+\frac{2t}{n-2})^{n/2}}dt = \frac{1}{(1+\frac{x^2}{n-2})^{(n-2)/2}},$$

and

$$\frac{1}{2}P\{F_{1,n-2} > x^2\} = P\{T_{n-2} > x\},$$

Let (3.13) and (3.14) equal, we have

$$P\{y < T_{n-2} < x\} = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{(n-2)\pi}\Gamma(\frac{n-2}{2})}\int_y^x (1+\frac{t^2}{n-2})^{-\frac{n-1}{2}}dt = \frac{1}{4}\frac{1}{(1+\frac{x^2}{n-2})^{(n-2)/2}},$$

By mean value theorem, there exists a $t^* \in [y, x]$ such that

$$\frac{\Gamma(\frac{n-1}{2})}{\sqrt{(n-2)\pi}\Gamma(\frac{n-2}{2})(1+\frac{{t^*}^2}{n-2})^{(n-3)/2}}\int_y^x \frac{1}{(1+\frac{t^2}{n-2})}dt = \frac{1}{4}\frac{1}{(1+\frac{x^2}{n-2})^{(n-2)/2}},$$

which is equivalent to

$$\int_{y/\sqrt{n-2}}^{x/\sqrt{n-2}} \frac{du}{1+u^2} = A\frac{(1+\frac{{t^*}^2}{n-2})^{(n-3)/2}}{(1+\frac{x^2}{n-2})^{(n-3)/2}},$$

Let $B(x) = A\frac{(1+\frac{{t^*}^2}{n-2})^{(n-3)/2}}{(1+\frac{x^2}{n-2})^{(n-3)/2}}$, then we have

$$\frac{y}{\sqrt{n-2}} = \frac{\frac{x}{\sqrt{n-2}} - \tan B(x)}{1 + \frac{x}{\sqrt{n-2}} \tan B(x)},$$

Then

$$y = \frac{x - \sqrt{n-2}\tan B(x)}{1 + \frac{x}{\sqrt{n-2}} \tan B(x)},$$

where

$$B(x) \leq \frac{A}{\sqrt{1+\frac{x^2}{n-2}}}$$

Therefore,

$$2y - x = \frac{x - 2\sqrt{n-2}\tan B(x) - \frac{x^2}{\sqrt{n-2}}\tan B(x)}{1 + \frac{x}{\sqrt{n-2}} \tan B(x)},$$

Let

$$\bar{g}(x) = \frac{x}{\sqrt{n-2}} - 2\tan\frac{A}{\sqrt{1+\frac{x^2}{n-2}}} - \frac{x^2}{n-2}\tan\frac{A}{\sqrt{1+\frac{x^2}{n-2}}},$$

which is equivalent to $g(z) = z - 2\tan\frac{A}{\sqrt{1+z^2}} - z^2\tan\frac{A}{\sqrt{1+z^2}}$, where $z = \frac{x}{\sqrt{n-2}}$.

Then $2y - x \geq \frac{\bar{g}(x)}{\frac{1}{\sqrt{n-2}} + \frac{x}{n-2}\tan B(x)}$ when $g(\bullet)$ attains lower bound. Since $g(z)$ is increasing and $z(\alpha)$ is a decreasing function of α. From the table that is given in (Tu, 1995), we know when $\alpha/2 \in (0, 0.218], g(z) = \bar{g}(x) \geq 0$ and thus $2y - x \geq 0$. Therefore, we get the proof for lemma 4.

3.2 Comparison between confidence intervals with or without restrictions

As we mentioned before, the comparison is based on lemmas discussed in the previous section. Therefore, we only do the comparison in mixed signs cases.

Suppose $x_{01} > 0, x_{02} < 0$, when we combine the U and L of mixed signs restricted case, we may find that it is very complicated because of the uncertainty of x_{01} and x_{02}. Therefore, we can not give a specific formula to discuss which confidence interval is better, however, when we are given these two values, we can identify which condition $\hat{\beta}_1$, $\hat{\beta}_2$ are in so that we can get our restricted confidence interval and compare it with the unrestricted confidence interval. For now, we can only compare those intervals when $\hat{\beta}$ is in some certain regions.

Though we cannot get the formulas easily, it does not mean the way we do the comparison does not work. For example, when σ is unknown, we replace σ with s and when $\hat{\beta}_1 \geq \frac{x_{02}\hat{\beta}_2 + (\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}})E_{\alpha/2}s}{\frac{S_{x_1}^2}{x_{01}}(\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n})}$, $\hat{\beta}_2 < \frac{1}{x_{02}}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}}E_{\alpha/2}s$ and $\frac{1}{x_{01}}\sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}K_{\alpha/2}s$ is relatively small, then we can conclude that our U will be $\hat{\beta}_0 + x_{01}\hat{\beta}_1 - \sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}K_{\alpha/2}s$ and our L is $\hat{\beta}_0 + x_{01}\hat{\beta}_1 - (\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}})E_{\alpha/2}s$. Note that $K_{\alpha/2} = E_{\alpha/2}$, we use $C_{\alpha/2}$ instead. The difference between restricted confidence interval and unrestricted one will be $D = \hat{\beta}_0 + x_{01}\hat{\beta}_1 - \sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}C_{\alpha/2}s - (\hat{\beta}_0 + x_{01}\hat{\beta}_1 - (\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}})C_{\alpha/2}s) - 2t_{\alpha/2}\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}s = (\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}} - \sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{1}{n}} - \sqrt{\frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}})C_{\alpha/2}s - 2t_{\alpha/2}s\sqrt{\frac{x_{02}^2}{S_{x_2}^2} + \frac{x_{01}^2}{S_{x_1}^2} + \frac{1}{n}}$. Now we apply lemma 4, since $C_{\alpha/2} \leq 2t_{\alpha/2}$, then D is negative, our restricted confidence interval is better. We are still working on this part and will show the other comparisons in the future work.

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