Calculus III Note-taking Guide Booklet

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These open-source mathematics note taking guides are created to help students with note-taking and correspond to chapters in OpenStax Calculus Volume I[1], II[2], and III[3]. They were created through the Worcester Polytechnic Institute Women's Impact Network EMPOwER grant program.

[1] Herman, Edwin, Gilbert Strang, Joseph Lakey, Elaine A. Terry, Alfred K. Mulzet, Sheri J. Boyd, Joyati Debnath et al. "Calculus Volume 1." (2016).

[2] Herman, Edwin, Gilbert Strang, William Radulovich, Erica A. Rutter, David Smith, Kirsten R. Messer, Alfred K. Mulzet et al. "Calculus Volume 2." (2016).

[3] Herman, Edwin, Gilbert Strang, Nicoleta Virginia Bila, Sheri J. Boyd, David Smith, Elaine A. Terry, David Torain et al. "Calculus Volume 3." (2016).

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1 4.8 L'Hôpital's Rule

Problem Set 1.1. *Find the following limits*

1. $$\lim_{x\to 0} \frac{x^3}{x} =$$

2. $$\lim_{x\to 0} \frac{x}{x^3} =$$

3. $$\lim_{x\to 0} \frac{x}{x} =$$

4. $$\lim_{x\to 0} \frac{x+x^3}{2} =$$

5. $$\lim_{x\to 0} \frac{x}{x+4x^2} =$$

Theorem 1.2. ***L'Hôpital's Rule***

Problem Set 1.3. *Evaluate the following limits.*

1.

$$\lim_{x \to 0} \frac{\sin(x)}{x} =$$

2.

$$\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x}} =$$

3.

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} =$$

Other Indeterminant Forms

-
-
-

Example 1.4.

$$\lim_{x\to 0^+} \frac{1}{x^2} - \frac{1}{\tan(x)}$$

Indeterminant Powers

If $\lim_{x\to a} \ln(f(x)) = L$, then

Example 1.5. *Evaluate*

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

.

2 3.7 Improper Integrals

Definition 2.1. ***Integrating over an Infinite Interval***

1. *If $f(x)$ is continuous on $[a,\infty)$, then*

2. *If $f(x)$ is continuous on $(-\infty,b]$, then*

3. *If $f(x)$ is continuous on $(-\infty,\infty)$, then*

In each cases, if the limit exists, then the improper integral is said to __________________________. *Otherwise, if the limit does not exist, then the improper integral is said to* __________________________.

Example 2.2. *We evaluate*

$$\int_1^\infty \frac{1}{x}dx =$$

Problem Set 2.3. *Evaluate*

$$\int_{-\infty}^0 \frac{1}{x^2+4}dx$$

Definition 2.4. ***Integrating a Discontinuous Integrand***

1. *If $f(x)$ is continuous on $[a, b)$, then*

2. *If $f(x)$ is continuous on $(a, b]$, then*

3. *If $f(x)$ is continuous on $[a, b]$ except at c in (a, b), then*

In each case, if the limit exists and is finite, then the improper integral is said to ________________________.
Otherwise, the improper integral is said to ________________________.

Example 2.5. *We evaluate*

$$\int_{-1}^{1} \frac{1}{x^3} dx =$$

Theorem 2.6. ***The Direct Comparison Test*** *Let f, g be continuous on $[a, \infty)$ and assumme that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then*

1. *If $\int_a^\infty g(x)dx$* ________________________, *then $\int_a^\infty f(x)dx$ also* ________________________.
2. *If $\int_a^\infty f(x)dx$* ________________________, *then $\int_a^\infty g(x)dx$ also* ________________________.

Example 2.7. *Consider for $p < 1$*

$$\int_{1}^{\infty} (x+7)^p dx$$

3 5.1 Sequences

Definition 3.1. *An* ________________________ *is an ordered list a of numbers of the form*

Each of the numbers is called a ________________________. *The symbol n is called the* ________________________ *for the sequence.*
We also use the notation

Example 3.2. *Examples of sequences:*

We sometimes would like to write sequences using its ____________________.

Problem Set 3.3. *Write each sequences given using its explicity formula. We will do the second one together:*

- $1, 2, 3, 4, ...$
- $2, 4, 6, 8, 10$
- $1, -1, 1, -1, ...$
- $1, 1, 2, 3, 5, 8, ...$

3.1 Limit of a Sequence

Definition 3.4. *Given a sequence $\{a_n\}$, if the terms of a_n become ____________________ to a __________ ______________ as _________________________, we say $\{a_n\}$ is a ____________________ and L is the ______________. In this case, we write*

If a sequence is not convergent, we say it is _________________.

More formally, we can instead use the definition:

Definition 3.5. *A sequence $\{a_n\}$ ____________________ to the number ______________ if for every $\varepsilon > 0$ there corresponds an integer N such that if $n \geq N$,*

The number L is the ____________________ and we write

Example 3.6. *Let* $\{a_n\} = \{\frac{1}{n}\}$ *and* $\{b_n\} = \{(-1)^n\}$*. We investigate the convergence or divergence of each.*

3.2 Calculating Limits of Sequences

Theorem 3.7. ***Limit of a Sequence Defined by a function*** *Consider a sequence* $\{a_n\}$ *such that* $a_n = f(n)$*. If* ____ *such that*

then $\{a_n\}$ *converges and*

.

Theorem 3.8. ***Algebraic Limit Laws:*** *Given sequences* $\{a_n\}$ *and* $\{b_n\}$ *and a real number* C*, if there exist constants* A, B *such that* $\lim_{n\to\infty} a_n = A, \lim_{n\to\infty} b_n = B$*. Then*

1.

2.

3.

4.

5.

Theorem 3.9. *Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values an for $n \geq N$, and the sequence*

This allows us to use things like L'Hôpital's rule for sequences.

Theorem 3.10. ***The Squeeze Theorem for Sequences*** *Consider sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and suppose that* ______________________ *for all $n \geq N$ for some N. If*

then

Problem Set 3.11. *If possible, find the limits of the following sequences.*

1. $1, 2, 3, 4, ...$

2. $5, 19, 5, 19, 5, 19, ...$

3. $\{\frac{1}{n^2}\}$

3.3 Bounded and Monotonic Sequences

Definition 3.12. *A sequence $\{a_n\}$ is* ______________________ *if there exists a number M so that* ______________________ *for all n.*
A sequence $\{a_n\}$ is ______________________ *if there exists a number m so that* ______________________ *for all n.*
A sequence $\{a_n\}$ is a ______________________ *if it is bounded above and bounded below.*
If a sequence is not bounded, it is an ______________________

Definition 3.13. *A sequence $\{a_n\}$ is* ______________________ *if* ______________________ *for all n. It is* ______________________ *if* ______________________ *for all n. A sequence is* ______________________ *if is either* ______________________ *or* ______________________

Theorem 3.14. ***Montone Convergence Theorem*** *If $\{a_n\}$ is a* ______________________ *sequence and there exists a positive integer n_0 such that $\{a_n\}$ is* ______________________ *for all $n \geq n_0$, then $\{a_n\}$* ______________________

Problem Set 3.15. *Classify each sequence as bounded or not and monotonic or not. Then using that information, decide if we know if the sequence converges.*

1. $1, 2, 3, 4, \ldots$

2. $5, 19, 5, 19, 5, 19, \ldots$

3. $\{(-1)^n\}$

4. $\{\frac{1}{n}\}$

5. $a_n = a_1$, *where* $a_1 = 7$

4 5.2 Infinite Series

Definition 4.1. *An* ____________________________ *is a sum of infinitely many terms and is written in the form*

For each k*,* S_k *is* ____________________________

If we can describe the convergence of a series to S*, we call* S *the* ____________________________*, and we write*

If the sequence of partial sums diverges, we have the ____________________________________

Example 4.2. *Decide whether each sum converges or diverges.*

- $$\sum_{n=1}^{\infty} 1$$

- $$\sum_{n=1}^{\infty} 0$$

Example 4.3. *Find the sum of the* ***telescoping series***

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Theorem 4.4. *Let $\sum a_n$, $\sum b_n$ be convergent series. Then we have:*

1. *Sum/Difference Rule:*

2. *Constant Multiple Rule:*

4.1 Geometric Series

Definition 4.5. *A* ______________________________ *is any series that we can write in the form*

where a, r are fixed and $a \neq 0$.

Problem Set 4.6. *Identify if each is a geometric series. If it is, what are a, r?*

1. $1 + \frac{1}{2} + \frac{1}{4} + ... + \left(\frac{1}{2}\right)^{n-1} + ...$

2. $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + ...$

3. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$

4. $5 + 5 + 5 + 5 + ...$

4.1.1 Convergence of the Geometric Series

Goal: Write S_n in terms of a, r. This way we know what the partial sum is of any geometric series. Consider

$$S_n = a + ar + ... + ar^{n-1}$$

Theorem 4.7. *If ______________________ in a geometric series, then*

If ______________________ in a geometric series, then it ______________________.

Problem Set 4.8. *Decide whether each geometric series converges or diverges. If it converges, what is its sum?*

1. $1 + \frac{1}{2} + \frac{1}{4} + ... + \left(\frac{1}{2}\right)^{n-1} + ...$
2. $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + ...$
3. $5 + 5 + 5 + 5 + ...$

5 5.3 The Divergence and Integral Tests

Theorem 5.1. ***Divergence Test*** *If $\lim_{n\to\infty} a_n = c \neq 0$ or does not exist, then $\sum_{n=1}^{\infty} a_n$ ______________________.*

Example 5.2. *Consider*

$$\sum_{n=1}^{\infty} \frac{1}{n}, \int_{1}^{\infty} \frac{1}{x} dx.$$

Let $f(x) = \frac{1}{x}$. *Then*

Theorem 5.3. ***Integral Test*** *Suppose* $\sum_{n=1}^{\infty} a_n$ *is a series with positive terms. Suppose there exists a function* f *and a positive integer* N *such that the following three conditions are satisfied:*

1.

2.

3.

Then

both ______________________________ *or both* ______________________________.

Problem Set 5.4. *Use the integral test to decide whether each series converges or diverges.*

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 1$

Definition 5.5. *For any real number* p, *the series*

is called a ______________________________.

6 5.4 Comparison Tests

Theorem 6.1. *1. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If* ____________________________, *then* ____________________________

2. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If ____________________________, *then* ____________________________

Example 6.2. *We investigate the convergence or lack thereof of $\sum \frac{1}{n^3+3n+1}$.*

Problem Set 6.3. *Investigate the convergence or lack thereof of $\sum \frac{1}{2^n-1}$.*

Theorem 6.4. ***Limit Comparison Test*** *Let $a_n, b_n > 0$ for all $n \ge 1$.*

1.

2.

3.

Example 6.5. *We use LCT to determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$.*

Problem Set 6.6. *Using LCT, determine whether or not each series converges or diverges.*

1. $\sum \frac{2n+1}{n^2+2n+1}$

2. $\sum \frac{5^n}{3^n+2}$

7 5.5 Alternating Series

Definition 7.1. *Any series whose terms alternate between positive and negative values is called an ______________ ______________. An ______________________________ can be written in the form*

Theorem 7.2. ***Alternating Series Test*** *An alternating series converges if*

1.

2.

Example 7.3. *Consider*

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}.$$

Theorem 7.4. ***Remainders in Alternating Series*** *Consider an alternating series that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the Nth partial sum. For any integer $N \geq 1$, the remainder* ________________________ *satisfies*

Definition 7.5. *A series $\sum a_n$ exhibits* ________________________ *if* ________________________*. A series $\sum a_n$ exhibits* ________________________ *if*________________________*, but*________________________

Problem Set 7.6. *Decide whether or not $\sum \frac{\cos(n\pi)}{n^2}$ and $\sum \frac{\cos(n\pi)}{n}$ are alternating series and whether they converge or diverge. If they converge, does they converge absolutely or conditionally?*

8 5.6 Ratio and Root Tests

Theorem 8.1. ***Ratio Test*** *Let $\sum a_n$ be any series be a series with nonzero terms. Let*

1. *If* __________, *then* ____________________________
2. *If* __________, *then* ____________________________
3. *If* __________, *then* ____________________________

Note: This extends the knowledge we already had for geometric series.

Theorem 8.2. ***Root Test*** *Consider the series $\sum a_n$. Let*

1. *If* __________, *then* ____________________________
2. *If* __________, *then* ____________________________
3. *If* __________, *then* ____________________________

Problem Set 8.3. *Determine if the following series converge absolutely.*

1. $\sum \frac{2^n}{n!}$

2. $\sum \frac{(-1)^n (n!)^2}{(2n)!}$

3. $\sum \left(\frac{1}{n}\right)^n$

9 6.1 Power Series and Functions

Definition 9.1. *A series of the form*

is a ____. *A series of the form*

is a ____.

Example 9.2.

$$\sum_{n=0}^{\infty} x^n$$

Theorem 9.3. ***Convergence of a Power Series*** *Consider the power series* $\sum_{n=0}^{\infty} c_n (x-a)^n$. *The series satisfies exactly one of the following properties:*

1.

2.

3.

Definition 9.4. *Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the* ______________________________. *If there exists a real number $R > 0$ such that the series* ____________________________________ *and* ____, *then R is the* ____. *If the series converges only at $x = a$, we say the radius of convergence is* ____________________. *If the series converges for all real numbers x, we say the radius of convergence is* ____________________.

How to Test a Power Series for Convergence

1. Use the ____ to find the largest open interval where the series converges

2.

3.

Example 9.5. *Determine where the Power Series $\sum(-1)^{n-1}\frac{x^n}{n}$ converges or diverges.*

Problem Set 9.6. *Determine where the power series below converge or diverge.*

1. $\sum \frac{x^n}{n!}$

2. $\sum n!x^n$

10 6.2 Properties of Power Series

Theorem 10.1. ***Combining Power Series*** *Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.*

1.

2.

3.

Theorem 10.2. *Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let*

Then

and

Theorem 10.3. ***Term-by-Term Differentiation and Integration of Power Series*** *Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ converges on the interval $(a-R, a+R)$ for some $R > 0$. Let f be the function defined by the series*

Then f is ________________ on the interval $(a-R, a+R)$ and we can find f' by differentiating the series term-by-term:

for $|x-a| < R$. Also, to find ________________, we can integrate the series term-by-term. The resulting series converges on $(a-R, a+R)$, and we have

for $|x-a| < R$.

Warning! This may not work for series that are not Power Series.

Example 10.4. *Let*

$$f(x) = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \ -1 \leq x \leq 1$$

We identify this as a function we more commonly know.

11 6.3 Taylor and Maclaurin Series

Definition 11.1. *If f has derivatives of all orders at $x = a$, then the* ____ *is*

The Taylor series for f at ________ *is known as the* ____

Definition 11.2. *If f has n derivatives at $x = a$, then the nth* ______________________________ *for f at a is*

Example 11.3. *We find the Taylor series generated by $1/x^2$ at $a = 1$.*

Problem Set 11.4. *Find the Taylor Series generated by* $f(x) = x^3$ *at* $x = 3$.

Example 11.5. *We find the Taylor Polynomial of degree* n *of* e^x.

Theorem 11.6. ***Taylor's Theorem with Remainder*** *Let* f *be a function that can be differentiated* $n+1$ *times on an interval* I *containing the real number* a. *Let* p_n *be the* n*th Taylor polynomial of* f *at* a *and let*

be the ______________________________. *Then for each* x *in the interval* I, *there exists a real number* c *between* a *and* x *such that*

If there exists a real number M *such that* ______________________________ *for all* $x \in I$, *then*

for all x *in* I.

Example 11.7. *We find the Taylor Series of* $\sin(x)$.

12 6.4 Working with Taylor Series

Note: We can use the following from here on without proof

$$e^x =$$
$$\cos(x) =$$
$$\sin(x) =$$

Example 12.1. *We express* $\int e^{-x^2} dx$ *as an infinite series.*

Problem Set 12.2. *Find the Taylor Series for* $\sinh(x)$. *Hint: Recall* $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

13 7.1 Parametric Equations

Definition 13.1. *If x and y are continuous functions of t on an interval I, then the equations*

are called ____________________ *and* __________ *is called the* ____________________.
The set of points (x, y) obtained as t varies over the interval I is called the __________________________.
The graph of parametric equations is called a ____________________ *or plane curve, and is denoted by C.*

Example 13.2. *Consider $x = \sin \frac{\pi t}{2}, y = 2t + 4, 0 \leq t \leq 4$.*

Problem Set 13.3. *Sketch the curve*

$$x = 3t + 2,\ y = t^2 + 1,\ -\infty < t < \infty.$$

Example 13.4. *We find two different parametric equations to represent the graph of $y = 2x^2 + 3$.*

13.1 Cycloids

Example 13.5. *A wheel of radius a rolls along a horizontal straight line. We find parametric equations for the path traced by a point on the wheel. The path is called a* ____________________________.

14 7.2 Calculus of Parametric Curves

14.1 Derivatives of Parametric Equations

Parametric Formula for $\frac{dy}{dx}$

Parametric Formula for $\frac{d^2y}{dx^2}$

Example 14.1. *Find the tangent line to the plane curve defined by the parametric equations*

$$x(t) = t^2 - 3, \;\; y(t) = 2t - 1, \;\; t \geq 0$$

at $t = 0$.

Problem Set 14.2. *Find* $\frac{d^2y}{dx^2}$ *as a function of* t *if* $x = 1 - t^2$, $y = t - t^2$.

Example 14.3. *We set up, but do not evaluate, an integral that gives the area under the curve of the cycloid defined by the equations*

$$x = t - \sin(t),\ \ y = 1 - \cos(t),\ \ 0 \le t \le 2\pi.$$

Definition 14.4. *Consider the plane curve defined by the differentiable parametric equations* $x = x(t)$, $y = y(t)$, $t_1 \le t \le t_2$. *Then the* __________________________ *is given by*

$$L =$$

Problem Set 14.5. *Find the length of the circle of radius* r *defined parametrically by*

$$x = r\cos t,\ \ y = \sin t,\ \ 0 \le t \le 2\pi.$$

Areas of Surface of Revolution for Parametrized Curves

1. **Revolution about the x-axis:**

2. **Revolution about the y-axis:**

15 7.3 Polar Coordinates

Definition 15.1. *The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has ____________________. The ____________________*

____.

This is the basis of the _______________________. In the _______________________, each point also has two values associated with it: ____________________.

Example 15.2. *We find all of the polar coordinates for the point $P(2, \pi/6)$.*

Theorem 15.3. ***Converting Points between Coordinate Systems:*** *Given a point P in the plane with ___________________________ and ___________________________ the following conversion formulas hold true:*

Some curves are easier to express in polar and some are easier to express in Cartesian.

Example 15.4. *We find the polar equation for the circle* $x^2 + (y-3)^2 = 3^2$ *(circle centered at (0,3) with radius 3).*

15.1 Polar Curves

Example 15.5. *We graph* $r = 4\sin\theta$*.*

Problem Set 15.6. *Graph the curve* $r = 1 - \cos(\theta)$*.*

Example 15.7. *Polar objects can have multiple representations*

-
-

Problem Set 15.8. *Graph the sets of points using the conditions:*

- $1 \leq r \leq 3$ *and* $0 \leq \theta \leq \pi/2$

- $-3 \leq r \leq 2$ *and* $\theta = \pi/4$

- $2\pi/3 \leq \theta \leq 5\pi/6$

15.2 Transforming Polar Equations to Rectangular Coordinates

Problem Set 15.9. *Write the polar equation as a Cartesian equation.*

- $r\cos(\theta) = 2$

- $r = 6\cos\theta - 8\sin\theta$

- $r = 1 - \cos(\theta)$

16 7.4 Area and Arc Length in Polar Coordinates

16.1 Slope of Polar Curves

Recall: Slope of a curve in Cartesian is $\frac{dy}{dx}$. This is **not** true in polar. When $r = f(\theta)$:

Areas of Regions Bounded by Polar Curves

Theorem 16.1. *Suppose f is continuous and nonnegative on the interval $\alpha \leq \theta \leq \beta$ with ______________________. The area of the region bounded by the graph of ______________________ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is*

Problem Set 16.2. *Find the area of the region enclosed by $r = 1 - \cos\theta$.*

Example 16.3. *We find the area of the region that lies outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.*

16.2 Arc Length in Polar Coordinates

Theorem 16.4. *Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \beta$ to $\theta = \beta$ is*

Problem Set 16.5. *Find the arc length of the $r = 2 + 2\cos\theta$.*

17 2.1 Vectors in the Plane

Definition 17.1. *A* ____________________ *is a quantity that has both* ____________________ *and* ____________________.

Definition 17.2. *Vectors are said to be* ____________________ *vectors if they have the* ________________________________.

Definition 17.3. *The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as*

The scalars x and y are called the ____________________ *of* ***v***.

Example 17.4. *Consider $\boldsymbol{v} = \overrightarrow{\mathrm{PQ}}$ with $P(-3, 4)$ and $Q(-5, 2)$. The vector $\boldsymbol{v}$ has components*

-
-

So the component form is ____________________. *The length is*

17.1 Combining Vectors

Definition 17.5. *Let k be a* ***scalar*** *(a real number). Then if* ***u,v*** *are vectors then we have*

- *Addition/Subtraction:*
- *Scalar Multiplication:*

Problem Set 17.6. *If $\boldsymbol{u} = \langle -1, 3\rangle$ and $\boldsymbol{u} = \langle 4, 7\rangle$ find:*

- $2\boldsymbol{u} - 3\boldsymbol{v}$
- $\|\frac{1}{2}\boldsymbol{u}\|$

- $\frac{1}{2}\|\boldsymbol{u}\|$

Properties of Vector Operations

1. $\mathbf{u} + \mathbf{v} =$
2. $\mathbf{u} + \mathbf{0} =$
3. $0\mathbf{u} =$
4. $a(b\mathbf{u}) =$
5. $(a + b)\mathbf{u} =$
6. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} =$
7. $\mathbf{u} - \mathbf{u} =$
8. $1\mathbf{u} =$
9. $a(\mathbf{u} + \mathbf{v}) =$

17.2 Unit Vectors

Definition 17.7. *A ________________________________ is a vector with ________________________________.*
For any nonzero vector $\boldsymbol{v}$, we can use scalar multiplication to find a unit vector $\boldsymbol{u}$ that has the same direction as v.

Problem Set 17.8. *Find a unit vector $\boldsymbol{u}$ in the direction of the vector from $P_1(1, 0)$ and $P_2(3, 2)$.*

18 2.2 Vectors in Three Dimensions

Definition 18.1. *The ________________________________ rectangular coordinate system consists of three perpendicular axes: the x-axis, the y-axis, ________________________________, and an origin at the point of intersection of the axes.*

Theorem 18.2. *The ________________________ between points ________________ and ________________ is given by the formula*

Definition 18.3. *A* ____________________ *is the set of all points in space* ________________________________ *from a fixed point, the* ____________________ *of the sphere. In a sphere, the distance from the center to a point on the sphere is called the* ____________________.
The sphere with center (a, b, c) and radius r can be represented by the equation

18.1 Graphing Other Equations in Three Dimensions

Example 18.4. *We describe the set of points in three-dimensional space that satisfies $(x-2)^2 + (y-1)^2 = 4$, and graph the set.*

18.2 Working with Vectors in 3D

Example 18.5. *Let $\overrightarrow{\mathrm{PQ}}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$. We express $\overrightarrow{\mathrm{PQ}}$ in both component form and using standard unit vectors.*

Problem Set 18.6. *If $\boldsymbol{u} = \langle -1, 3, 0 \rangle$ and $\boldsymbol{v} = \langle 4, 7, 11 \rangle$ find:*

- $2\boldsymbol{u} - 3\boldsymbol{v}$
- $\|\frac{1}{2}\boldsymbol{u}\|$
- $\frac{1}{2}\|\boldsymbol{u}\|$

19 2.3 The Dot Product

Definition 19.1. *The* ____________________ *of two vectors is* $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$

Theorem 19.2. *The* ____________________ *of two vectors is the product of the* ____________________ *of each vector and the* ____________________ *of the angle between them:*

Problem Set 19.3.

1. *Find the dot product of* $\boldsymbol{u} = \langle 1, -2, -2 \rangle$ *and* $\boldsymbol{v} = \langle -6, 2, -3 \rangle$*.*

2. *Find the angle between* $\boldsymbol{u} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ *and* $\boldsymbol{v} = 6\boldsymbol{i} + 3\boldsymbol{j} + 0\boldsymbol{k}$*.*

Theorem 19.4. *The nonzero vectors* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are* ____________________ *if and only if* ____________________.

Properties of the Dot Product

1. $\mathbf{u} \cdot \mathbf{v} =$
2. $c\mathbf{u} \cdot \mathbf{v} =$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) =$
4. $\mathbf{u} \cdot \mathbf{u} =$
5. $\mathbf{0} \cdot \mathbf{u} =$

19.1 Vector Projections

The ____ is the vector labeled ______________. It has the same ______________________ as **u** and **v** and the same ______________________, and represents the ______________________ that acts in the ______________________. If θ represents the angle between **u** and **v**, then the length of $proj_{\mathbf{u}}\mathbf{v}$ is __________. When expressing $\cos\theta$ in terms of the dot product, this becomes

We now multiply by a unit vector in the direction of **u** to get $proj_u v$

The length of this vector is also known as the ____ and is denoted by

Problem Set 19.5. *Find the vector projection of* $\boldsymbol{u} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ *onto* $\boldsymbol{v} = \boldsymbol{i} - 0\boldsymbol{j} - 0\boldsymbol{k}$.

20 2.4 The Cross Product

Definition 20.1. *The **cross product** of two vectors is*

where $\boldsymbol{n}$ *is the* ____

Parallel Vectors Nonzero vectors $\mathbf{u}, \mathbf{v}$ are __________ if and only if ________________.

Properties of the Cross Product If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors and r, s are scalars, then

1. $r(\mathbf{u}) \times (s\mathbf{v}) =$

2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) =$

3. $\mathbf{v} \times \mathbf{u} =$

4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} =$

5. $\mathbf{0} \times \mathbf{u} =$

6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) =$

Example 20.2. ***Area of a Parallelogram***

Example 20.3. *We find the cross product of the three dimensional vectors* $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ *and* $\boldsymbol{v} = -4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$.

Problem Set 20.4. *Find the cross product of the three dimensional vectors* $\boldsymbol{u} = \boldsymbol{i}$ *and* $\boldsymbol{v} = \boldsymbol{j}$.

21 2.5 Equations of Lines and Planes in Space

Recall:

- Slope-intercept form of a line: $y = mx + b$
- A parametric form of a line: $x(t) = m_1 t + x_0,\ y(t) = m_2 t + b, -\infty < t < \infty$

Definition 21.1. *A* ________________ *parallel to vector* ________________ *and passing through point* ________________ *can be described by the following parametric equations:*

If the constants $a, b,$ and c are all nonzero, then L can be described by the symmetric equation of the line:

Problem Set 21.2. *Find parametric and symmetric equations of the line passing through points $(1, 0, -2)$ and $(-3, 5, 0)$.*

Example 21.3. *A mouse travels from its home (the origin) to a piece of cheese in the direction of the point (1,1,1) at a speed of 60 cm per second. What is its position after 10 seconds?*

21.1 Distance between a Point and a Line

21.2 Equations for a Plane

Definition 21.4. *Given a point P and vector $\boldsymbol{n}$, the set of all points Q satisfying the equation* __________________ *forms a* ______________. *The equation*

is known as the ____.
The ____ *containing point $P = (x_0, y_0, z_0)$ with normal vector $\boldsymbol{n} = \langle a, b, c\rangle$ is*

This equation can be expressed as ____ *where* ___________________________.
This form of the equation is sometimes called the ____

Problem Set 21.5. *Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to*

$$\boldsymbol{n} = 2\boldsymbol{j} - \boldsymbol{k}.$$

Example 21.6. *We find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.*

Problem Set 21.7. *Find a vector parallel to the line of intersection of the planes* $3x-6y-2z=15$ *and* $2x+y-2z=5$. *Hint: This line of intersection is perpendicular to both planes normal vectors.*

Example 21.8. *We find the point where the line*

$$x(t)=\frac{8}{3}+2t,\ \ y(t)=-2t,\ \ z(t)=1+t$$

intersects the plane $3x+2y+6z=6$.

22 3.1 Vector-Valued Functions and Space Curves

Definition 22.1. *A* ______________________________ *is a function of the form*

where the ______________________________ $f,g,h,$ *are real-valued functions of the parameter* t. *Vector-valued functions are also written in the form*

Example 22.2. *We sketch* $\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j} + t\boldsymbol{k}$.

Problem Set 22.3. *Describe how the following compare to* $\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j} + t\boldsymbol{k}$.

- $\boldsymbol{r}(t) = \cos(2t)\boldsymbol{i} + \sin(2t)\boldsymbol{j} + t\boldsymbol{k}$.
- $\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j} + 2t\boldsymbol{k}$.

Definition 22.4. *A vector-valued function r approaches the* ____________________ *as t approaches a, written*

provided

Theorem 22.5. *Let f, g, and h be functions of t. Then the limit of the vector-valued function* $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ *as t approaches a is given by*

provided the limits exist.

Problem Set 22.6. *Let* $\boldsymbol{r}(t) = \frac{2t-4}{t+1}\boldsymbol{i} + \frac{t}{t^2+1}\boldsymbol{j} + (4t-3)\boldsymbol{k}$. *Find* $\lim_{t\to 3} \boldsymbol{r}(t)$.

Definition 22.7. *Let f, g, h functions of t. Then, the vector-valued function $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ is* ______________________________ *if the following three conditions hold:*

-
-
-

23 3.2 Calculus of Vector-Valued Functions

Theorem 23.1. *Let f, g, h be differentiable functions of t and let $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$. Then*

Problem Set 23.2. *Let $\boldsymbol{r}(t) = t\ln(t)\boldsymbol{i} + 5e^t\boldsymbol{j} + \cos(t)\boldsymbol{k}$. Find $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$.*

23.1 Tangent Vectors and Unit Tangent Vectors

Definition 23.3. *Let C be a curve defined by a vector-valued function $\boldsymbol{r}$, and assume that $\boldsymbol{r}'(t)$ exists when $t = t_0$. A* ______________________________ *$\boldsymbol{v}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\boldsymbol{r}(t_0)$ on the graph, vector $\boldsymbol{v}$ is* ______________ *to curve C. Vector* __________ *is an example of a tangent vector at point $t = t_0$. The* ______________________________ *at t is defined to be*

Problem Set 23.4. *Find the a tangent vector and the unit tangent vector for each of $\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j}$.*

23.2 Integrals of Vector-Valued Functions

Definition 23.5. *The* ______________________________ *of a vector-valued function $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ is*

The ____ *of the vector-valued function is*

Problem Set 23.6. *Calculate* $\int_1^3 \left((2t+4)\boldsymbol{i} - t^2\boldsymbol{j}\right) dt$.

24 3.3 Arc Length and Curvature

Recall: Arc Length of a Parametric Curve

Theorem 24.1. *Given a smooth curve C defined by the function $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where t lies within the interval $[a, b]$, the* ______________________ *of C over the interval is*

Problem Set 24.2. *Calculate the arc length for $\boldsymbol{r}(t) = \sin(t)\boldsymbol{i} + \cos(t)\boldsymbol{j} + (10 - t)\boldsymbol{k}$, from $t = 0$ to $t = 2\pi$.*

Theorem 24.3. *Let $\boldsymbol{r}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by*

Furthermore, ____.

24.1 Curvature

Definition 24.4. *Let C be a smooth curve in the plane or in space given by $\boldsymbol{r}(s)$, where s is the arc-length parameter. The* ______________________________________ *is*

Theorem 24.5. *If C is a smooth curve given by $\boldsymbol{r}(t)$, then the curvature κ of C at t is given by*

or

If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature at point (x, y) is given by

We show the first formula:

Problem Set 24.6. *Find the curvature for each of the following curves at the given point:*

$$\boldsymbol{r}(t) = 4\cos t\boldsymbol{i} + 4\sin t\boldsymbol{j} + 3t\boldsymbol{k}, t = 4\pi/3$$

24.2 The Normal and Binormal Vectors

Definition 24.7. *Let C be a three-dimensional smooth curve represented by $\boldsymbol{r}$ over an open interval I. If $\boldsymbol{T}(t) \neq 0$, then the* ______________________________________ *is*

The ______________________________________ *is*

where $\boldsymbol{T}(t)$ is the unit tangent vector.

Note:

Problem Set 24.8. *Find the principal unit normal vector and the binormal vector for $\boldsymbol{r}(t) = 4\cos t\boldsymbol{i} + 4\sin t\boldsymbol{j} + 3t\boldsymbol{k}$.*

Definition 24.9. *Suppose we form a circle in the osculating plane of C at point P on the curve. Assume that the circle has the* ______________________________________ *as the curve does at point P and let the circle have* ______________________. *Then, the* ______________________________________ *is given by* ______________________________________.

We call ______________________________ *of the curve, and it is equal to the reciprocal of the curvature. If this circle lies on the* ______________________________ *side of the curve and is* ______________________________________ *at point P, then this circle is called the* ______________________________________.

Example 24.10. *Find the equation of the osculating circle of the curve defined by the vector-valued function $y = x^2$ at the origin.*

25 3.4 Motion in Space

Definition 25.1. *Let $\boldsymbol{r}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the ____________________ of an object as a function of time. The ____________________________________ is*

The ____________________________________ is

The ____________________________________ is

Problem Set 25.2. *Find the velocity, speed, and acceleration of a particle whose path is*

$$\boldsymbol{r}(t) = t^2\boldsymbol{i} + (t+2)\boldsymbol{j} + 3t\boldsymbol{k}.$$

25.1 Components of the Acceleration Vector

Theorem 25.3. *The acceleration vector* $\boldsymbol{a}(t)$ *of an object moving along a curve traced out by a twice-differentiable function* $\boldsymbol{r}(t)$ *lies in the plane formed by the* ____________________ *and the* ____________________ *to* C*. Furthermore,*

The coefficients of $\boldsymbol{T}(t)$ *and* $\boldsymbol{N}(t)$ *are referred to as the* ____________________ *and the* ____________________ *respectively.*

Problem Set 25.4. *A particle moves in a path defined by the vector-valued function* $\boldsymbol{r}(t) = t^2\boldsymbol{i} + (2t-3)\boldsymbol{j} + (3t^2-3t)\boldsymbol{k}$*, where* t *measures time in seconds and distance is measured in feet. Find* a_T *and* a_N*.*