Worcester Polytechnic Institute

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Major Qualifying Project

Limit Cases of the p-Laplace Operator via Mosco Convergence

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Contents

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Abstract

In the classic theory, p-Laplace operator $(1 < p < +\infty)$ joined several main parts of the mathematics in a fruitful way, and one important principle of mathematics is that extreme cases reveal interesting structure. Looking at p-Laplace operator as subgradients of a sequence of convex functionals $\{E_p\}$, as p goes to 1 and to infinity, we study the connection of the dual problem between 1-Laplace operator and infinity-Laplace operator using tools from convex analysis and the notion of Mosco convergence.

1 Introduction

This paper is a study about the limit cases of the family of Laplace operators. With parameter p, the **(strong)** p**-Laplace operator**, $\triangle_p$ is defined as

$$\begin{aligned}\triangle_p u &= \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\\ &= |\nabla u|^{p-4}\left\{|\nabla u|^2\triangle u + (p-2)\sum_{i,j=1}^{N} u_{x_i}u_{x_j}u_{x_ix_j}\right\}.\end{aligned}$$

for

$$1 < p < 2, \quad p = 2, \quad 2 < p < \infty.$$

When $p = 2$, it is the classic Laplace operator, which is the sum of the second order partial derivatives. For the limit cases, the parameter p goes to 1 and to ∞,

1. **The 1-Laplace operator.** Setting $p = 1$,

$$\begin{aligned}\triangle_1 u :&= \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\\ &= \sum_{i,j=1}^{N}\frac{1}{|\nabla u|}\left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right)u_{x_ix_j}.\end{aligned}$$

2. **The ∞-Laplace operator.** Letting $p \to \infty$,

$$\triangle_\infty u = \sum_{i,j=1}^{N} \frac{1}{|\nabla u|^2} u_{x_i} u_{x_j} u_{x_i x_j}.$$

It is derived formally by dividing the p-Laplace equation $-\triangle_p u = 0$ by $(p-2)|\nabla u|^{p-2}$ then letting p tends to infinity.

Both limit cases have many applications in fields such as image processing, game theory, etc [9, 10]. This paper will also introduce one of the applications in modeling growing sandpile via Laplace operator and Mosco convergence [1].

The notion of Mosco-convergence plays an important role in many aspects of mathematics such as Functional Analysis, Convex Analysis, mathematical modeling [4, 5]. In 1971, Mosco established that the "sequential bicontinuity" of convex conjugate in reflexive spaces: if $\{f_n\}$ is a sequence of closed proper convex functions on a Banach space, then f_n Mosco converges to f if and only if $\{f_n^*\}$ Mosco converges to f^* [3]. In 1977, Attouch established that a sequence of closed proper convex functions Mosco converges to a convex functions if and only if the functions' subdifferentials graph converge to the subdifferential of the limiting function [6, 7]. The definition of **Mosco convergence** is given later in Section 4, and the **graph convergence of the subdifferentials** will be explained in Section 5.

2 Sobolev Space

Some basic facts of the Sobolev space are introduced, the (weak) p-Laplace operator and the energy functionals are also defined in this section.

Let Ω be a bounded open set in $\mathbb{R}^N$ and let $1 \leq p \leq \infty$. With $C_c^1(\Omega)$ denoting functions with compact support and continuous first order derivatives, recall the definition of Sobolev Space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \exists g_i \in L^p(\Omega) \text{ such that } \int_\Omega u\phi_{x_i} = -\int_\Omega g_i \phi \quad \forall \phi \in C_c^1(\Omega) \right\},$$

$$||u||_{W^{1,p}}^p = ||u||_{L^p}^p + \sum_{i=1}^{N} ||u_{x_i}||_{L^p}^p.$$

The following norm is equivalent to the one from above,

$$||u||_{W^{1,p}}^p = \int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx$$

where

$$|\nabla u| = \left(\sum_{i=1}^{N} (u_{x_i})^2 \right)^{\frac{1}{2}}.$$

For $1 < p < \infty$, $W^{1,p}(\Omega)$ is separable, reflexive. The space

$$W_0^{1,p}(\Omega) = \text{ the closure of } C_c^\infty(\Omega) \text{ in the space } W^{1,p}(\Omega)$$

is a closed subspace of $W^{1,p}(\Omega)$, hence it is a Banach Space under the same norm and it is a reflexive space as well.

The dual space $\left(W_0^{1,p}(\Omega)\right)^*$ is denoted by $W^{-1,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. It is equipped with the dual norm

$$||u^*||_* = \sup\left\{\langle u^*, v\rangle \mid v \in W_0^{1,p}(\Omega) \text{ and } ||v||_{W_0^{1,p}} = 1\right\}.$$

With the setup from above, the weak p-Laplace operator ($1 < p < \infty$) "$-A_p$" is an operator from the Sobolev Space $W_0^{1,p}(\Omega)$ to its dual $W^{-1,q}(\Omega)$,

$$-A_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$$

$$\langle -A_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx, \text{ for all } u, v \in W_0^{1,p}(\Omega).$$

$\langle -A_p u, \cdot\rangle$ is linear by construction and

$$\begin{aligned} |\langle -A_p u, v\rangle| = |\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx| &\le \int_\Omega |\nabla u|^{p-1}|\nabla v| \, dx \\ &\le \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla v|^p \, dx\right)^{\frac{1}{p}} \\ &\le ||u||_{W_0^{1,p}}^{p-1} ||v||_{W_0^{1,p}} \quad , \end{aligned}$$

thus $\langle -A_p u, \cdot\rangle \in W^{-1,q}(\Omega)$.

With the following two properties,

1. The space $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ is a Hilbert spaces with the inner product

$$(u, v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, dx;$$

2. $W_0^{1,p_1}(\Omega) \subset W_0^{1,p_2}(\Omega)$ whenever $p_1 > p_2$,

we now define the energy functionals.
For $1 < p < \infty$,

$$E_p : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$$

$$E_p(u) = \begin{cases} \int_\Omega \frac{1}{p}|\nabla u|^p dx & \text{if } u \in H_0^1(\Omega) \cap W_0^{1,p}(\Omega) \\ +\infty & \text{if } u \in H_0^1(\Omega) \backslash W_0^{1,p}(\Omega), \end{cases}$$

and for $p = \infty$,

$$E_\infty : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$$

$$E_\infty(u) = \begin{cases} 0 & \text{if } |\nabla u| \le 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

3 Gâteaux Derivative

Gâteaux derivative is the generalized concept of directional derivative. From different authors, the definition may vary. To be consistent, the following definition will be used through this paper.

Definition. Let X be a Banach space and $f : X \to, \mathbb{R} \cup \{+\infty\}$, for all $v \in X$

$$\delta f(u; v) = \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t} \qquad u \in X$$

where $\delta f(u; v)$ is a linear functional with respect to v, $\delta f(u; \cdot)$ is called the Gâteaux differential of f at u, and the linear operator is called the **Gâteaux derivative.** A function is differeniable in the Gâteaux sense in an open subset $S \subset X$ if it has a Gâteaux derivative at every point of S.

Remark: Here, the Gâteaux derivative is only required to be linear; *it is not necessarily in the dual space.* Some authors require the Gâteaux derivative to be both linear and continuous; some require neither.

Lemma: E_p is Gâteaux differentiable on the subspace $H_0^1(\Omega) \cap W_0^{1,p}(\Omega)$, and

$$\delta E_p(u; \cdot)\Big|_{H_0^1(\Omega) \cap W_0^{1,p}(\Omega)} = \langle -A_p u, \cdot \rangle \qquad u \in H_0^1(\Omega) \cap W_0^{1,p}(\Omega).$$

Proof. The proof related to the Laplace operator has two parts, $2 \leq p < \infty$ and $1 < p < 2$. The Gâteaux differentiability of E_p is followed since the Laplace operator is continuous linear and $\delta E_p(u; v) = \infty$ in $H_0^1(\Omega)/W_0^{1,p}(\Omega)$.

$\mathbf{2 \le p < \infty}$: $H_0^1(\Omega) \cap W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$, with $u, v \in W_0^{1,p}(\Omega)$

$$\begin{aligned}
\delta E_p(u;v) &= \lim_{t\to 0} \frac{E_p(u+tv) - E_p(u)}{t} \\
&= \lim_{t\to 0} \frac{1}{t} \left(\int_\Omega \frac{1}{p} |\nabla(u+tv)|^p dx - \int_\Omega \frac{1}{p} |\nabla u|^p dx \right) \\
&= \lim_{t\to 0} \frac{1}{t} \left(\int_\Omega \int_0^1 \frac{d}{ds} \left[\frac{1}{p} |\nabla(u+stv)|^p \right] ds\, dx \right) \\
&= \lim_{t\to 0} \frac{1}{t} \left(\int_\Omega \int_0^1 \frac{d}{ds} \left[\frac{1}{p} \left(\sum_{i=1}^N (u_{x_i} + stv_{x_i})^2 \right)^{\frac{p}{2}} \right] ds\, dx \right) \\
&= \lim_{t\to 0} \frac{1}{t} \int_\Omega \int_0^1 \frac{1}{p} \frac{p}{2} \left(\sum_{i=1}^N (u_{x_i} + stv_{x_i})^2 \right)^{\frac{p-2}{2}} \left(\sum_{i=1}^N 2(u_{x_i} + stv_{x_i}) tv_{x_i} \right) ds\, dx \\
&= \lim_{t\to 0} \int_\Omega \int_0^1 \underline{\left(\sum_{i=1}^N (u_{x_i} + stv_{x_i})^2 \right)^{\frac{p-2}{2}}} \left(\sum_{i=1}^N (u_{x_i} + stv_{x_i}) v_{x_i} \right) ds\, dx \\
&= \int_\Omega \int_0^1 \left(\sum_{i=1}^N (u_{x_i})^2 \right)^{\frac{p-2}{2}} \left(\sum_{i=1}^N u_{x_i} v_{x_i} \right) ds\, dx \\
&= \int_\Omega \int_0^1 |\nabla u|^{p-2} \nabla u \cdot \nabla v\, ds\, dx \\
&= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v\, dx \\
&= \langle -A_p u, v \rangle
\end{aligned}$$

Remark: The underlined term $\left(\sum_{i=1}^N (u_{x_i} + stv_{x_i})^2\right)^{\frac{p-2}{2}}$ might take the form $0^{\frac{p-2}{2}}$ and is not well defined when $p < 2$.

The passing of the limit is adjusted using Lebesgue Dominated Convergence theorem. Let

$$f_t(x,s) = \left(\sum_{i=1}^N (u_{x_i} + stv_{x_i})^2 \right)^{\frac{p-2}{2}} \left(\sum_{i=1}^N (u_{x_i} + stv_{x_i}) v_{x_i} \right),$$

apply Cauchy-Schwartz inequality to the second product term,

$$\sum_{i=1}^N (u_{x_i} + stv_{x_i}) v_{x_i} \le \left(\sum_{i=1}^N (u_{x_i} + stv_{x_i})^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^N (v_{x_i})^2 \right)^{\frac{1}{2}},$$

then

$$f_t(x,s) \le \left(\sum_{i=1}^N (u_{x_i}+stv_{x_i})^2\right)^{\frac{p-2}{2}} \left(\sum_{i=1}^N (u_{x_i}+stv_{x_i})^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^N (v_{x_i})^2\right)^{\frac{1}{2}}$$
$$= \left(\sum_{i=1}^N (u_{x_i}+stv_{x_i})^2\right)^{\frac{p-1}{2}} \left(\sum_{i=1}^N (v_{x_i})^2\right)^{\frac{1}{2}}$$

Using the fact that

$$|a+b| \le |a|+|b| \le 2\max\{|a|,|b|\}$$
$$|a+b|^c \le 2^c \max\{|a|,|b|\}^c \le 2^c(|a|^c+|b|^c) \quad \forall a,b,c\in\mathbb{R}, c\ge 0,$$
$$\left(\sum_{i=1}^N |a_i|\right)^c \le N^c \sum_{i=1}^N |a_i|^c \quad \forall a_i, c\in\mathbb{R}, c\ge 0,$$

since by assumption, $2 \le p < \infty \;\; \Rightarrow \;\; 0 \le p-1, 0 \le \frac{p-1}{2}$,

$$f_t(x,s) \le \left(\sum_{i=1}^N (u_{x_i}+stv_{x_i})^2\right)^{\frac{p-1}{2}} \left(\sum_{i=1}^N (v_{x_i})^2\right)^{\frac{1}{2}}$$
$$\le N^{\frac{p-1}{2}} \left(\sum_{i=1}^N |u_{x_i}+stv_{x_i}|^{p-1}\right) N^{\frac{1}{2}} \left(\sum_{i=1}^N |v_{x_i}|\right)$$
$$\le N^{\frac{p-1}{2}} \left(\sum_{i=1}^N 2^{p-1}\big(|u_{x_i}|^{p-1}+|stv_{x_i}|^{p-1}\big)\right) N^{\frac{1}{2}} \left(\sum_{i=1}^N |v_{x_i}|\right)$$
$$\le N^{\frac{p}{2}} 2^{p-1} \left(\sum_{i=1}^N \big(|u_{x_i}|^{p-1}+|stv_{x_i}|^{p-1}\big)\right) \left(\sum_{i=1}^N |v_{x_i}|\right),$$

and since $t \to 0$, we can assume $|t| \le 1$. Let us take $t=1$ and define the function

$$g(x,s) := N^{\frac{p}{2}} 2^{p-1} \left(\sum_{i=1}^N \big(|u_{x_i}|^{p-1}+|sv_{x_i}|^{p-1}\big)\right) \left(\sum_{i=1}^N |v_{x_i}|\right)$$
$$= N^{\frac{p}{2}} 2^{p-1} \sum_{i=1}^N \sum_{j=1}^N \big(|u_{x_i}|^{p-1}|v_{x_j}| + |s|^{p-1}|v_{x_i}|^{p-1}|v_{x_j}|\big)$$

We show that g is integrable,

$$\begin{aligned}
&\int_\Omega \int_0^1 g(x,s)\,ds\,dx \\
&= \int_\Omega \int_0^1 N^{\frac{p}{2}} 2^{p-1} \sum_{i=1}^N \sum_{j=1}^N \left(|u_{x_i}|^{p-1} |v_{x_j}| + |s|^{p-1} |v_{x_i}|^{p-1} |v_{x_j}| \right) ds\,dx \\
&= N^{\frac{p}{2}} 2^{p-1} \sum_{i=1}^N \sum_{j=1}^N \int_\Omega \int_0^1 |u_{x_i}|^{p-1} |v_{x_j}| + s^{p-1} |v_{x_i}|^{p-1} |v_{x_j}|\,ds\,dx \\
&= N^{\frac{p}{2}} 2^{p-1} \sum_{i=1}^N \sum_{j=1}^N \int_\Omega |u_{x_i}|^{p-1} |v_{x_j}| + \frac{1}{p} |v_{x_i}|^{p-1} |v_{x_j}|\,dx \\
&= N^{\frac{p}{2}} 2^{p-1} \sum_{i=1}^N \sum_{j=1}^N \left[\int_\Omega |u_{x_i}|^{p-1} |v_{x_j}|\,dx \right] + \frac{1}{p} \left[\int_\Omega |v_{x_i}|^{p-1} |v_{x_j}|\,dx \right],
\end{aligned}$$

Since $u, v \in W_0^{1,p}(\Omega)$, all the integrals over Ω in "[ ]" are finite for $i, j = 1, 2, ..., N$ by Hölder's Inequality,

$$\begin{aligned}
\int_\Omega |u_{x_i}|^{p-1} |v_{x_j}|\,dx &\le \left(\int_\Omega \left(|u_{x_i}|^{p-1} \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |v_{x_j}|^p\,dx \right)^{\frac{1}{p}} \\
&= \left(\int_\Omega |u_{x_i}|^p\,dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |v_{x_j}|^p\,dx \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
\int_\Omega |v_{x_i}|^{p-1} |v_{x_j}|\,dx &\le \left(\int_\Omega \left(|v_{x_i}|^{p-1} \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |v_{x_j}|^p\,dx \right)^{\frac{1}{p}} \\
&= \left(\int_\Omega |v_{x_i}|^p\,dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |v_{x_j}|^p\,dx \right)^{\frac{1}{p}},
\end{aligned}$$

which means

$$N^{\frac{p}{2}} 2^{p-1} \sum_{i=1}^N \sum_{j=1}^N \left[\int_\Omega |u_{x_i}|^{p-1} |v_{x_j}|\,dx \right] + \frac{1}{p} \left[\int_\Omega |v_{x_i}|^{p-1} |v_{x_j}|\,dx \right]$$

is finite, thus g is integrable. From how we defined g, we have $|f_t(x,s)| \le g(x,s)$. By Lebesgue Dominated Convergence Theorem, we can pass the limit $t \to 0$ inside the integral.

$\boldsymbol{1 < p < 2}$, as mentioned before, the term $\left(\sum_{i=1}^N (u_{x_i} + stv_{x_i})^2 \right)^{\frac{p-2}{2}}$ is not well defined when $\left(\sum_{i=1}^N (u_{x_i} + stv_{x_i})^2 \right)^{\frac{1}{2}} = |\nabla(u + stv)| = 0$.

Again, let us look at

$$\lim_{t\to 0}\int_\Omega\int_0^1 \left(\sum_{i=1}^N (u_{x_i}+stv_{x_i})^2\right)^{\frac{p-2}{2}} \left(\sum_{i=1}^N (u_{x_i}+stv_{x_i})v_{x_i}\right) ds\, dx$$

$$=: \lim_{t\to 0}\int_\Omega\int_0^1 f_t(x,s) ds\, dx.$$

In order to solve this problem, we first assume $u \in C_0^1(\Omega)$. Let $\Omega_1, \Omega_2 \subset \Omega$ defined as below

$$\Omega_1 := \{x \in \Omega \ : \ |\nabla u| = 0\}$$

$$\Omega_2 := \{x \in \Omega \ : \ |\nabla u| > 0\}.$$

Since $|\nabla u|$ is the sum and product of measurable functions, it is also a measurable function. The pre-image at zero is a measurable set, thus Ω_1 is measurable, which means Ω_2 is measurable as well.

$$\lim_{t\to 0}\int_\Omega\int_0^1 f_t(x,s)\, ds\, dx = \lim_{t\to 0}\int_{\Omega_1}\int_0^1 f_t(x,s)\, ds\, dx + \lim_{t\to 0}\int_{\Omega_2}\int_0^1 f_t(x,s)\, ds\, dx.$$

The second integral over Ω_2 can be calculated similarly as before, since the bounding function $g \geq 0$,

$$\int_{\Omega_2}\int_0^1 g(x,s)\, ds\, dx \leq \int_\Omega\int_0^1 g(x,s)\, ds\, dx < \infty.$$

For the first integral over Ω_1, $u_{x_i} \equiv 0$, then it becomes

$$\lim_{t\to 0}\int_{\Omega_1}\int_0^1 \left(\sum_{i=1}^N (stv_{x_i})^2\right)^{\frac{p-2}{2}} \left(\sum_{i=1}^N stv_{x_i}^2\right) ds\, dx = \lim_{t\to 0}\int_{\Omega_1}\int_0^1 |st|^{p-2}st\left(\sum_{i=1}^N (v_{x_i})^2\right)^{\frac{p}{2}} ds\, dx$$

Again using Lebesgue Dominated Convergence Theorem,

$$|st|^{p-2}st\left(\sum_{i=1}^N (v_{x_i})^2\right)^{\frac{p}{2}} \leq |st|^{p-1}\left(\sum_{i=1}^N (v_{x_i})^2\right)^{\frac{p}{2}}$$
$$\leq |st|^{p-1}N^{\frac{p}{2}}\left(\sum_{i=1}^N |v_{x_i}|^p\right).$$

Take $|t| = 1$ since $t \to 0$, and define

$$g(x,s) := |s|^{p-1}N^{\frac{p}{2}}\left(\sum_{i=1}^N |v_{x_i}|^p\right).$$

To show $g(x,s)$ is integrable,

$$\begin{aligned}
\int_{\Omega_1}\int_0^1 g(x,s)\,ds\,dx &\le \int_{\Omega}\int_0^1 g(x,s)\,ds\,dx \\
&= \int_{\Omega}\int_0^1 |s|^{p-1}N^{\frac{p}{2}}\left(\sum_{i=1}^N |v_{x_i}|^p\right)\,ds\,dx \\
&= N^{\frac{p}{2}}\sum_{i=1}^N \int_{\Omega}\int_0^1 s^{p-1}|v_{x_i}|^p\,ds\,dx \\
&= N^{\frac{p}{2}}\sum_{i=1}^N \int_{\Omega}\frac{1}{p}|v_{x_i}|^p\,dx = N^{\frac{p}{2}}\frac{1}{p}\sum_{i=1}^N\left[\int_{\Omega}|v_{x_i}|^p\,dx\right].
\end{aligned}$$

Since $v \in W_0^{1,p}(\Omega)$, the above is finite and we could pass the limit inside the integral and get

$$\lim_{t\to 0}\int_{\Omega_1}\int_0^1 f_t(x,s)\,ds\,dx = \int_{\Omega_1}\int_0^1 0\,ds\,dx = 0,$$

which means

$$\delta E(u;v) = 0 = \langle -A_p u, v\rangle \text{ on } \Omega_1$$
$$\delta E(u;v) = \langle -A_p u, v\rangle \text{ on } \Omega_2,$$

thus

$$\delta E(u;v) = \langle -A_p u, v\rangle \text{ on } \Omega.$$

Now in general for any $u \in H_0^1(\Omega)$, let $u_n \in C_0^1(\Omega)$ converge strongly to u in $H_0^1(\Omega)$. We have

$$\langle -A_p u_n, v\rangle \to \langle -A_p u, v\rangle,$$

using the fact that the p-Laplace operator is demicontinuous and the space is reflective.

Here we will give the definition of demicontinuous,

Definition. Let X be a reflexive Banach Space, an operator $A : X \to X^*$ is demicontinuous if $x_n \to x$ strongly in X, then $Ax_n \xrightarrow{w^*} Ax$ weakly* in X^*.

□

4 Mosco Convergence

Definition. Let X be a reflexive Banach space and $f_n : X \to \mathbb{R}\cup\{+\infty\}$. The sequence $\{f_n\}$ **Mosco converges** to f ($f_n \xrightarrow{M} f$) provided for each $x \in X$

$$\begin{cases} \forall x_n \xrightarrow{w} x & f(x) \le \liminf_n f_n(x_n) \\ \exists x_n \xrightarrow{s} x & f(x) \ge \limsup_n f_n(x_n), \end{cases}$$

where "w" and "s" denote the weak and the strong topology of X.

Recall that for $p < \infty$

$$E_p(u) = \begin{cases} \int_\Omega \frac{1}{p} |\nabla u|^p dx & \text{if } u \in H_0^1(\Omega) \cap W_0^{1,p}(\Omega) \\ +\infty & \text{if } u \in H_0^1(\Omega) \backslash W_0^{1,p}(\Omega), \end{cases}$$

and for $p = \infty$,

$$E_\infty : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$$
$$E_\infty(u) = \begin{cases} 0 & \text{if } |\nabla u| \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 4.1. E_p Mosco converges to E_∞.

Before the proof, recall the definition of essential domain and the following diagonalization lemma by H. Attouch [8],

Definition. The **essential domain** of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is the set $dom(f)$, given by

$$dom(f) := \{x \in X \quad : \quad f(x) < +\infty\}.$$

Lemma (Attouch)**.** Let $a_{n,m}$ be a doubly indexed family in $\overline{\mathbb{R}}$. Then, there exists a mapping $n \mapsto m(n)$ increasing to $+\infty$, such that:

$$\limsup_{n\to\infty} a_{n,m(n)} \leq \limsup_{m\to\infty} (\limsup_{n\to\infty} a_{n,m}).$$

Proof. 1. We first show that

$$\forall u \in dom(E_\infty) \ \exists u_p \in dom(E_p) \ : \ u_p \xrightarrow{H_0^1} u \text{ and } E_\infty(u) \geq \limsup_{n\to\infty} E_p(u_p).$$

It suffices to show $\forall u \in dom(E_\infty)$ instead of $\forall u \in H_0^1(\Omega)$ is because that the inequality is trivial for $u \notin dom(E_\infty)$.

First assume $u \in dom(E_\infty) \cap C_0^1(\Omega)$, construct $u_p := u f_p$ for a sequence of smooth functions $\{f_p\}$ with bounded derivatives. By product rule, the derivative is

$$|\nabla u_p| = |(\nabla u)(f_p) + (u)(\nabla f_p)|,$$

and we will see that each $u_p \in dom(E_p)$ since its derivative is bounded by construction. By assumption $u \in dom(E_\infty) \cap C_0^1(\Omega)$, we get $||\nabla u||_\infty = \sup_\Omega |\nabla u| \leq 1$ and $E_\infty(u) = 0$, in order to prove the strong limsup inequality, we need

$$\int_\Omega \frac{1}{p} |(\nabla u)(f_p) + (u)(\nabla f_p)|^p dx \to 0 \text{ as } p \to \infty,$$

thus we want $\big|(\nabla u)(f_p) + (u)(\nabla f_p)\big|^p$ to be bounded.

Now $||\nabla u||_\infty \leq 1 \Rightarrow u$ is bounded. To see this, fix an $x \in \Omega$ such that $u(x) < \infty$, then for each $y \in \Omega$, we have

$$|u(x) - u(y)| \leq ||\nabla u||_\infty |x - y|,$$

since Ω is bounded, $|x - y|$ is finite, thus u must be a bounded function on Ω.

Then we have the following inequality

$$|\nabla u_p| = \Big|(\nabla u)(f_p) + (u)(\nabla f_p)\Big|^p \leq \Big(|f_p| + C|\nabla f_p|\Big)^p,$$

we want $f_p \to 1$ pointwise and $\nabla f_p \to 0$ pointwise as $p \to \infty$. To construct f_p, define g to be a smooth function that takes the value 1 in $B_1(0)$, value 0 in $\mathbb{R}^N/B_2(0)$, and $0 < g < 1$ in $B_2(0)/B_1(0)$ with $0 < ||\nabla g||_\infty < 2$ in $B_2(0)/B_1(0)$. Let $f_p(x) = g(x/p)$, we see that

$$0 \leq f_p \leq 1 \text{ and } |\nabla f_p| \leq \frac{2}{p}.$$

Continue from the inequality above, we have

$$|\nabla u_p| \leq \Big(|f_p| + C|\nabla f_p|\Big)^p \leq \Big(1 + \frac{2C}{p}\Big)^p$$

Thus, we get

$$\begin{aligned}
\limsup_{p\to\infty} E_p(u_p) &= \limsup_{p\to\infty} \int_\Omega \frac{1}{p}\Big|(\nabla u)(f_p) + (u)(\nabla f_p)\Big|^p dx \\
&= \left(\limsup_{p\to\infty} \frac{1}{p}\right)\left(\limsup_{p\to\infty} \int_\Omega \Big|(\nabla u)(f_p) + (u)(\nabla f_p)\Big|^p dx\right) \\
&\leq \left(\limsup_{p\to\infty} \frac{1}{p}\right)\left(\limsup_{p\to\infty} \int_\Omega \left(1 + \frac{2C}{p}\right)^p dx\right) \\
&= \left(\limsup_{p\to\infty} \frac{1}{p}\right)\left(\limsup_{p\to\infty} \left(1 + \frac{2C}{p}\right)^p \int_\Omega 1 dx\right) \\
&= \left(\limsup_{p\to\infty} \frac{1}{p}\right) e^{2C}|\Omega| \\
&= 0 = E_\infty(u).
\end{aligned}$$

To show that $u_p = uf_p \to u$ in $W_0^{1,2}(\Omega)$,

$$\begin{aligned}
\int_\Omega |\nabla u - \nabla u_p|^2 dx &= \int_\Omega |\nabla u - \big((\nabla u)(f_p) + (u)(\nabla f_p)\big)|^2 dx \\
&= \int_\Omega |\nabla u(1 - f_p) - u\nabla f_p|^2 dx \\
&\leq \int_\Omega \Big(|\nabla u(1 - f_p)| + |u\nabla f_p|\Big)^2 dx \\
&\leq \int_\Omega \Big(1(1 - f_p) + C\frac{2}{p}\Big)^2 dx
\end{aligned}$$

By **Bounded Convergent Theorem**,

$$\lim_{p\to\infty}\int_\Omega |\nabla u - \nabla u_p|^2 dx \le \lim_{p\to\infty}\int_\Omega \Big(1(1-f_p)+C\frac{2}{p}\Big)^2 dx = \int_\Omega \lim_{p\to\infty}\Big(1(1-f_p)+C\frac{2}{p}\Big)^2 dx = 0$$

For general $u \in dom(E_\infty)$, first approximate u with $u_n \in dom(E_\infty)\cap C_0^1(\Omega)$ strongly in $H_0^1(\Omega)$, then construct the sequence $u_{p,n}$ in $dom(E_p)\cap C_0^1(\Omega)$ converging to u_n strongly in $H_0^1(\Omega)$ as above. By the diagonalization lemma, we have

$$\begin{aligned}\limsup_{p\to\infty} E_p(u_{p,n(p)}) &\le \lim_{n\to\infty}(\limsup_{p\to\infty} E_p(u_{p,n}))\\ &\le \limsup_{n\to\infty} E_\infty(u_n)\\ &= \limsup_{n\to\infty} 0\\ &= 0 = E_\infty(u),\end{aligned}$$

which is the desired inequality.

2. Now we show that

$$\forall u \in H_0^1(\Omega), \text{whenever } u_p \in dom(E_p) \text{ and } u_p \rightharpoonup u \text{ in } H_0^1(\Omega), \text{ then } \liminf_{p\to\infty} E_p(u_p) \ge E_\infty(u).$$

Once again, it suffices to work with $\forall u_p \in dom(E_p)$ instead of $H_0^1(\Omega)$ because else the inequality becomes trivial.

Let $x \in \Omega$ be a **Lebesgue point** (Lebesgue points are almost everywhere in Ω) for $|\nabla u| \in L^2(\Omega) \subset L^1(\Omega)$, and r sufficiently small such that $B_r(x) \subset \Omega$.

$$\begin{aligned}\int_{B_r(x)} |\nabla u_p| dy &\le \Big(\int_{B_r(x)} |\nabla u_p|^p dy\Big)^{\frac{1}{p}} |B_r(x)|^{\frac{p-1}{p}}\\ &= \Big(p\int_{B_r(x)} \frac{1}{p}|\nabla u_p|^p dy\Big)^{\frac{1}{p}} |B_r(x)|^{\frac{p-1}{p}}\\ &= (pE_p(u_p))^{\frac{1}{p}} |B_r(x)|^{\frac{p-1}{p}}\end{aligned}$$

Now if $\liminf_{p\to\infty} E_p(u_p)$ is not bounded, the proof for the inequality

$$\liminf_{p\to\infty} E_p(u_p) \ge E_\infty(u)$$

is done. Else if $\liminf_{p\to\infty} E_p(u_p) < \infty$, we have $\liminf_{p\to\infty}(pE_p(u_p))^{\frac{1}{p}} \le 1$. And

$$\liminf_{p\to\infty}\int_{B_r(x)} |\nabla u_p| dy \quad\le\quad \liminf_{p\to\infty} (pE_p(u_p))^{\frac{1}{p}} |B_r(x)|^{\frac{p-1}{p}} \quad\le\quad |B_r(x)|.$$

A well known theorem about the boundedness of weakly converging sequences states that suppose

$$\nabla u_p \rightharpoonup \nabla u \text{ in } L^1(\Omega),$$

then

$$||\nabla u||_{L^1} \leq \liminf_{p\to\infty} ||\nabla u_p||_{L^1}.$$

Using this theorem and the fact that the weak convergence in L^2 implies weak convergence in L^1, we have

$$\int_{B_r(x)} |\nabla u| dy \quad \leq \quad \liminf_{p\to\infty} \int_{B_r(x)} |\nabla u_p| dy \quad \leq \quad |B_r(x)|,$$

or

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |\nabla u| dy \leq 1.$$

The **Lebesgue differentiation theorem** states that for almost every point, the value of an integrable function is the limit of infinitesimal averages taken about the point (Lebesgue points are almost everywhere). Taking the limit as $r \to 0$, for almost everywhere

$$|\nabla u(x)| \quad = \quad \lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |\nabla u| dy \quad \leq \quad 1.$$

The desired inequality follows since $u \in dom(E_\infty)$,

$$\liminf_{p\to 0} E_p(u_p) \geq 0 = E_\infty(u).$$

□

5 Subdifferential

Definition. Let $f : X \to (-\infty, +\infty]$ be a convex function defined on a Banach space X, a functional x^* in the dual space X^* is called **subgradient** at $x_0 \in dom(f)$ in X if

$$f(x) - f(x_0) \geq \langle x^*, x - x_0 \rangle \qquad \forall x \in X.$$

The set of all subgradients at x_0 is called the **subdifferential** at x_0 and is denoted with $\partial f(x_0)$. If $f(x_0) = \infty$, $\partial f(x_0) = \emptyset$. The subdifferential can also be seen as an operator $\partial f(\cdot) : X \to 2^{X^*}$.

The reason for studying the subdifferential is because of the following theorem due to Attouch [6],

Theorem 5.1 (Attouch)**.** Let X be a reflexive Banach space and let $f, f_n : X \to \mathbb{R}\cup\{+\infty\}$ be proper lower semicontinuous convex functions. Then the following assertions are equivalent:

(i) $f_n \xrightarrow{M} f$

$$(ii) \begin{cases} \partial f_n \xrightarrow{G} \partial f \\ \text{Normalization condition: there exist } (a, a^*) \in \partial f \text{ and a sequence } (a_n, a_n^*) \in \partial f_n \\ \text{such that } a_n \to a \text{ strongly in } X, a_n^* \to a^* \text{ strongly in } X^*, f(a_n) \to f(a). \end{cases}$$

The **graph convergence of the subdifferentials** is defined as

- for any convergent sequence $\{(a_n, a_n^*) \in X \times X^* \mid a_n^* \in \partial f_n(a_n)\}$ with (a, a^*) as its limit, one has $a \in \partial f(a)$;
- for any (a, a^*) with $a^* \in \partial f(a)$, there exists at least one such sequence $\{(a_n, a_n^*) \in X \times X^* \mid a_n^* \in \partial f_n(a_n)\}$ converging to it.

For $p < \infty$ and $u \in H_0^1(\Omega) \cap W_0^{1,p}(\Omega)$, the Gâteaux derivative $\delta E_p(u; \cdot)$ is not necessarily continuous for $p > 2$. However, when restricting to the subspace, $\delta E_p(u;\cdot)\big|_{H_0^1(\Omega)\cap W_0^{1,p}(\Omega)} = \langle -A_p u, \cdot\rangle$ is a continuous linear functional.

Theorem 5.2 (Hahn-Banach)**.** Let X be a normed vector space, and let Y be a subspace of X. Then any continuous linear functional $u^* \in Y^*$ on Y can be extended to a continuous linear functional $\hat{u}^* \in X^*$ on X. If Y is a dense subspace of X, then there exists a unique element $\hat{u}^* \in X^*$ such that the restriction of $\hat{u}^*$ to Y is u^*. That is, u^* has a unique continuous extension to all of X.

By **Hahn-Banach Theorem**, there should be a unique continuous extension$\langle -\hat{A}_p u, \cdot\rangle \in H^{-1}(\Omega)$ such that

$$\langle -\hat{A}_p u, \cdot\rangle\Big|_{H_0^1(\Omega)\cap W_0^{1,p}(\Omega)} = \langle -A_p u, \cdot\rangle \quad \text{for each } 1 < p < \infty.$$

This is still a work in progress to give an explicit expression for the continuous extension $-\hat{A}_p(u)$. It is easy to show that $-\hat{A}_p u \in \partial E_p(u)$,

$$E_p(v) - f(u) \geq \langle -\hat{A}_p u, v - u\rangle.$$

The right hand side is ∞ whenever $v \in H_0^1(\Omega)/W_0^{1,p}(\Omega)$; for $v \in H_0^1(\Omega) \cap W_0^{1,p}(\Omega)$, by convexity,

$$\begin{aligned} E_p(u + tv) - E_p(u) &= E_p(tu + tv + (1-t)u) - E_p(u) \\ &\leq tE_p(u + v) + (1-t)E_p(u) - E_p(u) \\ &= t\big(E_p(u + v) - E_p(u)\big), \end{aligned}$$

and we have

$$\begin{aligned} \langle -\hat{A}_p u, v\rangle = \langle A_p u, v\rangle = \delta E_p(u; v) &= \lim_{t\to 0} \frac{E_p(u + tv) - E_p(u)}{t} \\ &\leq \lim_{t\to 0} \frac{t\big(E_p(u + v) + E_p(u)\big)}{t} \\ &= E_p(u + v) - E_p(u), \end{aligned}$$

this is just the subdifferential inequality in a different form.

For the infinity case, it is more delicate. The subdifferential of E_∞ is the **normal cone** over the convex set $||\nabla u||_\infty \leq 1$, defined as

$$\partial E_\infty(u) = \{v^* \mid \langle v^*, v - u\rangle \leq 0 \ \ \forall |\nabla v| \leq 1 \text{ a.e. } \}.$$

The connection with $-A_\infty$ is still being studied as part of the future work.

6 Convex Conjugate

Definition. Let X be a reflexive Banach space, and let X^* be its dual space, for the functional

$$f : X \to (-\infty, +\infty],$$

the **convex conjugate**

$$f^* : X^* \to (-\infty, +\infty]$$

is defined by

$$f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}.$$

Recall that Ω is a bounded open set in $\mathbb{R}^N$, and

$$E_p : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$$
$$E_p(u) = \begin{cases} \int_\Omega \frac{1}{p}|\nabla u|^p dx & \text{if } u \in H_0^1(\Omega) \cap W_0^{1,p}(\Omega) \\ +\infty & \text{if } u \in H_0^1(\Omega) \backslash W_0^{1,p}(\Omega) \end{cases}$$

$${E_p}^*(v^*) = \sup_{u \in H_0^1(\Omega)} \{\langle v^*, u\rangle - E_p(u)\}.$$

When $1 < q \leq 2 \leq p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} {E_p}^*(v^*) &= \sup_{u \in H_0^1(\Omega)} \{\langle v^*, u\rangle - E_p(u)\} \\ &= \sup_{u \in H_0^1(\Omega)} \{(v, u)_{H_0^1} - E_p(u)\} \\ &= E_q(v), \end{aligned}$$

where

$$\begin{cases} -A_2 v = v^* \\ -A_p u = v^*. \end{cases}$$

For the calculation, since $H_0^1(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad u, v \in H_0^1(\Omega),$$

the convex conjugate

$$\begin{aligned} {E_p}^* &: H^{-1}(\Omega) \to \mathbb{R} \cup \{+\infty\} \\ {E_p}^*(v^*) &= \sup_{u \in H_0^1(\Omega)} \{\langle v^*, u\rangle - E_p(u)\} \\ &= \sup_{u \in W_0^{1,p}(\Omega)} \left\{\langle v^*, u\rangle - \int_\Omega \frac{1}{p}|\nabla u|^p dx\right\}. \end{aligned}$$

For each fixed v^*, we calculate the Gâteaux derivative of $F(u) = \langle v^*, u\rangle - \int_\Omega \frac{1}{p}|\nabla u|^p dx$. Let $\phi \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}
\delta F(u;\phi) &= \lim_{t\to 0} \frac{F(u+t\phi) - F(u)}{t} \\
&= \lim_{t\to 0} \frac{\langle v^*, u+t\phi\rangle - \langle v^*, u\rangle}{t} - \lim_{t\to 0} \frac{\int_\Omega \frac{1}{p}|\nabla(u+t\phi)|^p - \frac{1}{p}|\nabla u|^p dx}{t} \\
&= \langle v^*, \phi\rangle - \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx \\
&= \langle v^*, \phi\rangle - \langle -A_p u, \phi\rangle,
\end{aligned}$$

and setting $\delta F(u;\phi) = 0$, we get

$$-A_p u = v^*.$$

By **Reisz's Theorem**, for each v^*, there exists a unique $v \in H_0^1(\Omega)$ such that $\langle v^*, \cdot\rangle = (v, \cdot)_{H_0^1}$, that is

$$\langle v^*, u\rangle = \int_\Omega \nabla v \cdot \nabla u dx,$$

which can be seen as

$$\langle v^*, u\rangle = \langle -A_2 v, u\rangle$$

or

$$v^* = -A_2 v.$$

Since $v^* = -A_2 u = -A_p v$, which means $\nabla u = |\nabla v|^{p-2}\nabla v$,

$$\begin{aligned}
\int_\Omega \nabla v \cdot \nabla u - \frac{1}{p}|\nabla u|^p dx &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u - \frac{1}{p}|\nabla u|^p dx \\
&= \int_\Omega |\nabla u|^p - \frac{1}{p}|\nabla u|^p dx \\
&= \int_\Omega \frac{p-1}{p}|\nabla u|^p dx \\
&= \int_\Omega \frac{p-1}{p}\left(|\nabla v|^{\frac{1}{p-1}}\right)^p dx \\
&= \int_\Omega \frac{p-1}{p}|\nabla v|^{\frac{p}{p-1}} dx \\
&= \int_\Omega \frac{1}{q}|\nabla v|^q dx \\
&= E_q(v),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $1 < q \le 2 \le p < \infty$.

The convex conjugate of E_∞ is more delicate and we will only present the basic form from the definition,

$$E_\infty{}^*(v^*) = \sup_{\substack{u\in H_0^1 \\ |\nabla u|\le 1 \text{ a.e.}}} \{\langle v^*, u\rangle\}.$$

Due to a theorem by U. Mosco [3], we know

$$E_p \xrightarrow{M} E_\infty \quad \text{iff} \quad {E_p}^* \xrightarrow{M} {E_\infty}^*.$$

As part of the future work, we will study the convergence of $E_q \xrightarrow{?} E_1$, and especially the connection between ${E_\infty}^*$ and E_1.

7 Application

In this section, we will take a look at the model of growing sandpile studied by M. Bocea, M. Mihailescu, M. Perez-Llanos and J.D. Rossi [1]. Let us look at the following quasilinear parabolic problem

$$\begin{cases} \frac{\partial v_p(t)}{\partial t} - \triangle_p v_p = f(t) & \text{a.e. } t \in (0,T) \\ v_p(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

f is nonnegative, and can be interpreted physically as a source term that adds material to an evolving system whithin which mass particles are continually rearranged by diffusion. Let us consider the following functionals

$$F_p : L^2(\mathbb{R}^N) \to [0, +\infty]$$
$$F_p(u) = \begin{cases} \int_\Omega \frac{1}{p} |\nabla u|^p dx & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N) \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \backslash W^{1,p}(\mathbb{R}^N). \end{cases}$$

The quasilinear parabolic problem above has the standard reformulation

$$\begin{cases} f(t) - \frac{\partial v_p(t)}{\partial t} \in \partial F_p(v_p(t)) & \text{a.e. } t \in (0,T) \\ v_p(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

When u_0 and f satisfy certain conditions, it is shown that there exists a sequence $p \to \infty$ and a limit function v_∞ such that, for each $T > 0$,

$$\begin{cases} v_p \to v_\infty & \text{a.e. and in } L^2((0,T) \times \mathbb{R}^N), \\ Dv_p \rightharpoonup Dv_\infty, v_{p,t} \rightharpoonup v_{\infty,t} & \text{weakly in } L^2((0,T) \times \mathbb{R}^N), \end{cases}$$

where D is the weak derivative. Moreover, v_∞ is a solution to the problem

$$\begin{cases} f(t) - \frac{\partial v_\infty(t)}{\partial t} \in \partial F_\infty(v_\infty(t)) & \text{a.e. } t \in (0,T) \\ v_\infty(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$F_\infty : L^2(\mathbb{R}^N) \to [0, +\infty]$$
$$F_\infty(u) = \begin{cases} 0 & \text{if } |\nabla u| \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

The limit problem governs the movement of a sandpile, with v_∞(x,t) describing the amount of sand at the point x at time t, under the assumption that the sandpile is stable if the slope is less than or equal to one and unstable otherwise.

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