PRICE IMPACT IN THE VIX FUTURES MARKET AND MEAN-FIELD GAMES IN TWO ORDER BOOKS

by

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Abstract

This Ph.D. thesis deals with the price impact in the VIX futures market from a statistical and mathematical perspective. The CBOE volatility index, VIX, is known by investors as the fear index. It was introduced to measure the investors' view on the future expected volatility of the S&P 500 stock index. Investors cannot trade the VIX index directly; however, one can trade VIX futures, which gauge the market's expectation of the 30-day implied volatility. Market volatility spiked on February 8, 2018, drawing wide attention to volatility-based products. On that day, the VIX went up more than 100% in intraday trading. The XIV, one of the VIX-based exchange-traded products (ETPs), dropped more than 80%, triggering an "acceleration event." As a consequence, the XIV issuer had to terminate this product. One of the factors contributing to this event was the architecture of the ETPs written on VIX: a daily contracts rolling where the short-term (mid-term) ETPs roll every day to maintain a weighted average of one month (five months) to expiration. Therefore, a large number of shares is expected to be acquired and liquidated every day before the market closes.

We study the effect of VIX ETPs on the price of VIX futures by investigating the impact curves at different times of the trading day. We find that the impact curve corresponding to the time before market close is the lowest. Our empirical results show that impact curves exhibit a power-law. This is theoretically justified by using dimensional analysis to show that if the immediate price impact is a function of the trade size, it is given by a power function.

We propose a mean-field game framework for the VIX futures market to complement our empirical study, where traders can trade in a regular limit order book (RLOB) and a trade-at-settlement order book (TASLOB). We assume that there are many high-frequency traders (HFTs) in the market, and they trade in both order books. We investigate the case where the number of HFTs tends to infinity. While transactions in RLOB suffer from a temporary price impact, transactions in TASLOB do not, but they trade at an unknown price, the daily settlement price that is only determined at the end of the trading day. We use a extended mean-field games approach where interactions between agents happen through controls instead of states, to solve an optimal liquidation problem in two order books. We solve the optimal liquidation problem for three cases. In the first case, there are only two types of traders: HFTs and noise traders. We introduce the exchange-traded funds' issuer (ETFs) into the market in the second case. They provide a large amount of additional liquidity on one side of the TAS limit order book (TASLOB). In the third case, we solve the optimal liquidation problem of HFT, where ETFs buy a limited number of shares in TASLOB.

Chapter 1

INTRODUCTION

On February 5, 2018, the S&P 500 stock index plunged by roughly 4 percent, and the VIX index moved up most in a single day in the 25 years of index history. The XIV, the exchange-traded note that tracks the inverse daily return of the VIX index, had to be terminated before its actual redemption date due to a greater than 80 percent drop in a single day. Several factors could have contributed to the development. One of them was the architecture of VIX Exchange-traded products (ETPs), where constant-maturity rolling and leveraged exposure needed to be achieved daily. Consequently, a large volume of shares had to be acquired or liquidated before the market closed.

In this thesis, we will study the effect of the VIX ETPs architecture on VIX futures by looking at the immediate price impact of VIX futures at different times of the trading day. Bouchaud (2009) defines price impact as the correlation between an incoming order (to buy or to sell) and the subsequent price change. Price impact is essential in both practical and theoretical studies. In the theoretical studies of mean-field games, both temporary and permanent price impacts are crucial in modeling price formulation. In empirical research, especially algorithmic trading, price impact is critical for designing algorithmic trade execution strategies since their trades also move prices.

Price impact has been studied extensively in the equity market. Mantegna, Lillo and Farmer (2003) studied the price impact of stocks traded in the New York Stock Exchange for four different years, Lim and Coggins (2005) examined the immediate price impact of stocks traded in the Australian Securities Exchange during 2001–2004, and Wilinski, Cui, Brabazon and Hamill (2015) analyzed the price impact of individual trades of stocks in the London Stock Exchange during 2011–2012. They have found that the price impact of individual trades is a concave function of trade size, e.g., $I(\omega) = \alpha \cdot \omega^{\beta}$ where $0 < \beta < 1$ and ω is the trade size, and $I(\omega)$ is the subsequent change in mid-quote resulting from trade. Toth, Lemperiere, Deremble, de Lataillade, Kockelkoren and Bouchaud (2001) discovered by using proprietary data that the price impact of futures contracts follows the square-root law. That is $\Delta(Q) = Y\sigma\sqrt{\frac{Q}{V}}$, where Q is the size of metaorder, $\Delta(Q)$ is the price change between and first and last trade of metaorder, and V is the total traded volume.

The first part of the thesis aims to address three issues. First, we want to examine the immediate price impact of VIX futures using trade and quote data and estimate the possible price impact models out of the available data. Second, we investigate the VIX futures' price impact at different trading times to uncover the effect of the trade from ETFs/ETNs' issuers on VIX futures. Third, we show that the price impact model obtained from empirical data is consistent with the formula obtained theoretically using dimensional analysis with some critical assumptions on the price process.

To complement our empirical study, the second part of this thesis examines the optimal trading strategies of high-frequency traders in the VIX futures market. In this market, traders can trade in two order books, the regular limit order book (RLOB) and the trade-at-settlement limit order book (TALSOB). While transactions in RLOB suffer from a temporary price impact, transactions in TASOB do not. However, trades in TASLOB are transacted at an unknown future price, the daily settlement price that is only determined at the end of the trading day. We assume that there are infinitely many high-frequency traders (HFTs) in the market, and our work will take place in a mean-field game framework where each trader interacts via the mean-field. Specifically, we use the extended mean-field games approach where interactions between agents are through controls instead of states to solve an optimal trading problem in two order books. We also expand our framework to include ETFs/ETNs that provide liquidity on one side of TASLOB.

The next chapter reviews the literature on price impact and the optimal execution problems in both the classical and mean-field game approaches.

Chapter 2

LITERATURE REVIEW

Price impact has been a topic of interest to empirical and theoretical research within the area of finance. Kraus and Stoll (1972) conducted the earliest empirical study of price impact, where they found that block trades affect a temporary price change as well as a change in the underlying value of a stock. Several empirical studies of the aspect of immediate price deviation related to trade size and market liquidity have examined in various stock exchanges, including New York Stock Exchange (NYSE), Australian Securities Exchange (ASX), and London Stock Exchange (LSE).

In this thesis, we want to examine the impact of the transaction size on the price in the VIX futures market. The most relevant previous empirical works are Mantegna, Lillo and Farmer (2003), Lim and Coggins (2005), and Wilinski, Cui, Brabazon and Hamill (2015). Mantegna, Lillo and Farmer (2003) proposed a method to estimate individual price impact by observing the price change as a response to a single trade and measuring time in units of transactions rather than seconds. They measured price impact by taking the difference between the mid-price before and after a single trade on a logarithmic scale. This method is applied to study the 1000 largest firms on NYSE traded between 1995-1998. They analyzed the price impact based on the firm's size by grouping them into 20 groups based on their market capitalization. They find that the slope of price impact as function of the normalized transaction size vary roughly from 0.1–0.5 on a log-log scale. A group with higher transactions and lower market capitalization has a smaller slope than those with lower transactions and higher market capitalization.

Lim and Coggins (2005) studied the price impact of trades in the Australian market where they examined four years of data of the top 300 stocks also sorted by market capitalization. They adapted the idea and methodology from Mantegna, Lillo and Farmer (2003). The difference is the way they defined the normalized transaction size, which is defined as the normalized daily-normalized volume to filter out the effects of intraday liquidity variation where they argue that this filtration makes any comparison between stocks more meaningful. Lim and Coggins (2005) found that a group of stocks with a lower market capitalization has a higher impact than a group of stocks with a higher market capitalization, which is similar to the results found by Mantegna, Lillo and Farmer (2003). However, the impact curves of NYSE stock are increasing throughout the entire transaction size range for all groups. In contrast, the ASX's impact curves show decreasing slightly first and start to grow after a certain transaction size. Lim and Coggins (2005) argued that this is because NYSE is a quote-driven market while ASX is a purely order-driven market. Besides, they find that the relationship between stock market capitalization and its liquidity is non-linear. Wilinski, Cui, Brabazon and Hamill (2015) studied and analyzed the price impact of six stocks traded on the London Stock exchange and investigated whether the impact function is invariant in trading time. They found that the impact curves of the six stocks are consistent with the findings of Lim and Coggins (2005), namely that the curves decrease for small trade sizes and start to increase at about $\omega = 0.1$. They also found that price impact is highest in the first sixty minutes of the trading day and lowest in the last ninety minutes before the market closes.

On the theoretical side, Kyle and Obizhaeva (2016) use the dimensional analysis method commonly used often in physics to obtain scaling laws expressing the general transaction cost function in terms of trading volume and volatility. They find formulas for special cases of the transaction cost function based on leverage neutrality and market microstructure invariance assumptions. They also provide empirical evidence by testing the formula found in one of the cases using Russian and U.S. stock data. Pohl, Ristig, Schachermayer and Tangpi (2018) follow the method presented by Kyle and Obizhaeva (2016) to show that up to a constant, the market impact of a meta-order is proportional to the square root of the size of the meta-order. This result is the consequence of the assumption on the dimensionallity of volatility. However, the empirical analysis is challenging since meta-orders are hard to access. Instead, they apply a similar dimensional analysis method to predict the number of trades per day from the observable variables, including volume of transactions, price of shares, volatility, and cost per trade. In our study, we will adapt the method presented by Pohl, Ristig, Schachermayer and Tangpi (2018) to find the price impact formula in the VIX futures market.

Price impact is the main component of modeling optimal execution problems; thus, the natural step is to investigate the optimal execution problem for traders in the VIX futures market. The optimal execution problem is the problem of finding the best trading strategy that maximizes or minimizes an objective function over a given trading period. The optimal execution problem is critical to traders, especially those who have to acquire or liquidate a large number of shares by a given time in an illiquid market. For instance, a trader who needs to liquidate a large number of shares needs to find the best trading strategy where she needs to trade fast enough to be close to her target price and, at the same time, make sure that her trades do not pressure the price too much as her trading has market impact.

There are many works of literature focusing on optimal execution problems. Bertsimas and Lo (1988) is among the first to solve the optimal execution problem using the dynamic programming principle to obtain the optimal strategies that provide the minimum expected cost of trading over a given period of execution time. Almgren and Chriss (2000) generalized the work of Bertsimas and Lo (1988) by including the variance of cost in the equation. Their objective is to minimize the expected transaction cost arising from the permanent and temporary price impact and volatility risk. Since then, several authors have extended the work of Almgren and Chriss (2000). For example, Cheng, DiGiacinto and Wang (2017) added to Almgren and Chriss (2000)'s model the penalty term that captures the risk of orders to be filled or overfilled at the end of the execution period. They obtained the closed-form expression for the optimal strategies in linear cases, and the classical Almgren–Chriss strategies were recovered in limiting cases. Vaes and Hauser (2018) extended Almgren and Chriss (2000)'s model by solving the optimal acquisition problem that accounts for volume uncertainty as in their model, the volume becomes known only at the end of the execution period. Kyle and Obizhaeva (2016) argued that the usual model of optimal execution problem that includes temporary and permanent price impact fails to capture the intertemporal nature of the supply/demand of security. They overcame the issue by having a dynamic model that captures the supply/demand of security and found that if the trading time is chosen optimally, supply and demand dynamics are the key factor in determining the optimal execution strategy.

Many of the works on execution problems, including Almgren and Chriss (2000) assume that trading is limited to one (traditional) venue. Kratz and Schöneborn (2014) added another trading venue to the optimal liquidation model by including what is known as a dark pool. In a dark pool, the available liquidity is not displayed to the public for this trading venue, so the trade execution is uncertain. A trader may have to wait until a matching order arrives for her order to execute. The problem of optimal liquidation that focuses on alternative trading venues instead of the traditional one was also investigated by Guéant and Lehalle (2015). They studied optimal liquidation problems with limit orders instead of market orders, which had been developed by Bayraktar and Ludkovski (2014). Cartea and Jaimungal (2015) investigated optimal execution problems involving limit and market orders using a target volume schedule strategy proposed by Almgren–Chriss and found that using both limit and market orders outperforms the Almgren–Chriss classical strategy. Readers can find more literature reviews on optimal execution problems in Laruelle and Lehalle (2014), Alfonsi, Fruth and Schied (2010) and Jaimungal and Kinzebulatov (2014) among others.

Most literature reviews on optimal execution problems study optimal trading in a single agent setting. There has been some work investigating multiple agent settings. For instance, Brunnermeier and Pedersen (2005) studied large traders who need to liquidate their position, and other traders know their need to liquidate, and Moallemi, Park and Roy (2012) who considered an agent who aims to liquidate a large position while facing an arbitrageur who tries to profit from her trades. Using standard techniques, the problems that account for multiple agents are challenging and intractable. To overcome the difficulty of multi-agent issues, one can consider solving the optimal execution within the mean-field game framework. A Mean-field game in optimal trading deals with the optimization problem of infinitely many players, where the interaction between players is via the mean-field. Each player's action is only affected by the mean-field of all other players, not by each particular player. The terminology and ideas of mean-field theory model were developed initially by Huang, Malhamé and Caines (2006) and Huang, Caines and Malhamé (2007), and around the same time, similar models were also developed independently by Lasry and Lions (2007). Carmona and Delarue (2018a) and (2018b) provides a comprehensive discussion of mean-field game theory and applications from a probabilistic point of view. After the pioneering works of Huang, Malhamé and Caines (2006) and Lasry and Lions (2007), literature on optimal execution using a mean-field game framework has been expanding, e.g., Gomes, Patrizi and Voskanyan (2014), Guéant, Lasry and Lions (2011), Cardaliaguet and Lehalle (2017), and Huang, Jaimungal and Nourian (2019) among others.

Chapter 3

PRICE IMPACT OF VIX FUTURES

This chapter reviews financial products based on/related to the VIX and the methodology for estimating the price impact of VIX futures from the actual data set. The chapter is organized as follows: section 3.1 gives an introduction to the VIX index and related products. Section 3.2 describes the data set used in this study. Section 3.3 presents the methods used to analyze price impact. Section 3.4 describes the empirical findings. Section 3.5 outlines a method to obtain a price impact formula using dimensional analysis. Section 3.6 summarizes and concludes.

3.1 VIX Index, VIX Futures, and VIX ETPs

The CBOE Volatility Index (VIX Index) was first presented in 1993 as a measurement of the market's expectation of 30-day implied volatility based on the S&P 100 Index (OEX Index). In 2003, the Chicago Board Options Exchange (CBOE) and Goldman Sachs introduced a new method to determine the VIX, which is now based on the S&P 500 index (SPX) to track the expected volatility based on the SPX put and call options over a range of strikes. It was not until March 24, 2004, that the CBOE launched volatility as a tradable asset product, introducing VIX futures as a trading vehicle that allows investors to trade on VIX. In February 2006, CBOE launched another volatility product, VIX options. VIX options are European style options, i.e., they can be exercised only on the expiration date. Many investors trade VIX products as instruments to hedge their portfolios as they generally correlate negatively with stocks.

Nowadays, several other VIX-related products have been introduced. The primary way to trade on VIX is to buy VIX exchange-traded funds (ETFs) and VIX exchange-traded notes (ETNs). However, one has to keep in mind that VIX ETFs and ETNs are not VIX spot; they are the collections of VIX futures that only roughly approximate the performance of VIX. VIX ETFs and ETNs have short-term and mid-term types. The construction of the two types is similar, except for the monthly contracts they hold. The short-term type holds the first- and second-month VIX futures contracts that roll daily, while the mid-term type holds positions in fourth- to seventh-month VIX futures that also roll daily. The most popular VIX exchange-traded product is VXX, the iPath S&P 500 VIX Short-Term Futures ETN issued by Barclays PLC. This product matured/expired on January 30, 2019, and a similar product known as VXXB was issued to replace the expired VXX.

Figure 3.1 and 3.2 demonstrate the daily rolling of VXX shares. Assuming that short-term ETPs compose the nearest two months of VIX futures, on December 17, 2019, the VXX portfolio would contain 100% January 2020 VIX futures contracts. There are 22 trading days before the expiration of January 2020 VIX futures. To shift the portfolio to the next day, VXX should contain a mix of 21/22 of January 2020 VIX futures and 1/22 of February 2020 VIX futures. Figure 3.1 displays the daily position VXX portfolio containing January and February VIX futures between December 17, 2019 - January 22, 2020. Figure 3.2 displays the number of shares of January VIX futures that need to be sold and the number of shares of February VIX futures that need to be bought. These two figures show how many shares the issuer has to buy or sell certain contracts by the end of the day, and the number of these shares is large. Consequently, the trades of these shares should move the price.

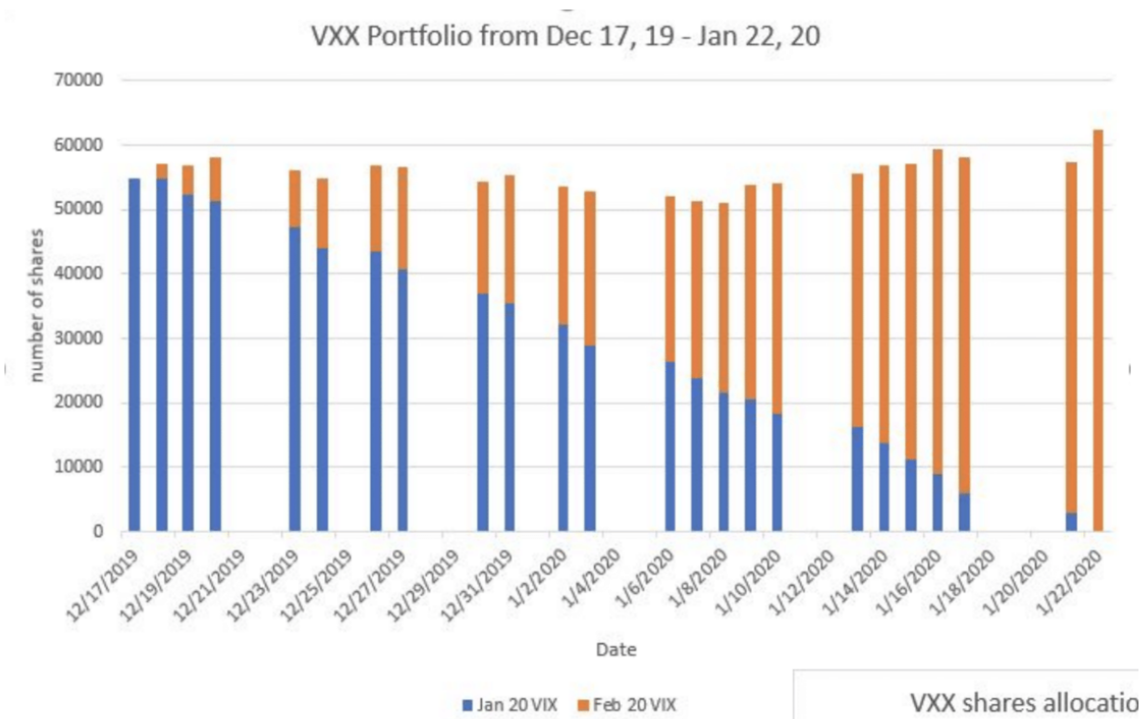

Figure 3.1: VXX portfolio between December 17, 2019 - January 22, 2020.

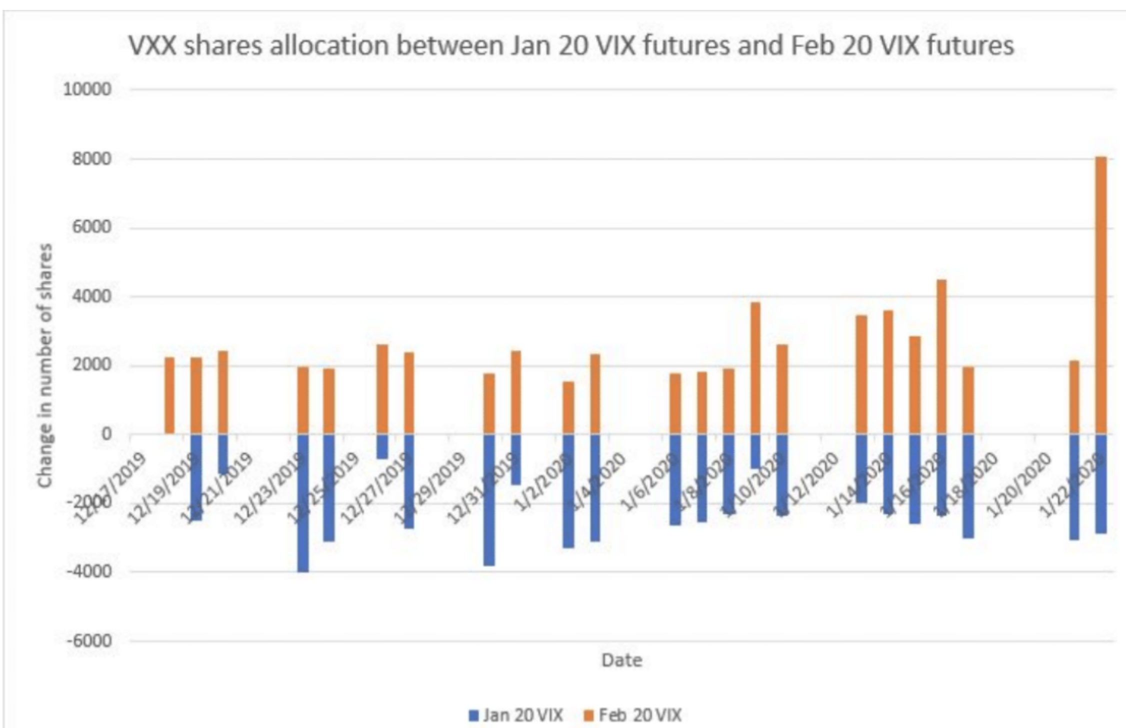

Figure 3.2: The daily VXX shares allocation between December 17, 2019 - January 22, 2020.

3.2 Data

3.2.1 Data

This study is based on Trades and Quotes (TAQ) data (TAQ contains the prices for all transactions as well as the best bid and ask price with timestamp) of June-December 2019 VIX futures contracts obtained from "Quote Recap (QR)" data from BMC. We use these data to investigate immediate price impact of trades of VIX futures by applying method similarly to Mantegna, Lillo and Farmer (2003) used to examine the price impact of stocks in the NYSE. Since futures have an expiration, we treat each month VIX futures contract as one individual asset. Each VIX futures contract is available to trade up to 9 months to expiry. However, the most active trading days are roughly between 2 months to 1-day to expiration. We can see from figure 3.3 that the total daily traded volume of January 2020 VIX futures is slowly increasing and has become more significant within the last two months before the contract expires. We presume that VIX exchange-traded products (ETPs) trade activities contribute to this increase in daily trade volume. Hence we are only interested in the data between two months to a day before expiration. For example, June 2019 VIX futures expired on Wednesday, June 19, 2019. We use June VIX futures traded between April 18, 2019, and June 18, 2019. Besides, investors can trade VIX futures during these sessions: regular trading hours from 8:30 a.m. to 3:00 p.m. and extended trading hours, which are before and after the regular trading hours session, specifically from 5:00 p.m. of the previous day to 8:30 a.m. and from 3:00 p.m. to 4:00 p.m. For our study, we consider only the regular trading session as this is the session where most activity occurs. Thus, all records that are not occur during regular trading session are ignored. Trading that happens on Saturday and Sunday is aso not taken into account. Note that an order by a single party may be traded with multiple counterparties; therefore, trades that occur in succession with the same timestamp will be lumped together as a single trade.

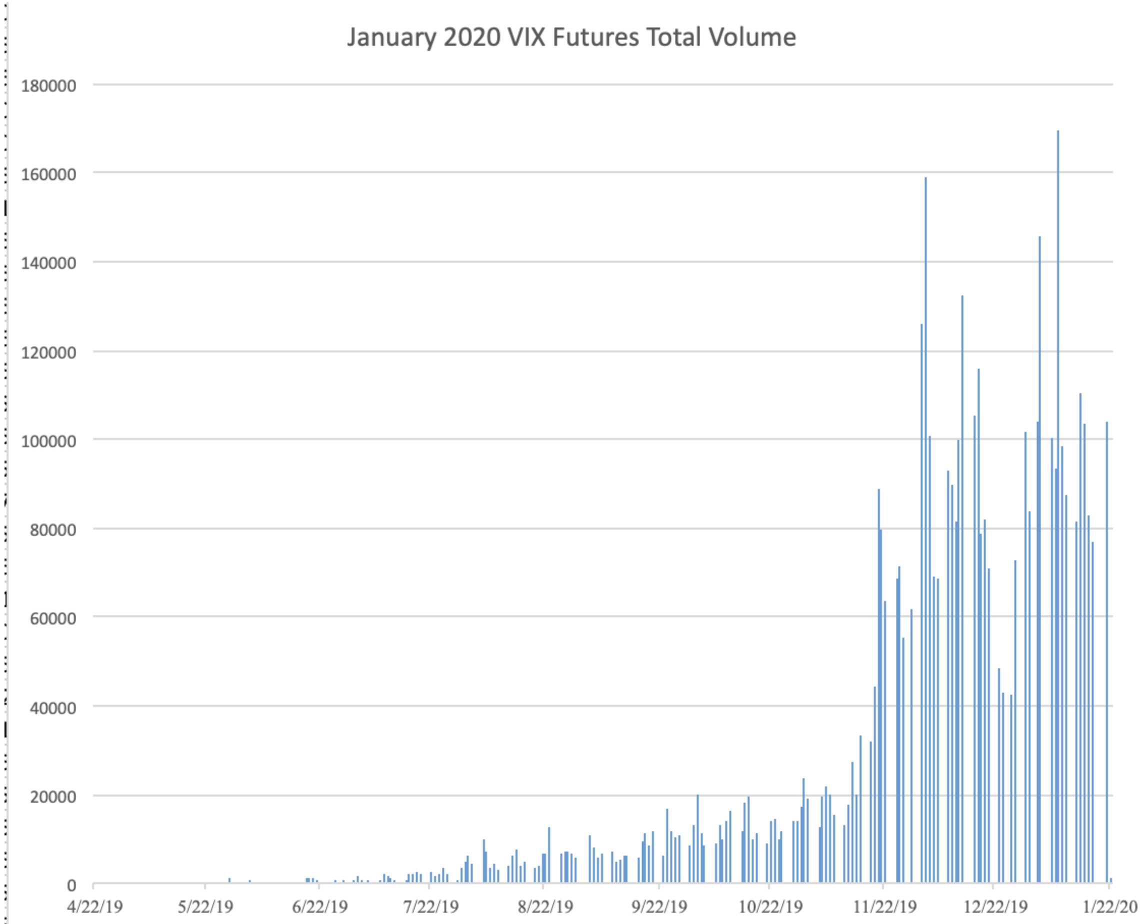

Figure 3.3: January 2020 VIX futures daily total traded volume from April 22, 2019 - January 22, 2020.

3.2.2 Trades Volume Observations

We observe trading volume and transaction size at different times of the trading day before reviewing the methodology for estimating price impact. Figure 3.4 exhibits the histogram of the total traded volume of each fifteen minutes time interval. The histogram displays a U-shape, where a large volume occurs at the beginning and the end of trading hours, and less volume occurs in the middle of the day. Figure 3.5 displays the frequency of transactions with a size greater than 100 shares that happen in the last seventy-five minutes of regular trading hours. Each window corresponds to a fifteen minutes time interval. The highest traded volume occurs fifteen minutes before the market closes, 4:00 p.m. - 4:15 p.m.

EST. Moreover, the last fifteen minutes contain more large-size trades than any other time intervals; some traders have to liquidate/purchase a specified amount of shares before the market closes, so they have to send larger orders to execute.

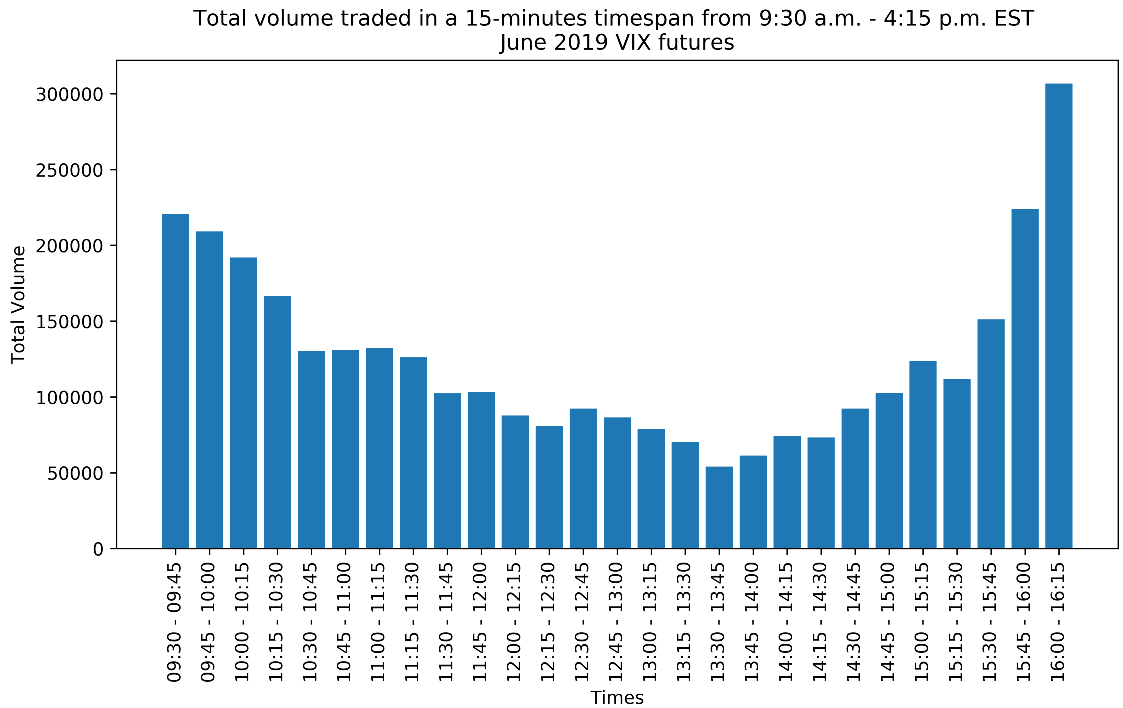

Figure 3.4: June 2019 VIX futures total traded volume in different timespan traded between April 17 - June 18, 2019.

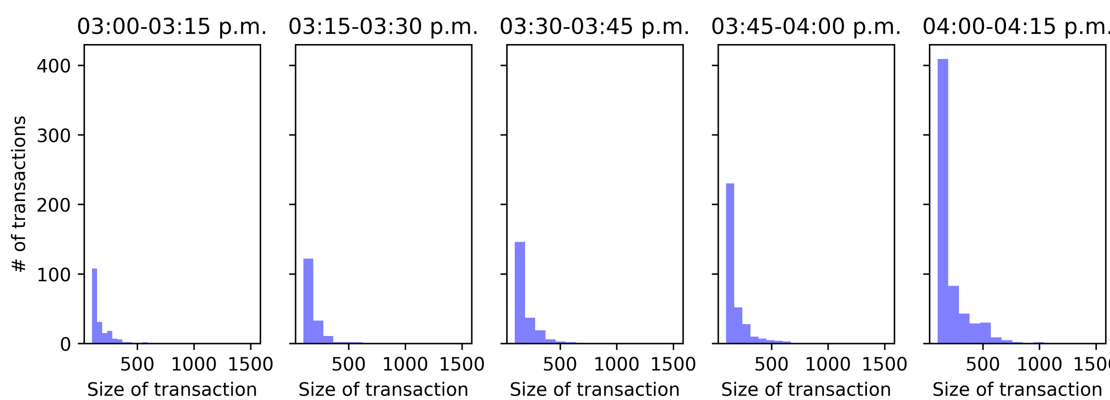

Figure 3.5: The frequency of transactions with size greater than 100 that occur between 3:00 p.m. - 4:15 p.m.

3.3 Method

In this section, we describe the method used to study price impact. We adopt the approach proposed by Mantegna, Lillo and Farmer (2003) for calculating price impact.

3.3.1 Estimating Available Variables

Based on the TAQ data, we can obtain the estimates of three variables; price impact (Δp), transaction size (q), and volatility (σ). We detail the method to calculate each variable next.

Estimate Price Impact (Δp)

Bouchaud (2009) defines price impact as the correlation between an incoming order and the subsequent price change. In this study, we measure price impact, Δp, as the consequence of a trade transaction. That is, for each transaction of volume q trading at time t, if the next event is a quote revision, we define

$$\Delta p(t_{i+1}) = p(t_{i+1}) - p(t_i) \tag{3.1}$$

where $p(t) = \log\left(\frac{BP_t + AP_t}{2}\right)$. BP_t is the best bid price, and AP_t is the best ask price at time t. t_i is the time of pre-trade price, and t_{i+1} is the time immediately before the next trade occurs. We calculate the logarithm of the mid-price as we want to obtain the relative price change resulting from the trade. The intuition is that, if we let $P_t = \frac{BP_t + AP_t}{2}$, then $\Delta p(t_{i+1}) = \log P_{t_{i+1}} - \log P_{t_i} = \log\left(\frac{P_{t_{i+1}}}{P_{t_i}}\right) = \log\left(1 + \frac{P_{t_{i+1}} - P_{t_i}}{P_{t_i}}\right)$.

Daily Normalized Volume (ω)

For each transaction size q, we define the *daily normalized volume* (ω) as

$$\omega = \frac{q}{V} \tag{3.2}$$

where V is the total daily traded volume. Note that in this paper, V is calculated only from data recorded during regular trading hours since we exclude data recorded during extended hours from the analysis.

Daily volatility (σ)

The volatility, σ, is the daily volatility calculated using the method discussed by Kyle and Obizhaeva (2016): it is calculated as the sum of squared one-minute changes in the mid-quote of the best bid and best ask at the end of each minute during the trading hours.

3.3.2 Getting the Impact Curve

Before fitting the impact curve, we define the *normalized impact*, $I(\omega)$, as

$$I(\omega) = \frac{\Delta p}{\sigma}.$$

We want to take into account that some days are significantly more volatile. Thus we divide each relative price impact by the volatility from that day.

We sort the data according to ω and bin them into 20 groups so that each group has approximately the same number of points/transactions. Then we calculate the average value of normalized price impact and the average ω of each group. We denote them as $I(\omega_i)$ and ω_i for $i = 1, ..., 20$ respectively. These 20 points represent the mean impact function for each VIX futures contract.

3.4 Results

We first discuss the price impact function of five months VIX futures contracts from June 2019 to October 2019. Then we examine whether the price impact function differs across the trading day by dividing each day into several time intervals and analyzing the price impact function during each period. The following sections present the results of analyzing price impact functions of the five VIX futures contracts investigated in our study.

3.4.1 Analysis of the Price Impact Function of five VIX futures Contracts

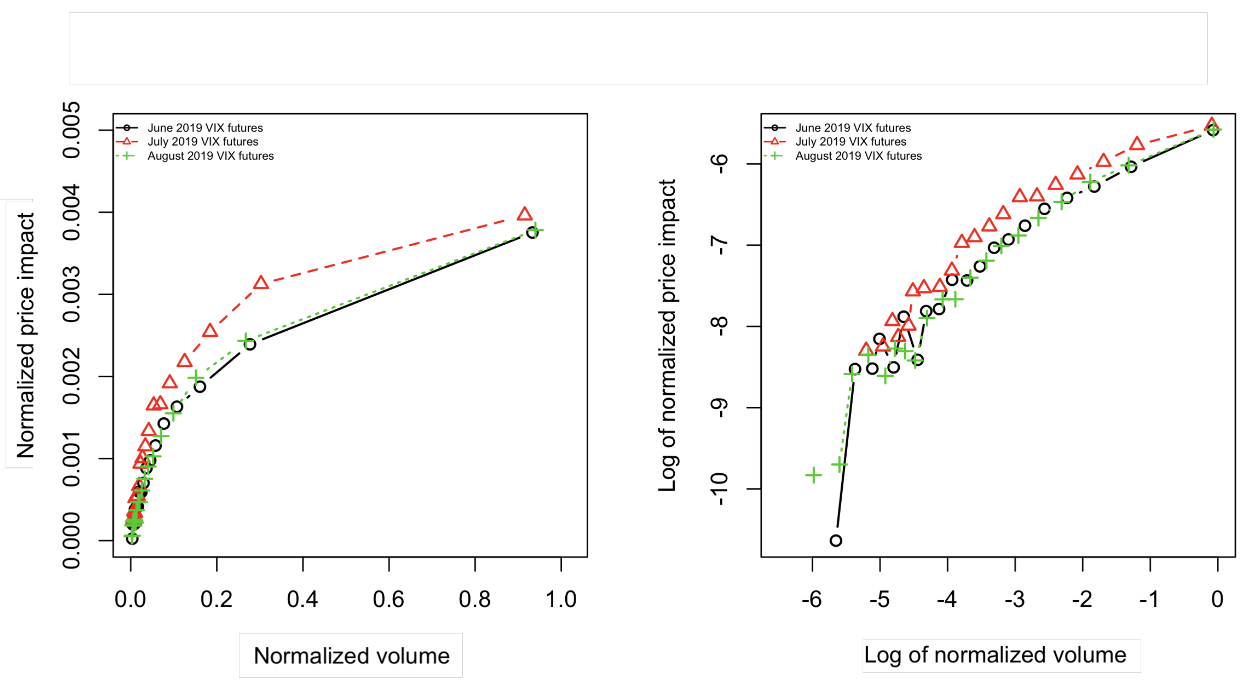

Figure 3.6: The impact curves of June, July, and August 2019 VIX futures traded from 9:30 AM - 4:15 PM EST (left), and the corresponding log-log plot (right).

This section investigates the impact curve of each month VIX futures contract traded two months to one day to maturity. For instance, June VIX futures 2019 expire on Wednesday, June 19, 2019; we use the data of these futures traded between April 18, 2019 - June 18, 2019.

We plot the average price shifts $I(\omega)$ in absolute value against the normalized transaction size (ω). Figure 3.6 shows the impact curves of June, July, and August 2019 VIX futures (three out of five contracts) traded from 9:30 AM - 4:15 PM EST. One can see that all impact curves are concave, which is consistent with the findings for US and Australian stocks in previous studies by Mantegna, Lillo and Farmer (2003); Lim and Coggins (2005); Wilinski, Cui, Brabazon and Hamill (2015). The other two contracts' impact curves(September 2019 and October 2019 VIX futures) also illustrate similar results.

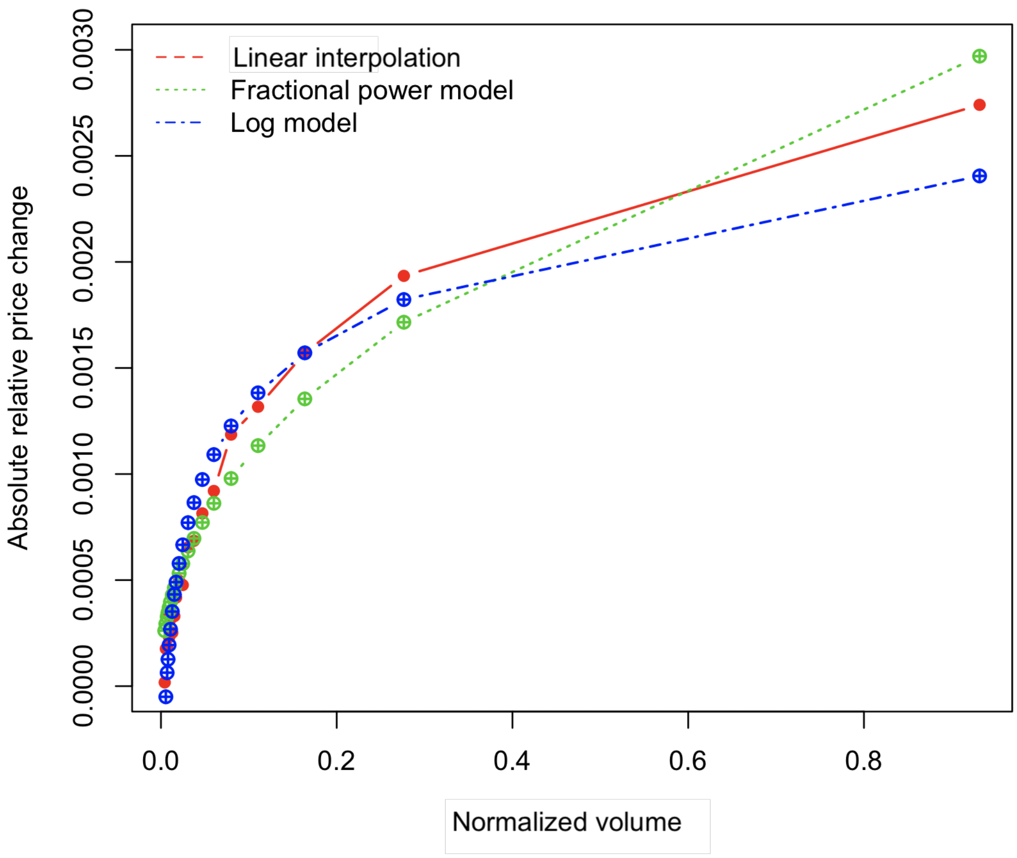

Figure 3.7: Impact curve of June 2019 VIX futures. The red dots are the scatter plot of average absolute price change corresponding to the norm of average normalized volume. The green line is the fitted line of the function $\frac{I(\omega)}{\sigma} = \alpha \cdot (\omega)^{\beta}$. The blue line is the fitted line of the function $\frac{I(\omega)}{\sigma} = a + b \cdot \log(\omega)$. We use the red, blue, and green dots to highlight the difference price impact value corresponding to the same normalized volume when fitting the different models to the data.

2019 VIX futures contracts	α	95 % confident interval of α	β	95 % confident interval of β	Goodness of fit	H value
June	0.0031	(0.0027, 0.0034)	0.4516	(0.4018, 0.5014)	0.9791	0.4491
July	0.0041	(0.0034, 0.0047)	0.3757	(0.3053, 0.4461)	0.9435	0.4528
August	0.0032	(0.0029, 0.0035)	0.4790	(0.4297, 0.5281)	0.9828	0.4582
September	0.0024	(0.0021, 0.0026)	0.4338	(0.3849, 0.4851)	0.9741	0.4514
October	0.0022	(0.0019, 0.0025)	0.3553	(0.2990, 0.4134)	0.9482	0.4843

Table 3.1: *Parameter estimates of price impact function $G = \alpha \cdot \omega^{\beta}$, where G is the normalized absolute price change. ω is the normalized trade volume and the corresponding Hurst value based on 1 minute interval.*

We see from the linear interpolation that connects two red dots (the calculated absolute relative price impact corresponds to the normalized volume) that the impact curve is concave. Thus, we fit the data using the fractional power and log models, and we find that both models fit the data well. We fit the model $\frac{I(\omega)}{\sigma} = \alpha \cdot (\omega)^{\beta}$ (fractional power model) to the June 2019 VIX futures contract data and find that $\alpha = 0.0031$, $\beta = 0.4516$ and the goodness of fit (a statistical test that determines how well a model fits a given set of data) is 0.9791. When fitting the fractional power model to other months VIX futures contracts, we obtain a similar curve with a different value of β's and slightly different value of α's as seen in Table 3.1. In addition to fractional power model, we also fit the log model, $\frac{I(\omega)}{\sigma} = a + b \cdot \log(\omega)$, to the data. The log model also fits the data well with the coefficients $a = 0.00244$, $b = 0.00048$, and the goodness of fit is 0.9749. The blue line in Figure 3.7 illustrates log model fit. Table 3.2 shows the parameter estimates of the log model for the price impact function of the other month's VIX futures.

2019 VIX futures contracts	a	95 % confident interval of a	b	95 % confident interval of b	Goodness of fit
June	0.00244	(0.00223, 0.00264)	0.00048	(0.00043, 0.00053)	0.9749
July	0.00363	(0.00347, 0.00378)	0.00069	(0.00064, 0.00073)	0.9924
August	0.00244	(0.00216, 0.00272)	0.00048	(0.00041, 0.00055)	0.9573
September	0.00193	(0.00180, 0.00206)	0.00038	(0.00034, 0.00041)	0.9833
October	0.00197	(0.00187, 0.00207)	0.00037	(0.00034, 0.00039)	0.9885

Table 3.2: *Parameter estimates of price impact function $G = a + b \cdot \log(\omega)$, where G is the normalized absolute price change. ω is the normalized trade volume.*

3.4.2 Analysis of VIX futures price impact at different times of trading day

We estimate the price impact of trades during the specific trading time interval. We group the data according to their timestamp, and the transaction volume is normalized the same way as discussed earlier. We first group the data into eight groups: 9:30 AM - 10:00 AM, 10:00 AM - 11:00 AM, 11:00 AM - 12:00 PM, 12:00 PM - 1:00 PM, 1:00 PM - 2:00 PM, 2:00 PM - 3:00 PM, 3:00 PM - 4:00 PM, and 4:00 PM - 4:15 PM.

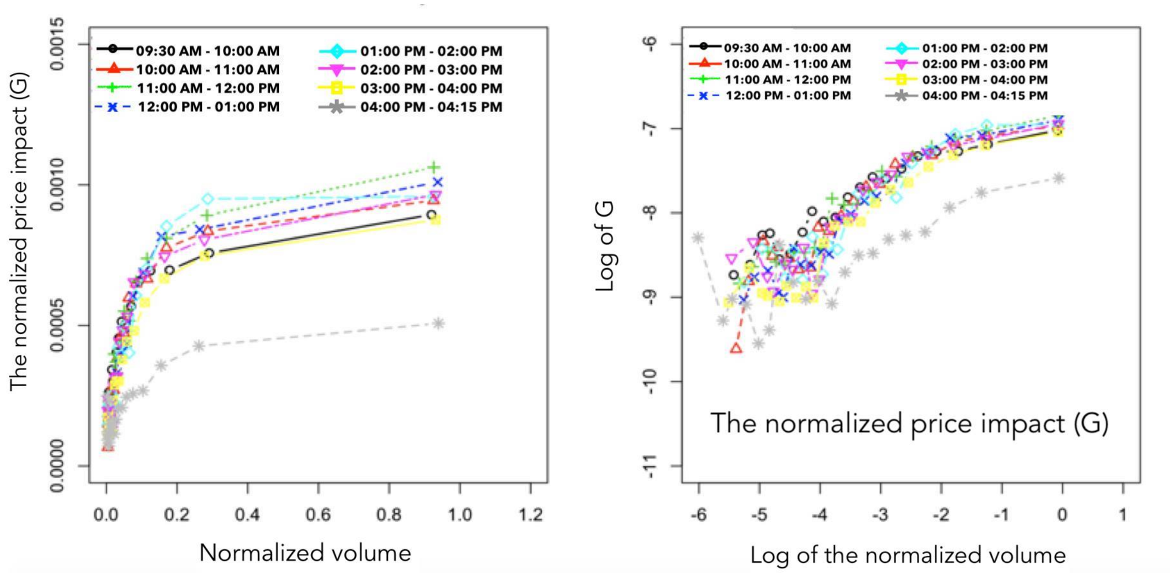

Figure 3.8: Impact curves of June 2019 VIX futures traded at different times of the day. The left plot is the plot of the price impact against the normalized transaction price, and the right plot is the corresponding log-log plot of the left curves.

Figure 3.8 shows the price impact curves for different trading times of the day of June 2019 VIX futures contracts. The reader can find the impact curve of other months contract in Appendix A. The results show that trades between 9:30 AM - 4:00 PM EST have a similar price impact. However, the period after 4:00 PM produces the lowest price impact of all.

We test the significant difference of each curve from the others using model

$$y = b_0 + b_1 x + b_2 1_{x \in A},$$

where A is the specific time interval we want to test. Our hypothesis tested are $H_0 : b_2 = 0$ and $H_1 : b_2 \neq 0$. If $b_2 = 0$, the average impact of trade occurring during A is different from trades during the other time.

VIX futures contracts	b_0 estimate	b_1 estimate	b_2 estimate	p-value for b_2
June 2019 VIX futures	-6.6073	0.4138	-0.4772	2.0e-11
July 2019 VIX futures	-6.3718	0.5417	-0.7042	4.6e-11
August 2019 VIX futures	-6.7585	0.2524	-0.6542	1.9e-11
September 2019 VIX futures	-6.5602	0.4227	-0.4564	1.06e-4
October 2019 VIX futures	-6.4094	0.4435	-0.7113	8.5e-16

Table 3.3: *The estimates of b_0, b_1, b_2 of the last fifteen-minute impact curves of June, July, and August 2019 VIX futures and the p-value resulting from the significant difference test of the last fifteen minutes impact curve from the others.*

The results in table 3.3 show that our test rejects H_0 with p-value almost 0. Thus, the average impact of trade occurring from 4:00 PM - 4:15 PM is significantly lower than in other periods. We argue that this is because ETPs' issuers are actively buying/selling a large amount of VIX futures shares during the last fifteen minutes, consequently yielding high liquidity during this period compared to the other times, which results in a lower impact compared to other trading times.

3.4.3 Price impact of VIX futures vs. stocks

This section compares the immediate price impact of VIX futures and stocks. We select five stocks from different market capitalization sizes for the comparison analysis, Twitter Inc (TWTR), American Airlines Group Inc (AAL), Overstock.com Inc (OSTK), Rite Aid Corporation (RAD), and Vaxart Inc (VXRT). The data of stocks in table 3.4 are extracted from TAQ data from April 6, 2021 – May 13, 2021, trading during regular trading hours (9:30 am – 4:00 pm EST), and for the two VIX futures, we use data of two months – a day before futures expire. We apply the estimating price impact method discussed of Section 3.3 to the chosen stocks and compare the results with June and July 2019 VIX futures. Figure 3.9 displays the impact curves of the five stocks and two VIX futures.

Stocks/VIX futures	Avg. price per share	Avg. daily traded volume
Large cap		
TWTR	$60.83 per shares	18,377,151
AAL	$21.86 per shares	33,650,575
Medium cap		
OSTK	$73.85 per shares	2,257,180
Small cap		
RAD	$18.43 per shares	1,941,356
VXRT	$8.45 per shares	28,302,133
VIX futures		
June 2019 VIX futures	17.09 index points	78,730
July 2019 VIX futures	16.25 index points	54,153

Table 3.4: *Data of five stocks from April 6, 2021 – May 13, 2021 trading during regular trading hours (9:30am – 4:00 pm EST).*

Figure 3.9 displays the impact curves of the five stocks and two VIX futures. By comparing the impact curves between stocks, we find that stocks with larger market capitalization exhibit lower impact curves than those with market capitalization, which is consistent with the result found by Mantegna, Lillo and Farmer (2003). When compared with the impact curves of these stocks with the impact curves of the VIX futures, we find that the impact curves of VIX futures are higher than the impact curves of stocks. The results imply that trades in the VIX futures market have more effect on the price than those in the stock market. We argue the difference in the level of the price impact curves of stocks and VIX futures is the result of the tick size [1]. In the stock market, the tick size is $0.01 per share [2], but in the VIX futures market, the tick size is 0.05 index points or $50 per share. Therefore, it is more expensive to buy and immediately sell a share of VIX futures than a stock share.

[1] The minimum price movement of a trading instrument in a market.

[2] The minimum tick size of $0.01 per share is for stocks over $1.00, while stocks under $1.00 can be quoted in increments of $0.0001 per share. For some less liquid stocks, the increment can be higher than $0.01.

In the next section, we provide the theoretical motivation for why the immediate price impact formula unveils the form of fractional power.

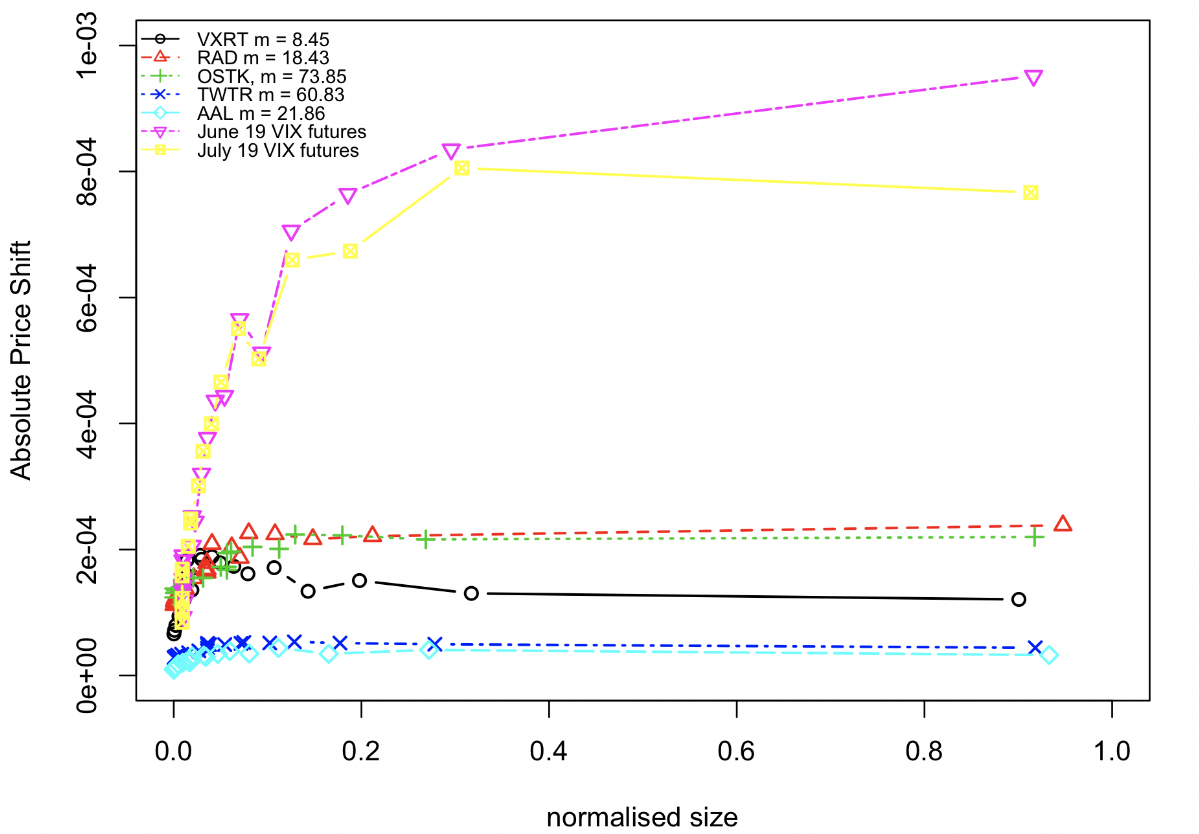

Figure 3.9: Price impact curves of VIX futures and stocks

3.5 Dimensional Analysis for Impact Formula

In this section, we will use the dimensional analysis technique discussed in Pohl, Ristig, Schachermayer and Tangpi (2018) and Kyle and Obizhaeva (2016) to find the relationship between price impact and other variables available from empirical data. Their analysis finds that the market impact is proportional to the square root of the size of meta-order. Our empirical research of VIX futures price impact results shows that the price impact of VIX futures is proportional to the fractional power of the order size, where those fractional power are about 0.3 – 0.5.

One assumption in Pohl, Ristig, Schachermayer and Tangpi (2018) analysis is that the price processes are driven the standard Brownian motion. This assumption leads to the fact that the dimension of σ^2 is $\frac{1}{\mathbb{T}}$. We suspect that the price process in our case may be driven by fractional Brownian motion (fBm) instead of standard Brownian motion. Since fBm scales differently in time compared to standard Brownian motion, it will influence the dimension of σ^2. As a result, we will get a different proportion between price impact and order size.

We follow a similar method discussed in Pohl, Ristig, Schachermayer and Tangpi (2018) with the main difference in the assumption that the price processes are driven by fBm.

3.5.1 Variables and their dimensions

We are interested in explaining the price impact of a given security. So we start with identifying variables and their dimensions ($[\cdot]$) that could influence the size of the market impact. Let

- Q be the size of the individual order, measured in units of shares, $[Q] = \mathbb{S}$,
- P be the price of security, measured in units of money per shares $[P] = \frac{\mathbb{U}}{\mathbb{S}}$,
- V be the traded volume of security, measured in units of shares per time $[V] = \frac{\mathbb{S}}{\mathbb{T}}$,
- $\sigma^2 = (\sigma^2)_t^{t+T} = \mathrm{Var}\Bigg(\log(P_{t+T}) - \log(P_t)\Bigg)$ be the squared volatility of security, and it is assumed to have $[\sigma^2] = \frac{1}{\mathbb{T}^{2H}}$.

The following is why we assume the dimensionality of sigma as above. Consider the model

$$P_t = P_0 e^{\sigma W_t^H} \tag{3.3}$$

where $(W_t^H)_{t\geq 0}$ is a fractional Brownian motion with Hurst parameter H, starting at $W_0^H = 0$, and σ is a constant. To estimate σ^2, we rearrange (3.3) as

$$\begin{aligned}
\log(P_t) - \log(P_0) &= \sigma(W_t^H - W_0^H) \\
\mathrm{Var}\Bigg(\log(P_t) - \log(P_0)\Bigg) &= \sigma^2 \,\mathrm{Var}\left(W_t^H - W_0^H\right) \\
&= \sigma^2 \mathbb{E}\Bigg[\Bigg(W_t^H - W_0^H\Bigg)^2 - \mathbb{E}\Bigg[W_t^H - W_0^H\Bigg]^2\Bigg] \\
&= \sigma^2 t^{2H}.
\end{aligned}$$

Thus,

$$\sigma^2 = \frac{\mathrm{Var}\Bigg(\log(P_t) - \log(P_0)\Bigg)}{t^{2H}}.$$

Since, $\mathrm{Var}\Bigg(\log(P_t) - \log(P_0)\Bigg)$ is dimensionless, we have $[\sigma^2] = \frac{1}{\mathbb{T}^{2H}}$.

Now we have four explanatory variables that could influence price impact and three fundamental dimensions; $\mathbb{S}$, $\mathbb{T}$, and $\mathbb{U}$. Next, we provide a formal definition for dimensional invariance and make a key assumption necessary for further analysis.

Definition 1 (dimensional invariance). A function $g : \mathbb{R}_+^n \to \mathbb{R}_+$ relating a quantity of interest Y to the explanatory variables $X_1, ...X_n$, i.e,

$$Y = g(X_1, ..., X_n),$$

is called dimensionally invariant if it is invariant under rescaling the involved dimensions.

Assumption 1 The market impact G depends only on the above four explanatory variables, i.e.,

$$G = g(Q, P, V, \sigma^2),$$

where the function $g : \mathbb{R}_+^4 \to \mathbb{R}_+$ and the quantity G are invariant under change of the units chosen to measure the dimensions $\mathbb{S}$, $\mathbb{T}$, and $\mathbb{U}$.

Since we have only three equations resulting from the scaling invariance of the dimensions: $\mathbb{S}$, $\mathbb{T}$, and $\mathbb{U}$, and four explanatory variables: Q, P, V, and σ^2, we need one more equation to obtain a unique solution. The remedy to this is to introduce leverage neutrality introduced in this context by Kyle and Obizhaeva (2016). The idea is traced back to Modigliani-Miller equivalence, which tells us which quantities do or do not affect when leverage is changed, and we discussed them in detail next.

We identify which quantities change if there is some change in leverage: denote by $\mathbb{M}$, the Modigliani-Miller dimension. It is a dimension measurement of the leverage L, where $L = \frac{\text{total assets}}{\text{equity}}$. Therefore, $[L] = \mathbb{M}$. Based on the change of leverage, which we make, following Pohl, Ristig, Schachermayer and Tangpi (2018) the assumption:

Assumption 2 (Leverage neutrality) Scaling the Modigliani-Miller dimension $\mathbb{M}$ by a factor $A \in R_+$ implies that

- Q and V remain constant,
- P changes by a factor A^{-1}
- σ^2 change by a factor of A^2
- G changes by a factor A.

To summarize, suppose we multiply leverage by a factor of $A > 1$. This corresponds to paying out a cash dividend of $(1 - \frac{1}{A})P$ per share. This change in leverage does not affect the trading size, Q, and the traded volume, V, of stock. However, after paying the dividend, the stock price will change to $\frac{1}{A}P$ since the value of the share plus dividend has to be conserved, i.e., $(1 - \frac{1}{A})P + \frac{1}{A}P = P$. Each stock share continues to carry the same risk σP. Therefore the standard deviation σ will change to $A\sigma$. For instance, $dP_t = \sigma P_t dW_t = (\sigma A)(\frac{1}{A}P_t)dW_t$.

Thus, the return variance of σ^2 will change to $A^2\sigma^2$. G is measured by the change in price over size of the order. Only P changes by a factor of A, So, G has to multiply A.

Now we have four scaling relations and four explanatory variables. This leads to four linear equations in four unknowns, give us a unique solution. We review the dimensional analysis results, following Bluman and Kumei (1989) Chapter 1.

Assumption 3 (Dimensional Analysis)

(A1) Let u be the quantity that can be determined by n measurable quantities $W_1, W_2, ..., W_n$, i.e.,

$$u = f(W_1, W_2, ..., W_n). \tag{3.4}$$

for some function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$.

(A2) The quantities $u, W_1, W_2, ..., W_n$ are measured in terms of m fundamental dimensions labelled by $L_1, L_2, ..., L_m$.

(A3) For any quantities X, the dimension of X is denoted by [X], and

$$[X] = L_1^{x_1} L_2^{x_2} ... L_m^{x_m}. \tag{3.5}$$

If $[X] = 1$, the quantity X is said to be dimensionless. The dimensions of the quantities $u, W_1, W_2, ..., W_n$ are known and given in the form of vectors; a, the dimension vector of u; and b_i, the dimension vector of $W_i, i = 1, 2, ..., n$. So,

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \quad b_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{mi} \end{bmatrix}.$$

And let

$$B = \begin{bmatrix} b_{1i} & b_{12} & ... & b_{1n} \\ b_{2i} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{mi} & b_{m2} & ... & b_{mn} \end{bmatrix}$$

be the $m \times n$ dimension matrix of the given problem.

(A4) For a given set of fundamental dimensions $L_1, L_2, ..., L_m$, one can choose the system of units to measure quantities. Changing from one system of units to another involves scaling all considered quantities. Scaling does not affect the dimensionless quantity as its value is invariant under arbitrary scaling of the fundamental dimensions.

Theorem (Pi-Theorem)

Under assumptions (A1) – (A4), let $x^{(i)} := (x_{1i}, ..., x_{ni})^T$, $i = 1, ..., k := n - \text{rank}(B)$ be a basis of the solutions to the homogeneous system $Bx = 0$ and $y := (y_1, ..., y_n)^T$ a solution to the inhomogeneous system $By = a$ respectively. Then, there is a function $Q : \mathbb{R}_+^k \to \mathbb{R}_+$ such that

$$u \cdot W_1^{-y_1} \cdots W_n^{-y_n} = Q(\pi_1, ..., \pi_k), \tag{3.6}$$

where $\pi_i := W_1^{x_1} \cdots W_n^{x_n}$ are dimensionless quantities, for i = 1,...,k.

Corollary 1 Under assumption (A1) – (A4), suppose that $\text{rank}(B) = n$ and let $y := (y_2, ..., y_n)^T$ be the unique solution to the linear system $By = a$. Then there is a constant $c > 0$ such that

$$U = c \cdot W_1^{y_1} \cdots W_n^{y_n}.$$

Theorem 1 Under assumption 1 and 2, the market impact is of the form

$$G = c\sigma(\frac{Q}{V})^H \tag{5}$$

	Q	P	V	σ^2	G
$\mathbb{S}$	1	-1	1	0	0
$\mathbb{U}$	0	1	0	0	0
$\mathbb{T}$	0	0	-1	-2H	0
$\mathbb{M}$	0	-1	0	2	1

Table 3.5: *The matrix B related to the dimensions of the quantities Q,P,V,and σ^2, as well as the vector a related to the dimensions of G.*

for some constant $c > 0$.

Proof. By assumption 1 and 2 with the dimensions of Q, P, V and σ^2, we have table 3.3, and the corresponding matrix B and the vector a are

$$B = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2H \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

$$a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Notice that B has full rank, $\text{rank}(B) = 4$. Thus $By = a$ has a unique solution. Solving the system of equation above yield $y^T = (H, 0, -H, \frac{1}{2})$. By corollary 1, we have

$$\begin{aligned} G &= c \cdot Q^{y_1} \cdot P^{y_2} \cdot V^{y_3} \cdot (\sigma^2)^{y_4} \\ &= c \cdot Q^H \cdot P^0 \cdot V^{-H} \cdot (\sigma^2)^{\frac{1}{2}} \\ &= c \cdot \sigma \cdot \left(\frac{Q}{V}\right)^H . \end{aligned}$$

□

Using dimensional analysis, we obtain the relationship that the normalised impact of individual trade is proportional to the fractional power of the normalised trade size. That is

$$\frac{G}{\sigma} = c \cdot \left(\frac{Q}{V}\right)^{H}.$$

Amazingly, this formula agrees with the price impact model of VIX futures we obtained in section 3.4.

3.6 Conclusion

In this study, we examined the price impact on the VIX futures market for futures comprising five months VIX futures contracts. The results show that all impact curves are concave with points cluster together when normalized volumes are small. The period after 4:00 p.m. exhibits the lowest price impact compared to all other curves. Each impact curve fits the model $I(\omega) = \alpha \cdot \omega^{\beta}$ well. We also found that the price impact of VIX futures is the lowest in the last fifteen minutes of regular trading hours. This time concurs with the period where VIX ETPs should rebalance their positions, which results in supplying additional liquidity to the market, making the price impact around this time lower than the other trading time. We deduce that the lower price impact at the end of regular trading time is the evidence of the consequences of the architecture of VIX ETPs on VIX futures.

By following a similar argument as discussed by Kyle and Obizhaeva with the different assumption on the dimension of the σ^2, the general formula for price impact obtained theoretically agrees with our price impact model obtained from the data when setting $c = \alpha$, and $\beta = H$, where H is the Hurst exponent.

Chapter 4

MEAN-FIELD GAMES IN THE VIX FUTURES MARKET

In this chapter, we will discuss the optimal trading of futures by high-frequency traders in the VIX futures market where a trader can buy and sell in two order books in a given interval of time (from $t = 0$ to $t = T$) using a mean-field game framework. For most optimal trading problems, the goal is to optimize a value function of a trader that has three components: the state of the cash account at terminal time T, a terminal execution for the remaining inventory, and the terminal penalty in case the execution target is not met. Our project discusses trading strategies in two limit order books, where traders use both order books to balance their positions. Traders benefit more if they trade in two order books to incorporate more information into their trading strategies. For instance, if one order book becomes illiquid at one point, they have a choice to trade in the other order book. The two order books in the VIX futures market are the regular limit order book (RLOB) and the trade-at-settlement limit order book (TASLOB). Trades that occur in the RLOB will suffer from a temporary price impact. On the other hand, trades in the TASLOB do not suffer from a temporary price impact; however, they are transacted only at maturity for an unknown future price. We want to investigate the optimal trading strategies of traders in this market using the mean-field game framework, where our setup accounts for permanent price impact stemming from all traders in the market. We assume that there are three types of traders in the VIX futures market; high-frequency traders (HFTs) who want to liquidate their positions by the end of the trading day, exchange-traded funds' issuers (ETFs) who have to adjust their inventory daily to meet their objective requirements, and noise traders who buy and sell at random.

This chapter is organized as follows: Section 4.1 gives an overview of VIX futures order books. Section 4.2 investigates the optimal liquidation problem of HFTs using a mean-field game approach in two order books, and we assume that there are only two types of traders in this market, the HFTs, and noise traders. Section 4.3 introduces another type of trader into the market, the exchange-traded funds' issuer (ETFs), providing a large amount of additional liquidity on one side of the TAS limit order book (TASLOB). Section 4.4 uses the mean-field game framework to solve the optimal liquidation problem of HFT in the VIX futures market where ETFs sell a limited number of shares in TASLOB.

4.1 VIX Futures Order Books

The CBOE introduced VIX futures in March 2004, and traders can trade them electronically via a limit order book, which we will call the regular limit order book. In this order book, investors can trade at any time during the trading hours, and they can trade or quote in the regular limit order book at a known futures price. Traders can place an order during the regular trading hours, 8:30 a.m. to 3:15 p.m. CST on Monday to Friday, and during the extended trading hours, 5:00 p.m. (previous day) to 8:30 a.m. CST, and 3:30 p.m. to 4:00 p.m. CST.

On November 4, 2011, the CFE presented an additional VIX futures trading venue, the trade at settlement (TAS) limit order book. In this order book, a trader has an opportunity to trade VIX futures close to the daily settlement price. The price range investors are allowed to place for all types of TAS transactions in TASLOB is called the permissible price range, which is from 0.50 index points below the daily settlement price to 0.50 index points above the daily settlement price. The minimum price increment for TAS transactions is 0.01 index points. Investors trade or quote in the TAS limit order book in reference to an unknown futures price since the daily settlement price is unknown when the trade or quote occurs. Traders can place an order from 5:00 p.m. (previous day) to 2:58 p.m. on Monday-Friday[1]. The market is closed from 2:58 p.m. to 4:45 p.m., and TAS Orders are accepted again from 4:45 p.m. to 5.00 p.m. during Queuing Period[2] (Monday – Thursday). Huskaj and Nordén (2015) found that the TAS transactions account for approximately 10% of the total VIX futures trading volume. The number of transactions in the TAS limit order books is less than 1% of the total number of the VIX futures transaction. This finding indicates that traders utilize the TAS limit order book to execute an order with a large volume. Many institutional investors, such as VIX futures exchange-traded notes and exchange-traded funds' issuer, trade in the TAS limit order book as they have incentives to execute their trades as close to the settlement price as possible (Bayraktar and Ludkovski, 2014).

Figure 4.1 taken from Huskaj and Nordén (2015) illustrates the bid and ask quotes in the regular limit order book and the TAS limit order book on June 6, 2013, for the June VIX futures contract quoted at 3:00 p.m. The regular limit order book displays the liquidity in terms of volume to offer to buy and sell. The best bid (offer to buy) is \$16.65 per contract, and the best ask (offer to sell) is \$16.70 per contract. Thus the (best) bid-ask spread is 0.05, equal to the minimum futures price increment in the regular limit order book. The TAS limit order book also displays the liquidity in volume to offer to buy and sell at 3:00 p.m. The best bid (offer to buy) is 0.02 less than the daily futures settlement price per contract, and

[1]Before October 7, 2020, the daily settlement price of a VIX futures contract is determined at 3:15 p.m. CST, and the trading hours for TAS transactions in VIX futures end at 3:13 p.m. on a normal business day.

[2]The Queuing Period is when the system accepts orders for queuing but not transacting at that time.

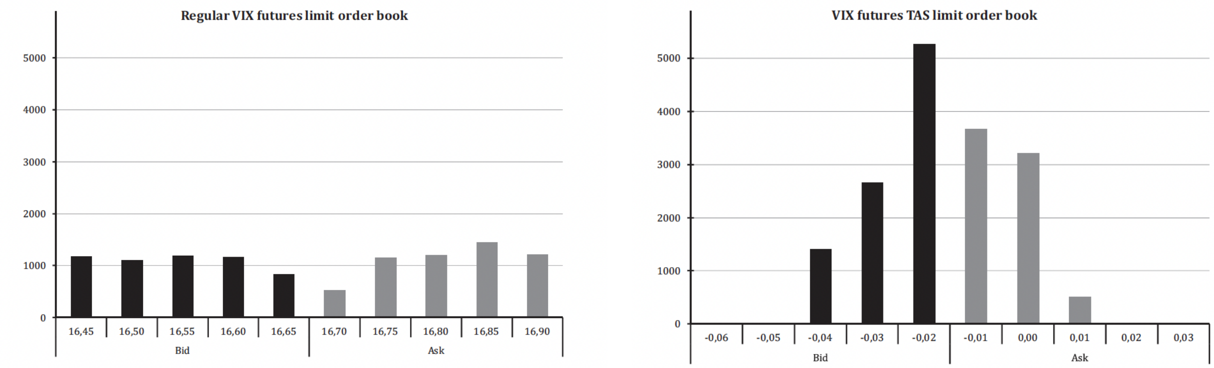

Figure 4.1: The snapshot of the regular limit order book (left), and the TAS limit order book (right), for the June 2013 contract on June 6, 2013 at 3:00 p.m.

the best ask (offer to sell) is 0.01 less than the daily futures settlement price per contract. The best bid-ask spread is 0.01, equal to the minimum price increment allowed in the TAS limit order book.

We are interested in studying the optimal trading strategies of traders in this market, precisely the high-frequency traders (HFTs), in different settings. We use a mean-field game approach to solve the problem instead of the classical one, which we will discuss next.

4.2 Optimal Liquidation in the VIX Futures Market using MFG Approach

In this section, we consider a dynamic system of N high frequency traders (HFTs) in the VIX futures market where each HFT can trade trade in the regular limit order book (RLOB) and the TAS limit order book (TASLOB). We study the problem using a mean-field game approach where we assume that there are two types of traders in the VIX futures market, that is the HFTs and noise traders. Each HFT is either a buy-side or sell-side trader. We use the index i to represent an arbitrary HFT agent from N HFTs. We study how the representative agent i liquidates her initial $Q_i(0)$ shares optimally between $t = 0$, and T in the VIX futures market using both order books.

4.2.1 The Model

We work with the following set-up. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a complete filtered probability space that is rich enough to support all following constructions.

Dynamics in the regular limit order book

We consider a large number of HFTs, indexed by $i \in \{1, ..., N\}$. We will use superscript v to represent action that occurs in the RLOB. In this order book, we assume that the representative agent i trades at rate $v_i(t)$, which can be positive or negative depending on whether the agent is buying or selling, respectively, at time t. Thus the *inventory process* of representative agent i is

$$dQ_i^v(t) = v_i(t)dt, \qquad Q^v(0) = q_0^v.$$

where $i \in \{1, ..., N\}$ and $t \in [0, T]$, and the set of initial inventories of HFTs in the regular order book is $\{Q_i^v(0)\}_{i\in\{1,...,N\}}$. We denote the *fundamental price process* as $F^{\boldsymbol{v}}(t)$ for $t \in [0, T]$, where $\boldsymbol{v} = \{v_1, ..., v_N, w_1, ...w_N\}$ is a vector representing the trading rate of the N agents in the market. Here $\{v_1, ..., v_N\}$ are trading rates of the N agents in the RLOB, and $\{w_1, ..., w_N\}$ are trading rates of the N agents in the TASLOB, which we will discuss later. Thus, the notation $F^{\boldsymbol{v}}$ denotes the dependence of this fundamental price on the trading rates of all agents in the market. We assume that the representative agent i's own trades do not affect the mid-price of the asset. However, the agent's trades have temporary impact on her own *execution price*, the price at which the agent i trades the asset for at time t. When trading at a rate $v_i(t)$ at time t in RLOB, the execution price of agent i reads

$$S_i^v(t) = F^{\boldsymbol{v}}(t) + av_i(t), \qquad t \in [0, T].$$

Here $a > 0$ is the (linear) temporary price impact that the agent i has on the execution price. In general, the execution price process has a bid-ask spread. For simplicity, we assume for the bid-ask spread to be zero, that is, $F^{\boldsymbol{v}}(t)$ represent the best bid price (ask price) if the strategy $v_i(t)$ is to sell (buy). The *cash process* at time t of agent i reads

$$dX_i^v(t) = -S_i^v(t)v_i(t)dt = -\Big(F^{\boldsymbol{v}}(t) + av_i(t)\Big)v_i(t)dt, \qquad t \in [0,T],$$

with initial cash $\{X_i(0)\}_{i\in 1,..,N}$, which usually is assumed zero for each agent. From the cash process above, if the strategy of agent i at time t is to sell the asset, then $v_i(t)$ is negative. Consequently, there will be positive change in cash process. On the other hand, if the strategy of agent i at time t is to buy the asset, then $v_i(t)$ is positive. Therefore, the change in cash process is negative at time t.

Dynamics in the trade at settlement limit order book

In this section, we describe the state processes in the TASLOB. The superscript w is used to represent action occurs in this limit order book. The *inventory process* of representation agent i in the TASLOB is

$$dQ_i^w(t) = w_i(t)dt,$$

where $w_i(t)$ is the trading speed of the representation agent at time $0 \le t \le T$. The set of initial inventories of HFTs in the TAS order book is $\{Q_i^w(0)\}_{i\in\{1,\ldots,N\}}$. A trader in the TASLOB can place the order at the permissible price range around an unknown price, the daily settlement price, with the permissible minimum increment 0.01 index points. Thus, we assume that if a trader wants to buy in the TASLOB, she will buy at δ above the daily settlement price, and if she wants to sell in the TASLOB, she will sell at δ below the daily settlement price. Note that traders in TASLOB do not suffer from temporary price impacts.

Noise Traders

We assume that the average inventory process of noise traders satisfies the SDE

$$dU(t) = \mu(t)dt + \sigma_\mu dW_t^\mu, \qquad t \in [0, T].$$

with the given initial inventory of all noise traders $U(0) = U_0$, μ is the overall trading speed of noise traders, and W^μ is assumed to be a Brownian motion independent of $W^{\mathbb{P}}$, which will be introduced next.

Fundamental Price Dynamics

Since there are two types of traders in the market, HFTs and noise traders, the fundamental price process is assumed to be driven by the actions of these traders. Let $\mathbb{P}$-Brownian motion $W_t^{\mathbb{P}}$ be background noise other than noise traders. The price process is of the form

$$\begin{aligned} dF^{\boldsymbol{v}}(t) &= \lambda\left(\frac{1}{N}\sum_{i=1}^{N} dQ_i(t)\right) + \lambda_\mu dU(t) + \tilde{\sigma} dW_t^{\mathbb{P}}, \qquad t \in [0, T] \\ &= \lambda v^{(N)}(t)dt + \lambda_\mu\Big(\mu(t)dt + \sigma_\mu dW_t^\mu\Big) + \tilde{\sigma} dW_t^{\mathbb{P}}, \\ &= \Big(\lambda v^{(N)}(t) + \lambda_\mu \mu(t)\Big)dt + \sigma dW_t. \end{aligned} \tag{4.1}$$

W_t is a Brownian motion, where $\sigma = \sqrt{\lambda_\mu^2 \sigma_\mu^2 + \tilde{\sigma}^2}$, since $W_t^{\mathbb{P}}$ and W_t^μ are two independent Brownian motions. $F^{\boldsymbol{v}}(0) = f_0$ is the initial condition for the mid-price process, $\lambda > 0$ is HFTs' permanent impact, and $v^{(N)}$ is the average trading speed of N HFTs, μ is the overall trading speed of noise traders, $\lambda_\mu > 0$ is the price impact the noise traders have on the fundamental price process.

We assume

$$\mathbb{E}^{\mathbb{P}}\left[\exp \frac{1}{2}\int_0^T \left(\frac{\lambda v^N(t) + \lambda_\mu \mu}{\sigma}\right) dt\right] < \infty,$$

then by Carmeron-Martin-Girsanov theorem, there exist an equivalent measure $\mathbb{Q}$, and the corresponding $\mathbb{Q}$-Brownian motion $\tilde{W}_t$ such that

$$dF^{\boldsymbol{v}}(t) = \sigma\left(\frac{\lambda v^N(t) + \lambda_\mu \mu}{\sigma} dt + dW_t\right) = \sigma d\tilde{W}_t.$$

Thus, under the $\mathbb{Q}$-measure, the daily settlement price at time t can be written as

$$\mathbb{E}^{\mathbb{Q}}[F^{\boldsymbol{v}}(T)|\mathcal{F}_t] = F^{\boldsymbol{v}}(t). \tag{4.2}$$

Note that traders can buy and sell in TASLOB. If they choose to buy, they must buy at δ above the daily settlement price, and they must sell at δ below the daily settlement price if they sell. Since we consider the optimal liquidation problem; thus, the representative agent i is a sell trader. Then the corresponding execution price at the TASLOB reads

$$S_i^w(t) = \mathbb{E}^{\mathbb{Q}}[F^{\boldsymbol{v}}(T)|\mathcal{F}_t] - \delta \mathbb{1}_{\{w_i(t)<0\}}$$

The corresponding *cash process* in the TASLOB then reads

$$dX_i^w(t) = -\Big(\mathbb{E}^{\mathbb{Q}}[F^{\boldsymbol{v}}(T)|\mathcal{F}_t] - \delta \mathbb{1}_{\{w_i(t)<0\}}\Big) w_i(t)dt.$$

By (4.2), the execution price process in the TASLOB is

$$S_i^w(t) = F^{\boldsymbol{v}}(t) - \delta \mathbb{1}_{\{w_i(t)<0\}},$$

and the corresponding *cash process* in the TASLOB then reads

$$dX_i^w(t) = -\left(F^{\boldsymbol{v}}(t) - \delta \mathbb{1}_{\{w_i(t)<0\}}\right) w_i(t)dt.$$

Here $F^{\boldsymbol{v}}(t) - \delta \mathbb{1}_{\{w_i(t)<0\}}$ represents a sell price at time t. That is the agent has to sell δ less than the mid-price per share in the TASLOB.

As $v^{(N)}$ is the average trading speed of N HFTs, we have

$$v^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}(v_i(t) + w_i(t)).$$

That is, the average trading of HFTs is the average trading rate of actions from both order books as traders can observe both order books from the exchange as discussed in section 4.1. We will use the mean-field game approach to solve the optimal trading problem for HFTs in the VIX futures market. That is in the large N population limit of HFTs, each HFT has an asymptotically negligible impact on the fundamental price. However, the aggregated effect of the trades of all HFTs on the fundamental price is non-negligible.

The Value Function

The aim of agent i is to optimize her wealth on finite-time horizon T, subject to an inventory penalty and trading activity in two order books.

$$J_i(v_i, w_i; v_{-i}, w_{-i}) := \mathbb{E}\left[X_i(T) + Q_i(T)\Big(F^{\boldsymbol{v}}(T) - \beta Q_i(T)\Big) - \psi \int_t^T \Big(v_i(u) + w_i(u)\Big)^2 du\right] \tag{4.3}$$

where $v_{-i} := (v_1, ..., v_{i-1}, v_{i+1}, ..., v_N)$ and $w_{-i} := (w_1, ..., w_{i-1}, w_{i+1}, ..., w_N)$ is the collection of trading rates in RLOB and TASLOB excluding agent i. $Q_i(T)\Big(F^{\boldsymbol{v}}(T) - \beta Q_i(T)\Big)$ is the value of closing her position at the end of the trading horizon, and $\beta > 0$ is the terminal liquidation penalty parameter. The penalty parameter $\psi > 0$ is used to penalize in case agent i does not trade synchronously in two order books. The agent's aim is to maximize the performance criterion over a class of admissible trading strategies $\mathcal{A}$ defined as

$$\mathcal{A} := \left\{\nu \;\middle|\; \nu \quad \text{is} \quad \mathcal{G}-\text{adapted} \quad \& \quad \mathbb{E}\Big[\int_0^T \nu_t^2 dt < \infty\Big]\right\}$$

Here $\mathcal{G} = (\mathcal{G}_t)_{t\in[0,T]}$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma((Q_i(0))_{\{1,...,N\}})$. That is the filtration $\mathcal{G}$ is the filtration $\mathcal{F}$ enlarged by the inventories of all HFTs.

A Mean-Field Game Approach

As mentioned before, the aim of each agent i ($i \in \{1, ..., N\}$) is to optimize the value function (4.3), and this problem can be seen as the stochastic game where the interaction between agents is via controls in which their aggregated control will have an influence on the fundamental asset price dynamics. Therefore, the problem falls within a class of stochastic dynamic games with mean-field couplings in the fundamental asset price dynamics. However, the N-players game is very difficult to solve as each agent need to track the other agents' information, which is usually infeasible. Thus we develop the mean-field game framework to solve this problem. We are interested in large games, where $N \to \infty$. Here we define the mean trading speed of HFTs as $\bar{v} = (\bar{v}(t))_{t\in[0,T]}$

$$\bar{v}(t) := \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (v_i(t) + w_i(t)), \tag{4.4}$$

and assume this limit exists, and the mean inventory of HFT agents corresponding to this mean trading speed is $Q^{\bar{v}} = (Q^{\bar{v}}(t))_{t\in[0,T]}$ where

$$Q(t)^{\bar{v}} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (Q_i^v(t) + Q_i^w(t)). \tag{4.5}$$

Infinite Number of HFTs

Problem: Next we will reformulate the N players game problem into the mean-field game problem by replacing $v^{(N)}$ with $\bar{v}$. Therefore, we obtain the following system:

$$\begin{aligned} dQ_i(t) &:= dQ_i^v(t) + dQ_i^w(t) \\ &= (v_i(t) + w_i(t))dt, \qquad \text{with} \quad Q(0) = q_0^v + q_0^w. \end{aligned} \tag{4.6}$$

$$d\bar{F}(t) = \Big(\lambda\bar{\nu}(t) + \lambda_\mu \mu(t)\Big)dt + \sigma dW_t, \qquad \text{with} \quad \bar{F}(0) = f_0 \tag{4.7}$$

$$\begin{aligned} d\bar{X}_i(t) &:= d\bar{X}_i^v(t) + d\bar{X}_i^w(t) \\ &= -\Big((\bar{F}(t) + av_i(t))v_i(t) - (\bar{F}(t) - \delta \mathbb{1}_{\{w_i(t)<0\}})w_i(t)\Big)dt \\ &\qquad \text{with} \quad \bar{X}(0) = \bar{x}_0. \end{aligned} \tag{4.8}$$

The aim of agent i is to optimize her finite-time horizon wealth, subject to an inventory penalty and trading activity in two order books, in the mean-field game frame work reads.

$$\bar{J}_i(v_i, w_i; \bar{v}) := \mathbb{E}\Big[\bar{X}_i(T) + Q_i(T)\Big(\bar{F}(T) - \beta Q_i(T)\Big) - \psi \int_t^T \Big(v_i(u) + w_i(u)\Big)^2 du\Big] \tag{4.9}$$

Thus the HFT-i's control problem is to obtain

$$\bar{J}_i(\bar{v}) = \sup_{v_i, w_i \in \mathcal{A}} \bar{J}_i(v_i, w_i; \bar{v}).$$

If the supremum is attained in $\mathcal{A}$, then we denote the optimal strategy by (v_i^*, w_i^*).

Solution to Limiting Stochastic Control Problem for HFTs

Solution to Problem: This problem falls within the class of optimal execution problems for a single agent with a time-dependent fundamental price with drift, and permanent impact. For $x_i, f, q_i, y, u \in \mathbb{R}$ and $t \in [0, T]$, we define the performance criterion for a generic HFT agent $i = 1, ..., N$ under the arbitrary trading strategy $(v_i, w_i) \in \mathcal{A}$ as

$$
\begin{aligned}
&H_i^{v_i,w_i,\bar{v}}(t, x_i, f, q_i, y, u) = \\
&\qquad \mathbb{E}_{t,x_i,f,q_i,y,u}\left[\bar{X}_i(T) + Q_i(T)\Big(\bar{F}(T) - \beta Q_i(T)\Big) - \psi\int_t^T \Big(v_i(s) + w_i(s)\Big)^2 ds\right] \\
&\qquad\quad = \mathbb{E}_{t,x_i,f,q_i,y,u}\left[\bar{X}_i(T) + Q_i(T)\Big(\bar{F}(T) - \beta Q_i(T)\Big) + \int_t^T K_i(s, v_i, w_i) ds\right]
\end{aligned}
\tag{4.10}
$$

where $\mathbb{E}_{t,x_i,f,q_i,y}\Big[\cdot\Big] = \mathbb{E}_{t,y}\Big[\cdot\,|\bar{X}_i(t) = x_i, \bar{F}(t) = f, Q_i(t) = q_i, U(t) = u\Big]$.
We define the value function as:

$$
H_i^{\bar{v}}(t, x_i, f, q_i, y, u) := \sup_{v_i, w_i \in \mathcal{A}} H_i^{v_i,w_i,\bar{v}}(t, x_i, f, q_i, y, u) \tag{4.11}
$$

Note that $H_i^{\bar{v}}(t, x_i, f, q_i, y, u)$ is equivalent to $\bar{J}_i(\bar{v})$ in the previouse section. We use the notation H to denote the value function instead of J to highlight the explicit dependence on the state variables. The value function J depends on control variables.

4.2.2 Heuristic solution

This section derives a solution to the problem (4.11) by following the similar arguments to the ones in Huang, Jaimungal and Nourian (2019) and Cartea, Jaimungal and Penalva (2015), Chapter 5. By employing a standard dynamic programming principle and the use of Ito's formula, we obtain the following HJB equation with terminal condition

$$\begin{cases} \partial_t H_i(t,x_i,f,q_i,y,u) + \sup_{v_i,w_i\in\mathcal{A}}(\mathcal{L}^{v_i,w_i}H_i(t,x_i,f,q_i,y,u) & +K_i(v_i,w_i)) = 0 \\ H_i(T,x_i,f,q_i,y,u) & = x_i + q_i f - \beta q_i^2 \end{cases} \tag{4.12}$$

where $K_i(v_i,w_i) = -\psi(v_i+w_i)^2$, and $\mathcal{L}^{v_i,w_i}$ is the infinitesimal generator of $\Big(X_i(t), F(t), Q_i(t), y(t), U(t)\Big)$ and acts as follow:

$$\begin{aligned} \mathcal{L}^{v_i,w_i}H_i = \Big[& -(f+av_i)v_i\partial_{x_i} - (f - \delta\mathbb{1}_{\{w_i<0\}} + \delta\mathbb{1}_{\{w_i\geq 0\}})w_i\partial_{x_i} \\ & + (\lambda\bar{\nu} + \lambda_\mu\mu)\partial_f + (v_i+w_i)\partial_{q_i} + \bar{\nu}\partial_y + \mu\partial_u + \varsigma\Big] H_i \end{aligned}$$

where ς is the operator that encompasses the higher moments of the fundamental price dynamic f in the HJB equation. Note that we obtain the terminal condition directly from (4.10) when $t=T$.

Let

$$L^{v_i,w_i} = \mathcal{L}^{v_i,w_i}H_i + K_i(v_i,w_i).$$

Next, we calculate the optimal control in the feedback form, v_i and w_i.

Since,

$$\begin{aligned} L^{v_i,w_i} = \Big[& -(f+av_i)v_i\partial_x - (f - \delta\mathbb{1}_{\{w_i<0\}} + \delta\mathbb{1}_{\{w_i\geq 0\}})w_i\partial_x \\ & + (\lambda\bar{\nu} + \lambda_\mu\mu)\partial_f + (v_i+w_i)\partial_{q_i} + \bar{\nu}\partial_y + \mu\partial_u + \varsigma\Big] H_i - \psi(v_i+w_i)^2, \end{aligned} \tag{4.13}$$

for a fixed w_i, the optimal control in the feedback form of v_i is

$$v_i^* := \arg\sup_{v_i \in \mathcal{A}} L^{v_i, w_i} = \frac{\partial_{q_i} H_i - f\partial_x H_i - 2\psi w_i}{2a\partial_x H_i + 2\psi}, \tag{4.14}$$

and for a fixed v_i^* the optimal control in the feedback form, w_i

$$w_i^* := \arg\sup_{v_i \in \mathcal{A}} L^{v_i, w_i} = \frac{1}{2\psi}\Bigg(- \Bigg(f - \delta \mathbb{1}_{\{w_i^* < 0\}} + \delta \mathbb{1}_{\{w_i^* \geq 0\}}\Bigg)\partial_{x_i} H + \partial_{q_i} H\Bigg) - v_i^*. \tag{4.15}$$

Substitute (4.15) for w_i in (4.14) yields

$$v_i^* = -\frac{\delta \mathbb{1}_{\{w_i < 0\}} - \delta \mathbb{1}_{\{w_i \geq 0\}}}{2a\partial_x H_i}. \tag{4.16}$$

Next, substitute v^* in (4.16) into (4.15) yields

$$w_i^* = \frac{1}{2\psi}\Bigg(- \Bigg(f - \delta \mathbb{1}_{\{w_i^* < 0\}} + \delta \mathbb{1}_{\{w_i \geq 0\}}\Bigg)\partial_{x_i} H + \partial_{q_i} H\Bigg) + \frac{\delta \mathbb{1}_{\{w_i^* < 0\}}}{2a\partial_x H_i}. \tag{4.17}$$

Substitution of (4.16) and (4.17) into (4.13) gives

$$\begin{aligned} L^{v_i^*, w_i^*} = \Bigg[&- (f + av_i^*)v_i^*\partial_x - \Bigg(f - \delta \mathbb{1}_{\{w_i^* < 0\}} + \delta \mathbb{1}_{\{w_i^* \geq 0\}}\Bigg) w_i^* \partial_x \\ &+ (\lambda\bar{\nu} + \lambda_\mu \mu)\partial_f + (v_i^* + w_i^*)\partial_{q_i} + \bar{\nu}\partial_y + \mu\partial_u + \varsigma \Bigg] H_i - \psi(v_i^* + w_i^*)^2. \end{aligned}$$

Finally, the HJB equation in equation (4.12) becomes

$$\begin{aligned} &\partial_t H_i(t, x_i, f, y, q_i, u) - (f + av_i^*)v_i^*\partial_{x_i} H_i - \Bigg(f - \delta \mathbb{1}_{\{w_i^* < 0\}} + \delta \mathbb{1}_{\{w_i^* \geq 0\}}\Bigg) w_i^* \partial_{x_i} H_i \\ &\quad + (\lambda\bar{\nu} + \lambda_\mu \mu)\partial_f H_i + (v_i^* + w_i^*)\partial_{q_i} H_i + \bar{\nu}\partial_y H_i + \mu\partial_u H_i + \varsigma H_i - \psi(v_i^* + w_i^*)^2 = 0 \end{aligned} \tag{4.18}$$

subject to the terminal condition $H_i(T, x_i, f, y, q_i) = x_i + q_i f - \beta q_i^2$. This terminal condition suggests an ansatz of the form

$$H_i(t, x_i, f, y, q_i, u) = x_i + q_i f + h(t, q_i, y, u), \tag{4.19}$$

subject to the terminal condition $h(T, q_i, y, u) = -\beta q_i^2$. As q_i appears polynomial of quadratic order, we next try the ansatz

$$h(t, q_i, y, u) = h_0(t, y, u) + h_1(t, y, u)q_i + h_2(t)q_i^2, \tag{4.20}$$

subject to the following terminal conditions $h_0(T, y, u) = 0$, $h_1(T, y, u) = 0$, and $h_2(T) = -\beta$. With the ansatz (4.20), v_i^* in (4.16) and w_i^* in (4.17) become

$$v_i^* = -\frac{\delta \mathbb{1}_{\{w_i^*<0\}}}{2a} + \frac{\delta \mathbb{1}_{\{w_i^*\geq 0\}}}{2a}, \tag{4.21}$$

and

$$w_i^* = \frac{1}{2\psi}(\delta \mathbb{1}_{\{w_i^*<0\}} - \delta \mathbb{1}_{\{w_i^*\geq 0\}} + h_1 + 2h_2 q_i) + \frac{\delta \mathbb{1}_{\{w_i^*<0\}}}{2a} - \frac{\delta \mathbb{1}_{\{w_i^*\geq 0\}}}{2a}. \tag{4.22}$$

We see that the optimal trading strategy in RLOB depends only on δ and the temporary price impact in that limit order book. However, the optimal strategy in TASLOB depends also on h_1, and h_2 functions. By substituting v_i^*, w_i^*, and $h(t, q_i, y, u)$ into (4.18) and grouping them in terms of q_i^0, q_i^1, q_i^2 yields

$$\begin{aligned}
\partial_t h_0 + \bar{\nu}\partial_y h_0 + \mu\partial_\mu h_0 + \frac{h_1\delta}{2\psi}\Big(\mathbb{1}_{\{w_i^*(t)<0\}} - \mathbb{1}_{\{w_i^*(t)\geq 0\}}\Big) + \frac{h_1^2}{4\psi} + \delta^2\Big(\frac{1}{4\psi} + \frac{1}{4a}\Big) \\
+ \left\{\partial_t h_1 + \frac{h_1 h_2}{\psi} + \frac{\delta h_2}{\psi}\Big(\mathbb{1}_{\{w_i^*(t)<0\}} - \mathbb{1}_{\{w_i^*(t)\geq 0\}}\Big) + \bar{v}\partial_y h_1 + \mu\partial_\mu h_1 + \lambda\bar{\nu} + \lambda_\mu\mu\right\} q_i \\
+ \left\{\partial_t h_2 + \frac{1}{\psi}h_2^2\right\} q_i^2 = 0.
\end{aligned}$$

Therefore, we obtain the following system of of ODEs to solve:

$$\begin{aligned}
\partial_t h_0 + \bar{\nu}\partial_y h_0 + \mu\partial_\mu h_0 + \frac{h_1\delta}{2\psi}\Big(\mathbb{1}_{\{w_i^*(t)<0\}} - \mathbb{1}_{\{w_i^*(t)\geq 0\}}\Big) + \frac{h_1^2}{4\psi} + \delta^2\Big(\frac{1}{4\psi} + \frac{1}{4a}\Big) &= 0 \\
\text{with} \quad h_0(T,y,u) &= 0, \qquad (4.23) \\
\partial_t h_1 + \frac{h_1 h_2}{\psi} + \frac{\delta h_2}{\psi}\Big(\mathbb{1}_{\{w_i^*(t)<0\}} - \mathbb{1}_{\{w_i^*(t)\geq 0\}}\Big) + \bar{v}\partial_y h_1 + \mu\partial_\mu h_1 + \lambda\bar{\nu} + \lambda_\mu \mu &= 0 \\
\text{with} \quad h_1(T,y,u) &= 0, \qquad (4.24) \\
\partial_t h_2 + \frac{1}{\psi}h_2^2 &= 0 \\
\text{with} \quad h_2(T) &= -\beta. \qquad (4.25)
\end{aligned}$$

Since h_1 depends on the average inventory and the inventory of noise traders, we assume further that it has the form

$$h_1(t,y,u) = h_1^0(t) + h_1^1(t)y + h_1^2(t)u, \tag{4.26}$$

subject to the terminal conditions $h_1^0(T) = h_1^1(T) = h_1^2(T) = 0$.

Therefore, the value function (4.19) reads

$$H_i(t,x_i,f,q_i,y,u) = x_i + q_i f + h_0(t,y,u) + \Big(h_1^0(t) + h_1^1(t)\cdot y + h_1^2(t)\cdot u\Big)\cdot q_i + h_2(t)\cdot q_i^2. \tag{4.27}$$

and the optimal controls (4.21) and (4.22) become

$$v_i^* = -\frac{\delta\mathbb{1}_{\{w_i<0\}}}{2a} + \frac{\delta\mathbb{1}_{\{w_i\geq 0\}}}{2a}, \tag{4.28}$$

and

$$w_i^* = \frac{1}{2\psi}\Big(\delta\mathbb{1}_{\{w_i<0\}} - \delta\mathbb{1}_{\{w_i\geq 0\}} + h_1^0(t) + h_1^1(t)y + h_1^2(t)u + 2h_2 q_i\Big) + \frac{\delta\mathbb{1}_{\{w_i<0\}}}{2a} - \frac{\delta\mathbb{1}_{\{w_i\geq 0\}}}{2a}. \tag{4.29}$$

Mean Field Game (MFG) Consistency condition

We define the difference in the percentage of sellers and buyers in the VIX futures market as

$$\eta := \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{w_i<0\}} - \mathbb{1}_{\{w_i\geq 0\}}. \tag{4.30}$$

We assume in this project that the limit exists, and each trader is on only one side of the market. That is if she is a buy-side in the market, she will not change to be a sell-side. Thus η does not depend on t, as the number of buyers or sellers in this market are already decided at time $t = 0$. Note that η ranges between -1 and 1. If η takes -1, then all HFTs are buyers, and if η takes 1, then all HFTs are sellers. If $\eta = 0$, this implies that the percentage of HFT buyers is the same as HFT sellers. By (4.4) and (4.5), we have

$$\bar{\nu}(t) = \lim_{N\to\infty} \sum_{i=1}^{N} \Big(v_i^*(t) + w_i^*(t)\Big) = \frac{1}{2\psi}\Big(\delta\eta + h_1^0(t) + h_1^2(t)u(t)\Big) + \frac{1}{2\psi}\Big(h_1^1(t) + 2h_2(t)\Big)y(t). \tag{4.31}$$

Since $w_i^*(t)$ depends on h_1 and h_2, we have to solve for h_1 in equation (4.24). To do that we plug in $\bar{\nu}(t)$ and (4.26) into (4.24) and then group them in term of y, and u we have

$$\begin{aligned}\partial_t h_1^0 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_1^1}{2\psi} - \frac{h_1^2}{2\psi} + \frac{h_2}{\psi}\Big)h_1^0 + \frac{\delta}{2\psi}\Big(\mathbb{1}_{\{w_i(t)<0\}} - \mathbb{1}_{\{w_i(t)\geq 0\}}\Big) + \frac{\delta\eta}{2\psi}\Big(\lambda - \lambda_\mu + h_1^1 - h_1^2\Big)\\ + \left\{\partial_t h_1^1 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{2h_2}{\psi} - \frac{h_1^2}{2\psi}\Big)h_1^1 + \frac{1}{2\psi}(h_1^1)^2 + \frac{1}{\psi}\Big(\lambda h_2 - \lambda_\mu h_2 - h_2 h_1^2\Big)\right\} y\\ + \left\{\partial_t h_1^2 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_2}{\psi} + \frac{h_1^1}{2\psi}\Big)h_1^2 - \frac{1}{2\psi}(h_1^2)^2\right\} u = 0,\end{aligned} \tag{4.32}$$

which, by the comparison of coefficients, yields the following system of ODEs

$$\begin{aligned}\partial_t h_1^0 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_1^1}{2\psi} - \frac{h_1^2}{2\psi} + \frac{h_2}{\psi}\Big)h_1^0 + \frac{\delta}{2\psi}\Big(\mathbb{1}_{\{w_i(t)<0\}} - \mathbb{1}_{\{w_i(t)\geq 0\}}\Big) + \frac{\delta\eta}{2\psi}\Big(\lambda - \lambda_\mu + h_1^1 - h_1^2\Big) = 0,\\ \partial_t h_1^1 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{2h_2}{\psi} - \frac{h_1^2}{2\psi}\Big)h_1^1 + \frac{1}{2\psi}(h_1^1)^2 + \frac{1}{\psi}\Big(\lambda h_2 - \lambda_\mu h_2 - h_2 h_1^2\Big) = 0,\\ \partial_t h_1^2 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_2}{\psi} + \frac{h_1^1}{2\psi}\Big)h_1^2 - \frac{1}{2\psi}(h_1^2)^2 = 0,\end{aligned}$$

with $h_1^0(T) = h_1^1(T) = h_1^2(T) = 0$. We are interested in the optimal liquidation problem where agent i aims to liquidate her inventory; thus, we are only interested in the region where $w_i(t) < 0$. Therefore, we have the following ODEs.

$$\partial_t h_1^0 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_1^1}{2\psi} - \frac{h_1^2}{2\psi} + \frac{h_2}{\psi}\Big)h_1^0 + \frac{\delta}{2\psi} + \frac{\delta\eta}{2\psi}\Big(\lambda - \lambda_\mu + h_1^1 - h_1^2\Big) = 0, \quad h_1^0(T) = 0, \tag{4.33}$$

$$\partial_t h_1^1 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{2h_2}{\psi} - \frac{h_1^2}{2\psi}\Big)h_1^1 + \frac{1}{2\psi}(h_1^1)^2 + \frac{1}{\psi}\Big(\lambda h_2 - \lambda_\mu h_2 - h_2 h_1^2\Big) = 0, \quad h_1^1(T) = 0, \tag{4.34}$$

$$\partial_t h_1^2 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_2}{\psi} + \frac{h_1^1}{2\psi}\Big)h_1^2 - \frac{1}{2\psi}(h_1^2)^2 = 0, \quad h_1^2(T) = 0 \tag{4.35}$$

The system (4.33) – (4.35) is a system of coupled Riccati equations, which is difficult to solve in general. However, (4.35) is a homogeneous differential equation subject to the terminal condition $h_1^2(T) = 0$, thus

$$h_1^2(t) = 0 \qquad \text{for} \qquad 0 \le t \le T. \tag{4.36}$$

Solving the ODE (4.25) for $h_2(t)$, we obtain

$$h_2(t) = -\frac{\psi}{T - t + \frac{\psi}{\beta}} \qquad \text{for} \qquad 0 \le t \le T. \tag{4.37}$$

Substitution (4.36) and (4.37) into (4.34) yields the following Riccati equation

$$\partial_t h_1^1 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} - \frac{2}{T - t + \frac{\psi}{\beta}}\Big)h_1^1 + \frac{1}{2\psi}(h_1^1)^2 - \frac{(\lambda-\lambda_\mu)}{T - t + \frac{\psi}{\beta}} = 0 \tag{4.38}$$

subject to terminal condition $h_1^1(T) = 0$. The solution to (4.38) obtained using the Mathematica reads

$$h_1^1(t) = \begin{cases} -\frac{(\lambda-\lambda_\mu)}{2} + \frac{2\psi}{T-t+\frac{\psi}{\beta}} + \frac{(\lambda-\lambda_\mu)}{2}\tanh(At + B_1), & \text{for} \quad \beta \le \frac{(\lambda-\lambda_\mu)}{2} \\ -\frac{(\lambda-\lambda_\mu)}{2} + \frac{2\psi}{T-t+\frac{\psi}{\beta}} + \frac{(\lambda-\lambda_\mu)}{2\tanh(At+B_2)}, & \text{for} \quad \beta > \frac{(\lambda-\lambda_\mu)}{2} \end{cases} \tag{4.39}$$

where

$$
\begin{aligned}
A &:= \frac{(\lambda - \lambda_\mu)}{4\psi}, \\
B_1 &:= -\frac{(\lambda - \lambda_\mu)}{4\psi}T + \tanh^{-1}(1 - \frac{4\beta}{(\lambda - \lambda_\mu)}), \\
B_2 &:= -\frac{(\lambda - \lambda_\mu)}{4\psi}T + \frac{1}{2}\log(1 - \frac{(\lambda - \lambda_\mu)}{2\beta}).
\end{aligned}
$$

Once h_1^1 is determined, we can inject it (4.39) into the ODE (4.33) which yields:

$$
\partial_t h_1^0(t) = \begin{cases} A(1 + \tanh(At + B_1))h_1^0(t) + \delta A\eta(1 + \tanh(At + B_1)) + \frac{\delta(1-\eta)}{T-t+\frac{\psi}{\beta}}, & \text{for} \quad \beta \le \frac{(\lambda-\lambda_\mu)}{2} \\ A(1 + \frac{1}{\tanh(At+B_2)})h_1^0(t) + \delta A\eta(1 + \frac{1}{\tanh(At+B_2)}) + \frac{\delta(1-\eta)}{T-t+\frac{\psi}{\beta}}, & \text{for} \quad \beta > \frac{(\lambda-\lambda_\mu)}{2}. \end{cases} \tag{4.40}
$$

By substitution of h_1 and h_2, we obtain the following optimal controls for agent i who wants to liquidate $Q_i(0)$ shares by time T as

$$
v_i^*(t) = -\frac{\delta}{2a} \tag{4.41}
$$

$$
w_i^*(t) = \frac{1}{2\psi}\left(\delta + h_1^0(t) + h_1^1(t)y(t) - \frac{2q_i(t)}{T - t + \frac{\psi}{\beta}}\right) + \frac{\delta}{2a}. \tag{4.42}
$$

Finally, for $w_i(t) < 0$, we get the lemma below.

Lemma 4.2.1. *Assume that the value function h_0 and h_1 for equation (4.20) are C^1 functions of the variables t, y and u. If $w_i(t) < 0$, then the HFTs' value functions (4.23) and (4.24) can be written as*

$$
\begin{aligned}
h_1(t, y, u) &= h_1^1(t) + h_1^1(t)y + h_1^2(t)u \\
h_0(t, y, u) &= h_0^0(t) + h_0^1(t)y + h_0^2(t)u + h_0^3(t)yu + h_0^4(t)y^2 + h_0^5(t)u^2,
\end{aligned}
$$

with $h_1^0, h_1^1, h_1^2, h_0^0, h_0^1, h_0^2, h_0^3, h_0^4$ and h_0^5 being solutions to the following system of ODEs,

$$
\begin{aligned}
\partial_t h_1^0 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_1^1}{2\psi} - \frac{h_1^2}{2\psi} + \frac{h_2}{\psi}\Big)h_1^0 + \frac{\delta}{2\psi} + \frac{\delta\eta}{2\psi}\Big(\lambda - \lambda_\mu + h_1^1 - h_1^2\Big) &= 0, \quad h_1^0(T) = 0,\\
\partial_t h_1^1 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{2h_2}{\psi} - \frac{h_1^2}{2\psi}\Big)h_1^1 + \frac{1}{2\psi}(h_1^1)^2 + \frac{1}{\psi}\Big(\lambda h_2 - \lambda_\mu h_2 - h_2 h_1^2\Big) &= 0, \quad h_1^1(T) = 0,\\
\partial_t h_1^2 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_2}{\psi} + \frac{h_1^1}{2\psi}\Big)h_1^2 - \frac{1}{2\psi}(h_1^2)^2 &= 0, \quad h_1^2(T) = 0,\\
\partial_t h_0^0 + \frac{(h_1^0)^2}{4\psi} + \mu h_0^2 + h_0^1\left(\frac{\delta\eta}{2\psi} + \frac{h_1^0}{2\psi}\right) + \frac{\delta h_1^0}{2\psi} &= 0, \quad h_0^0(T) = 0,\\
\partial_t h_0^1 + \left(\frac{h_1^1}{2\psi} + \frac{h_2}{\psi}\right)h_0^1 + \left(\frac{\delta}{2\psi} + \frac{h_1^0}{2\psi}\right)h_1^1 + \left(\frac{h_1^0}{\psi} + \frac{\delta\eta}{\psi}\right)h_0^4 &= 0, \quad h_0^1(T) = 0,\\
\partial_t h_0^2 + \left(\frac{h_1^0}{2\psi} + \frac{\delta\eta}{2\psi} + \mu\right)h_0^3 + \left(\frac{\delta}{2\psi} + \frac{h_1^0}{2\psi} + \frac{h_1^2}{2\psi}\right)h_1^2 + 2h_0^5\mu &= 0, \quad h_0^2(T) = 0,\\
\partial_t h_0^3 + \left(\frac{h_1^1}{2\psi} + \frac{h_2}{\psi}\right)h_0^3 + \left(\frac{h_0^4}{\psi} + \frac{h_1^1}{2\psi}\right)h_1^2 &= 0, \quad h_0^3(T) = 0,\\
\partial_t h_0^4 + \left(\frac{h_1^1}{\psi} + \frac{2h_2}{\psi}\right)h_0^4 + \frac{(h_1^1)^2}{4\psi} &= 0, \quad h_0^4(T) = 0,\\
\partial_t h_0^5 + \frac{h_1^2 h_0^3}{2\psi} + \frac{(h_1^2)^2}{4\psi} &= 0, \quad h_0^5(T) = 0.
\end{aligned}
\tag{4.43}
$$

We have derived the solution to the optimal liquidation problem under the mean-field game approach using the heuristic arguments. Theorem 4.2.2 below states the main result of the optimal liquidation problem we obtained from the heuristic derivation.

Theorem 4.2.2. *The solution to the stochastic control problems (4.6) – (4.9) is*

$$H_i^{\bar{v}}(t, x_i, f, q_i, y, u) = x + fq_i + h_0(t, y, u) + h_1(t, y, u)q_i + h_2(t)q_i^2, \tag{4.44}$$

where the functions $h_0, h_1,$ and h_2 are

$$\begin{aligned}
h_0(t, y, u) &= h_0^0(t) + h_0^1(t)y + h_0^2(t)u + h_0^3(t)yu + h_0^4(t)y^2 + h_0^5(t)u^2, \\
h_1(t, y, u) &= h_1^0(t) + h_1^1(t)y + h_1^2(t)u, \\
h_2(t) &= -\frac{\psi}{T - t + \frac{\beta}{\psi}}.
\end{aligned}$$

and where $h_0^1, h_0^2, h_0^3, h_0^4, h_0^5, h_1^0, h_1^1,$ and h_1^2 satisfy the system of coupled ODEs given by (4.43), and the optimal trading rate $\{v_i^*\}_{i\in\{1,\ldots,N\}}$ and $\{w_i^*\}_{i\in\{1,\ldots,N\}}$ are as the following:

$$v_i^*(t) = -\frac{\delta}{2a}, \tag{4.45}$$

$$w_i^*(t) = \frac{1}{2\psi}\Big(\delta + h_1(t, y, u) + 2h_2(t)q_i(t)\Big) + \frac{\delta}{2a}. \tag{4.46}$$

for $w_i < 0$,

Proof. By employing a standard dynamic programming principle and the use of Ito's formula, we obtain HJB equation for the problem (4.6) – (4.9) as in (4.12). By substituting H_i in (4.44) into (4.12) in case where $w_i < 0$, we see that (4.44) is the solution to that HJB equation. We have $H_i \in C^{12}([0,T) \times \mathbb{R}^5) \cap C([0,T] \times \mathbb{R}^5)$, and H_i is at most quadratic in state variables. Moreover, as the HFTs' strategies are $(\mathcal{F})_t$-predictable, and $\bar{v}$ is linear in the state variables, the model's assumptions are satisfied, and the verification theorem then holds by applying the Theorem 2.6 in Nisio (2015) or Theorem 3.1 in Fleming and Soner (2016). Therefore, H_i is the optimal solution and the optimal controls are given by (4.45) and (4.46). $\square$

4.2.3 Numerical results

This section shows numerical results of agent i who wants to liquidate 1000 shares by the end of the day in the VIX futures order books, with the initial mean inventory at 200 shares.

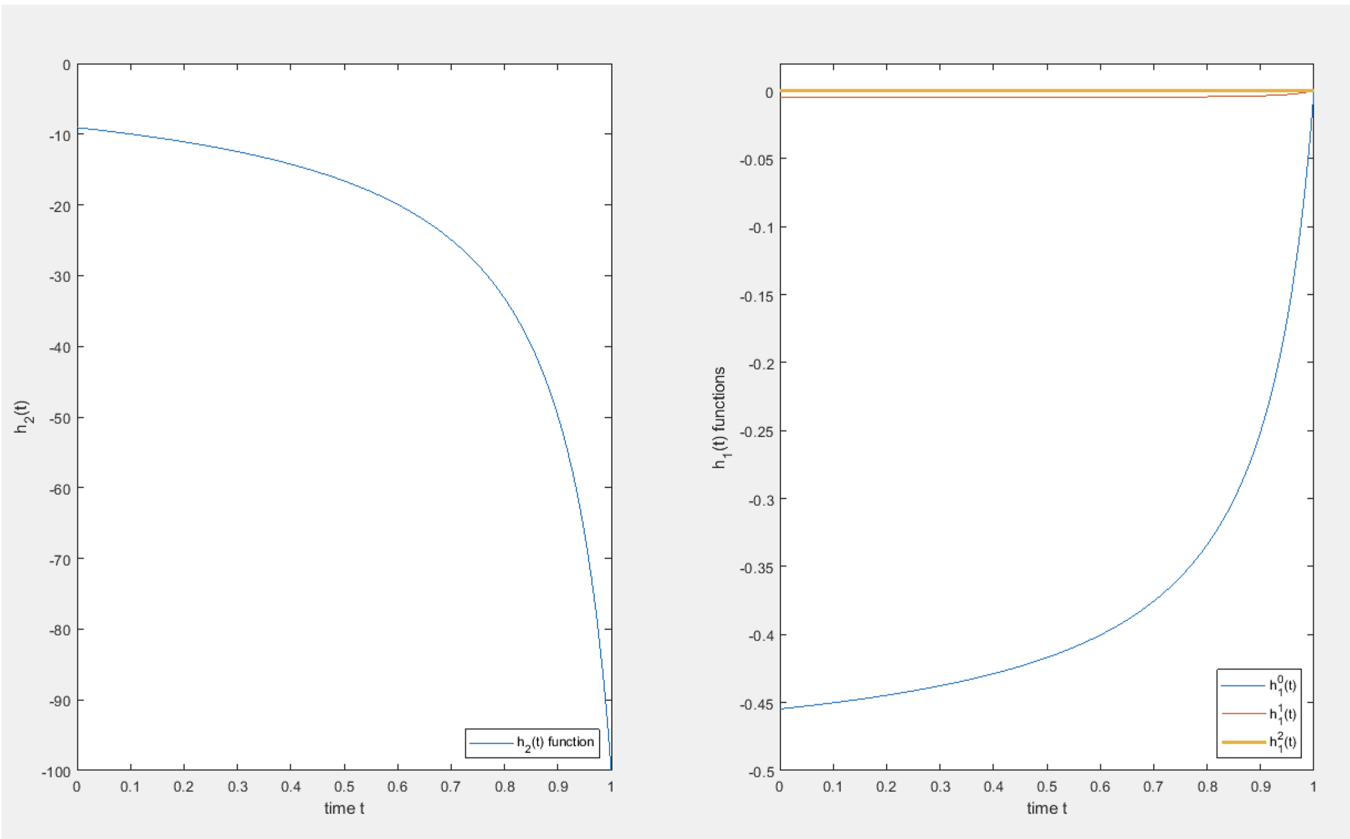

Figure 4.2: Left: plot of h_2 function, right: plot of h_1^0, h_1^1, and h_1^2 functions for the following set of parameters: $T = 1, \beta = 100, \psi = 10, \lambda = 0.2, \lambda_\mu = 0.19$, and $\delta = 0.5$.

Figure 4.2 shows the dynamics of $h_2(t)$ (left panel), $h_1^0(t)$, $h_1^1(t)$ and $h_1^2(t)$ (right panel) for typical values of the parameters. From our optimal control

$$w_i^*(t) = \frac{1}{2\psi}\left(\delta + h_1^0(t) + h_1^1(t)y + 2h_2(t)q_i(t)\right) + \frac{\delta}{2a},$$

gwe observe that $h_2(t)$ is the coefficient of $q_i(t)$. The larger the number of remaining shares to sell $q_i(t)$, the faster one has to trade, namely proportionally to $h_2(t)$. Notice that the influence of the mean field to the optimal control of agent i is via $h_1^1(t)$, and this influence varies with $\lambda - \lambda_\mu$. The reason is that all traders are connected via $(\lambda - \lambda_\mu)\bar{v}$ which is the drift term of the fundamental price process. Thus the smaller the magnitude of $\lambda - \lambda_\mu$, the more disconnected agent i from the mean field. In our simulation, $\lambda - \lambda_\mu = 0.01$, and we have $h_1^1(t)$ very close to zero as displayed in Figure 4.2 (right panel).

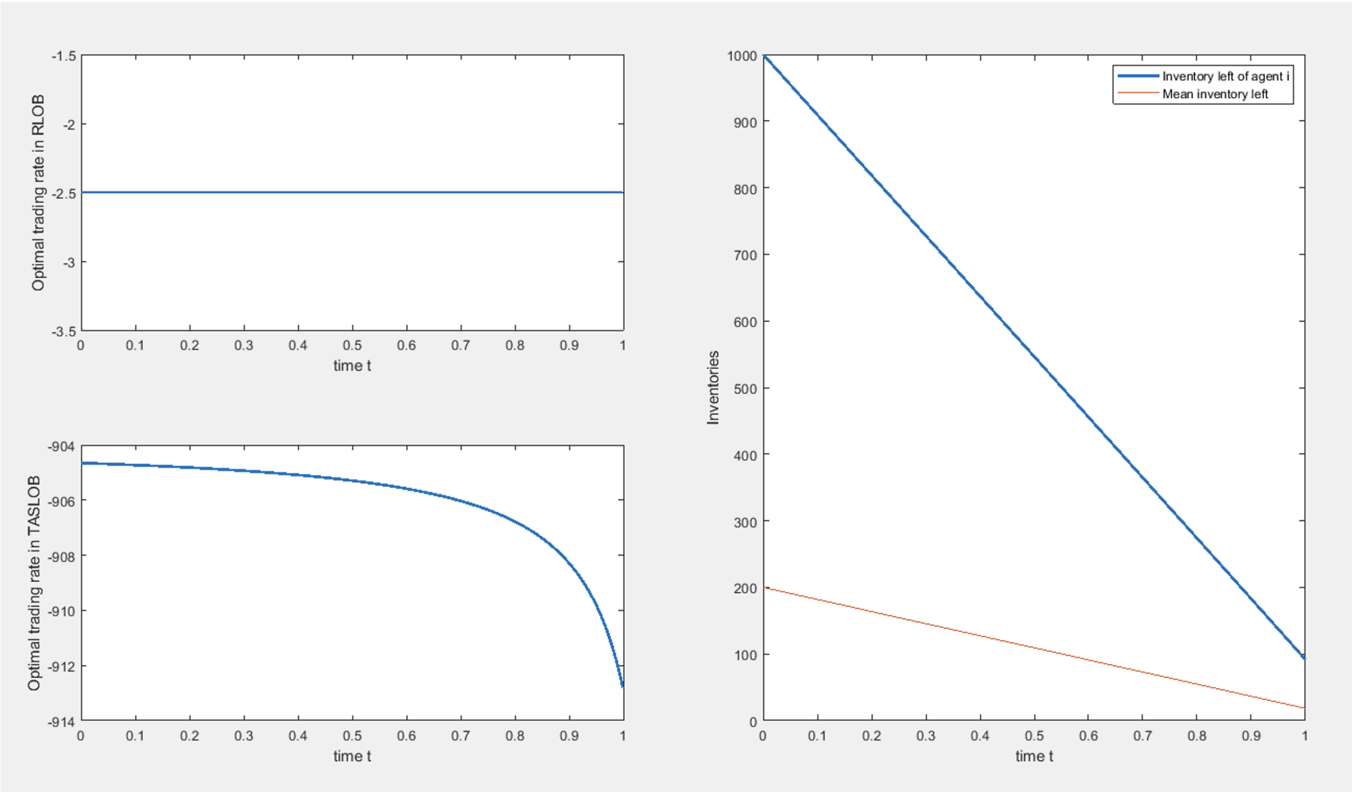

Figure 4.3: Optimal trading rate in RLOB (top left) and TASOB (bottom left), and the Inventory left of agent i and mean inventory left at time $0 \leq t \leq T$ (right) using the set of parameters: $T = 1, \beta = 100, \psi = 10, \lambda = 0.2, \lambda_\mu = 0.19$, and $\delta = 0.5$.

Figure 4.3 (right panel) shows the dynamics of inventory of agent i who wants to liquidate 1000 shares using RLOB and TASOB by time $T = 1$, with the initial mean inventory starting at 200 shares. Note that we use the same set of parameters as the one we simulate functions in figure 4.2. The optimal trading rate in the RLOB is constant at rate $\frac{\delta}{2a}$. The optimal trading rate in the TASLOB is proportional to $h_1^0(t)+h_1^1(t)y(t)+2h_2(t)q_i(t)$. As $Q_i(0) > y(0)$ and $h_1^0(t), h_1^1(t)$, and $h_2(t)$ are all negative with $h_2(t)$ dominates $h_1^0(t)$ and $h_1^1(t)$ in absolute values, thus $2h_2(t)q_i(t)$ is large compared to $h_1^0(t) + h_1^1(t)y(t)$. Therefore, the trading rate $w_i(t)$ has the similar shape as h_2 function. As expected, the inventories go toward 0 as t approaches T.

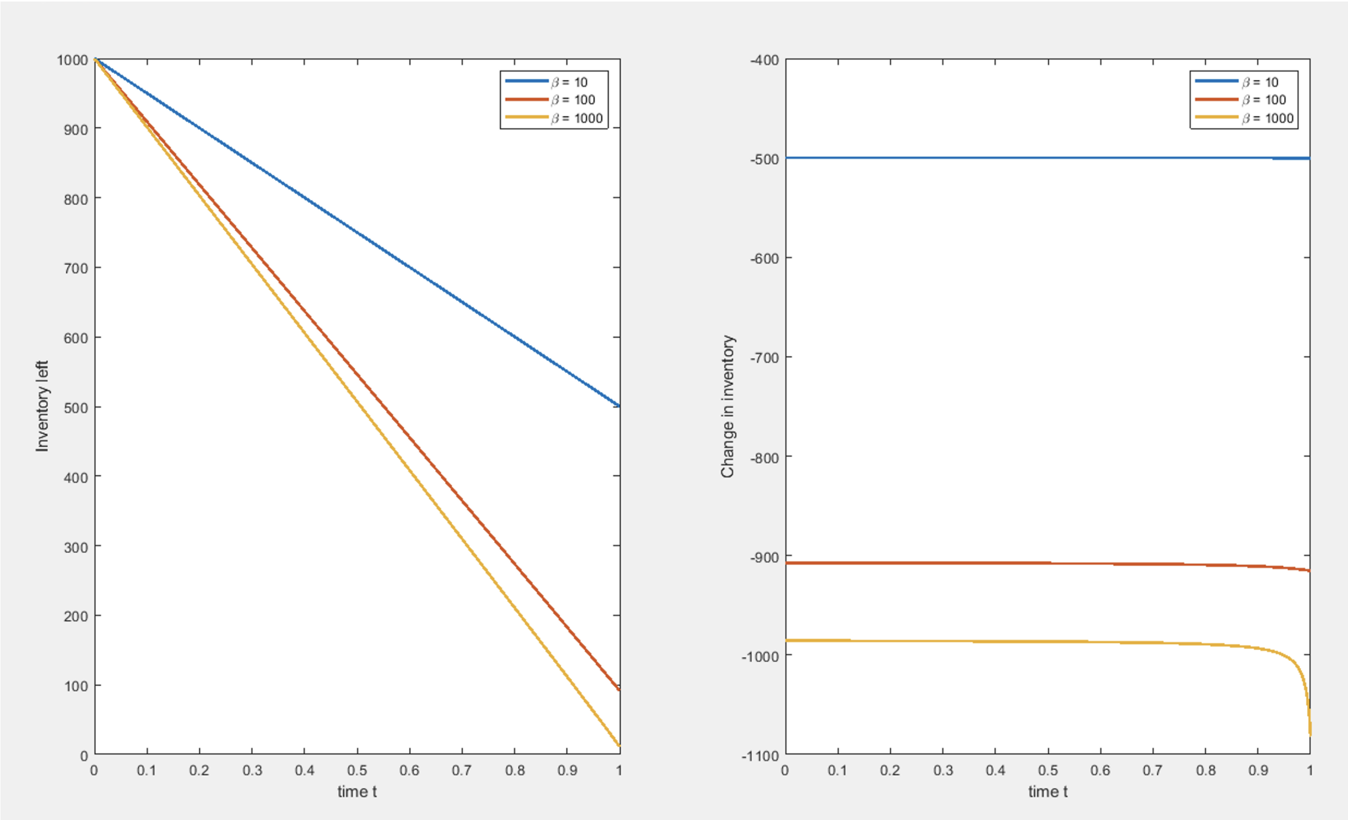

Figure 4.4: Inventory paths of agent i when varying β and other parameters are fixed: $T = 1, \beta = 100, \psi = 10, \lambda = 0.2, \lambda_\mu = 0.19$, and $\delta = 0.5$.

Note that β is the penalty for non-zero terminal inventory. Figure 4.4 shows the inventory paths (left) when we vary β. We see clearly that as β increases the terminal inventory is closer to zero. In addition, the right panel of figure 4.4 shows that agent i trades faster as t is closer to T, especially when β is large. Thus, we can say that the larger the terminal penalization, the faster one has to trade when time is close to terminal for a given remaining number of shares to sell.

4.3 Optimal Liquidation of HFTs in the VIX Futures Market with the Existence of ETFs

Recall that traders can place an order in two order books in the VIX futures market, the RLOB and the TASLOB. Let us assume another type of trader in the VIX futures market other than HFTs and noise traders: the exchange-traded fund's issuers, which we call ETFs. Recall from the first part of the thesis that the VIX ETF issuers have to roll their position daily to meet their constant-maturity requirement and achieve their daily leveraged exposure. Thus, a large volume of shares has to be acquired or liquidated before the market closes. This circumstance implies that the ETF issuers are major buyers or sellers in the VIX futures market.

In this section, we further assume that the ETFs place orders only in the TASLOB, and they are the buy-side of the TASLOB. That is, they provide liquidity on that side of the order book. HFTs can place orders in both order books. They can place an order at any time during the regular trading hour in the TASLOB, and the order is always filled. Assume that there are N HFTs in the VIX futures market. We want to discuss the optimal strategies of the representative agent i to liquidate $Q_i(0)$ shares by the terminal time T provided that ETFs are on the buy-side in TASLOB.

4.3.1 The Model

We work with the following set-up. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a complete filtered probability space that is rich enough to support all following constructions.

Dynamics in the regular limit order book

For the setting in this section, the dynamics of the state variables in the regular limit order book are the same as in the previous section. We consider a large number of HFTs, indexed by $i \in \{1, ..., N\}$. We will use superscript v to represent action in the regular limit order book. In this order book, the representative HFT agent i trades at a rate $v_i(t)$ at time t, which can be positive or negative depending on whether the agent is buying or selling. Thus the *inventory process* of representation agent i is

$$dQ_i^v(t) = v_i(t)dt, \qquad Q^v(0) = q_0^v,$$

where $i \in \{1, ..., N\}$ and $t \in [0, T]$, and the set of initial inventories of HFTs in the regular order book is given by $\{Q_i^v(0)\}_{i\in\{1,...,N\}}$. The execution price of agent i reads

$$S_i^v(t) = F^{\boldsymbol{v}}(t) + av_i(t), \qquad t \in [0, T].$$

where $F^{\boldsymbol{v}}(t)$ for $t \in [0, T]$ denote the fundamental price process discussed in the previous section, and $a > 0$ is the (linear) temporary price impact that the agent i trading has on the mid-price of the asset, and $\boldsymbol{v} = \{v_1, ..., v_N, w_1, ...w_N\}$ is, as mentioned in previous section, a vector representing the trading rate of the N agents in the market. The *cash process* at time t of agent i reads

$$dX_i^v(t) = -S_i^v(t)v_i(t)dt = -(F^{\boldsymbol{v}}(t) + av_i(t))v_i(t)dt, \qquad t \in [0, T],$$

with given initial cash $\{X_i(0)\}_{i\in 1,..,N}$.

Dynamics in the trade at settlement limit order book

We use superscript w to represent action occurs in the TASLOB. Thus, the *inventory process* of representation agent i in the TAS limit order book is

$$dQ_i^w(t) = w_i(t)dt,$$

where $w_i(t)$ is the trading speed of the representative agent i at time $0 \leq t \leq T$. The set of initial inventories of HFTs in the TAS order book is $\{Q_i^w(0)\}_{i\in\{1,\dots,N\}}$.

As in the previous section, we let $F^{\boldsymbol{v}}(T)$ be the price at settlement at time T. As (4.1), the fundamental price process has the form:

$$dF^{\boldsymbol{v}}(t) = \Big(\lambda v^{(N)}(t) + \lambda_\mu \mu(t)\Big)dt + \sigma dW_t, \qquad t \in [0, T], \tag{4.47}$$

with the $\mathbb{P}-$Brownian motion W_t. If we assume

$$\mathbb{E}^{\mathbb{P}}\left[\exp\frac{1}{2}\int_0^T\left(\frac{\lambda v^N(t) + \lambda_\mu \mu}{\sigma}\right)dt\right] < \infty,$$

then by Carmeron-Martin-Girsanov theorem, there exist an equivalent measure $\mathbb{Q}$, and the corresponding $\mathbb{Q}$-Brownian motion $\tilde{W}_t$ such that

$$dF^{\boldsymbol{v}}(t) = \sigma\left(\frac{\lambda v^N(t) + \lambda_\mu \mu}{\sigma}dt + dW_t\right) = \sigma d\tilde{W}_t.$$

Thus, under the $\mathbb{Q}$-measure, the daily settlement price at time t can be written as

$$\mathbb{E}^{\mathbb{Q}}[F^{\boldsymbol{v}}(T)|\mathcal{F}_t] = F^{\boldsymbol{v}}(t).$$

Note that in this section, the ETF issuers provide the liquidity on the buy-side of TASLOB. Therefore, they will buy at $\delta > 0$ above the mid-price. Thus, the HFTs can sell to the ETFs at $\delta > 0$ above the mid-price. The corresponding risk neutral execution price at the TAS limit order book reads

$$S_i^w(t) = F^{\boldsymbol{v}}(t) + \delta \mathbb{1}_{\{w_i(t)<0\}}.$$

The corresponding risk neutral *cash process* in the TASLOB then reads

$$dX_i^w(t) = -(F(t) + \delta \mathbb{1}_{\{w_i(t)<0\}})w_i(t)dt.$$

Again, in the mean-field game setting, a trade of one HFT does not affect the mid-price process, but the overall trading of all HFTs affects the mid-price process. Thus, the average trading speed of N HFTs, $v^{(N)}$, in (4.47) is

$$v^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}(v_i(t) + w_i(t)).$$

We will again solve the optimal trading problem of HFTs in the VIX futures market using the mean-field game approach. In the large N population limit of HFTs, each individual agent has an asymptotically negligible impact on the fundamental price. However, the aggregated effect of the trades of all individuals on the fundamental price is non-negligible.

The Value Function

The aim of agent i is to optimize her wealth, on a finite-time horizon T subject to an inventory penalty and trading activity in two order books.

$$J_i(v_i, w_i; v_{-i}, w_{-i}) := \mathbb{E}\left[X_i(T) + Q_i(T)\Big(F^{\boldsymbol{v}}(T) - \beta Q_i(T)\Big) - \psi \int_t^T \Big(v_i(u) + w_i(u)\Big)^2 du\right] \tag{4.48}$$

where $v_{-i} := (v_1, ..., v_{i-1}, v_{i+1}, ..., v_N)$ and $w_{-i} := (w_1, ..., w_{i-1}, w_{i+1}, ..., w_N)$ is the collection of trading rates excluding agent i. $Q_i(T)(F^{\boldsymbol{v}}(T) - \beta(Q_i(T))^2)$ is the value of closing her position at the end of the trading horizon, with the penalty parameter $\beta > 0$. The penalty parameter $\psi > 0$ is used to penalize in case agent i does not trade synchronously in two order books. The agent's aim is to maximize the performance criterion over a class of admissible trading strategies $\mathcal{A}$ defined as

$$\mathcal{A} := \left\{ \nu \quad \middle| \quad \nu \quad \text{is} \quad \mathcal{G}-\text{adapted} \quad \& \quad \mathbb{E}\left[\int_0^T \nu_t^2 dt < \infty \right] \right\}.$$

Here $\mathcal{G} = (\mathcal{G}_t)_{t\in[0,T]}$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma((Q_i(0))_{\{1,\ldots,N\}})$, that is the filtration $\mathcal{G}$ is the filtration $\mathcal{F}$ expanded by the initial inventories of all HFTs.

A Mean-Field Game Approach

As mentioned before, the aim of each agent i ($i \in \{1, ..., N\}$) is to optimize the value function (4.48), and this problem can be seen as the stochastic game where the interaction between agent is via controls in which their aggregated control will have an influence on the fundamental asset price dynamics. We are interested in large games, where $N \to \infty$. Here we define the mean trading speed of HFTs as $\bar{v} = (\bar{v}(t))_{t\in[0,T]}$

$$\bar{v}(t) := \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (v_i(t) + w_i(t)),$$

and assume this limit exists, and the mean inventory of HFT agents corresponding to this mean trading speed is $Q^{\bar{v}} = (Q^{\bar{v}}(t))_{t\in[0,T]}$ where

$$Q(t)^{\bar{v}} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (Q_i^v(t) + Q_i^w(t)).$$

Infinite Number of HFTs

Problem: Similar to the previous section, we reformulate the N players game problem into the mean-field game problem by replacing $v^{(N)}$ with $\bar{v}$. Therefore, we obtain the following system:

$$
\begin{aligned}
dQ_i(t) &:= dQ_i^v(t) + dQ_i^w(t) \\
&= (v_i(t) + w_i(t))dt, && \text{with} \quad Q(0) = q_0^v + q_0^w, && (4.49) \\
d\bar{F}(t) &= \Big(\lambda\bar{v}(t) + \lambda_\mu \mu(t)\Big)dt + \sigma dW_t, && \text{with} \quad \bar{F}(0) = f_0, && (4.50) \\
d\bar{X}_i(t) &:= d\bar{X}_i^v(t) + d\bar{X}_i^w(t) \\
&= -\bigg((\bar{F}(t) + av_i(t))v_i(t) - (\bar{F}(t) + \delta\mathbb{1}_{\{w_i(t)<0\}})w_i(t)\bigg)dt \\
& && \text{with} \quad \bar{X}(0) = \bar{x}_0. && (4.51)
\end{aligned}
$$

The aim of agent i is to optimize her wealth, at finite-time horizon subject to an inventory penalty and trading activities in two order books. In the mean-field game frame work, her finite-time wealth reads

$$\bar{J}_i(v_i, w_i; \bar{v}) := \mathbb{E}\left[\bar{X}_i(T) + Q_i(T)(\bar{F}(T) - \beta(Q_i(T))^2) - \psi \int_t^T (v_i(u) + w_i(u))^2 du\right]. \quad (4.52)$$

Thus the HFT-i's control problem is the obtain

$$\bar{J}_i(\bar{v}) = \sup_{v_i, w_i \in \mathcal{A}} \bar{J}_i(v_i, w_i; \bar{v}). \quad (4.53)$$

If the supremum is attained in $\mathcal{A}$, then we denote the optimal strategy by (v_i^*, w_i^*).

Solution to Limiting Stochastic Control Problem for HFTs

Solution to Problem: This problem falls within the class of optimal execution problems for a single agent with a time-dependent fundamental price process with drift, and permanent impact. For $x_i, f, q_i, y, u \in \mathbb{R}$ and $t \in [0,T]$, we define the performance criterion for a generic HFT agent $i = 1, ..., N$ under the arbitrary trading strategy $(v_i, w_i) \in \mathcal{A}$ as

$$
\begin{aligned}
&H_i^{v_i,w_i,\bar{v}}(t, x_i, f, q_i, y, u) = \\
&\qquad \mathbb{E}_{t,x_i,f,q_i,y,u}\left[\bar{X}_i(T) + Q_i(T)\Big(\bar{F}(T) - \beta Q_i(T)\Big) - \psi \int_t^T \Big(v_i(s) + w_i(s)\Big)^2 ds\right] \\
&\qquad\qquad = \mathbb{E}_{t,x_i,f,q_i,y,u}\left[\bar{X}_i(T) + Q_i(T)\Big(\bar{F}(T) - \beta Q_i(T)\Big) + \int_t^T K_i(v_i, w_i) ds\right] \qquad (4.54)
\end{aligned}
$$

where $\mathbb{E}_{t,x_i,f,q_i,y,u}\,[\cdot] \; = \mathbb{E}_{t,y,u}\left[\,\cdot\,|\bar{X}_i(t) = x_i, \bar{F}(t) = f, Q_i(t) = q_i\right]$.
We define a value function as:

$$
H_i^{\bar{v}}(t, x_i, f, q_i, y, u) := \sup_{v_i, w_i \in \mathcal{A}} H_i^{v_i,w_i,\bar{v}}(t, x_i, f, q_i, y, u). \qquad (4.55)
$$

Note that $H_i^{\bar{v}}(t, x_i, f, q_i, y, u)$ is equivalent to $\bar{J}_i(\bar{v})$ in the previouse section. We use the notation H to denote the value function instead of J to highlight the explicit dependence on the state variables. The value function J depends on control variables.

4.3.2 Heuristic solution

This section derives a solution to the problem (4.55) by following the similar arguments to the ones in Huang, Jaimungal and Nourian (2019) and Cartea, Jaimungal and Penalva (2015), Chapter 5. By employing a standard dynamic programming principle and the use of Ito's formula, we obtain the following HJB equation with terminal condition

$$\begin{cases} \partial_t H_i(t,x_i,f,q_i,y,u) + \sup_{v_i,w_i\in\mathcal{A}}(\mathcal{L}^{v_i,w_i} H_i(t,x_i,f,q_i,y,u) & +K_i(v_i,w_i)) = 0 \\ H_i(T,x_i,f,q_i,y,u) & = x_i + q_i f - \beta q_i^2 \end{cases} \tag{4.56}$$

where $K_i(v_i,w_i) = -\psi(v_i+w_i)^2$, and $\mathcal{L}^{v_i,w_i}$ is the infinitesimal generator of $(X_i(t), F(t), Q_i(t), y(t), U(t))$ and acts as follow:

$$\mathcal{L}^{v_i,w_i} H_i = \Bigg[-(f+av_i)v_i\partial_{x_i} - (f+\delta\mathbb{1}_{\{w_i<0\}})w_i\partial_{x_i} \\ + (\lambda\bar{\nu} + \lambda_\mu \mu)\partial_f + (v_i+w_i)\partial_{q_i} + \bar{\nu}\partial_y + \mu\partial_u + \varsigma \Bigg] H_i$$

where ς is the operator that encompasses the higher moments of the fundamental price dynamic f in the HJB equation. Note that we obtain the terminal condition directly from (4.54) when $t = T$.

The optimal control in the feedback form, v_i, is

$$v_i^* = \arg\sup_{v_i\in\mathcal{A}} \left(\mathcal{L}^{v_i,w_i} H_i + K_i(v_i,w_i)\right) = \frac{\partial_{q_i} H - f\partial_x H - 2\psi w_i}{2a\partial_x H + 2\psi}, \tag{4.57}$$

and the optimal control in the feedback form, w_i

$$w_i^* = \arg\sup_{w_i\in\mathcal{A}} \left(\mathcal{L}^{v_i,w_i} H_i + K_i(v_i,w_i)\right) = \frac{1}{2\psi}\left(-\left(f+\delta\mathbb{1}_{\{w_i<0\}}\right)\partial_{x_i} H + \partial_{q_i} H\right) - v_i. \tag{4.58}$$

Finally, the HJB equation in equation (4.56) becomes

$$\begin{cases} \partial_t H_i(t,x_i,f,y,q_i,u) - (f+av_i^*)v_i^*\partial_{x_i}H_i \\ \qquad -(f+\delta\mathbb{1}_{\{w_i^*<0\}})w_i^*\partial_{x_i}H_i + (\lambda\bar{\nu}+\lambda_\mu\mu)\partial_f H_i \\ \qquad\quad +(v_i^*+w_i^*)\partial_{q_i}H_i + \bar{\nu}\partial_y H_i + \mu\partial_u + \varsigma - \psi(v_i^*+w_i^*)^2 & = 0 \\ \qquad\qquad H_i(T,x_i,f,y,q_i,u) & = x_i + q_i f - \beta q_i^2. \end{cases} \tag{4.59}$$

Again, by looking at the terminal condition in equation (4.59) we assume that

$$H_i(t,x_i,f,y,q_i,u) = x_i + q_i f + h(t,q_i,y,u),$$

where we assume as the previous section that

$$h(t,q_i,y,u) = h_0(t,y,u) + h_1(t,y,u)q_i + h_2(t)q_i^2, \tag{4.60}$$

subject to the following terminal conditions $h_0(T,y,u)=0$, $h_1(T,y,u)=0$, and $h_2(T)=-\beta$. With the ansatz (4.60), v^* in (4.57) and w^* in (4.58) become

$$v_i^*(t) = \frac{\delta\mathbb{1}_{\{w_i^*<0\}}}{2a}, \tag{4.61}$$

and

$$w_i^*(t) = \frac{1}{2\psi}\Big(-\delta\mathbb{1}_{\{w_i^*<0\}} + h_1(t,y) + 2h_2(t)q_i(t)\Big) - \frac{\delta\mathbb{1}_{\{w_i^*<0\}}}{2a}. \tag{4.62}$$

By substituting of $v_i^*, w_i^*, h(t,q_i,y,u)$ in (4.60) into (4.59) with $w_i^*<0$ and grouping them in terms of q_i^0, q_i, q_i^2 yields

$$\partial_t h_0 + \bar{\nu}\partial_y h_0 + \mu\partial_\mu h_0 - \frac{h_1\delta}{2\psi} + \frac{1}{4\psi}\Big(\delta^2 + h_1^2\Big) + \frac{\delta}{2a} - \frac{1}{4a} + \left\{\partial_t h_1 + \frac{h_1 h_2}{\psi} - \frac{\delta h_2}{\psi} + \bar{v}\partial_y h_1 + \mu\partial_\mu h_1 + \lambda\bar{\nu} + \lambda_\mu \mu\right\} q_i + \left\{\partial_t h_2 + \frac{1}{\psi}h_2^2\right\} q_i^2 = 0.$$

Therefore, we obtain the following system of of ODEs to solve:

$$\partial_t h_0 + \bar{\nu}\partial_y h_0 + \mu\partial_\mu h_0 - \frac{h_1\delta}{2\psi} + \frac{1}{4\psi}\Big(\delta^2 + h_1^2\Big) + \frac{\delta}{2a} - \frac{1}{4a} = 0 \quad \text{with} \quad h_0(T, y, u) = 0, \tag{4.63}$$

$$\partial_t h_1 + \frac{h_1 h_2}{\psi} - \frac{\delta h_2}{\psi} + \bar{v}\partial_y h_1 + \mu\partial_\mu h_1 + \lambda\bar{\nu} + \lambda_\mu\mu = 0 \quad \text{with} \quad h_1(T, y, u) = 0, \tag{4.64}$$

$$\partial_t h_2 + \frac{1}{\psi}h_2^2 = 0 \quad \text{with} \quad h_2(T) = -\beta. \tag{4.65}$$

Again, since h_1 depends on the average inventory and the inventory of noise traders, we assume further that it has the form

$$h_1(t, y, u) = h_1^0(t) + h_1^1(t)y + h_1^2(t)u, \tag{4.66}$$

subject to the terminal conditions $h_1^0(T) = h_1^1(T) = h_1^2(T) = 0$. With the above ansatz for $h_1(t, y, u)$, the optimal controls (4.61) and (4.62) becomes

$$v_i^* = \frac{\delta}{2a}, \tag{4.67}$$

and

$$w_i^*(t) = \frac{1}{2\psi}\Big(-\delta + h_1^0(t) + h_1^1(t)y + h_1^2(t)u + 2h_2(t)q_i(t)\Big) - \frac{\delta}{2a}. \tag{4.68}$$

Mean Field Game (MFG) Consistency condition

Recall that the mean trading rate is the average trading rate of all agent i for $i \in \{1, 2, ..., N\}$. By (4.67) and (4.68), we have

$$\begin{aligned}\bar{\nu}(t) &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\Big(v_i^*(t) + w_i^*(t)\Big)\\ &= \frac{1}{2\psi}\Big(-\delta + h_1^0(t)\Big) + \frac{1}{2\psi}\Big(h_1^1(t) + 2h_2(t)\Big)y(t) + \frac{h_1^2(t)}{2\psi}U(t).\end{aligned}$$

Again $w_i^*(t)$ depends on h_1 and h_2, we have to solve for h_1 in equation (4.64). To solve that, we plug in $\bar{\nu}(t)$ and (4.66) into (4.64) and then group them in term of y, and u, we have

$$\begin{aligned}&\Big\{\partial_t h_1^0 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{(h_1^1 - h_1^2)}{2\psi} + \frac{h_2}{\psi}\Big)h_1^0 - \delta\Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{(h_1^1 - h_1^2)}{2\psi} + \frac{h_2}{\psi}\Big)\Big\}\\ &\quad + \Big\{\partial_t h_1^1 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{2h_2}{\psi} - \frac{h_1^2}{2\psi}\Big)h_1^1 + \frac{1}{2\psi}(h_1^1)^2 + \frac{h_2}{\psi}\Big(\lambda - \lambda_\mu - h_1^2\Big)\Big\}\, y\\ &\qquad + \Big\{\partial_t h_1^2 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_1^1}{2\psi} + \frac{h_2}{\psi}\Big)h_1^2 - \frac{1}{2\psi}(h_1^2)^2\Big\}\, u = 0.\end{aligned}$$

By the comparison of coefficients, we have the following ODEs to solve

$$\partial_t h_1^0 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{(h_1^1 - h_1^2)}{2\psi} + \frac{h_2}{\psi}\Big)h_1^0 - \delta\Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{(h_1^1 - h_1^2)}{2\psi} + \frac{h_2}{\psi}\Big) = 0$$

$$\text{with} \quad h_1^0(T) = 0,. \tag{4.69}$$

$$\partial_t h_1^1 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{2h_2}{\psi} - \frac{h_1^2}{2\psi}\Big)h_1^1 + \frac{1}{2\psi}(h_1^1)^2 + \frac{h_2}{\psi}\Big(\lambda - \lambda_\mu - h_1^2\Big) = 0$$

$$\text{with} \quad h_1^1(T) = 0, \tag{4.70}$$

$$\partial_t h_1^2 + \Big(\frac{(\lambda-\lambda_\mu)}{2\psi} + \frac{h_2}{\psi} + \frac{h_1^1}{2\psi}\Big)h_1^2 - \frac{1}{2\psi}(h_1^2)^2 = 0$$

$$\text{with} \quad h_1^2(T) = 0. \tag{4.71}$$

Observe that the ODE (4.70) and ODE (4.71) are exactly the same as ODE (4.34) and (4.35), respectively. Since we have found the closed-form expressions for the two ODEs as well as $h_2(t)$, we substitute these closed-form expressions into (4.69) yields

$$\partial_t h_1^0 = \begin{cases} -A(1+\tanh(At+B_1)) + \delta A(1+\tanh(At+B_1)), & \text{for} \quad \beta \leq \frac{(\lambda-\lambda_\mu)}{2} \\ -A(1+\frac{1}{\tanh(At+B_1)}) + \delta A(1+\frac{1}{\tanh(At+B_1)}), & \text{for} \quad \beta > \frac{(\lambda-\lambda_\mu)}{2} \end{cases}$$

The closed-form solution to $h_1^0(t)$ above is obtained using the Mathematica reads

$$h_1^0(t) = \begin{cases} k_1 e^{At}\cosh(At+B_1) - \delta e^{2(At+B_1)}, & \text{for} \quad \beta \leq \frac{(\lambda-\lambda_\mu)}{2} \\ k_1 e^{At}\sinh(At+B_2) + \delta e^{2(At+B_2)},, & \text{for} \quad \beta > \frac{(\lambda-\lambda_\mu)}{2} \end{cases}$$

where

$$\begin{aligned} A &:= \frac{(\lambda-\lambda_\mu)}{4\psi}, \\ B_1 &:= -\frac{(\lambda-\lambda_\mu)}{4\psi}T + \tanh^{-1}(1-\frac{4\beta}{(\lambda-\lambda_\mu)}), \\ B_2 &:= -\frac{(\lambda-\lambda_\mu)}{4\psi}T + \frac{1}{2}\log(1-\frac{(\lambda-\lambda_\mu)}{2\beta}), \\ k_1 &= \frac{\delta e^{(AT+2B_1)}}{\cosh(AT+B_1)}, \\ k_2 &= -\frac{\delta e^{(AT+2B_2)}}{\sinh(AT+B_2)}. \end{aligned}$$

Lemma 4.3.1. *Assume that the value function h_0 and h_1 for equation (4.60) are C^1 functions of the variables t, y and u. If $w_i(t) < 0$, then the HFTs' value functions (4.63) and (4.64) can be written as*

$$
\begin{aligned}
h_1(t,y,u) &= h_1^1(t) + h_1^1(t)y + h_1^2(t)u \\
h_0(t,y,u) &= h_0^0(t) + h_0^1(t)y + h_0^2(t)u + h_0^3(t)yu + h_0^4(t)y^2 + h_0^5(t)u^2,
\end{aligned}
$$

with $h_1^0, h_1^1, h_1^2, h_0^0, h_0^1, h_0^2, h_0^3, h_0^4$ and h_0^5 being solutions to the following system of ODEs,

$$
\begin{aligned}
\partial_t h_0^0 + \frac{(h_1^0)^2}{4\psi} + \mu h_0^2 + h_0^1\Big(-\frac{\delta}{2\psi} + \frac{h_1^0}{2\psi}\Big) - \frac{\delta h_0^1}{2\psi} - \frac{1}{4a} + \frac{\delta}{2a} + \frac{\delta^2}{4\psi} &= 0, \quad h_0^0(T) = 0, \\
\partial_t h_0^1 + \left(\frac{h_1^1}{2\psi} + \frac{h_2}{\psi}\right) h_0^1 + \left(-\frac{\delta}{2\psi} + \frac{h_1^0}{2\psi}\right) h_1^1 + \left(\frac{h_1^0}{\psi} - \frac{\delta}{\psi}\right) h_0^4 + h_0^3\mu &= 0, \quad h_0^1(T) = 0, \\
\partial_t h_0^2 + \left(\frac{h_1^0}{2\psi} - \frac{\delta}{2\psi}\right) h_0^3 + \left(-\frac{\delta}{2\psi} + \frac{h_1^0}{2\psi} + \frac{h_0^1}{2\psi}\right) h_1^2 + 2h_0^5\mu &= 0, \quad h_0^2(T) = 0, \\
\partial_t h_0^3 + \left(\frac{h_1^1}{2\psi} + \frac{h_2}{\psi}\right) h_0^3 + \left(\frac{h_0^4}{\psi} + \frac{h_1^1}{2\psi}\right) h_1^2 &= 0, \quad h_0^3(T) = 0, \\
\partial_t h_0^4 + \left(\frac{h_1^1}{\psi} + \frac{2h_2}{\psi}\right) h_0^4 + \frac{(h_1^1)^2}{4\psi} &= 0, \quad h_0^4(T) = 0, \\
\partial_t h_0^5 + \frac{h_1^2 h_0^3}{2\psi} + \frac{(h_1^2)^2}{4\psi} &= 0, \quad h_0^5(T) = 0.
\end{aligned}
\tag{4.72}
$$

We have derived the solution to the optimal liquidation problem under the mean-field game approach using the heuristic arguments. Theorem 4.3.2 below states the main result of the optimal liquidation problem we obtained from the heuristic derivation.

Theorem 4.3.2. *The solution to the stochastic control problems* (4.49) – (4.52) *is*

$$H_i^{\bar{v}}(t, x_i, f, q_i, y, u) = x + fq_i + h_0(t, y, u) + h_1(t, y, u)q_i + h_2(t)q_i^2,$$

where the functions $h_0, h_1,$ *and* h_2 *are*

$$\begin{aligned} h_0(t, y, u) &= h_0^0(t) + h_0^1(t)y + h_0^2(t)u + h_0^3(t)yu + h_0^4(t)y^2 + h_0^5(t)u^2, \\ h_1(t, y, u) &= h_1^0(t) + h_1^1(t)y + h_1^2(t)u, \\ h_2(t) &= -\frac{\psi}{T - t + \frac{\beta}{\psi}}. \end{aligned}$$

and where $h_0^1, h_0^2, h_0^3, h_0^4, h_0^5, h_1^0, h_1^1,$ *and* h_1^2 *satisfy the system of coupled ODEs given by (4.72), and the optimal trading rate* $\{v_i^*\}_{i\in\{1,\ldots,N\}}$ *and* $\{w_i^*\}_{i\in\{1,\ldots,N\}}$ *are as the following:*

$$\begin{aligned} v_i^*(t) &= \frac{\delta}{2a}, \\ w_i^*(t) &= \frac{1}{2\psi}\bigg(- \delta + h_1(t, y, u) + 2h_2(t)q_i(t)\bigg) - \frac{\delta}{2a}. \end{aligned}$$

for $w_i < 0$*,*

Proof. The proof is similar to previous section that the verification theorem then holds by applying the Theorem 2.6 in Nisio (2015) or Theorem 3.1 in Fleming and Soner (2016). □

4.3.3 Numerical results

This section shows numerical results for agent i who wants to liquidate 1000 shares by the end of the day in the VIX futures order books, with the initial mean inventory at 200 shares, and the ETFs is buying in TASLOB.

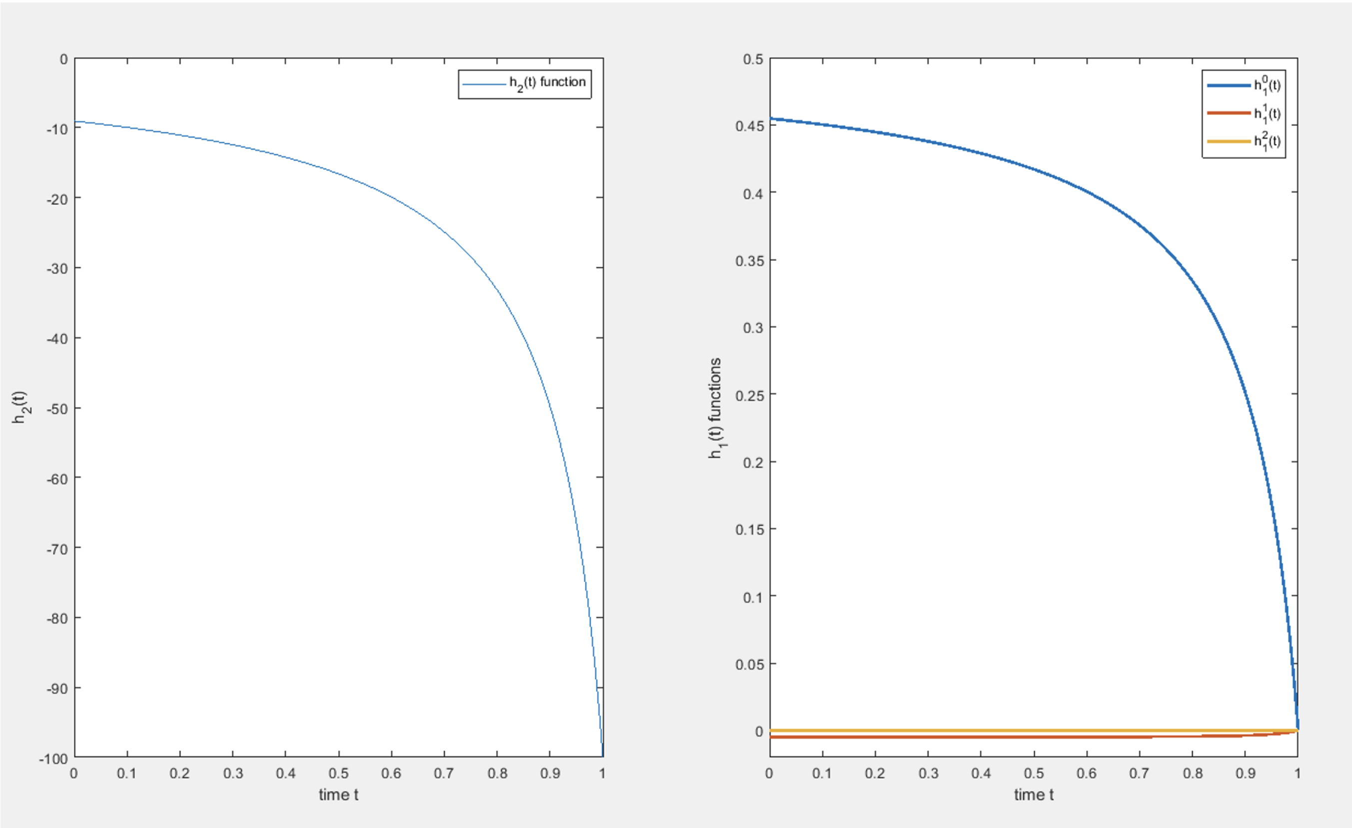

Figure 4.5: Left: plot of $h_2(t)$ function, right: plot of h_1^0, h_1^1, and h_1^2 functions for the following set of parameters: $T = 1, \beta = 100, \psi = 10, \lambda = 0.2, \lambda_\mu = 0.19$, and $\delta = 0.5$.

Figure 4.5 shows the dynamics of $h_2(t)$ (left panel), $h_1^0(t)$, $h_1^1(t)$ and $h_1^2(t)$ (right panel) for the set of the parameters: $T = 1, \beta = 100, \psi = 10, \lambda = 0.2, \lambda_\mu = 0.19$, and $\delta = 0.5$ when ETFs provides liquidity on the buy-side on TASLOB. Compared to figure 4.2, the only h function that is different is $h_1^0(t)$. Without ETFs in the market, $h_1^0(t)$ is negative. On the other hand, with ETFs in the market, $h_1^0(t)$ takes positive values. However, the influence of h_2 on the optimal trading in the TASLOB is still large compared to $h_1^0(t)$. Figure 4.6 shows the optimal trading rate of agent i in both order books (left panel), and the inventory path of agent i resulting from the trading activities from both order book (right panel, blue line) when ETFs provides liquidity on the buy-side of TASLOB.

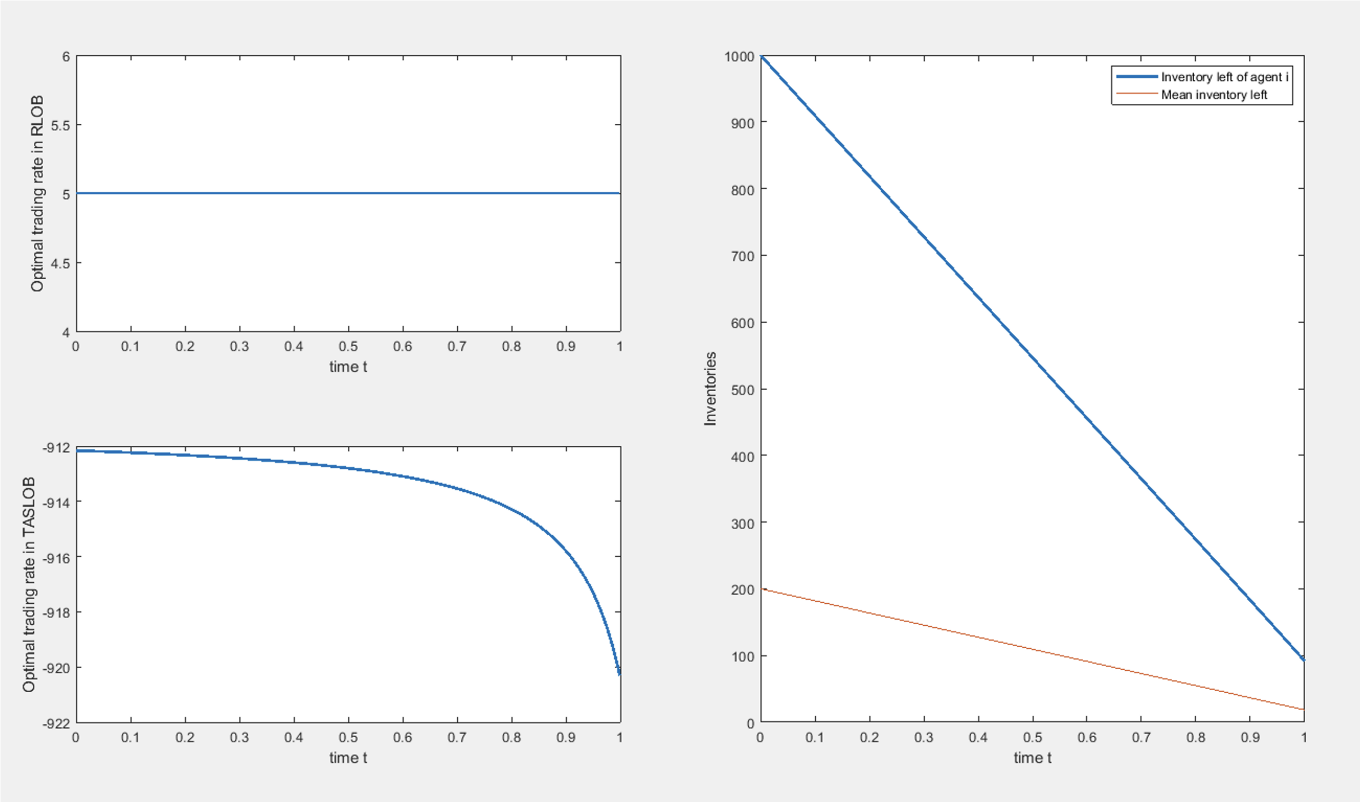

Figure 4.6: Optimal trading rate in RLOB (top left) and TASOB (bottom left), and the Inventory left of agent i and mean inventory left at time $0 \leq t \leq T$ (right) using the set of parameters: $T = 1, \beta = 100, \psi = 10, \lambda = 0.2, \lambda_\mu = 0.19$, and $\delta = 0.5$.

Figure 4.7 (left) compares optimal trading rates in RLOB and TASLOB of agent i with and without having ETFs in the market. One can see that if there are ETFs acquiring shares in TASLOB, agent i will buy in RLOB and sell in TASLOB. We argue that this is because HFTs know that ETFs will buy δ above mid-price in TASLOB, thus they can take advantage of this information by buying some shares in RLOB and sell them δ more than the mid-price in TASLOB to pocket some profit while liquidating their original inventory in TASLOB. Consequently, there will be more shares for agent i to liquidate in TASLOB. That is why the trading rate in TASLOB is faster when there is ETFs in the market.

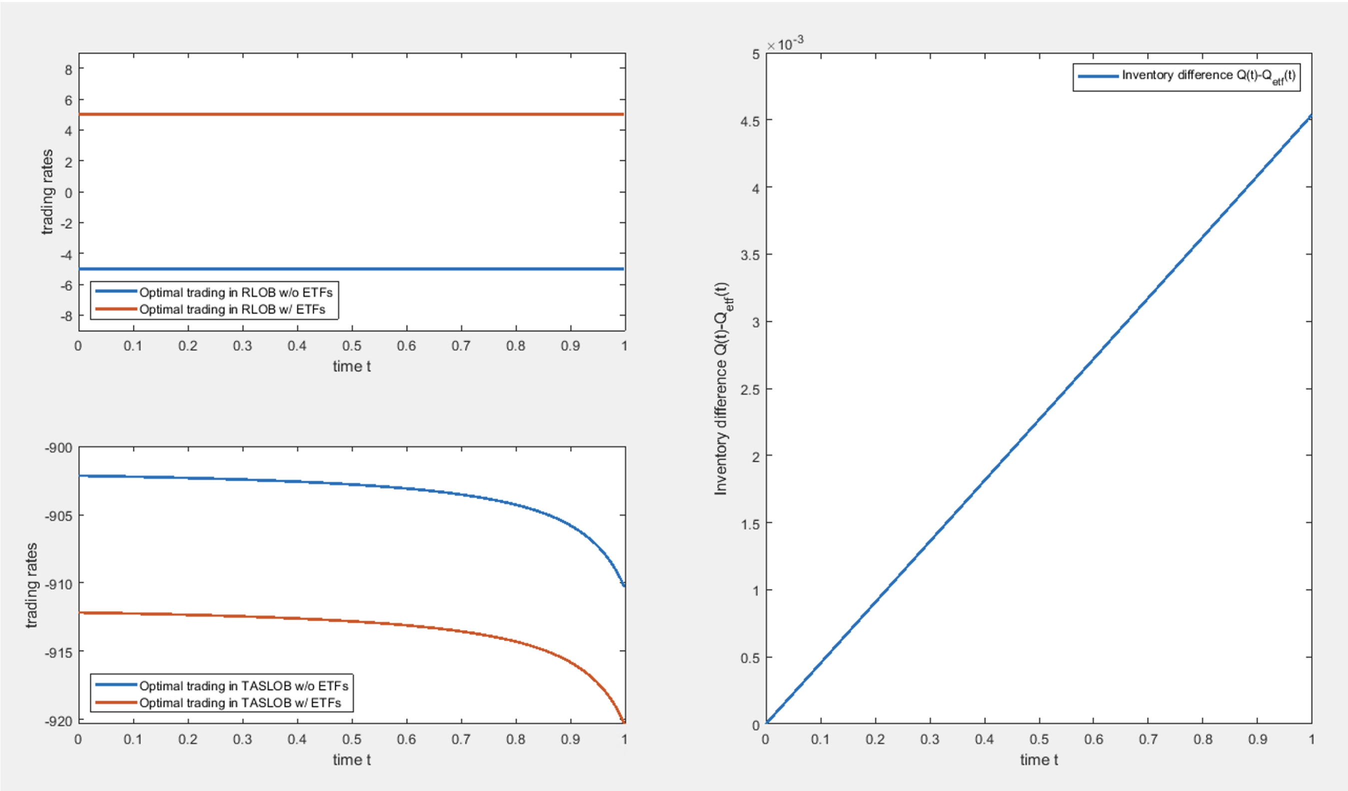

Figure 4.7: Optimal trading with and without ETFs using the set of parameters: $T = 1, \beta = 100, \psi = 10, \lambda = 0.2, \lambda_\mu = 0.19$, and $\delta = 0.5$.

4.4 Optimal Liquidation of HFTs in the VIX Futures Market with Limited Supply in the Trade-at-Settlement Order Book by ETFs

In this section, we investigate the optimal trading strategies of a representative agent i who wants to liquidate $Q_i(0)$ shares by the end of the trading day in the VIX futures market, where ETFs are acquiring shares in the TASLOB. However, the number of shares ETFs are acquiring in the TASLOB is limited. In other words, the ETFs want to acquire Π_0 shares in the TASLOB by the end of the day, says at time $t \leq T$, and we assume that each agent i can sell their inventory to ETFs up to $\frac{\Pi_0}{N}$ shares in the TASLOB. We further assume that the ETFs have priority over other traders in the TASLOB; that is, when ETFs acquire their shares, the HFTs can only place an order on one side of this order book, which is to sell in the TASLOB. In this setting, HFTs can place an order on both sides of the TAS limit order book only after Π_0 shares of ETFs are eaten up.

4.4.1 The Model

Dynamics in the regular limit order book

Again will use superscript v to represent action that occurs in the regular limit order book. The *inventory process* of representation agent i is

$$dQ_i^v(t) = v_i(t)dt \qquad \text{with} \qquad Q_i^v(0) = q_0^v,$$

where $v_i(t)$ is the trading speed of the representation agent i at time t.

The *execution price* is

$$S_i^v(t) = F^{\boldsymbol{v}}(t) + av_i(t)$$

where $F^{\boldsymbol{v}}(t)$ is the mid-price process at time t and a is the temporary price impact in the regular limit order book. The corresponding *cash process* is

$$dX_i^v(t) = -S_i^v(t)\nu_i dt = -(F^{\boldsymbol{v}}(t) + av_i(t))v_i(t)dt, \qquad t \in [0, T],$$

with initial cash $\{X_i(0)\}_{i \in 1,\ldots,N}$.

Dynamics in the trade at settlement limit order book

We assume that the number of shares the ETFs want to sell in the TASLOB is Π_0. If there are N HFTs in the VIX futures market, each can buy approximately $\frac{\Pi_0}{N}$ shares in the TASLOB before ETFs liquidate all of their shares in that order book. Thus, the *inventory* *process* of representation agent i in the TAS limit order book is

$$dQ_i^w(t) = w_i(t)dt \qquad \text{with} \qquad Q_i^w(0) = q_0^w$$

where $w_i(t)$ is the trading speed of the representation agent at time $0 \le t \le T$. Let $\int_0^t w_i(s)^- ds$ be the accumulated shares agent i sold in the TAS limit order book up to time t. For convenience, set

$$\mathcal{S} := \left\{ t > 0 : \int_0^t w_i(s)^- ds \le \frac{\Pi_0}{N} \right\}.$$

Similar to the previous section, we let $F^{\boldsymbol{v}}(T)$ be the price at the settlement at time T. The corresponding execution price reads

$$S_i^w(t) = F^{\boldsymbol{v}}(t) + \delta \mathbb{1}_{\{w_i(t)<0\}} \mathbb{1}_{\mathcal{S}} + \Big(\delta \mathbb{1}_{\{w_i(t)\ge 0\}} - \delta \mathbb{1}_{\{w_i(t)<0\}} \Big) \mathbb{1}_{\mathcal{S}^c}$$

Here $F^{\boldsymbol{v}}(t) + \delta \mathbb{1}_{\{w_i(t)<0\}} \mathbb{1}_{\mathcal{S}}$ represents the execution price at time t of agent i when her accumulated shares is less than or equal to Π_0/N. That is the HFTs benefit from ETFs' acquisition by selling δ per share more than the mid-price in this region. $F^{\boldsymbol{v}}(t) + \delta \mathbb{1}_{\{w_i(t)\ge 0\}} - \delta \mathbb{1}_{\{w_i(t)<0\}}$ represents whether the order is a buy or a sell order at time t when the accumulated shares of agent i is greater than Π_0/N. In other words, the HFTs no longer benefit from ETF's acquisition in region $\mathcal{S}^c$. Specifically in $\mathcal{S}^c$, if agent i wants to buy shares, she has to buy at δ more than the mid-price per share, and if she wants to sell, she has to sell at δ less than the mid-price per share.

The corresponding *cash process* in the TASLOB then reads

$$dX_i^w(t) = -\left(F^{\boldsymbol{v}}(t) + \delta \mathbb{1}_{\{w_i(t)<0\}} \mathbb{1}_{\mathcal{S}} + \Big(\delta \mathbb{1}_{\{w_i(t)\ge 0\}} - \delta \mathbb{1}_{\{w_i(t)<0\}} \Big) \mathbb{1}_{\mathcal{S}^c} \right) w_i(t)dt.$$

We assume the same fundamental price dynamics as (4.47) in previous section:

$$dF^{\boldsymbol{v}}(t) = \Big(\lambda v^{(N)}(t) + \lambda_\mu \mu(t) \Big) dt + \sigma dW_t, \qquad t \in [0, T],$$

where $F^{\boldsymbol{v}}(0) = f_0$ and

$$v^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}(v_i(t) + w_i(t)).$$

For each representative agent i, we have the combined inventory process at time t as

$$dQ_i(t) = v_i(t)dt + w_i(t)dt.$$

The cash process of the representative agent i in the VIX futures market is

$$\begin{aligned} dX_i(t) &= -(F^{\boldsymbol{v}}(t) + av_i(t))v_i(t)dt \\ &\quad - \left(F^{\boldsymbol{v}}(t) + \delta\mathbb{1}_{\{w_i(t)<0\}}\mathbb{1}_{\mathcal{S}} + \left(\delta\mathbb{1}_{\{w_i(t)\geq 0\}} - \delta\mathbb{1}_{\{w_i(t)<0\}}\right)\mathbb{1}_{\mathcal{S}^c}\right)w_i(t)dt. \end{aligned}$$

A Mean-Field Game Approach

As mentioned, we are interested in large games, where $N \to \infty$. Here we define the mean trading speed of HFTs as

$$\bar{v}(t) := \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(v_i(t) + w_i(t)\right),$$

and assume this limit exists. The mean inventory of HFT agents corresponding to this mean trading speed is $Q^{\bar{v}} = (Q^{\bar{v}}(t))_{t\in[0,T]}$ where

$$Q^{\bar{v}}(t) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(Q_i^{v}(t) + Q_i^{w}(t)\right).$$

Infinite Number of HFTs

Next we will reformulate the N players game problem into the mean-field game problem by replacing $v^{(N)}$ with $\bar{v}$. Therefore, we obtain the following system:

$$\begin{aligned}
dQ_i(t) &= dQ_i^v(t) + dQ_i^w(t) \\
&= (v_i(t) + w_i(t))dt \qquad \text{with} \quad Q(0) = q_0^v + q_0^w, \qquad (4.73)\\
d\bar{F}(t) &= \Big(\lambda\bar{v}(t) + \lambda_\mu \mu(t)\Big)dt + \sigma dW_t \qquad \text{with} \quad \bar{F}(0) = f_0, \qquad (4.74)\\
d\bar{X}_i(t) &= d\bar{X}_i^v(t) + d\bar{X}_i^w(t) \\
&= -\bigg(\Big(\bar{F}(t) + av_i(t)\Big)v_i(t) \\
&\quad + \Big(\bar{F}(t) + \delta\mathbb{1}_{\{w_i(t)<0\}}\mathbb{1}_{\mathcal{S}} + (\delta\mathbb{1}_{\{w_i(t)\geq 0\}} - \delta\mathbb{1}_{\{w_i(t)<0\}})\mathbb{1}_{\mathcal{S}^c}\Big)w_i(t)\bigg)dt, \\
&\qquad \text{with} \quad \bar{X}(0) = \bar{x}_0. \qquad (4.75)
\end{aligned}$$

The aim of agent i is to optimize her finite-time horizon wealth, subject to an inventory penalty and trading activities in two order books, in the mean-field game frame work reads,

$$\bar{J}_i(v_i, w_i; \bar{v}) := \mathbb{E}\bigg[\bar{X}_i(T) + Q_i(T)\Big(\bar{F}(T) - \beta Q_i(T)\Big) - \psi\int_t^T \Big(v_i(u) + w_i(u)\Big)^2 du\bigg]. \qquad (4.76)$$

Thus the HFT-i's control problem is to maximize

$$\bar{J}_i(\bar{v}) = \sup_{v_i, w_i \in \mathcal{A}} \bar{J}_i(v_i, w_i; \bar{v}). \qquad (4.77)$$

If the supremum is attained in $\mathcal{A}$, then we denote the optimal strategy by (v_i^*, w_i^*).

Solution to Limiting Stochastic Control Problem for HFTs

Solution to Problem: This problem falls within the class of optimal execution problems for a single agent with a time-dependent fundamental price with drift, and permanent impact. For $x_i, f, q_i, y, u \in \mathbb{R}$ and $t \in [0,T]$, we define the performance criterion for a generic HFT agent $i = 1, ..., N$ under the arbitrary trading strategy $(v_i, w_i) \in \mathcal{A}$ as

$$
\begin{aligned}
&H_i^{v_i,w_i,\bar{v}}(t, x_i, f, q_i, y, u) = \\
&\qquad \mathbb{E}_{t,x_i,f,q_i,y,u}\left[\bar{X}_i(T) + Q_i(T)(\bar{F}(T) - \beta(Q_i(T))^2 - \psi\int_t^T (v_i(s) + w_i(s))^2 ds\right] \\
&\qquad = \mathbb{E}_{t,x_i,f,q_i,y,u}\left[\bar{X}_i(T) + Q_i(T)(\bar{F}(T) - \beta(Q_i(T))^2 + \int_t^T K_i(s, v_i, w_i) ds\right]
\end{aligned} \tag{4.78}
$$

where $\mathbb{E}_{t,x_i,f,q_i,y}\left[\cdot\right] = \mathbb{E}_{t,y}\left[\cdot\,|\bar{X}_i(t) = x_i, \bar{F}(t) = f, Q_i(t) = q_i, U(t) = u\right]$.
We define the value function as:

$$
H_i^{\bar{v}}(t, x_i, f, q_i, y, u) := \sup_{v_i, w_i \in \mathcal{A}} H_i^{v_i,w_i,\bar{v}}(t, x_i, f, q_i, y, u) \tag{4.79}
$$

Note that $H_i^{\bar{v}}(t, x_i, f, q_i, y, u)$ is equivalent to $\bar{J}_i(\bar{v})$ in the previouse section. We use the notation H to denote the value function instead of J to highlight the explicit dependence on the state variables. The value function J depends on control variables.

4.4.2 Heuristic solution

By employing a standard dynamic programming principle and the use of Ito's formula, we obtain the following HJB equation with terminal condition

$$\begin{cases} \partial_t H_i(t,x,f,q_i,y,u) + \sup_{v_i,w_i \in \mathcal{A}} (\mathcal{L}^{v_i,w_i} H_i(t,x,f,q_i,y,u) + K_i(v_i,w_i)) = 0 \\ H_i(T,x,f,q_i,y,u) = x + q_i f - \beta q^2 \end{cases} \tag{4.80}$$

where $K_i(v_i,w_i) = -\psi(v_i+w_i)^2$ and $\mathcal{L}^{v_i,w_i}$ is the infinitesimal generator of $\Big(X(t), \bar{F}(t), Q(t), y(t), U(t)\Big)$ and acts as follow:

$$\mathcal{L}^{v_i,w_i} H_i = \Bigg[-(f+av_i)v_i\partial_x - \Bigg(f + \delta \mathbb{1}_{\{w_i<0\}} \mathbb{1}_{\mathcal{S}} + (\delta \mathbb{1}_{\{w_i \geq 0\}} - \delta \mathbb{1}_{\{w_i<0\}}) \mathbb{1}_{\mathcal{S}^c} \Bigg) w_i(t)\partial_x \\ (\lambda\bar{\nu} + \lambda_\mu \mu)\partial_f + (v_i+w_i)\partial_{q_i} + \bar{\nu}\partial_y + \mu\partial_u + \varsigma \Bigg] H_i$$

where ς is the operator that encompasses the higher moments of the fundamental price dynamic f in the HJB equation.

Let

$$L^{v_i,w_i} = \mathcal{L}^{v_i,w_i} H_i + K_i(v_i,w_i)$$

Note that $\mathcal{S}$ and $\mathcal{S}^c$ are disjoint, thus we split the problem into two cases and solve them separately. The two cases are (i) $t \in \mathcal{S}$, and (ii) $t \notin \mathcal{S}$. We let t^* be the point where t switches from $\mathcal{S}$ to $\mathcal{S}^c$. For $t \in \mathcal{S}$, then $0 \leq t \leq t^*$, and for $t \in \mathcal{S}^c$, then $t^* < t \leq T$. Assuming that $t^* < T$ exists, we find the solution as follows.

For $t \in \mathcal{S}^c$

In this case, we obtain the following HJB equation

$$\partial_t H_i(t, x_i, f, y, q_i, u) - (f + av_i^*)v_i^*\partial_{x_i} H_i - \Big(f - \delta \mathbb{1}_{\{w_i^*<0\}}\Big) w_i^* \partial_{x_i} H_i \\ + (\lambda\bar{\nu} + \lambda_\mu \mu)\partial_f + (v_i^* + w_i^*)\partial_{q_i} + \bar{\nu}\partial_y + \mu\partial_u + \varsigma] H_i - \psi(v_i^* + w_i^*)^2 = 0$$

subject to the terminal condition $H_i(T, x_i, f, y, q_i) = x_i + q_i f - \beta q_i^2$. This is exactly the same HJB equation in (4.18) for the case $w < 0$, previously solved in case where HFTs trade in both order books without ETFs. The following are optimal trading strategies for agent i who is liquidating initial shares $Q_i(t^*) > 0$:

For $w_i(t) < 0$ **and** $t^* \le t \le T$

$$v_i^*(t) = -\frac{\delta}{2a},$$
$$w_i^*(t) = \frac{1}{2\psi}\Big(\delta + h_1^0(t) + h_1^1(t)y + 2h_2(t)q_i(t)\Big) + \frac{\delta}{2a},$$

where

$$h_2(t) = -\frac{\psi}{T - t + \frac{\psi}{\beta}},$$

$$h_1^1(t) = \begin{cases} -\frac{(\lambda-\lambda_\mu)}{2} + \frac{2\psi}{T-t+\frac{\psi}{\beta}} + \frac{(\lambda-\lambda_\mu)}{2}\tanh(At + B_1) & \text{for} \quad \beta \le \frac{\lambda-\lambda_\mu}{2} \\ -\frac{(\lambda-\lambda_\mu)}{2} + \frac{2\psi}{T-t+\frac{\psi}{\beta}} + \frac{(\lambda-\lambda_\mu)}{2\tanh(At+B_2)} & \text{for} \quad \beta > \frac{\lambda-\lambda_\mu}{2}, \end{cases}$$

and $h_1^0(t)$ solves

$$\partial_t h_1^0(t) = \begin{cases} A\Big(1 + \tanh(At + B_1)\Big)h_1^0(t) + \delta A\eta\Big(1 + \tanh(At + B_1)\Big) + \frac{\delta(1-\eta)}{T-t+\frac{\psi}{\beta}} & \text{for} \quad \beta \le \frac{(\lambda-\lambda_\mu)}{2}, \\ A\Big(1 + \frac{1}{\tanh(At+B_2)}\Big)h_1^0(t) + \delta A\eta\Big(1 + \frac{1}{\tanh(At+B_2)}\Big) + \frac{\delta(1-\eta)}{T-t+\frac{\psi}{\beta}} & \text{for} \quad \beta > \frac{(\lambda-\lambda_\mu)}{2}, \end{cases}$$

$$
\begin{aligned}
A &:= \frac{(\lambda-\lambda_\mu)}{4\psi},\\
B_1 &:= -\frac{(\lambda-\lambda_\mu)}{4\psi}T + \tanh^{-1}\left(1-\frac{4\beta}{(\lambda-\lambda_\mu})\right),\\
B_2 &:= -\frac{(\lambda-\lambda_\mu)}{4\psi}T + \frac{1}{2}\log\left(1-\frac{(\lambda-\lambda_\mu)}{2\beta}\right).
\end{aligned}
$$

Moreover, the mean trading speed is

$$
\bar{v}(t) = \frac{1}{2\psi}\left(\delta\eta + h_1^0(t) + \Big(h_1^1(t) + 2h_2(t)\Big)y(t)\right).
$$

For $w_i(t) \geq 0$ and $t^* \leq t \leq T$

$$
\begin{aligned}
v_i^*(t) &= 0,\\
w_i^*(t) &= 0.
\end{aligned}
$$

For $t \in \mathcal{S}$

In this case, the ETFs is buying shares. Since buying larges share will push the price up, we assume that they will buy at δ more than the daily settlement price. As previously assumed, the execution price in the TASLOB at time t is $F(t)+\delta$ no matter the transaction on the TASLOB is a sell or a buy order. In this case, we obtain the following HJB equation

$$
\begin{cases}
\partial_t H_i(t,x_i,f,y,q_i,u) - (f+av_i^*)v_i^*\partial_{x_i}H_i & \\
\qquad -(f+\delta\mathbb{1}_{\{w_i^*<0\}})w_i^*\partial_{x_i}H_i + (\lambda\bar{v}+\lambda_\mu\mu)\partial_f H_i & \\
\qquad\quad +(v_i^*+w_i^*)\partial_{q_i}H_i + \bar{v}\partial_y H_i + \mu\partial_u + \varsigma - \psi(v_i^*+w_i^*)^2 & = 0,\\
\qquad\qquad H_i(T,x_i,f,y,q_i,u) & = x_i + q_i f - \beta q_i^2.
\end{cases}
$$

In this case, we recover the same HJB equation (4.59), previously solved in case where the HFTs can place an order in only the opposite side of the ETFs in the TASLOB. The following are optimal trading strategies for agent i who is liquidating initial shares $Q_i(0) > 0$:

For $w_i(t) < 0$ **and** $0 \le t \le t^*$

$$\begin{aligned} v_i^*(t) &= \frac{\delta}{2a}, \\ w_i^*(t) &= \frac{1}{2\psi}\Big(-\delta + h_1^0(t) + h_1^1(t)y + 2h_2(t)q_i(t)\Big) - \frac{\delta}{2a}, \end{aligned}$$

where

$$h_1^1(t) = \begin{cases} -\frac{(\lambda-\lambda_\mu)}{2} + \frac{2\psi}{T-t+\frac{\psi}{\beta}} + \frac{(\lambda-\lambda_\mu)}{2}\tanh(At+B_1) & \text{for} \quad \beta \le \frac{(\lambda-\lambda_\mu)}{2}, \\ -\frac{(\lambda-\lambda_\mu)}{2} + \frac{2\psi}{T-t+\frac{\psi}{\beta}} + \frac{(\lambda-\lambda_\mu)}{2\tanh(At+B_2)} & \text{for} \quad \beta > \frac{(\lambda-\lambda_\mu)}{2}, \end{cases}$$

and

$$h_1^0(t) = \begin{cases} k_1 e^{At}\cosh(At+B_1) - \delta e^{2(At+B_1)} & \text{for} \quad \beta \le \frac{(\lambda-\lambda_\mu)}{2} \\ k_1 e^{At}\sinh(At+B_2) + \delta e^{2(At+B_2)} & \text{for} \quad \beta > \frac{(\lambda-\lambda_\mu)}{2}, \end{cases}$$

$$\begin{aligned} A &:= \frac{(\lambda-\lambda_\mu)}{4\psi}, \\ B_1 &:= -\frac{(\lambda-\lambda_\mu)}{4\psi}T + \tanh^{-1}\left(1 - \frac{4\beta}{(\lambda-\lambda_\mu)}\right), \\ B_2 &:= -\frac{(\lambda-\lambda_\mu)}{4\psi}T + \frac{1}{2}\log\left(1 - \frac{(\lambda-\lambda_\mu)}{2\beta}\right), \\ k_1 &= \frac{\delta e^{(AT+2B_1)}}{\cosh(AT+B_1)}, \\ k_2 &= -\frac{\delta e^{(AT+2B_2)}}{\sinh(AT+B_2)}. \end{aligned}$$

Moreover, the mean trading speed is

$$\bar{v}(t) = \frac{1}{2\psi}\Big(-\delta + h_1^0(t) + \Big(h_1^1(t) + 2h_2(t)\Big)y(t)\Big).$$

For $w_i(t) \geq 0$ **and** $0 \leq t \leq t^*$

$$v_i^*(t) = 0,$$

$$w_i^*(t) = 0.$$

Note that we want to find the optimal control for time $0 \leq t \leq T$; thus, we want to find t^* to switch the trading strategies from region $\mathcal{S}$ to region $\mathcal{S}^c$. If we have the closed-form expression for the optimal trading strategies for both regions, it is possible to solve for t^* explicitly. However, the optimal strategy $w_i^*(t)$ for $t^* < t \leq T$ is not in the closed-form expression as we do not have the closed-form solution for $h_1^0(t)$. Therefore, we propose the algorithm in the next section to numerically search for t^*.

4.4.3 Searching for t^*

Since we do not have the closed-form solution, we present the following algorithm to find t^*. The idea is that we generate the sequence of times that we treat them as our candidates for t^*, say $0 < \tau_1 < \tau_2 < ... < \tau_i < ... < \tau_{n-1} < \tau_n = T$. Then, we approximate the solution by testing each τ_i as t^*, and compare the value function of τ_i to find t^*.
Given the set of parameters $\{a, \delta, \beta, \psi, T, \lambda, \lambda_\mu\}$, and for $m < n \in \mathbb{N}$ with n is evenly divisible by m.

- Initialize $Q_i(t_0), y(t_0), \frac{\Pi_0}{N}$

- Set $\Delta_1 := \frac{T}{m}$, and $\Delta_2 := \frac{T}{n}$. m is the number of candidates for t^*; thus, Δ_1 is the distance between the two nearest candidates. Δ_2 is the time step size. Next, generate the Brownian motion from a normal distribution with a mean of 0 and a standard deviation of $\sqrt{\Delta_2}$. We generate the Brownian motion in this step to make sure that we fix the randomization procedure as we have to compare the results later.

- Let $\tau_{i+1} = \tau_i + \Delta_1$. For each $\tau_i \in \{\tau_1 = 0, \tau_2, ..., \tau_m = T\}$,

1. Let $t_{j+1} = t_j + \Delta_2$.
2. For $t_j \in [0, \tau_i]$, set $Q_i(t_0) = \frac{\Pi_0}{N}$, and do the following step using the solutions obtained when $t \in \mathcal{S}$
   - Calculate $h_1^0(t_j), h_1^1(t_j)$, and $h_2(t_j)$
   - Calculate $v_i^*(t_j), w_i^*(t_j)$, and $\bar{v}(t_j)$ to obtain $\bar{X}(t_j), \bar{F}(t_j), y(t_j)$, and $Q_i(t_j)$. This step we obtain $\bar{X}(\tau_i), \bar{F}(\tau_i), y(\tau_i)$, and $Q_i(\tau_i)$
3. For $t_j \in [\tau_i, T]$, set $Q_i(\tau_i) := Q_i(0) - Q_i(\tau_i)$, and do the following step using the solutions obtained when $t \in \mathcal{S}^c$
   - Calculate $h_1^0(t_j), h_1^1(t_j)$, and $h_2(t_j)$
   - Calculate $v_i^*(t_j), w_i^*(t_j)$, and $\bar{v}(t_j)$ to obtain $\bar{X}(t_j), \bar{F}(t_j), y(t_j)$, and $Q_i(t_j)$. This step we obtain $\bar{X}(T), \bar{F}(T), y(T)$, and $Q_i(T)$

- Calculate the value function for each τ_i and obtain τ_i to give the maximum value function. This τ_i is our t^*.

4.4.4 Numerical results

This section shows numerical results of agent i who wants to liquidate 1000 shares of VIX futures by the end of the day using RLOB and TASLOB, with the initial mean inventory at 200 shares, and the ETFs is buying in TASLOB. However, the maximum inventory agent i can sell in TASLOB is up to 900 shares during the time ETFs are still in the market.

Since the number of shares agent i can sell in TASLOB at the price ETFs are buying is less than the number of shares she plans to sell, her optimal strategy is that she will sell up to the number of shares available to sell at the price ETFs are buying in TASLOB using the strategy when $t \in \mathcal{S}$. She then switches to the strategy when $t \in \mathcal{S}^c$ to liquidate the remaining shares. The optimal time t^* to switch the strategies can be found using the algorithm discussed in searching for t^*. Figure 4.8 is a plot of terminal value against the time t_i^* when the agent i chooses to switch the strategy from optimal strategies in $0 \leq t \leq t_i^*$ to optimal strategies in

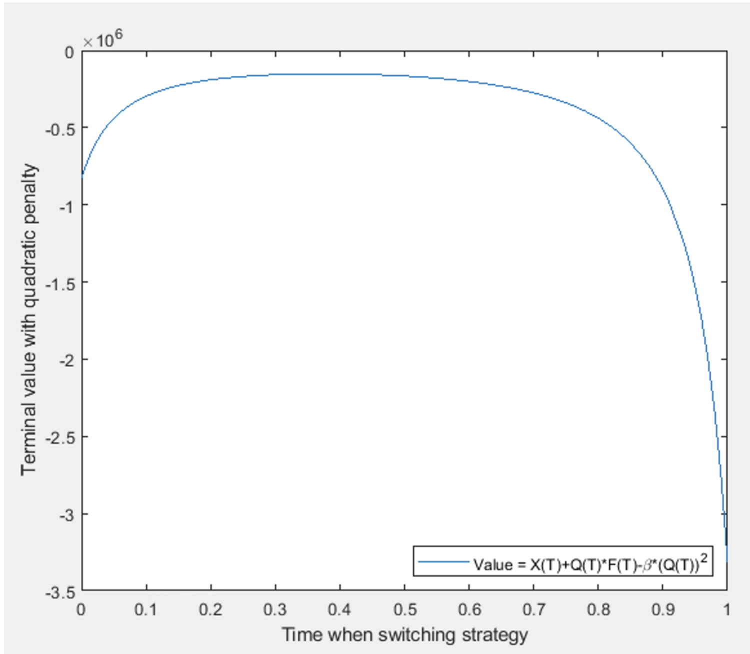

Figure 4.8: The terminal value function when we switch strategies at different time t^*

$t_i^* < t \leq T$ when liquidating 1000 shares by time $T = 1$. The plot shows that the optimal t_i^* that yields the maximum terminal value is when $t_i^* = 0.38$. Figure 4.9 displays the terminal

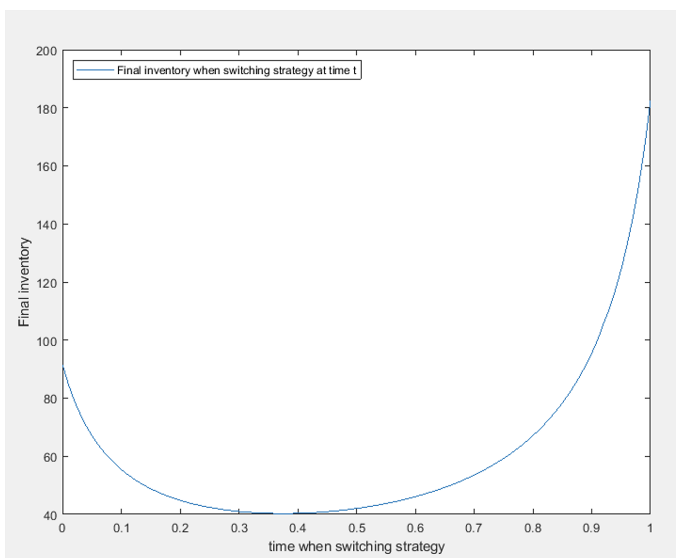

Figure 4.9: The terminal inventory when we switch strategies at different time t^*

inventory agent i has at terminal time $T = 1$ when switching strategies at time t_i^*. We find that the terminal inventory of agent i is minimum if she switches the strategies at time

$t_i^* = 0.38$. Thus, we conclude that the optimal time to switch strategies is when $t^* = 0.38$. Figure 4.10 illustrates the inventory paths of agent i when she chooses the optimal switching time t^* (blue curve) compared to times if she chooses to switches them before (red line) or after (yellow curve) t^*.

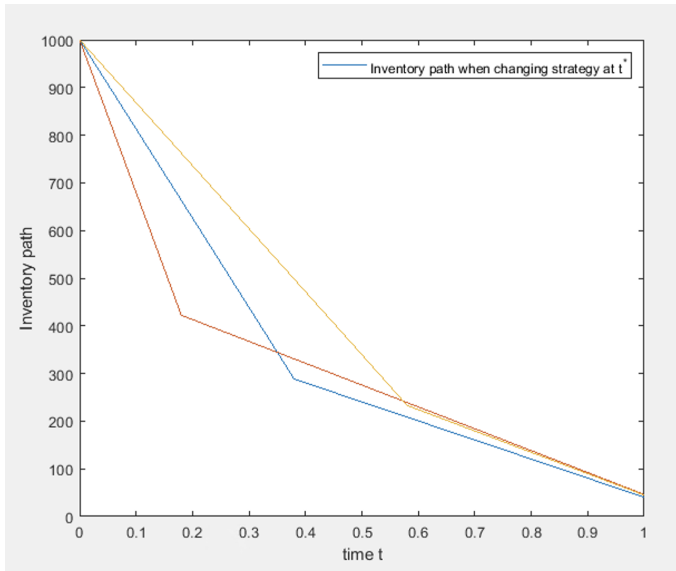

Figure 4.10: Inventory path of agent i when we switch strategy at different time

4.5 Conclusion

In this chapter, we solved the optimal liquidation problem in the VIX futures market using the framework of mean-field games, where traders can place an order in not only one but two order books. In our mean-field game setting, the interaction between traders is through $\bar{v}$, the price impact of all traders have on the drift of the fundamental price process. The impact each trader has on the fundamental price process is negligible, but the overall actions of all traders are not.

In section 4.2, we solved the optimal liquidation problem of a representative HFT agent i when there are only two types of traders in the VIX futures market, the HFTs, and noise traders. The optimal strategy for the representative agent i is liquidating her shares using both order books. The optimal trading rate in RLOB is constant, whereas the optimal trading rate in TASLOB is not. Although trades in TASLOB do not suffer from temporary price impact, they do suffer from the permanent price impact. As the trading time is far from the terminal time, the representative agent i will avoid the effect of price uncertainty by liquidating her positions slowly to ensure that she gets the best price possible. As the trading time is close to terminal time, she will trade faster to ensure that her terminal inventory is close to zero, and also, there is a lower risk of price uncertainty at that time.

In section 4.3, we solved the optimal liquidation problem of a representative agent i with ETFs in the market that provide liquidity on the buy-side in the TASLOB. We found that with the existence of the ETFs, the representative agent i will liquidate her shares using TASLOB while acquiring shares in RLOB. We argue that HFTs knew the information about acquiring a large number of shares in TASLOB by ETFs, where the transaction price is usually above mid-price. They benefited from this information by buying shares in RLOB at a constant rate and selling their positions to the ETFs in TASLOB.

In section 4.4, we solved the optimal liquidation problem of same representative agent i in a similar setting as the previous section, but where the liquidity provided by the ETFs is limited. We found that the representative agent i will use the optimal strategy found in section 4.3 to liquidate her positions, then switch to the optimal strategy found in section 4.2. The critical part of this problem is finding the optimal time to switch the strategy. We define the optimal switching time as the switching time that maximizes the value function at terminal time T, where we found the optimal time to switch the strategy using the proposed algorithm in 4.4.3.

Chapter 5

Conclusions and Future Work

5.1 Conclusions

This dissertation consists of two parts. In the first part, we conduct an empirical study investigating the price impact of individual trades in the VIX futures market. In the second part, we develop the mean-field game model for optimal liquidation in the VIX futures market, where traders can trade in two order books, RLOB and TASLOB.

In Chapter 3, we examined the price impact on the VIX futures market for futures comprising five months VIX futures contracts. The results show that all impact curves are concave, with points clustered together when normalized volumes are small. The period after 4:00 p.m. exhibits the lowest price impact compared to all other curves. Besides, each impact curve fits the model $I(\omega) = \alpha \cdot \omega^{\beta}$ well. We also found that the price impact of VIX futures is the lowest in the last fifteen minutes of regular trading hours. This time concurs with the period where VIX ETPs should rebalance their positions, which supplies additional liquidity to the market, making the price impact around this time lower than the other trading time. We deduce that the lower price impact at the end of regular trading time is evidence of the consequences of the architecture of VIX ETPs on VIX futures. Besides, by following a similar argument discussed by Kyle and Obizhaeva (2016) with the different assumption on the dimension of the σ^2, the general formula for price impact obtained theoretically agrees with our price impact model obtained from the data when setting $c = \alpha$, and $\beta = H$, where H is the Hurst exponent.

In chapter 4, we solved the optimal liquidation problem in the VIX futures market using the framework of the mean-field game, where traders can place an order in not only one but two order books. In our mean-field game setting, the interaction of traders is via $\bar{v}$, the price impact of all traders have on the drift of the fundamental price process. In section 4.2, we solved the optimal liquidation of representative HFT agent i when there are only two types of traders in the VIX futures market, the HFTs, and noise traders. The optimal strategy for the representative agent i is liquidating her shares using both order books. She trades more in TASLOB as there is no temporary price impact in this order book compared to the RLOB. In section 4.3, we solved the optimal liquidation of representative agent i when we introduced ETFs into the market, and they provide the liquidity on the buy-side in the TASLOB. We found that with the existence of the ETFs, the representative agent i will liquidate her shares using TASLOB while acquiring shares in RLOB. We argue that HFTs knew the information about acquiring a large number of shares in TASLOB by ETFs, where the transaction price is usually above mid-price. They benefited from this information by buying shares in RLOB and selling them to the ETFs in TASLOB. In section 4.4, we solved the optimal liquidation of representative agent i in a similar setting as in 4.3, but the liquidity provided by the ETFs is limited. We found that the representative agent i will use the optimal strategy found in section 4.3 to liquidate her positions, then switch to the optimal strategy found in section 4.2. The critical part of this problem is when to switch the strategy, and we found the optimal time to switch the strategy using the proposed algorithm in 4.4.3.

5.2 Future Work

The temporary price impact we used to model the execution price process in Chapter 4 is constant. However, in Chapter 3, we found that the temporary price impact in the VIX futures market is not constant. One can extend the optimal execution problem to incorporate the non-linear price impact into the model.

As mentioned in section 4.1, the permissible price range investors are allowed to place is $\pm$ 0.50 index point of the daily settlement price with a minimum price increment for TAS transactions is 0.01 index points. One can relax the assumption of modeling execution price in TASLOB to incorporate this increment into the model and introduce the probability of the order being filled depending on the order book depth. One could also consider modeling trades in TASLOB as dark pools trading. This trading venue does not display the bid and ask prices, but the trades may occur continuously as soon as there is a matching order.

We introduced the ETFs to the model as the liquidity providers in TASLOB. We can extend their role as a major agent in the VIX futures market, where they can also trade in both RLOB and TASLOB and solve the optimal execution problem for both ETFs and HFTs via the mean-field game framework.

Appendix A

The Impact Curves of June-October 2019 VIX Futures

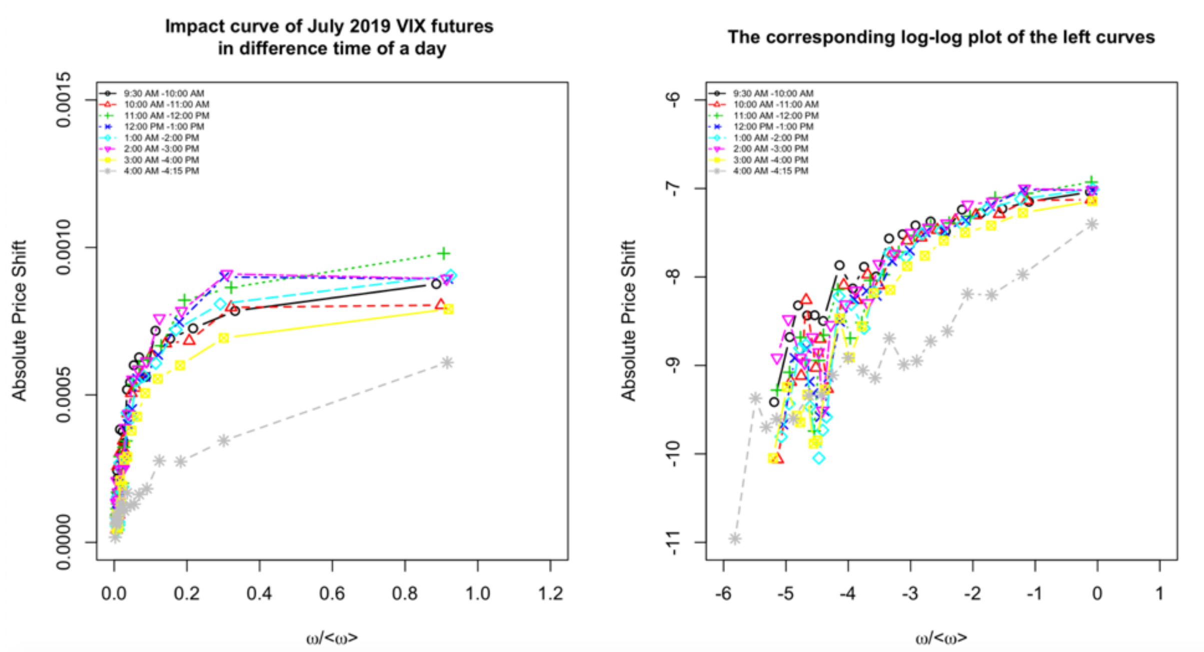

Figure A.1: Impact curves of July 2019 VIX futures traded at different times of the day. The left plot is the plot of the price impact against the normalized transaction price, and the right plot is the corresponding log-log plot of the left curves.

Trading times	b_0 estimate	b_1 estimate	b_2 estimate	p-value for b_2
09:00 AM - 10:00 AM	-6.5122	0.5422	0.4320	0.0001
10:00 AM - 11:00 AM	-6.4646	0.5461	0.1572	0.1681
11:00 AM - 12:00 PM	-6.4635	0.5468	0.1670	0.1430
12:00 PM - 01:00 PM	-6.4426	0.5470	0.0073	0.9492
01:00 PM - 02:00 PM	-6.4342	0.5467	-0.0671	0.5573
02:00 PM - 03:00 PM	-6.4736	0.5473	0.2628	0.0205
03:00 PM - 04:00 PM	-6.4144	0.5457	-0.2548	0.0248
04:00 PM - 04:15 PM	-6.3718	0.5417	-0.7042	4.56E-11

Table A.1: *The estimates of b_0, b_1, b_2 of using model $y = b_0 + b_1 x + b_2 1_{x \in A}$ to July 2019 VIX futures price impact data at different trading times.*

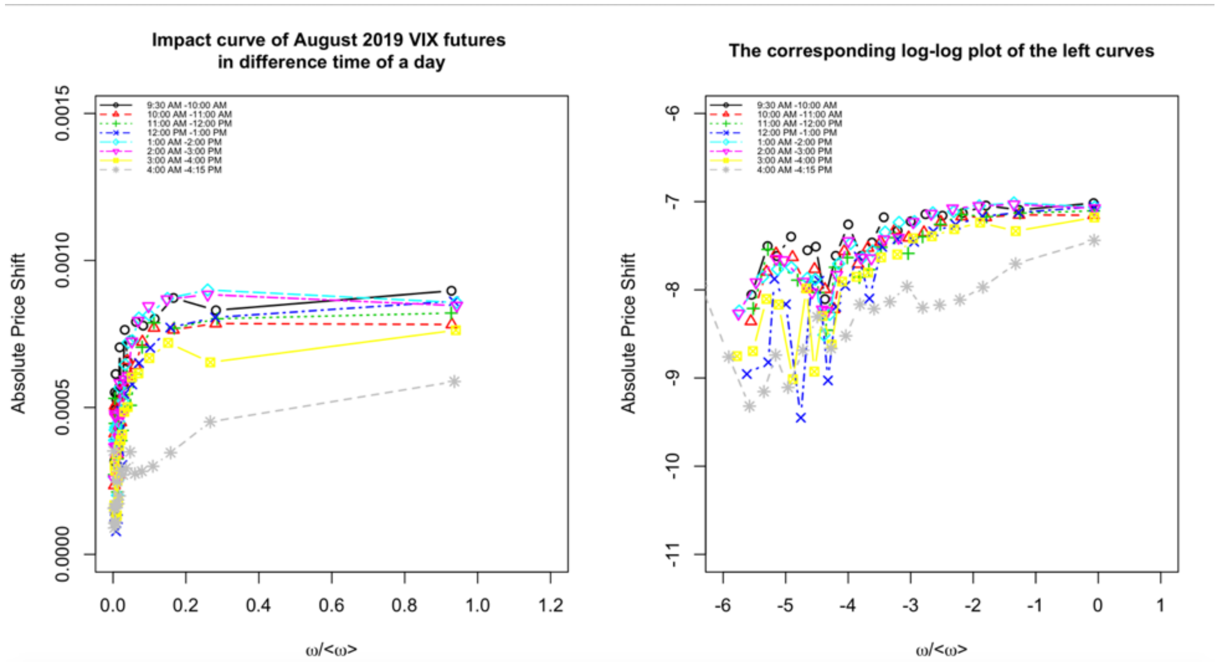

Figure A.2: Impact curves of August 2019 VIX futures traded at different times of the day. The left plot is the plot of the price impact against the normalized transaction price, and the right plot is the corresponding log-log plot of the left curves.

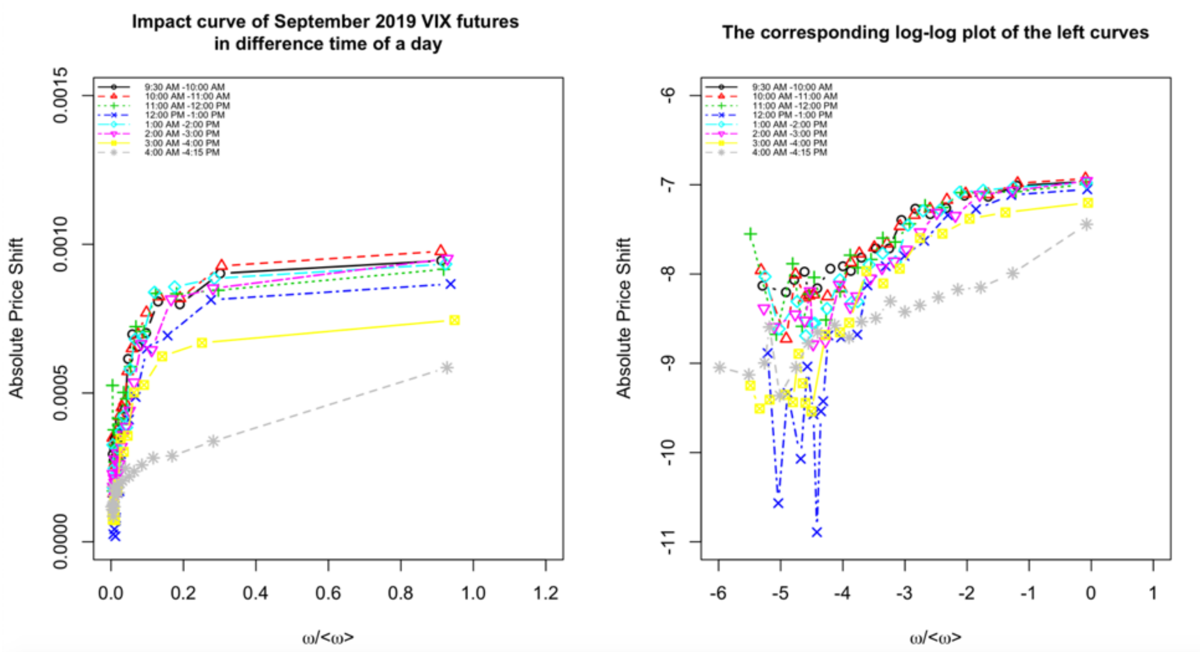

Figure A.3: Impact curves of September 2019 VIX futures traded at different times of the day. The left plot is the plot of the price impact against the normalized transaction price, and the right plot is the corresponding log-log plot of the left curves.

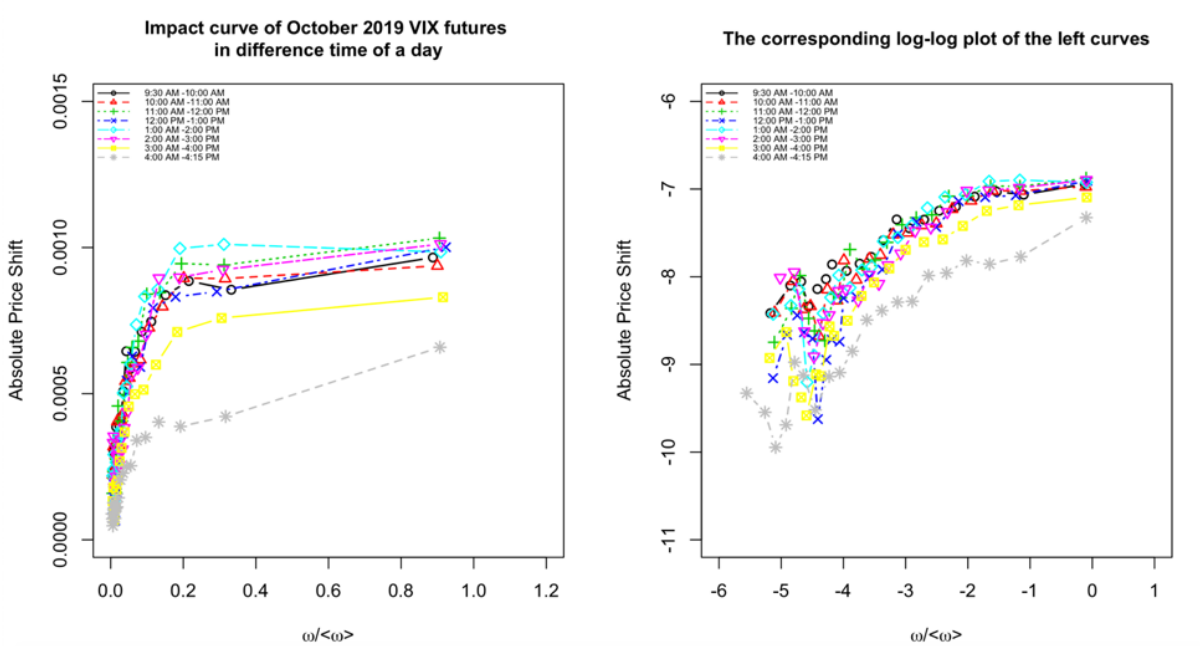

Figure A.4: Impact curves of October 2019 VIX futures traded at different times of the day. The left plot is the plot of the price impact against the normalized transaction price, and the right plot is the corresponding log-log plot of the left curves.

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