STRONG COMPARISON PRINCIPLE FOR $p-$HARMONIC FUNCTIONS IN CARNOT-CARATHEODORY SPACES

LUCA CAPOGNA AND XIAODAN ZHOU

ABSTRACT. We extend Bony's propagation of support argument [2] to C^1 solutions of the non-homogeneous sub-elliptic $p-$Laplacian associated to a system of smooth vector fields satisfying Hörmander's finite rank condition. As a consequence we prove a strong maximum principle and strong comparison principle that generalize results of Tolksdorf [7].

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and connected set, and consider a family of smooth vector fields $X_1, \cdots, X_m$ in $\mathbb{R}^n$ satisfying Hörmander's finite rank condition [6],

$$\text{rank Lie}[X_1, \cdots, X_m](x) = n, \tag{1.1}$$

for all $x \in \Omega$. We set $Xu = (X_1 u, \cdots, X_m u)$ for any function $u : \Omega \to \mathbb{R}$ for which the expression is meaningful.

In this paper we will prove a strong comparison principle for solutions of the class of quasilinear, degenerate elliptic equations

$$L_p u = \sum_{j=1}^{m} X_j^*(A_j(Xu)) = f(x, u), \tag{1.2}$$

satisfying the structure conditions (3.1), and which includes the $p-$Laplacian, in the range $p > 1$, associated to $X_1, ..., X_m$ and to the Lebesgue measure dx in $\mathbb{R}^n$. Note that in (1.2) we have let $X_j^* = -X_j + d_j(x)$ denote the L^2 adjoint of the operator X_j with respect to the Lebesgue measure. Here d_j is a smooth function obtained as the trace of X_j. We explicitly note that all the results in this paper continue to hold if one substitutes the Lebsgue measure dx with any other measure $d\mu = \lambda(x)dx$ with $\lambda \in C^1$ density function. In particular the results apply in any subRiemannian manifold, for solutions of the subelliptic $p-$Laplacian associated to a smooth volume form.

In addition to the structure conditions (3.1), our strong comparison principle holds under the following hypothesis:

$$\begin{aligned} &(i) \quad \partial_u f \leq 0 \text{ in } \Omega, \\ &(ii) \quad |f(x, u_2 + \epsilon) - f(x, u_2)| \leq L\epsilon, \text{ for any } \epsilon \in [0, \epsilon_0], x \in \Omega \end{aligned} \tag{1.3}$$

Date: July 21, 2017.
Key words and phrases. subelliptic PDE, strong comparison principle, p-harmonic functions, sub-Riemannian geometry
MSC: 35H20, 35B50
L.C. was partially funded by NSF awards DMS 1449143 and DMS 1503683. X. Z. was partially funded by an AMS-Simons Travel Grant.

for some positive constants L, ϵ_0. Our main result is the following

Theorem 1 (Strong Comparison Principle). *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and consider two weak solutions $u_1 \in C^1(\bar{\Omega})$, and $u_2 \in C^2(\bar{\Omega})$ of* (1.2) *in Ω, with $|Xu_2| \geq \delta$ in Ω. We assume that the structure conditions* (3.1)*, and the hypothesis* (1.3) *are satisfied. If*

$$u_1 \geq u_2 \text{ in } \Omega,$$

then either $u_1 = u_2$ or

$$u_1 > u_2 \text{ in } \Omega.$$

As it will be evident from the proof, the regularity assumptions and the lower bound on $|Xu_2|$ are required only in a neighborhood of the contact set. The lower bound is not required in the non-degenerate case $\kappa > 0$.

Bony's method can also be used to establish a non-homogenous strong maximum principle. We suppose that f satisfy the following conditions: for all $x \in \Omega$ and $u \in \mathbb{R}$,

$$\begin{aligned} &(i) \quad \partial_u f \leq 0, \\ &(ii) \quad |f(x,u)| \leq \bar{C}(\kappa + |u|)^{p-2}|u| \end{aligned} \tag{1.4}$$

for some positive constant $\bar{C}$ and κ as in the structure conditions (3.1).

Theorem 2 (Strong Maximum Principle). *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and consider a weak solution $u \in C^1(\bar{\Omega})$ of* (1.2) *in Ω. We assume that the structure conditions* (3.1) *and the hypothesis* (1.4) *hold. If*

$$u \geq 0 \text{ in } \Omega,$$

then either $u = 0$ or

$$u > 0 \text{ in } \Omega.$$

The proof of these results is at the end of Section 3. Theorem 1 and Theorem 2 extend to the subelliptic setting the strong maximum and comparison principles proved by Tolksdorff in [7, Propositions 3.2.2 and 3.3.2].

In the subelliptic setting Theorem 1 seems to be new even in the homogeneous case $f = 0$. In terms of previous literature on this subject: we recall that the case $p = 2$ was established through geometric methods by Bony in his landmark paper [2]. A proof of the strong maximum principle for the subelliptic $p-$Laplacian in $H-$type groups can be found in [8]. We note however that at the conclusion of that proof the authors claim that one can always fit a gauge ball tangentially at every point of the set where the solution attains the maximum. This statement is not proved in [8], and since gauge balls have zero curvature at the poles, we do not seem how it can be proved.

A strong comparison and maximum principle for smooth solutions of the subelliptic p-Laplacian and of the horizontal mean curvature operator has been recently proved by Cheng, Chiu, Hwang and Yang in their preprint [4]. Their proof is based on a linearization approach which is different from our arguments, however it also ultimately relies on Bony's argument, and holds in every subRiemannian manifold. In comparison to the present paper, on the one hand our results hold for solutions which do not have to be smooth necessarily[1], but for the

[1]We recall that in general $p-$harmonic functions do not enjoy more regularity than the Hölder continuity of their gradient.

comparison principle we require one of the two solutions to have non-vanishing horizontal gradient. On the other hand while we only deal with the $p-$Laplacian, in [4] the authors also establish far reaching results for the mean curvature operator, including some special cases where $|Xv_2|$ is allowed to vanish in a controlled fashion and still have a comparison principle.

The technical core of the proofs in the present paper is in Lemma 7 and consists in an adaptation of Bony's argument to our nonlinear setting. Note that, as in the Euclidean setting, one cannot relax the conditions on u_1, u_2 and f unless more hypothesis are added.

In closing we note that both in the elliptic and in the subelliptic case, a corresponding strong maximum principle for the homogenous problem, $f = 0$, can be established immediately from the Harnack inequality (see for instance [1], [5], [3]), as well as with small modifications of the argument presented here. However, while in the linear setting one can deduce the strong comparison principle from the strong maximum principle, this is no longer the case in the nonlinear setting, where a new approach is needed.

2. Bony's propagation of support technique

Tolksdorf's argument in [7, 3.3.2] breaks down in the subelliptic setting, due to the fact that the horizontal gradient of the barrier functions typically used in this proof may vanish. The same problem occurs also in the linear setting, for $p = 2$. To deal with this issue we follow the outline of the proof of the strong maximum principle for subLaplacians, from Bony's paper [2], and adapt it to our non-linear and non-homogeneous setting.

We begin by recalling from [2, Definition 2.1], the definition of a nonzero vector $\mathbf{v}$ orthogonal to a set $F \subset \mathbb{R}^n$ at a point $y \in \partial F$.

Definition 1. *Let F be a relatively closed subset of Ω. We say that a vector $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$ is (exterior) normal to F at a point $y \in \Omega \cap \partial F$ if*

$$\overline{B(y + \mathbf{v}, |\mathbf{v}|)} \subset (\Omega \setminus F) \cup \{y\}.$$

If this inclusion holds, we write $\mathbf{v} \perp F$ at y. Set

$$F^* = \{y \in \Omega \cap \partial F \colon \textit{there exists } \mathbf{v} \textit{ such that } \mathbf{v} \perp F \textit{ at } y\}.$$

Note when Ω is connected and $\emptyset \neq F \neq \Omega$, we have $F^* \neq \emptyset$.

We list in the following some of the results and definitions from [2] that play a role in our proof.

Definition 2. *Let X be vector field in Ω and $F \subset \Omega$ be a closed set. We say that X is tangent to F if, for all $x_0 \in F^*$ and all vectors v normal to F at x_0 one has that their Euclidean product vanishes, i.e. $\langle X(x_0), v\rangle = 0$.*

The following results are from [2, Theoreme 2.1], and [2, Theoreme 2.2]:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $F \subset \Omega$ a closed subset. Let X be a Lipschitz vector field in Ω. If X is tangent to F then all its integral curves that intersect F are entirely contained in F.*

Note that the converse of this result is also true, and follows from a direct computation.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $F \subset \Omega$ a closed subset. Let $X_1, ..., X_m$ be smooth vector fields in Ω. If $X_1, ..., X_m$ are tangent to F then so is the Lie algebra they generate.*

As a corollary, if $X_1, ..., X_m$ satisfy Hörmander finite rank condition (1.1) and are all tangent to F then every curve that touches F is entirely contained in F, so that either F is the empty set or $F = \Omega$.

3. A Hopf-type comparison principle and proof of Theorem 1

First we state precisely the structure conditions imposed on the left hand side of (1.2). The functions A_j satisfy the following ellipticity and growth condition: For $p > 1$, for a.e. $\xi \in \mathbb{R}^m$ and for every $\eta \in \mathbb{R}^m$,

$$\begin{aligned} \sum_{i,j=1}^m \frac{\partial A_j}{\partial \xi_i}(\xi)\eta_i\eta_j &\geq \beta(\kappa + |\xi|)^{p-2}|\eta|^2 \\ \sum_{i,j=1}^m |\frac{\partial A_j}{\partial \xi_i}(\xi)| &\leq \gamma(\kappa + |\xi|)^{p-2} \end{aligned} \tag{3.1}$$

for some positive constants β, γ, κ.

One can easily deduce that there exists positive constant λ, C such that for all $\xi \in \mathbb{R}^m$,

$$\langle A_j(\xi) - A_j(\xi'), \xi - \xi' \rangle \geq \lambda \begin{cases} (1 + |\xi| + |\xi'|)^{p-2}|\xi - \xi'|^2 & \text{if} \quad p \leq 2 \\ |\xi - \xi'|^p & \text{if} \quad p \geq 2, \end{cases} \tag{3.2}$$

and

$$|A_j(\xi)| \leq C(\kappa + |\xi|)^{p-2}|\xi|.$$

The subelliptic $p-$Laplacian

$$L_p u = \sum_{j=1}^m X_j^*(|Xu|^{p-2}X_j u),$$

corresponds to the choice $A_j(\xi) = |\xi|^{p-2}\xi_j$ for $j = 1, \cdots, m$.

We will need the following immediate consequence of the monotonicity inequality (3.2).

Lemma 5 (Weak Comparison Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and $v_1, v_2 \in C^1(\Omega)$ satisfy in a weak sense*

$$\begin{cases} L_p v_2 \leq f(x, v_2) & \text{in } \Omega \\ L_p v_1 \geq f(x, v_1) & \text{in } \Omega, \end{cases} \tag{3.3}$$

with A_j satisfying the structure conditions (3.1) *and $\partial_u f(x, u) \leq 0$. If $v_2 \leq v_1$ in $\partial\Omega$, then $v_2 \leq v_1$ in Ω.*

Proof. Given an arbitrary $\epsilon > 0$, we define $E_\epsilon = \{x \in \Omega | v_2(x) > v_1(x) + \epsilon\}$. Assume that $E_\epsilon \neq \emptyset$, then $\overline{E_\epsilon} \subset \Omega$. For all $\varphi \in C_c^1(\Omega)$, we have

$$\int_\Omega \langle A_j(Xv_2), X\varphi \rangle \leq \int_\Omega f(x, v_2)\varphi,$$

$$\int_\Omega \langle A_j(Xv_1), X\varphi \rangle \geq \int_\Omega f(x, v_1)\varphi.$$

Subtracting the above two inequalities and setting $\varphi(x) = \max\{v_2(x) - v_1(x) - \epsilon, 0\}$ then as a consequence of (ii) in (1.3), one has

$$\int_{E_\epsilon} \langle A_j(Xv_2) - A_j(Xv_1), X(v_2 - v_1)\rangle \leq \int_{\{v_2 > v_1 + \epsilon\}} (f(x, v_2) - f(x, v_1))(v_2 - v_1 - \epsilon) \leq 0.$$

By (3.2), this inequality holds if and only if $X(v_2 - v_1) = 0$. Thus, $v_2 = v_1 + C$ in E_ϵ. The fact that $v_2 = v_1 + \epsilon$ on ∂E_ϵ implies that $C = \epsilon$. It follows that $v_2 \leq v_1 + \epsilon$ in Ω. Let $\epsilon \to 0$, we get $v_2 \leq v_1$ in Ω. □

Next, we prove an analogue of the classical Hopf comparison principle: Given a subsolution v_2 and a supersolution v_1 such that $v_2 \leq v_1$, then every vector field $X_1, ..., X_m$ must be tangent to the contact set $F = \{v_2 = v_1\}$.

Lemma 6. *(A Hopf-type Comparison Principle) Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and $v_1 \in C^1(\Omega)$, $v_2 \in C^2(\Omega)$ with $|Xv_2| \geq \delta$ in Ω satisfy*

$$\begin{cases} v_2 \leq v_1 & \text{in } \Omega \\ L_p v_2 \leq f(x, v_2) & \text{in } \Omega \\ L_p v_1 \geq f(x, v_1) & \text{in } \Omega. \end{cases} \tag{3.4}$$

Set $F = \{x \in \Omega : v_2(x) = v_1(x)\}$. If the structure conditions (3.1) and hypothesis (1.3) are satisfied and $\emptyset \neq F \neq \Omega$, then for every $y \in F^*$ and $\mathbf{v} \perp F$ at y, it follows that

$$\langle X_i(y), \mathbf{v}\rangle = 0$$

for all $i = 1, \cdots, m$.

Proof. We argue by contradiction and suppose that there exists $y \in F^*$, a vector $\mathbf{v} \perp F$ at y, and $i \in \{1, \cdots m\}$ such that $\sigma_i(y) := \langle X_i(y), \mathbf{v}\rangle \neq 0$. We denote by $\sigma(x)$ the vector field $\sigma(x) = (\sigma_1(x), ..., \sigma_m(x))$, and note that for $\mathbf{v}$ fixed, this is a smooth vector field on Ω.

Let $z = y + \mathbf{v}$ and $r = |\mathbf{v}|$. We denote by $|x - z|$ the Euclidean distance between the points x, z and proceed to define $\tilde{b}(x) = e^{-\alpha|x-z|^2}$, and

$$b(x) = \alpha^{-2}(\tilde{b}(x) - e^{-\alpha r^2})$$

in Ω where the value of the positive constant α is to be determined later. Choose a neighborhood V of y such that $0 < |\sigma(x)|$ for $x \in \overline{V} \subset \Omega$ and denote by M_1, M_2, M_3, M_4 positive constants depending on v_2 and F, such that for every $x \in \overline{V}$ one has $|X_j\sigma_i(x)| \leq M_1$, $|X_jX_i(b + v_2)(x)| \leq M_2$, and $M_4 \leq |\sigma(x)| \leq M_3$ for $i, j = 1, \cdots, m$.

By a direct calculation, one can deduce

$$X_i b(x) = -2\alpha^{-1}\tilde{b}(x)\sigma_i(x),$$

$$|Xb(x)| = 2\alpha^{-1}\tilde{b}(x)|\sigma(x)| = 2\alpha^{-1}\tilde{b}(x)\big(\sum_{i=1}^m \sigma_i(x)^2\big)^{1/2},$$

$$X_jX_ib(x) = \tilde{b}(x)(4\sigma_j\sigma_i - 2\alpha^{-1}X_j\sigma_i(x)).$$

Substituting the identities above in the expression for $L_p b$ yields

$$L_p b(x) = -\sum_{j=1}^{m}\sum_{i=1}^{m} \frac{\partial A_j}{\partial \xi_i}(Xb)X_jX_ib + d_jA_j(Xb)$$
$$= -\tilde{b}(x)\sum_{i,j=1}^{m}\Big(4\frac{\partial A_j}{\partial \xi_i}(Xb)\sigma_j\sigma_i - 2\alpha^{-1}\frac{\partial A_j}{\partial \xi_i}(Xb)X_j\sigma_i\Big) + d_jA_j(Xb).$$

Applying the structure conditions (3.1) of A_j, it follows that for every $x \in \overline{V}$,

$$L_p b(x) = -\tilde{b}(x)\sum_{i,j=1}^{m}\Big(4\frac{\partial A_j}{\partial \xi_i}(Xb)\sigma_j\sigma_i - 2\alpha^{-1}\frac{\partial A_j}{\partial \xi_i}(Xb)X_j\sigma_i\Big) + d_jA_j(Xb)$$
$$\leq -\tilde{b}(x)\Big(4\beta(\kappa+|Xb|)^{p-2}|\sigma|^2 - 2\alpha^{-1}M_1\gamma(\kappa+|Xb|)^{p-2}\Big) + C(\kappa+|Xb|)^{p-2}|Xb|$$
$$= -\tilde{b}(x)(\kappa+|Xb|)^{p-2}\Bigg(4\beta|\sigma|^2 - 2\alpha^{-1}M_1\gamma - C\alpha^{-1}|\sigma(x)|)\Bigg).$$

Similarily,

$$\sum_{i,j=1}^{m}\frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_ib \geq \tilde{b}(x)(\kappa+|Xv_2|)^{p-2}\Bigg(4\beta|\sigma|^2 - 2\alpha^{-1}M_1\gamma\Bigg)$$
$$\geq \tilde{b}(x)(\kappa+|Xv_2|)^{p-2}\Bigg(4\beta M_4^2 - 2\alpha^{-1}M_1\gamma\Bigg).$$

In view of the non-vanishing hypothesis on $|Xv_2|$, there exist α_1 and a positive constant ϵ_1 such that for $\alpha \geq \alpha_1$ and $x \in \overline{V}$

$$|Xb(x)| \leq \frac{1}{2}|Xv_2(x)|,$$
$$L_p b(x) \leq 0,$$
$$\sum_{i,j=1}^{m}\frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_ib(x) \geq \epsilon_1\tilde{b}(x). \tag{3.5}$$

Since $A_j(\xi)$ is smooth in $\mathbb{R}^n \setminus \{0\}$, there exists positive constants C, ϵ_2 such that

$$\sum_{i,j=1}^{m}|\frac{\partial A_j}{\partial \xi_i}(X(b+v_2)) - \frac{\partial A_j}{\partial \xi_i}(Xv_2)| \leq C|Xb| \leq \epsilon_2\alpha^{-1}\tilde{b}(x) \tag{3.6}$$

for $x \in \overline{V}$. Thus,

$$
\begin{aligned}
L_p(b+v_2) &= -\sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(X(b+v_2))X_jX_i(b+v_2) + d_jA_j(Xb+Xv_2)\\
&= -\sum_{i,j=1}^{m} \Big(\frac{\partial A_j}{\partial \xi_i}(X(b+v_2)) - \frac{\partial A_j}{\partial \xi_i}(Xv_2) + \frac{\partial A_j}{\partial \xi_i}(Xv_2)\Big)X_jX_i(b+v_2) + d_jA_j(Xb+Xv_2)\\
&= -\sum_{i,j=1}^{m} \Big(\frac{\partial A_j}{\partial \xi_i}(X(b+v_2)) - \frac{\partial A_j}{\partial \xi_i}(Xv_2)\Big)X_jX_i(b+v_2)\\
&\quad - \sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_ib - \sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_iv_2 + d_jA_j(Xb+Xv_2)\\
&\le M_2\epsilon_2\alpha^{-1}\tilde{b}(x) - \epsilon_1\tilde{b}(x) + L_pv_2 - d_jA_j(Xv_2) + d_jA_j(Xb+Xv_2)\\
&\le (-\epsilon_1 + M_2\epsilon_2\alpha^{-1})\tilde{b}(x) + f(v_2) + |d_j||A_j(Xb+Xv_2) - A_j(Xv_2)|\\
&\le (-\epsilon_1 + M_2\epsilon_2\alpha^{-1} + C\alpha^{-1}|\sigma(x)|)\tilde{b}(x) + f(x,v_2)\\
&\le (-\epsilon_1 + M_2\epsilon_2\alpha^{-1} + C\alpha^{-1}|\sigma(x)|)\tilde{b}(x) + |f(x,b+v_2) - f(x,v_2)| + f(x,b+v_2)\\
\text{(By } (ii) \text{ in (1.3))} &\le (-\epsilon_1 + M_2\epsilon_2\alpha^{-1} + C\alpha^{-1}|\sigma(x)|)\tilde{b}(x) + L|b| + f(x,b+v_2)
\end{aligned}
$$

We can now choose $\alpha \ge \alpha_1$ such that $L_p(b+v_2) \le f(x,b+v_2)$ on $\overline{V}$.

Next, we let $U = V \cap B(z,r)$ and express its boundary as the union of two components

$$\partial U = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_1 = \overline{B(z,r)} \cap \partial V$ and $\Gamma_2 = \overline{V} \cap \partial B(z,r)$.

For $x \in \Gamma_1 \subset \Omega \setminus F$, we have $v_2(x) < v_1(x)$. Choose α be sufficiently large so that $v_2(x) + b(x) \le v_1(x)$ on Γ_1 and $L_p(v_2+b) \le f(x,b+v_2)$ on U. On the other hand, since $b(x) = 0$ when $x \in \Gamma_2$, then the estimate $v_2(x) + b(x) \le v_1(x)$ also holds on Γ_2. Thus one eventually obtains

$$
\begin{cases}
v_2 + b \le v_1 & \text{in } \partial U\\
L_p(v_2+b) \le f(x,b+v_2) & \text{in } U\\
L_pv_1 \ge f(x,v_1) & \text{in } U.
\end{cases}
\tag{3.7}
$$

The Weak Comparison Principle in Lemma 5 implies that $v_2 + b \le v_1$ in U. Since y is a maximum point of $v_2 - v_1$ in Ω, then necessarily its gradient at y must vanish, i.e. $\nabla(v_2 - v_1)(y) = 0$. Finally we invoke the C^1 regularity of v_1 near the contact set and we observe that

$$
\begin{aligned}
0 = \langle \mathbf{v}, \nabla(v_2 - v_1)(y)\rangle &= \lim_{t\to 0^+} \frac{v_2(y+t\mathbf{v}) - v_1(y+t\mathbf{v}) - (v_2(y) - v_1(y))}{t}\\
&\le -\langle \mathbf{v}, \nabla b(y)\rangle\\
&= -2\alpha^{-1}r^2e^{-\alpha r^2} < 0.
\end{aligned}
$$

Since we have arrived at a contradiction the proof is complete.

□

By a similar argument, a Hopf-type maximum principle can be established.

Lemma 7. *(A Hopf-type Maximum Principle) Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and $v \in C^2(\Omega)$ satisfy*

$$\begin{cases} v \geq 0 & \text{in } \Omega \\ L_p v \geq f(x,v) & \text{in } \Omega. \end{cases} \tag{3.8}$$

Set $F = \{x \in \Omega : v(x) = 0\}$. If the structure conditions (3.1) *and hypothesis* (1.4) *are satisfied and $\emptyset \neq F \neq \Omega$, then for every $y \in F^*$ and $\mathbf{v} \perp F$ at y, it follows that*

$$\langle X_i(y), \mathbf{v} \rangle = 0$$

for all $i = 1, \cdots, m$.

Proof. We argue by contradiction and suppose that there exists $y \in F^*$, a vector $\mathbf{v} \perp F$ at y, and $i \in \{1, \cdots m\}$ such that $\sigma_i(y) := \langle X_i(y), \mathbf{v} \rangle \neq 0$. We denote by $\sigma(x)$ the vector field $\sigma(x) = (\sigma_1(x), ..., \sigma_m(x))$, and note that for $\mathbf{v}$ fixed, this is a smooth vector field on Ω.

Let $z = y + \mathbf{v}$ and $r = |\mathbf{v}|$. We denote by $|x - z|$ the Euclidean distance between the points x, z and proceed to define $\tilde{b}(x) = e^{-\alpha|x-z|^2}$, and

$$b(x) = k(\tilde{b}(x) - e^{-\alpha r^2})$$

in Ω where the value of the positive constant k and α are to be determined later. Choose a neighborhood V of y such that $0 < |\sigma(x)|$ for $x \in \overline{V} \subset \Omega$ and denote by M_1, M_2, M_3 positive constants depending on v_2 and F, such that for every $x \in \overline{V}$ one has $|X_j \sigma_i(x)| \leq M_1$ and $M_2 \leq |\sigma(x)| \leq M_3$ for $i, j = 1, \cdots, m$.

Elementary calculations and hypothesis (1.4) show that for α sufficiently large,

$$\begin{aligned} L_p b(x) &= -\sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\partial A_j}{\partial \xi_i}(Xb) X_j X_i b + d_j A_j(Xb) \\ &\leq -k\tilde{b}(x)\alpha^2(\kappa + |Xb|)^{p-2} \left(4\beta|\sigma|^2 - 2\alpha^{-1}|X\sigma|\gamma - 2\sup_{\overline{V}} |d||\sigma(x)|\alpha^{-1} \right) \\ &= -k\tilde{b}(x)\alpha^2(\kappa + 2\alpha|\sigma(x)|k\tilde{b}(x))^{p-2} \left[4\beta M_2^2 - 2\alpha^{-1} M_1 \gamma - C M_3 \alpha^{-1} \right] \\ &\quad \text{(choosing } \alpha \text{ sufficiently large we may assume that the expression in brackets is larger than } M_2\beta) \\ &\leq -\alpha\beta|b(x)|(\kappa + |b(x)|)^{p-2} \\ &\leq -\bar{C}|b(x)|(\kappa + |b(x)|)^{p-2} \leq f(x, b(x)) \end{aligned}$$

for every $x \in \overline{V}$. Next, we let $U = V \cap B(z,r)$ and express its boundary as the union of two components

$$\partial U = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_1 = \overline{B(z,r)} \cap \partial V$ and $\Gamma_2 = \overline{V} \cap \partial B(z,r)$.

For $x \in \Gamma_1 \subset \Omega \setminus F$, we have $v(x) > 0$. Choose k be sufficiently small so that $b(x) \leq v(x)$ on Γ_1. On the other hand, since $b(x) = 0$ when $x \in \Gamma_2$, then the estimate $b(x) \leq v(x)$ also holds on Γ_2. Thus one eventually obtains

$$\begin{cases} b(x) \le v(x) & \text{in } \partial U \\ L_p(b) \le f(x,b) & \text{in } U \\ L_p v \ge f(x,v) & \text{in } U. \end{cases} \tag{3.9}$$

The Weak Comparison Principle in Lemma 5 implies that $b(x) \le v(x)$ in U. Since y is a minimum point of $v(x)$ in Ω, then necessarily its gradient at y must vanish, i.e. $\nabla v(y) = 0$. Finally we observe that in view of the C^1 regularity of v, one has

$$\begin{aligned} 0 = \langle \mathbf{v}, \nabla v(y) \rangle &= \lim_{t \to 0^+} \frac{v(y + t\mathbf{v}) - v(y)}{t} \\ &\ge \lim_{t \to 0^+} \frac{b(y + t\mathbf{v}) - b(y)}{t} \\ &= 2k\alpha r^2 e^{-\alpha r^2} > 0, \end{aligned}$$

arriving at a contradiction. □

In view of the Hopf-type comparison principle and of Theorem 4, we deduce that the contact set $F = \{v_2 = v_1\}$ must be either all of Ω or the empty set, thus completing the proof of the strong comparison principle in Theorem 1.

Likewise, the strong maximum principle theorem 2 follows from the Hopf-type maximum principle.

References

[1] Z. Balogh, I. Holopainen, J. Tyson, Singular solutions, homogeneous norms, and quasiconformal mappings in Carnot groups. *Math. Ann.* 324 (2002), no. 1, 159-186.

[2] J.-M. Bony, *Principe du maximum, inégalite de Harnack et unicité du probléme de Cauchy pour les operateurs elliptiques dégénérés.* (French) *Ann. Inst. Fourier (Grenoble)* 19 (1969), fasc. 1, 277-304 xii.

[3] L. Capogna, D. Danielli, N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, *Comm. Partial Differential Equations*,18 (1993), no. 9-10, 17651794.

[4] J-H Cheng, H-L Chiu, J-F Hwang, P. Yang, *Strong maximum principle for mean curvature operators on subriemannian manifolds*, arXiv:1611.02384 (2016), 1–43.

[5] J. Heinonen, I. Holopainen, *Quasiregular maps on Carnot groups*, *J. Geom. Anal.*, (1997), 7–109.

[6] L. Hörmander, *Hypoelliptic second order differential equations*, *Acta Math,* 119 (1967), 147–171.

[7] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points.* Comm. Partial Differential Equations 8 (1983), no. 7, 773-817.

[8] Z. Yuan, P. Niu, *A Hopf type principle and a strong maximum principle for the p-sub-Laplacian on a group of Heisenberg type*, *Journal of mathematical research and exposition*, 27 (2007), no. 3, 605-612.

Luca Capogna, Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA, lcapogna@wpi.edu

Xiaodan Zhou, Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA, xzhou3@wpi.edu