Theoretical Study of Variable Measurement Uncertainty h_I and Infinite Unobservable Entropy

by

Kevin Vanslette

A Thesis

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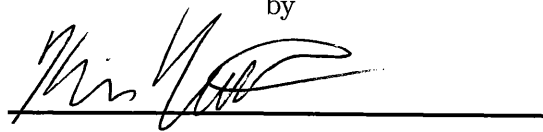
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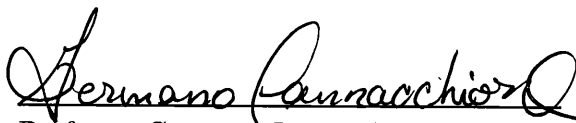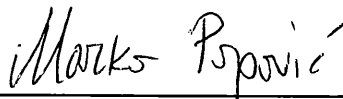
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Abstract

This paper examines the statistical mechanical and thermodynamical consequences of variable phase-space volume element $h_I = \triangle x_i \triangle p_i$. Varying h_I leads to variations in the amount of measured entropy of a system but the maximum entropy remains constant due to the uncertainty principle. By taking $h_u \to 0^+$ an infinite unobservable entropy is attained leading to an infinite unobservable energy per particle and an unobservable chemical equilibrium between all particles. The amount of heat fluxing though measurement apparatus is formulated as a function of h_I for systems in steady state equilibrium as well as the number of measured particles or sub-particles so any system can be described as unitary or composite in number. Some example systems are given using variable h_I.

Acknowledgements

I give great thanks and thoroughly appreciate Professor Germano Iannacchione for his support in me exploring an idea, advice/thoughts, and interest which helped make this thesis a reality. I also acknowledge Professor Marko Popovic for his help reading and clarifying this thesis as well as his support in the creative process during our time together at WPI. Finally I would like to give a special thanks to Melissa Rhodes, Andrew Vanslette, Philip Cox, Mason Ruffing, Stephen Hoffer, Anthony Gianfrancesco, and Rebecca Gaddis as well as my family who have provided advice, support, love, and ears that never got too sick of me constantly talking about information and entropy.

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Chapter 1

Introduction

Quantum Mechanics, statistical mechanics, and information physics have given great insight into the probabilistic nature of the universe at the fundamental level of particles and the microscopic behavior of large ensembles, respectively [3–6]. In quantum mechanics, a particle is described by its wavefunction, yielding a range of probabilistic particle states before measurement collapses the wavefunction into a specific state [3, 4]. In statistical mechanics, the probabilistic nature of particles arises from the propagation of uncertainties and practical unknowability (lack of exact knowledge) of the individual members of a large N ensemble [5, 6]. In both approaches, well-defined averages/expectation values exist despite the lack of exact knowledge or uncertainty about an individual particle or member of an ensemble [3–6]. The goal of this thesis is to combine well known notions of entropy with concepts of the uncertainty principle to formulate changes in the amount of information pertaining to the position and momentum of particles by changing measurement uncertainty, $h_I \geq \hbar/2$. Also a goal is to investigate the theory behind what we cannot measure due to the uncertainty principle.

1.1 Background

In the year 1900, Max Planck introduced the Planck's constant $h = 6.626 \times 10^{-34}$ J · s because he found the energy of light was proportional to it's frequency, $E = h\nu$ [7,8]. He also stated that to "interpret U_N [the vibrational energy of N oscillators] not as a continuous, infinitely divisible quantity, but as a discrete quantity composed of an integral number of finite equal parts.". Einstein later described a phenomena called the photoelectric effect by considering $E = h\nu$ to imply that light must be quantized. This gave rise to the wave-particle duality of light and particles [9], and the foundation of quantum mechanics. Heisenberg formulated the uncertainty principle

$$\frac{\hbar}{2} \leq \sigma_{x_i} \sigma_{p_{x_i}} \tag{1.1}$$

which states that the position and momentum in a given dimension cannot be measured to an absolute precision at the same instant in time [4]. The uncertainty principle lead to the development of quantum mechanics and the Schrdinger equation where Planck's constant appears explicitly in the momentum operator as well as it's solutions for the energy [3,4]. Recently Boyer has shown that $\hbar$ can be calculated from a classical formulation of the Casimir effect [**?**].

The entropy of a system of particles was first introduced into thermodynamics by Clausius as an extensive state function during his investigation of the Carnot Cycle [10]. Its purpose was to account for reversible and irreversible processes. The entropy, S, itself cannot be directly measured but can be inferred by measuring changes in heat, dQ, and the temperature, T,

$$dS = TdQ, \tag{1.2}$$

for a reversible process and is a statment of the 2nd Law of Thermodynamics **??**.

Entropy is defined in statistical mechanics, quantum mechanics, and information theory [3,6,11]. Gibbs and Boltzmann played a role in the development of the Gibb's Entropy,

$$S = -k_b \sum_{i=1}^{n} P_i \ln(P_i), \tag{1.3}$$

where P_i is the probability the system is in the ith microstate, n is the number of microstates, and k_b is the Boltzmann constant having the units of entropy [12]. The Gibbs entropy is generally thought of a measure of the amount of randomness pertaining to a system [5]. If we assume *a priori* that each microstate is equiprobable, $P_i = P = \frac{1}{\Omega}$ where Ω is the number of microstates, then the Gibbs entropy can be written in a reduced form

$$S = k_b \ln \left(\frac{1}{P} \right), \tag{1.4}$$

which is sometimes called the Boltzmann entropy. For a microcanonnical system the equiprobable assumption is $p_i = \frac{h}{V}$ where $V = V_q V_p$ is the phase-space volume and is a product of the spacial and momentum volumes where the momentum volume is bound by the energy of the system. In this case h is a phase-space element that breaks phase-space up into a number of possible states, $\Omega = \frac{V}{h}$, and is treated like an arbitrary constant [5]. Taking derivatives of this function with respect to some of it's arguments gives values for state variables like temperature, pressure, volume, or energy as seen by the equation for the classical change in entropy

$$dS = \left(\frac{\partial S}{\partial E} \right)_{V,N} dE + \left(\frac{\partial S}{\partial V} \right)_{E,N} dV + \left(\frac{\partial S}{\partial N} \right)_{E,V} dN; \tag{1.5}$$

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which is not a function of the phase-space element h [5]. Later the von Neumann entropy was introduced in the framework of quantum mechanics,

$$S = -\operatorname{Tr}(\rho \ln(\rho)),\tag{1.6}$$

where $\rho = \sum_{i=1}^{N} p_i |\Psi_i\rangle\langle\Psi_i|$ is the probability density matrix [3, 13]. Finally, the amount of attainable information (the "measure" of information) is the Shannon entropy [11],

$$H(x) = -\sum_{i=1}^{n} P(x_i) \log_b(P(x_i)),\tag{1.7}$$

denoted by an H to match Boltzmann's H-Theorem, $S = Nk_bH$ for N non-interacting particles [11]. In equation (1.7), b gives the units for the amount of information, for instance if $b = 2$ the Shannon entropy gives the number of bits, and for $b = e$, it gives the amount of information in units of nats (natural bits) [11, 14]. Shannon also shows that a fractional number of bits is possible [11]. The Shannon entropy vanishes, $H = 0$, when one probability state equal to unity and all other probability states equal 0 [11]. Since Shannon, other developments in information theory have been formulated.

The Shannon entropy is an entropy pertaining to information space that describes the lack of knowledge about a piece of information x_i (element) out of the set of possible states $x_i \,\epsilon\, X$. Consider the location of a finite volume particle in real space volume such that it is equally likely to be located anywhere in the volume. The piece of information pertaining to position are the coordinates $r_i = (x_i, y_i, z_i)$ and the span of information space is the real space volume $V = \triangle x \,\triangle y \,\triangle z$ which include all possible positions of the particle. Because the particle has finite size, consider there to be $N = \frac{V}{r_0}$ number of equiprobable positions for the particle. The

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entropy pertaining to this information space (positional) is the Shannon entropy, $H = \log_b(N)$, which is given in units of bits for $b = 2$ or nats for $b = e$. After measurement, the probabilities collapse to a particular state (for a stationary particle in real space volume) and the Shannon entropy is zero because the configuration of the system is known ($p_1 = 1$, $p_{\texttt{other}} = 0$).

One pertinent development in information theory is the relative entropy (also known as information gain or the Kullback-Leibler divergence),

$$D(p||q) = S_r = \sum_i p(x_i) \log \frac{p(x_i)}{q(x_i)}, \tag{1.8}$$

which is a measure of the "distance" between two probability mass functions p and q and a probability mass function is a discrete probability distribution with each possibility exactly equal to one value. I have set $D(p||q) = S_r$ for notational convenience that will be employed later in this document. By convention $0 \log \frac{0}{0} = 0$, $0 \log \frac{0}{q} = 0$, and $p \log \frac{p}{0} = \infty$. Also, for $q = p \to S_r = 0$ and $S_r \geq 0$. The relative entropy is not a true distance because it does not satisfy the triangle inequality, which is the sum of two sides is longer than the length of the third side. The above definitions have been been paraphrased from [14]. The probability distributions p and q must be defined on the same set or measurable space [15]. The relative entropy can thought of as the amount of information gained when one thought the distribution applying to a certain process was q but learns that the distribution is actually p [16].

Several other developments in information theory are pertinent to this paper. Blankenbecler and Partovi developed a description of the entropy as a joint property of the system and the measuring device [17]. Frank and Smith describe an information measure invariance by scaling the probability space and measurement

scale equally [18]. In their work they denote the measurement scale to be m_x use it to keep the amount of information invariant. In this work, they used a measurement value that scaled with the probability space to keep the information invariant. They give the example that the same amount of information exists for measurements of equal ratio; for instance if you measure distances to objects in your office with an "office" ruler or by the same ratio measure the distances to galaxies with a "galaxy" ruler that shares the same ratio to the galaxies as the office ruler shares to the office, then the amount of information is invariant. Infinite entropy and amount of information are discussed in several publications including [19, 20] but has never been mentioned as local and unobservable. The idea that a change in the amount of information changes probability distributions is not novel but what is novel is changing probability distributions by changing the measurement accuracy $h_I \geq \hbar/2$, where the maximum entropy remains constant.

1.2 Introduction to Variable h

Statistical mechanics and quantum mechanics both share the notion of h, a "cell" in phase-space containing only one microstate having units of action (J·s). A fundamental feature of quantum mechanics is the use of $\hbar = \frac{h}{2\pi}$ in the uncertainty principle $\hbar/2 \leq \sigma_{x_i} \sigma_{p_i}$ [3, 4], which is the combined uncertainty of the position and momentum of a particle. It should be emphasized that the inequality accounts for measurement uncertainties greater than or equal to the fundamental limit of $\hbar/2$, due to experimental imperfections. In statistical mechanics, h is the area of a cell in phase-space, $h = \triangle x_i \triangle p_i$, containing one state and is used to subdivide the accessible phase-space of a microcanonical ensemble to count the total number of accessible states [5]. Generally, h is treated as an arbitrary constant [5]. Further, h is assumed

a priori to be uniform in size spanning all of a systems's phase-space hyper-volume $V = V_q V_p$ [5], where V_q is the real space volume, V_p is the momentum space volume where the boundary of V_p is set by the total energy E. Here, h divides V into a "checkerboard" of phase-space elements all with size h, giving the probability of a particle occupying a particular state (a single square on the checkerboard) to be $P = \frac{h}{V}$. However, what if the uniform size constraint on h is relaxed allowing h to vary (homogeneously or inhomogeneously) throughout phase-space? The purpose of this thesis is to address this question and explore the microscopic and macroscopic consequences of varying h. This is an interesting issue because such an approach allows for changes in "viewing contrast" when observing a system, leading to different perspectives on the same system, much like how the length of the coast of Britain depends on the length of your measuring tool [21] or the description of a system of particles to be unitary or composite. Also, exploring this question allows for the possibility of understanding hidden states due to a lack of a perfect measurement as defined by the uncertainty principle. Originally, allowing a variable $h \to 0^+$ leading to infinite unobservable entropy was investigated because it seemed to validate the following presumption, "If entropy is how much we do not know about the state of a system and if the system is undefined, than I would expect this system to have infinite entropy because how can we begin to know anything about a system we cannot even define?! " .

There are two ways that the fundamental cell size h can vary for a specified system. In Case A, the cell size h is rescaled homogeneously across a system's phase-space volume, leading to a rescaling of the probability $P = \frac{h}{V} \to P' = \frac{h'}{V}$ with $h = ch'$, so the probability distribution remains uniform across the system. In Case B, h may vary in size across phase-space inhomogeneously while still spanning the entire hyper-volume resulting in the probability varying across the system. Both

Case A and B are illustrated in Figure 1.1.

Case A implies that a h_1 used to subdivide a phase-space volume could have nested h_2's (perhaps previously unknown) within it, further dividing phase-space. This would redefine the previous notion of a single particle state. An application of Case A is probability updating, where the multiplication rule for probability is used to update the probability [22], and decreasing h implies an increase in measurement accuracy (decrease in uncertainty). A simple example of Case A is throwing a dart at board 1 in Figure 1.1 and the observer noting four equi-probable places for the dart to land giving $P = 1/4$ of landing in a given square. When the observer approaches board 1 to remove the dart, he/she notices that the dart hit a particular quadrant of that square. The observer having acknowledged "nested" quadrants within board 1 transforms his/her landing probability to $P = \frac{1}{16}$; thus, the observer has updated the probability of a dart hitting a board partitioned like board 2 in Figure 1.1. The dart example translates to a particle's location in a system's phase-space hyper-volume and Case A can be used when changing the resolution homogeneously by changing $h = \triangle x_i \triangle p_i$. An example of Case B is given in the lower board of Figure 1.1 where the observer assigns different size single particle states depending on their location in phase-space. It will be shown that such variations in h does not effect the expected thermodynamic variables for a system of particles when averaged over the entire phase-space; however, it does have implications in information theory because the amount of attainable information $H(x)$ is a function of the probability of measurement $P(x_i)$ and therefore the number of elements that can be measured.

This thesis focuses mainly on Case A using variable h to partition the definition of the total entropy into the current amount of information, the relative entropy, and the unobservable entropy. Interestingly enough, the unobservable entropy, S_u, is infinite when allowing $h \to 0^+$, which infinitely subdivides phase-space beyond the

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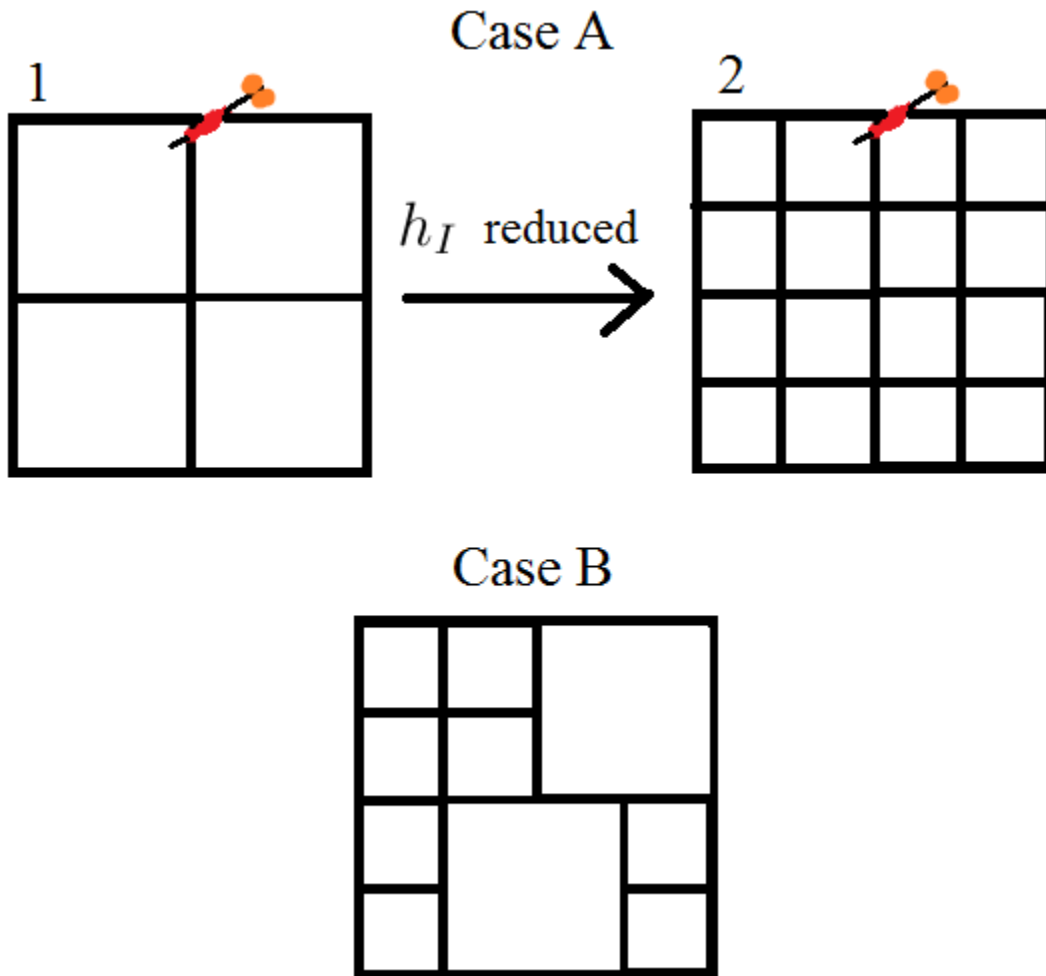
Figure 1.1: Illustration of Case A and Case B discussed in the text, where h is uniformly scaling the probability $P = \frac{h}{V}$ (nested elements inside elements) when reducing or increasing h and Case B where volume elements vary throughout a system's phase-space, respectively.

observable limit and implies infinite unobservable energy, $Q_u = TS_u$ per particle, at a fixed temperature.

Chapter 2 presents the formalism for homogeneous transformations of h followed by two example systems. Section 2.2 uses an example system of inhomogeneous h to contrast Case A and Case B. Chapter 3 discusses the number of particles as a function of the measurement uncertainty h_I. Chapter 4 discusses topics due to variable h including measurement energy, Q_I, a measurement scale transformation, unobservable chemical equilibrium due to S_u, and how the universe can be "viewed" as granular and non-granular. Finally, conclusions are drawn and future directions are discussed in Chapter 5.

Chapter 2

Case A – Homogeneous Transformations of h_I

"If the doors of perception were cleansed, everything would appear to man as it is, infinite." -William Blake

This section develops a model for homogeneous transformations of measurement accuracy. The dart example and Figure 2.1 show that the maximum number of places the dart could land before throwing it is 24. The maximum entropy of the system before throwing is then $S_{max} = k_b \ln(24)$ from equation (1.4), given equal landing probability. When the dart is thrown and observed to land in a particular region using a less than perfect measuring apparatus, the amount of information gained by observing the location of the dart is a function of the measurably *distinguishable* number of places the dart could land, $\Omega_I = 6$ because (in this case) there are 6 distinguishable squares for this measurement accuracy as seen in Figure 2.1, and the Shannon entropy, is the amount of information not known about the system (configurationally), but after measurement the amount of information gained
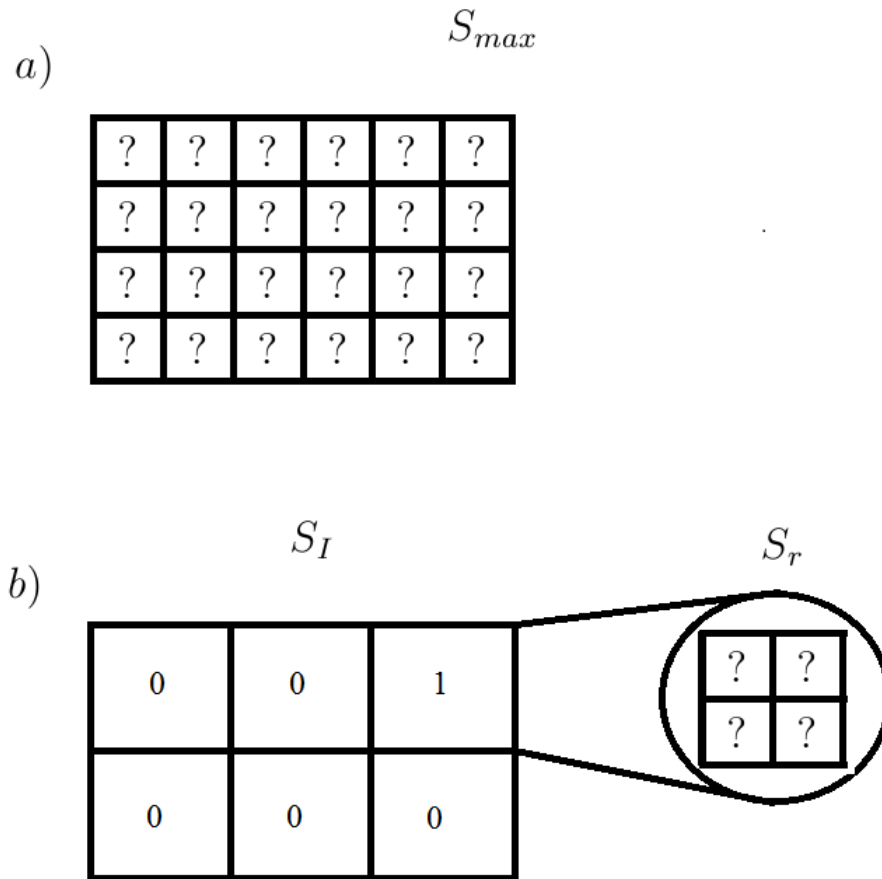
Figure 2.1: Cartoon depicting the differences between S_{max}, S_I, and S_r. Both the top and bottom panels are the same system, but the top is the system with zero measured entropy known about the location of the dart (depicted with ? marks) and the bottom is the system with an amount of measured entropy known about the location of the dart (depicted as 0's or 1's). This defines S_r as the remainder or relative entropy left over after an inaccurate measurement.

is $H = \ln(\Omega_I) = \ln(6)$ from equation (1.7). Multiplying H by k_b, an application of Boltzmann's H-theorum, gives what I will denote the measurable amount of entropy, $S_I = k_b \ln(6)$ when it pertains to the number of phase-space configurations. Because in reality the dart could land in a particular quadrant of the observed square, there are 4 nested states, which helps defines a remainder or relative entropy $S_r = k_b \ln(4)$, because there still exists positional uncertainty about the dart location.

Due to log rules, the maximum amount of entropy can be partitioned into the amount of entropy we are measuring S_I, plus the remaining entropy S_r, giving in general,

$$S_{max} = S_I + S_r, \tag{2.1}$$

which follows the notions of the example above. Using the uncertainty principle, $\frac{\hbar}{2} \leq \sigma_{x_i} \sigma_{p_i}$, the smallest measurable phase-space volume element is $h_{min} = \triangle x_i \triangle p_i = \hbar$, which is the smallest measurably distinguishable single-state volume element, giving

$$S_{max} = N k_b \ln\left(\frac{V}{h_{min}^D}\right) = N k_b \ln(\Omega), \tag{2.2}$$

as the maximum entropy from $[11, 23]$ or stems from the "maximum uncertainty principle" $[17]$ and is applicable to a system with D dimensional real volume V_q, large phase-space volume $V = V_q V_p$, and many N distinguishable particles. The usual thermodynamic limit applies, $V \to \infty$ and $N \to \infty$ remain finite resulting in arbitrarily accurate and well-defined state functions and parameters. It may be possible for $h_{min} > \hbar$ for systems with minimum uncertainty greater than $\hbar$, like the

infinite square well. Equations (2.1) and (2.2) lead to the following definitions:

$$S_I = NDk_b \ln(\Omega_I) = NDk_b \ln\left(\frac{V}{h_I}\right), \tag{2.3}$$

$$S_r = NDk_b \ln\left(\frac{h_I}{h_{min}}\right), \tag{2.4}$$

where h_I is the measurement uncertainty used by the observer (resolution of $\vec{p_x}$ and $\vec{q}$) notional to [17], or h_I can be thought of as a setting the information channel capacity [11]. The measurement entropy S_I is the amount of entropy measured by observing with h_I, The phase-space uncertainty used to obtain an amount of information from a system, h_I, has the quality that $h_{min} \leq h_I \leq V$ so that I is measurable and positive. The ratio $\frac{h_I}{h_{min}}$ is the number of internal accessible states within h_I and $N \leq \Omega_I = \frac{V}{h_I}$ because only one particle is allowed for each microstate (ie fermions). Decreasing h_I increases the number of possible measurably distinguishable states but decreases the relative entropy as in the illustrative example of subdividing the dart board into more squares shown in Figures 1.1 and 2.1. The *maximum* entropy S_{max} remains constant as h_I varies. The difference between S_I and S_{max} should be noted because in almost all cases (including macroscopic environments) $S_r \neq 0$.

The relative or remaining entropy, $S_r = k_b \ln(\frac{h_I}{h_{min}})$, is the relative entropy defined in (1.8) where $q = \frac{h_{min}}{V}$ and $p = \frac{h_I}{V}$ and S_r describes the difference in the amount of information when measuring with accuracy h_{min} and h_I under the equi-probable *a priori* assumption (times k_b for units of entropy).

Another way to formulate S_I and S_r consistent with a constant S_{max} is to consider these two quantities as a basis set describing S_{max}. A transformation of $S_{max} \rightarrow |e^{i\theta}|^2 S_{max}$ conserves the maximum entropy (and probability of S_{max}) where θ is the angle of S_{max} with respect to S_I as depicted in Figure 2.2. Applying this
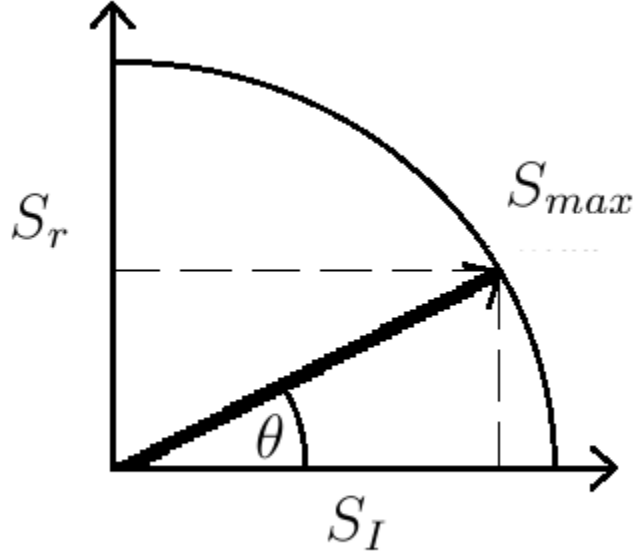
Figure 2.2: Graphical definition of the angle θ, where $0 \leq \theta \leq \pi/2$, between S_I and S_r, where the magnitude of S_{max} is conserved. The angle θ depicts changes in the observer's relative entropy S_r due to changes in the measurement entropy S_I when observing a system with absolute maximum entropy S_{max}. The relative entropy S_r of a system should be assigned on a per observer basis by the amount of entropy measured by the observer.

transformation gives

$$S_{max} = |e^{i\theta}|^2 S_{max} = \cos^2(\theta) S_{max} + \sin^2(\theta) S_{max}. \tag{2.5}$$

Taking equations (2.1) and (2.5), the measurement entropy can be assigned $S_I = \cos^2(\theta) S_{max}$ and the relative entropy $S_r = \sin^2 \theta S_{max}$. The relationship between Ω, and θ is now $\Omega_I = \Omega^{\cos^2(\theta)}$, $h_I/h_{min} = \Omega^{\sin^2(\theta)}$, and therefore

$$h_I = \frac{V}{\Omega^{\cos^2(\theta)}}. \tag{2.6}$$

From the perspective of either S_I or S_r, changing h_I is a scale transformation [24].

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An overall definition of these transformations will be discussed further in Chapter 4.

In "Statistical Interpretation of Quantum Mechanics" [26] Born mentions "...quantum mechanics is competent for dealing with the interaction of object and apparatus, it is seen that no arrangement is possible that will fulfill both requirements simultaneously". This statement implies the uncertainty principle only holds when *measuring* a system but does not consider unmeasured systems/unmeasurable parts of a system. Therefore, a phase-space element h_u can be chosen to be less than $\hbar/2$, given observers will not be able to measurably distinguish the momentum-position coordinates at the same instance due to the uncertainty principle. Although the infinitesimal paired phase-space coordinates maybe be measurably "blurred" does not mean their existence can be ruled out, much like how S_r must be included when deliberating between S_{max} and S_I. A view not limited by human interaction/measurement of the universe is the existence of possible states not measurable due to the lack of a perfect measuring apparatus. Thus, it is appropriate to introduce the total entropy of a system as

$$S_{tot} = S_{max} + S_u = S_m + S_r + S_u \qquad (2.7)$$

where the S_u is unobservable entropy due to h_u being less than $\hbar/2$ and describes *the amount of entropy we cannot measure due to the uncertainty principle.* If phase-space is assumed continuous then,

$$S_u = NDk_b \ln\left(\frac{h_{min}}{h_u}\right) = \infty \text{ as } h_u \to 0^+, \qquad (2.8)$$

which applies to every system with $V > 0^+$ and $N > 0$. Because $S_{tot} = \infty$ since $S_u = \infty$, $S_{max} = S_{tot} - S_u$ essentially "picks points" out of the infinite number of

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possible points in the system where the point picked out is the average momentum and position for the particular h_{min} cell in phase-space. This description of S_u differs from other formulations of infinite entropy, [19], because here h_u becomes infinitesimally small rather than V becoming infinitely large or varying at all. Because S_u is defined as unobservable, it also differs from the reference above. Because all systems share an underlying S_u, the total entropy difference between two systems with equal number of particles is,

$$\triangle S_{total} = S_{max1} + S_u - (S_{max2} + S_u) = \triangle S_{max} \tag{2.9}$$

and is depicted in Figure 2.3.

Note that a difference in unobservable entropy between two systems, $\triangle S_u$, is equal to zero only for systems that have equal number of particles but phase-space could be continuous independent of the number of particles. It is difficult to compare differences in unobservable entropy $\triangle S_u$ because it is unobservable and because there could exist infinite unobservable sub-sub-particles if $N = N(h_u)$. The number of particles as a function of h_I, $N_I = N_I(h_I)$ will be developed in Chapter 3.

One interpretation of S_u as a function of $\Omega_u = \frac{V}{h_u}$ is that it represents the lack of information associated with undefinable quantity because $h_u \to 0^+$. Logically, we have zero information about an undefined quantity so its entropy should be infinite. This is also logical because a definition is required for a quantity to be described (which requires measuring/transmitting information). A system having zero measured entropy because $h_I = V$ is completely unknown since h_I is the same size as the system and hence the system is being used to measure itself e.g. if the system is a particle in a box and a "phase-space ruler" (h_I) is used of that size then all that is measured is what is already known; the particle is in the box, so the
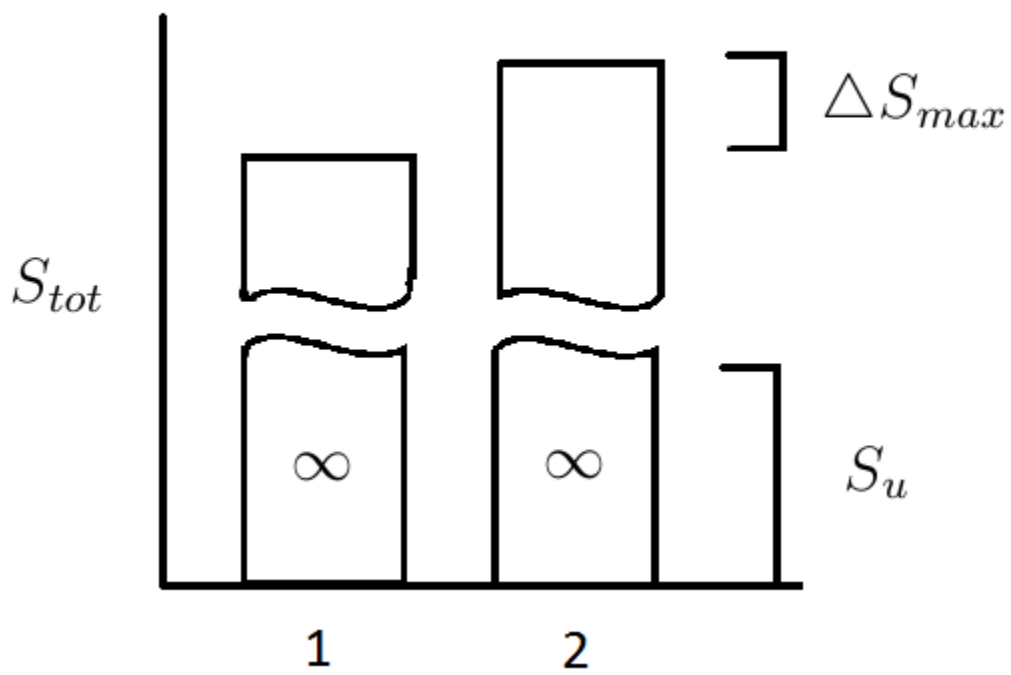
Figure 2.3: Behavior of the total entropy of system 1 verses system 2 where $\triangle S_{max}$ is calculated in equation (2.9).

amount of information pertaining to internal states is zero.

A classical change in the entropy dS is given by [5],

$$dS' = \left(\frac{\partial S}{\partial E}\right)_{V,N} dE + \left(\frac{\partial S}{\partial V}\right)_{E,N} dV + \left(\frac{\partial S}{\partial N}\right)_{E,V} dN; \qquad (2.10)$$

where E is the internal energy and here V is the real space volume; however, if h_I is considered variable, then

$$dS = dS' + dS_I = \left(\frac{\partial S}{\partial E}\right)_{V,N,h_I} dE + \left(\frac{\partial S}{\partial V}\right)_{E,N,h_I} dV + \left(\frac{\partial S}{\partial N}\right)_{E,V,h_I} dN$$
$$+ \left(\frac{\partial S}{\partial h_I}\right)_{E,V,N} dh_I. \quad (2.11)$$

For all considered microcanonical ensembles,

$$dS_I = \left(\frac{\partial S}{\partial h_I}\right)_{E,V,N} dh_I = -\frac{N k_b D}{h_I} dh_I \qquad (2.12)$$

where D is the dimension of the spatial volume. Equation (2.12) shows entropy increasing when there is a negative change in h_I; h_I becomes smaller.

An example that applies dS_I is measuring the same system with two different measuring apparatus, e.g. two microscopes with different resolution (and hence different values of h_I: h_1 and h_2). This change in entropy dS_I, when integrated, is a change in the measured entropy $\triangle S_I$ and is formulated in an example below:

The maximum entropy, S_{max}, remains constant because the same system is being measured with two different values of h, but the measured entropy S_I changes in accordance with the complement entropy S_r to satisfy Eq. (2.1) given that $h_1, h_2 >$

h_{min} and V is a constant phase-space volume. Integrating dS_I from h_2 to h_1 yields,

$$\int_{h_1}^{h_2} dS_I = -\int_{h_1}^{h_2} \frac{Nk_bD}{h_I} dh_I = Nk_bD \ln\left(\frac{h_1}{h_2}\right) = \triangle S_I, . \qquad (2.13)$$

where the equivalence to $\triangle S_I$ is derived in the following arguments:

$$S_{max1} = Nk_bD \ln\left(\frac{V}{h_{min}}\right) = S_{I1} + S_{r1}$$
$$= Nk_bD \left[\ln\left(\frac{V}{h_1}\right) + \ln\left(\frac{h_1}{h_{min}}\right)\right] \qquad (2.14)$$

$$S_{max2} = Nk_bD \ln\left(\frac{V}{h_{min}}\right) = S_{I2} + S_{r2}$$
$$= Nk_bD \left[\ln\left(\frac{V}{h_2}\right) + \ln\left(\frac{h_2}{h_{min}}\right)\right] \qquad (2.15)$$

leading to,

$$\triangle S_{max} = S_{max2} - S_{max1} = 0 = \triangle S_I + \triangle S_r$$
$$= Nk_bD \left[\ln\left(\frac{h_1}{h_2}\right) + \ln\left(\frac{h_2}{h_1}\right)\right] \qquad (2.16)$$

respectively and hence,

$$\triangle S_I = NDk_b \ln\left(\frac{h_1}{h_2}\right) = -\triangle S_r = \int_{h_1}^{h_2} dS_I \qquad (2.17)$$

proving (2.13). Adding $\triangle S_I$ to S_I and $\triangle S_r$ to S_r updates the entropy of the system. Integrating dS_I from V to 0^+ (due to the negative sign) yields the equation for S_{tot}.

Setting $dS' = -dS_I$ from Eq. (2.11) shows how the amount of measured entropy in a system can remain constant as macroscopic variables (E, V, N) are varied in conjunction with h_I. This can be visualized by imagining the stretching or com-

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pressing a rubber phase-space "checkerboard" sheet uniformly. Because the number of micro-states remains constant as h_I grows or shrinks in accordance to the other changes of macroscopic variables, S_I remains constant and S_r grows or shrinks. This is notional to the scale invariance of the amount of information described by [18] where both the volume and measuring unit are scaled equally but differs in that this definition requires S_r to grow or shrink.

A classical change in internal energy, dE, remains constant with changes in h_I. This is because $dS_I = \left(\frac{\partial S}{\partial h_I}\right)_{E,V,N} dh_I$ in (2.11), which when solving for dE (or dN, dV) leads to its cancellation, leaving the classical definition of dE in [5] unscathed, given the system is in equilibrium. Because classical macroscopic thermodynamic variables other than entropy do not depend on partial derivatives with respect to h_I, they are h-scale invariant [24].

An infinitesimal change in the *measured* heat by varying h_I is,

$$dQ_I = TdS_I = -T\frac{Nk_bD}{h_I}dh_I, \tag{2.18}$$

again for a system in steady state equilibrium because the internal energy and phase space volume V remain constant for changes in h_I. A finite change in measurable heat by varying h_I is

$$\triangle Q_I = T \triangle S_I. \tag{2.19}$$

Note that a change in Q_I is a change in the amount of heat measured by the apparatus because the max heat of the system $Q_{max} = TS_{max}$ remains constant with changes in h_I. Chapter 4 expands upon this stating that a change in heat leads to a measurement scale transformation of h_I.

Because $S_u = \infty$, a statement of energy from the first law of thermodynamics is

that, and assuming equilibrium

$$Q_u = TS_u = \infty \tag{2.20}$$

for $N, E, T, V \neq 0$, stating that S_u must be part of a larger system for Q_u and T to be correctly described. The heat Q_u is the amount of heat we cannot measure due to the uncertainty principle. Assuming no heat can be transfered from the Q_u bath of heat requires the temperature to be uniform across the entire system hence,

$$\begin{aligned}
Q_{tot} = \frac{2E}{Nk_bD}S_{tot} = \frac{2E}{Nk_bD}\int_{0^+}^{V}\frac{Nk_bD}{h}dh = 2E\ln\left(\frac{V}{0^+}\right) \\
= 2E\frac{S_{max}}{k_b} + 2E\frac{S_u}{k_b}, \tag{2.21}
\end{aligned}$$

stating the temperature should remain constant, and in this example, for a classical gas in a box where $E = \sum_i \frac{p_i^2}{2m}$. Suppose we calculate the temperature of a system with energy $E' = \frac{P^2}{2m} + c$. The temperature of the system remains constant because $\frac{\partial S}{\partial E'} = \frac{1}{T} \propto \frac{1}{E'-c} = \frac{1}{E}$. This shows that adding an infinite energy, c, to the internal energy of the system, E, does not change the temperature or the thermodynamics. The unobservable infinite energy, Q_u, is a underlying heat energy that does no work. The total entropy, S_{tot}, is in the framework of statistical mechanics because there h is treated like an arbitrary constant [5]. An interpretation of Q_u is the amount of energy a particle has within its respective h_{min} as it explores infinite unobservable internal degrees of freedom. Possibly, due to infinite unobservable internal degrees of freedom, a particle can broken into finner vibrating sub-particles, beyond elementary particles. Speculatively, the infinite unobservable internal degrees of freedom could cause unpredictable scattering angles mimicking more of a quantum interpretation of collision rather than a classical interpretation. Due to the energy being

unobservable, it could have relationships with other unobservables like dark energy, dark matter, or string theory.

Because the Helmholtz free-energy F is a measure of the useful available work at constant V [5], a new Helmholtz free-energy is equal to the original ,

$$F(T,V,N) = E + Q_u - T(S + S_u) = E - TS. \tag{2.22}$$

Because the unobservable part of the Helmholtz free-energy $F_u = Q_u - TS_u = 0$ is a minimum, it suggests that systems having S_u are in chemical equilibrium with one another, because a minimum in free energy is a maximum in entropy [5]. It is clear that Q_u is strictly a heat related energy not pertaining to potential work, as one would expect by it's definition in (2.20).

2.1 Application Examples of Case A

2.1.1 Classical Ideal Gas in a D-dimensional Box (Microcanonical)

Imagine an ideal classical gas in a closed and isolated box with dimensionality D (having side length L), the ensemble can be described microcanonically and the entropy is for indistinguishable particles is,

$$\begin{aligned}
S(N,V,E) = S_{max} &= Nk_b \ln\left(\frac{V_q(\pi 2mE)^{\frac{D}{2}})}{(D/2)!h_{min}^D N!}\right) \\
&= Nk_b\left[\ln\left(\frac{V_q(\pi 2mE)^{\frac{D}{2}})}{(D/2)!h_I^D N!}\right) + D\ln\left(\frac{h_I}{h_{min}}\right)\right]
\end{aligned} \tag{2.23}$$

and the second line is applying (2.1). Having variable h_I does not effect the equation of state because,

$$\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N,h} = \frac{Nk_b}{L^D}, \tag{2.24}$$

where T is specified by,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N,h} \tag{2.25}$$

$$T = \frac{2E}{k_b N D} \tag{2.26}$$

not depending on h_I. Consider a change in entropy using (2.11) for a classical gas in a box,

$$dS + dS_I = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN - \frac{Nk_b D}{h}dh_I. \tag{2.27}$$

Equation (2.27) assumes that the number of particles does not change as h_I is changed. Solving for dE in (2.27) gives

$$dE = TdS + TdI - PdV + \mu dN + \frac{Nk_b TD}{h_I}dh_I$$

$$= TdS - PdV + \mu dN, \tag{2.28}$$

which is the classical statement of dE because $TdI = -\frac{Nk_b TD}{h_I}dh_I = -\frac{PVD}{h_I}dh_I$ so a change in h_I does not change the energy of the system. The energy $TdI = dQ_I$ is discussed further in Chapter 4.

2.1.2 Quantum Mechanical Particles in a D-dimensional Infinite Potential Box

Consider this system to be in steady state equilibrium and the particles to be non-interacting. The entropy of N quantum mechanical particles in a D-dimensional infinite potential box of side length L is,

$$S(E,V,N) = k_b N \ln \left(L^D \left(\frac{2\pi m E}{\hbar^2} \right)^{D/2} \right) - k_b \ln \left(N! \Gamma \left[\frac{DN}{2} \right] \right)$$

$$= k_b N \ln \left(L^D \left(\frac{2\pi m E}{h_I^2} \right)^{D/2} \right) - k_b \ln \left(N! \Gamma \left[\frac{DN}{2} \right] \right) + N D k_b \ln \left(\frac{h_I}{\hbar} \right), \quad (2.29)$$

for large N, where the first line is solved explicitly in [25], the second line employs (2.1), and $S_r = N D k_b \ln \left(\frac{h_I}{\hbar} \right)$. The phase-space volume in equation (2.29) is derived from

$$E_n = \frac{\pi^2}{2mL^2} \sum_{i=1}^{D} \hbar^2 n_i^2. \quad (2.30)$$

In this case, changing h_I affects the accuracy of the observable energy levels in the system. Because changing h_I essentially is breaking up the box's phase-space into different number of phase-space cells, it is safe to assume that the box could have adjustable energy levels, n_I, inversely proportional to h_I so that the energy remains constant,

$$n_I h_I = \hbar n \quad (2.31)$$

$$h_I = \frac{n\hbar}{n_I} \quad (2.32)$$

where n_I acts as a "weighing" integer that when $n_I = 1$ reverts to the normal observable energy levels and h_I is the weighted Planck's constant. Once a scale has

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been set by choosing an h_I, it can be substituted it into (2.30) to obtain the scaled equation for the observable/distinguishable energy levels,

$$E_{n_I} = \frac{\pi^2}{2mL^2} \sum_{i=1}^{D} h_I^2 n_{Ii}^2 \tag{2.33}$$

where the observable energy levels only occur when $E_n = E_{n_I}$ and when each dimension's $n_i = \frac{h_I}{h} n_{Ii}$ in order to exclude false degeneracy scenarios. A general description is given in Chapter 4 with a derived unitary operator for scale transformation. A one dimensional example is choosing $h_I = 2\hbar$ giving $n_I = .5n$ from (2.32) which follows that $E_{n_I} = 4E_n = E_{.5n}$. The n_I solutions are the even energy levels from E_n, thus there are half as many observable/distinguishable energy levels because our step size has doubled in n-space, $n_I = .5n$, and particles at consecutive energy levels are lumped into one observable energy. This is shown in Figure 2.5. For this case, $S_r = NDk_b \ln(2)$ admitting 2ND indistinguishable states in the system after observation with measurement accuracy h_I. A shift in n-space by taking $\hbar \to h_I$ is depicted in Figure 2.4. It should be noted that E_{n_I} is bound by the total internal energy E.

Choosing an $h_u < \hbar/2$ requires the addition of unobservable energy levels n_u that would not change the maximum measurable amount of entropy in the system S_{max} but if $h_u = 0^+$ (continuous energy in this case), it forces $n_{ui} \to n * \infty$ and hence infinite unobservable energy levels between each observable energy level are accounted for in S_{tot} by S_u from equation (2.7) and in equation (2.29). These infinite unobservable energy levels between observable energy levels are explored by the particle as the particle evolves from one discrete observable energy state to another. Often when calculating the phase-space volume in 2.29 for large N, the sum over n is replaced by an integral over n when using the gamma function, which
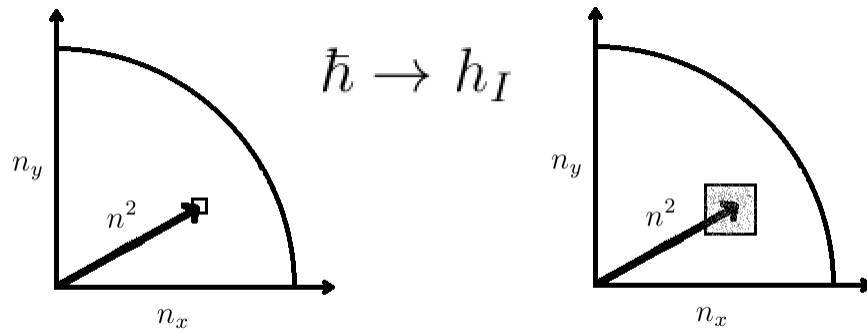
Figure 2.4: Above is a naive 2D example of the uncertainty in n-space (momentum space) increasing as $\hbar \to h_I > \hbar$ bound by the total energy E. When measuring with h_I, the n^2 vector is somewhere within the fuzzy gray box, where as when measuring with $\hbar$ the particle is in a definite state. As h_I increases, the possible directions and lengths for the n^2 vector increases until it could be anywhere in the entire quarter circle just as it is for $S_{max} = S_r$.
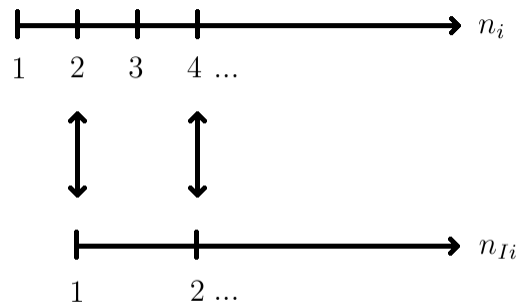
Figure 2.5: A line in number space is rescaled by $h_I = 2\hbar = \frac{n}{n_I}\hbar$ such that the first distinguishable energy level occurs at the second actual energy level and second distinguishable energy level at the fourth actual energy level.

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implies that n is a calculated as a continuous parameter as $n_u \rightarrow n * \infty$ implies and still does not change the energy of the system.

Just as the example in the previous section, taking partial derivatives of the entropy with respect to macroscopic variables and h_I one can obtain the fundamental equation for a change in entropy from equation (2.27).

2.2 Case B – Particles in a Box

Case B considers inhomogeneous distributions of h_I in a system that still "tiles" all of phase-space. Implications of Case B can be realized by an example where a system is partitioned into two areas of unequal h_I.

Consider a uniform classical gas in a box where half of the volume is measured with an apparatus having h_1 and the other half is measured with h_2, $h_{min} = h_1 < h_2$, and there is no wall separating phase-space. Take $V_1 = V_2 = V_q V_p$ and N distinguishable particles distributed uniformly random. Although $S_{m1} \neq S_{m2}$, the system is in equilibrium when $S_{max1} = S_{max2}$. The entropy of any system with varying h can be partitioned into areas of homogeneous h_i's and summed using Gibbs entropy formula,

$$I = k_b \sum_{i=1}^{n} p_i \ln \left(\frac{1}{p_i} \right) \tag{2.34}$$

where $n = \sum_j V_j / h_j$ is the number of elements and the dummy index j is over homogeneous partitions of phase-space. Taking $V_1 = V_2 = 1000 h_1^D = 500 h_2^D$, which are $j = 2$ homogeneous partitions of phase-space from $n = \sum_j V_j / h_j$, we find that $n = 1000 + 500 = 1500$ accounting for each probability p_i. The amount of

measurement entropy for this system is

$$S_I = Nk_b \sum_{j=1}^{1000} \frac{h_j^D}{V} \ln\left(\frac{V}{h_j^D}\right) + Nk_b \sum_{k=1}^{500} \frac{h_k^D}{V} \ln\left(\frac{V}{h_k^D}\right)$$

$$= Nk_b \ln\left[\left(\frac{V}{h_1^D}\right)^{V_1/V=1/2}\right] + Nk_b \ln\left[\left(\frac{V}{h_2^D}\right)^{V_2/V=1/2}\right] = S_{I1} + S_{I2}. \qquad (2.35)$$

By adding S_{r1} and S_{r2} to S_I we can attain S_{max} by (2.1),

$$S_{max} = \frac{1}{2}Nk_b \left[\ln(\frac{V}{h_1^D}) + D\ln(\frac{h_1}{h_{min}})\right] + \frac{1}{2}Nk_b \left[\ln(\frac{V}{h_2^D}) + D\ln(\frac{h_2}{h_{min}})\right]$$

$$= [S_{I1} + S_{r1}] + [S_{I2} + S_{r2}] \qquad (2.36)$$

where the first bracket includes the measured and complement entropy of V_1 and the second bracket includes the measured and complement entropy of V_2. Interestingly even knowing the system is homogeneous, it appears non-homogeneous because $S_{I1} > S_{I2}$. Observable consequences are that a particle will be more blurry on the S_{I2} side than the S_{I1} side and this could be shown experimentally.

It is a straight forward calculation to show that the chemical potentials $\mu_1 = \mu_2$, which making the two sides in equilibrium with one another on average. Side 1 and 2 would be out of thermal equilibrium if measuring device 1 and 2 created a temperature gradient between the two sides due to the measuring process.

Another example is to consider $h_I = h_I(x_i, p_{x_i})$, which represents measurement accuracy to be a function of position in phase-space volume V, such that the ith member of the probability distribution is $P_i = h_I(x_i, p_{x_i})/V$. For example, objects in your peripherals appear blurry compared to objects directly in your line of sight. A naive model of peripheral blurriness is taking $h_I(x, p_x) = h_I(x_i) = \sigma_{p_I}|x_i| + h_I$, where

σ_{p_I} is the standard deviation of the momentum belonging to h_I and $P(x_i) = A\frac{h_I(x_i)}{V}$. The normalization factor A is included in the probability distribution to guarantee that $V = A\sum_i^n h_I(x_i)$, and therefore

$$S_I(h_I(x_i)) = -\frac{Ak_b}{V}\sum_{i=1}^n [h_I(x_i)]\ln\left(A\frac{[h_I(x_i)]}{V}\right), \tag{2.37}$$

stating the amount of measurement entropy is a function of $h_I(x_i)$. The x_i coordinates are the position states located in the center of a the $h_I(x_i)$ cell. Because consecutive $h_I(x_i)$ values should not overlap in phase-space (to avoid double counting), one should "build" the values for x_i depending on the boundary conditions of the system and the number of distinguishable coordinates n.

Because the maximum entropy is $S_{max} = k_b\ln\left(\frac{V}{h_{min}}\right)$, the remaining entropy after observation is also dependent on the coordinate of observation such that $S_r = S_{max} - S_I(h_I(x_i))$. It is likely easier to find numerical solutions to (2.37) rather than analytic solutions for given values of x_i. The amount of measured entropy S_I can be generalized to include dependence on momentum as well as multiple dimensions by summing over the desired dependence as in (2.37) and summing the result to (2.37) due to entropy being additive. Note that $h_I(x_i, p_i)$ should span all of V.

Chapter 3

The Number of Observed Particles as a Function of h_I, N_I

"An open jar holds the entire universe, except itself."

This chapter generalizes the idea that as h_I decreases/increases, the number of resolved sub-particles can increase/decrease. An explanatory example is to consider an individual with poor eye sight looking at a bowl of rice. Because this individual's eyesight is poor, the rice looks like one congealed object, but putting on glassware the individual grains of rice become visible and hence the measurable number of objects in the bowl has increased.

3.1 The General Number of Configurations Ω_K

First, the number of microstate configurations are generalized. The ith level of phase-space contributes to the total number of phase-space elements by defining

$$\Omega_K = \prod_{i=0}^{K} m_i, \tag{3.1}$$

where $m_0 = \frac{V}{h_0}$ and $m_{i>0} = \frac{h_{i-1}}{h_i}$, the h_i's are positive integers, and K is the total number of levels. Each level slices the previous level into more phase-space elements ($m_i > 1$) or compresses ($m_i < 1$) the previous ith level . Because *a priori* $\Omega_K = \frac{1}{\text{Probability}}$, Ω_K should be ≥ 1 but an interpretation of $\Omega_K < 1$ is that of $m_i < 1$ because the phase-space volume V being evaluated is less than the size of the volume element used to measure phase-space. The only way one could know that $V < h_0$ is if the person took an additional $h_1 \leq V < h_0$ to measure the system finally yielding a positive entropy result.

Equation (3.1) is consistent with equation (2.1) when $\Omega_3 = \frac{V}{h_I} \frac{h_I}{h_{min}} \frac{h_{min}}{0^+}$ because

$$S_{tot} = k_b \ln(\Omega_3) = S_I + S_r + S_u, \tag{3.2}$$

stating the total entropy is equal to the amount measured entropy, relative entropy, and unobservable entropy respectively.

3.2 Changing the Number of Observed Particles, N_I

This formulation includes the notion of appearing/disappearing sub-particles when changing h_I. The main thrust of this thesis is that accuracy of a measurement that

determines the number of measurable particles in a system. Transformations of h_I thus change the number of measurable particles.

It is often useful to describe the maximum number of distinguishable particles for homogeneous transformations of h_I as

$$N_K = \prod_{i=0}^{K} n_i = N_{max} = N_I N_r, \tag{3.3}$$

where N_I is the notation for the number of measured particles in the system and N_r is the relative number of sub-particles per particle N_I and $N_{max} \geq N_I$. Here, $N_I = N_I(h_I)$ is a function of h_I that decreasing h_I increases N_I. The simplest example being $N_I \approx N_{max}(1 - \frac{h_I^D}{V})$ which forces $N_r = \frac{V}{V - h_I^D}$. Note that this equation is only valid for when all the particles in the system are particles of the same nature and measured with equal accuracy in all dimensions. Because for fermions a phase-space element is either occupied or unoccupied, the number of distinguishable particles must have the quality $N_I \leq \Omega_{max}$; however, for bosons $N_I \leq \Omega_{tot}$ given that some sub-particles may not be measurably distinguishable. Combining the notions of (3.1) and (3.3) yields

$$\Omega_{max}^{N_{max}} = \Omega_I^{N_{max}} \Omega_r^{N_{max}}, \tag{3.4}$$

or more suggestively

$$\Omega_{max}^{N_{max}} = \Omega_I^{N_I} \left(\Omega_r^{N_I} \Omega_{max}^{N_{max} - N_I} \right), \tag{3.5}$$

where $\Omega_I^{N_I}$ is the number of possible measurable states for the measured N_I particles, $\Omega_r^{N_I}$ is the number of possible internal states for those N_I particles, and $\Omega_{max}^{N_{max} - N_I}$ is the number of possible states for the $N_{max} - N_I = N_I(N_r - 1)$ unmeasured

particle/sub-particles. Equation (3.5) implies that members of the observed particles N_I differ in phase-space scale h_I than unobserved particles $N_{max} - N_I$, whose scale is set by h_{min}. It should be noted that N_I is not necessarily the number of particles but could be the number of like groups with zero measured internal information; for instance if a microscope was used to distinguish the number of plant cells on a leaf, but was not precise enough to observe the interior of said cells. Another example for N_I not being explicitly particles is a list of named animals at the zoo. Because the list reader does not know what animals are at the zoo, he/she has not measured the rest of the particle configurations belonging to the animal/animals N_r (positions of limbs). The amount of measured entropy for the system is,

$$S_I = k_b N_I \ln(\Omega_I) \tag{3.6}$$

and the remaining entropy (a deviation from the relative entropy) is,

$$S_r = k_b \ln(\Omega_r^{N_I} \Omega_{max}^{N_I(N_r-1)}). \tag{3.7}$$

Figure 3.1 gives an example of changing h leading to a change in N_I. For a case where $h \to 0^+$ it is possible to "define" a $N_u \to \infty$ given that observers may not be able to fully distinguish sub-particle positions and momentums. This is because the preciseness these coordinates attached to the N_u particles will be below the observable limit as S_u is unobservable.

For particles and configurations Ω_I,

$$S_I = k_b \ln(Z_I) \tag{3.8}$$

and the number of configurations is $Z_I \approx \Omega_I^{N_I D}/N_I!$ in the limit $\Omega_I >> N_I$ but can

be formally written as n choose k [27]

$$Z_I = \frac{\Omega_I!}{(\Omega_I - N_I)!N_I!};$$ (3.9)

per spacial dimension for $\Omega_I > N_I$. Equation (3.9) is the number of sub-particle configurations assuming that there are no physical constraints between the sub-particles that would guarantee the location of sub-particles within a specified phase-space radius of other composite sub-particles. The sub-particles are free to be anywhere in the system's phase-space for this formulation, which can be recognized by Figure 3.1.

$$h_{I1} = \frac{V}{2} \qquad\qquad h_{I2} = \frac{V}{4}$$
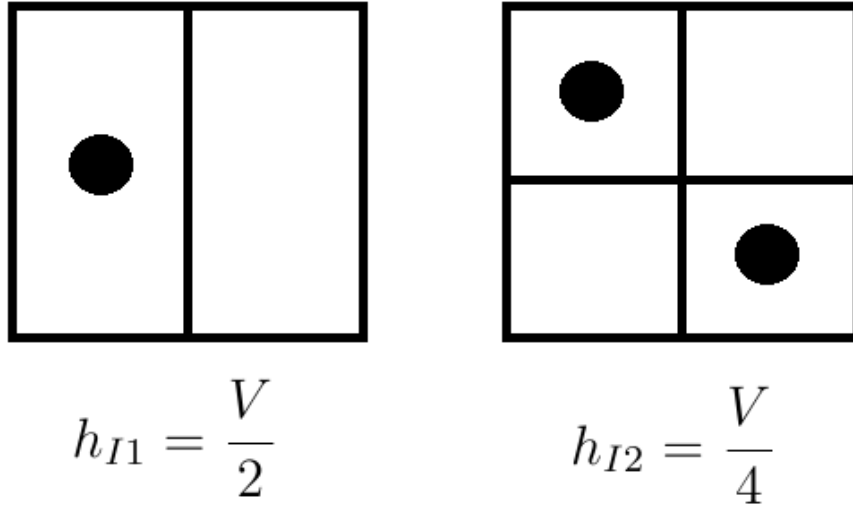
Figure 3.1: Depiction of a given system at different h_I values. The higher resolution perspective contains the sub-particles of the lower resolution perspective. The $\Omega_I^{N_I}$'s, with respect to equation (3.9), are $\Omega_{I_1}^1 = 2$ and $\Omega_{I_2}^2 = 4!/(4-2)!2! = 6$ assuming the particles are indistinguishable. This is the case of "free" sub-particles where possibly the lower right hand square's occupancy is possible at the $i = 1$ level because it's existence at the $i = 0$ level is rounded to zero, e.g. it occupies $< 50\%$ of the square.

3.3   Chemical Potential and N_I

To keep the formulation of N_I consistent with classical thermodynamics, a chemical potential should accompany N_I such that,

$$E = \mu N_{max} = \mu N_I N_r, \tag{3.10}$$

from (3.3). Suppose the introduction of a particle or sub-particle into the sample system $N_{max} \to N'_{max}$; a change in energy is, by the chain rule,

$$dE = \mu dN_{max} = \mu(N_r dN_I + N_I dN_r). \tag{3.11}$$

Because N_r is defined as the number of sub-particles per particle of N_I, for the addition of a particle of the same nature as N_{max}, yields $dN_r = 0$.

Calculating the change in the number of sub-particles is straight forward when knowing the change in energy and the value of μ. Solving for dN_I using equation (3.11) and $dN_r = 0$, gives

$$dN_I = \frac{dE}{\mu N_r} \tag{3.12}$$

In the current frame of h_I, dN_I must be an integer value, which occurs once enough sub-particles enter the system. This means that at the current frame of reference h_I, small dE from a change in the number of particles may go undetected for $-1/2 \leq dN_I \leq 1/2$.

3.3.1 Weighted sub-particles

The above discussion only evaluated sub-particles of equal chemical potential when in reality this may not be the case. To account for multiple types of particles (of same composition) a weighted chemical potential $\mu = \sum_{i=1}^{N_r} \mu_i$ is introduced to keep E constant, so

$$E = \sum_{i}^{N_r} N_i \mu_i, \tag{3.13}$$

where μ_i and N_i are assigned on a per sub-particle type basis and $N_{max} = \sum_i N_i$. For a change in energy,

$$dE = \sum_{i}^{N_r} dN_i \mu_i. \tag{3.14}$$

In the current frame of measurement reference h_I, only changes in energy appreciable to the change in energy from (3.12) may be measured meaning that some sub-particles may slip in or out of the system undetected due to the measurement accuracy being limited by h_I, thus the individual sub-particle chemical potentials of the ith species will not be measurable unless the observer knows he/she is adding a large number of sub-particles with the same μ_i to the system. To measure an individual μ_i the observer will be forced to average over a large number of like sub-particles. Because the observer cannot measure a sub-particle due to the value of h_I, the observer must have prior knowledge of sub-particles entering/leaving the system without explicitly measuring them. In this case the particles entering the system are moving from an area of lower h_I to an area of higher h_I (in the system). An example of this is if the observer had a proton gun shooting into the system, which likely should be set to speeds that change the system adiabatically. The observer

knows the flow rate of protons entering the system (lower h_I) but loses track of them in the system (higher h_I).

3.3.2 Constrained sub-particles

The previous sections have considered only free particles or sub-particles. If sub-particles are constrained to their respective particle, the allowed locations for the sub-particles in phase-space decreases the total number of possible configurations in the system, $Z_{Ic} = CZ_I < Z_I$ where C is a constraining coefficient. The value of C should be calculated on a per system basis. Figure 3.2 gives an example of how the total number of configurations and hence entropy grow as h_I is decreased even with the volume of the particle and the volume of phase-space both remaining constant.

3.3.3 Sub-particles Constrained to a Cubic Lattice in a Cube Volume Space

The goal is to calculate the value of C for sub-particles constrained in cube formation in cube volume space. The numerical example below calculates positional Ω_q from the relation $\Omega = \Omega_q \Omega_p$ where Ω is broken into the number of configurations within volume space times the number of configurations in momentum space respectively.

The MATLAB simulation in the Appendix 6.2 counts the number of phase-space configurations for a system as h_I decreases with the following constraints:

1. Both the 3D volume of space and the volume of the particle remains constant with $v = V/8$,

2. The number of phase-space elements $M_i = V/h_i^3$ grows for this simulation by
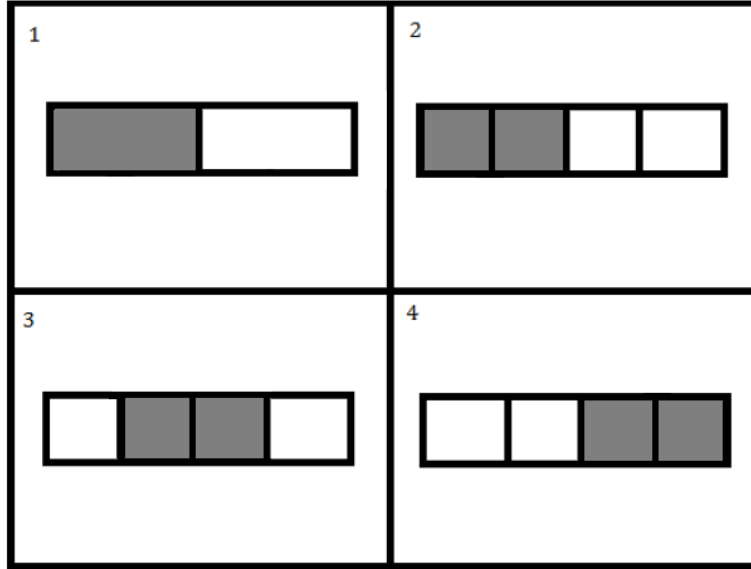
Figure 3.2: All four panels show one particle with volume equal to half of the volume of phase-space. Panel 1 shows there are two configurations because $h_I = V/2$. Panels 2-4 show that if we set $h_I = V/4$ that there are now three configurations which is half the total possible configurations $CZ = Z/2 = \frac{4!}{2!*2!*2} = 3$ because half of the configurations have non-adjacent sub-particles, giving the value for the constraining factor to be $C = 1/2$.

reducing h_I by a factor of 2 in each x, y, and z direction every i-loop from code and initially $h_1 = V/2$, $M_I = 8$,

3. The number of measurable sub-particles N_I is 1/8th the number of phase-space elements at the ith level $N_I = M_i/8$,

4. Generated sub-particles by reducing h_I (the measurement uncertainty) are constrained to be adjacent and in a cube formation in volume space while filling the previous requirement, giving the particle the appearance of having finite volume as well as uniform and continuous sub-particle density.

5. There are K total number of loops,

6. The MATLAB code and results are located in the Appendix.

The main recursion relation for the number of measurable configurations, where i is the loop number, is

$$\Omega_i = (2\Omega_{i-1}^{1/3} - 1)^3 \qquad (3.15)$$

and for very large number of phase-space elements ($i \to \infty$) it can be approximated by,

$$\Omega_i \approx 8\Omega_{i-1} \qquad (3.16)$$

where $\Omega_1 = 8$ for the first level. Equation (3.15) can be rewritten in the form of equation (3.1) by recursively applying (3.15) to reach the level $\Omega_1 = 8$ level giving the measurable number of configurations,

$$\Omega_I = (2^{K-1} + 1)^3. \qquad (3.17)$$

Recall that Ω_I is also related to the total number of volume space elements (3.1) $\Omega_K = \prod_{i=1}^{K} m_i^3 = 2^{3I} = 8^I$ where each $m_i = 2$ because h_I has reduced by half, by the expression

$$\Omega_I = \left(\frac{\Omega_K^{1/3}}{2} + 1\right)^3 = \left(\frac{1}{2}\prod_{i=1}^{K} m_i^K + 1\right)^3. \qquad (3.18)$$

If the particles are not constrained, an n choose k formulation may be used. The value of the constraining coefficient in this case is found by $C_K\Omega_{\mathrm{nk}} = \Omega_I$ where Ω_{nk}

is the number of configurations in the n choose k method yielding

$$C_K \frac{\Omega_K^{N_I}}{N_I!} = C_K \frac{\Omega_K^{\Omega_K/8}}{(\Omega_K/8)!} = (2^{K-1} + 1)^3 \tag{3.19}$$

$$\rightarrow C_K = \frac{(\frac{\Omega_K^{1/3}}{2} + 1)^3 (\Omega_K/8)!}{\Omega_K^{\Omega_K/8}} = \frac{(2^{K-1} + 1)^3 (8^{K-1})!}{8^{K(8^{K-1})}}. \tag{3.20}$$

This constraining coefficient approaches zero rapidly as K increases, in-fact MAT-LAB considered it "not a number" due to rounding a large number in the denominator to infinity for values $I \geq 4$. It makes sense that the C_K is so small because the number of configurations for an n choose k formulation would increase much faster than the number of cube constraint particle configurations because cube constraint particle configurations are scarce elements in the set of all possible configurations from n choose k formulation.

The MATLAB code in the appendix can be adjusted to fit other constraint types or conditions. For example, having h_i decrease by different lengths each i loop rather than uniformly by half. Using methods similar to that presented in this work, momentum constraints could also be considered if there is a known energy threshold binding sub-particles together in momentum space (i.e. the largest difference in momentum between sub-particles of a particle is $\triangle p$) or for possibly entangled spin states.

Chapter 4

Discussion

4.1 Measurement Energy Q_I and Measurement Scale Transformation

"Look at how a single candle can both defy and define the darkness."-Anne Frank

Equation (2.28) showed that a change in h_I does not change the internal energy E of the system to be measured. The statement $T dI = dQ_I$ implies some change in energy as measurement accuracy is varied, but not energy gained by the system to be measured; hence, this change in energy must occur outside the system. Consider a system in steady state equilibrium. Multiplying equation (2.1) by T gives,

$$Q_{max} = Q_I + Q_r \tag{4.1}$$

stating the the maximum measurable heat is equal to the measured heat Q_I plus the remaining heat Q_r. A scale transformation on the measurable number of accessible states Ω_I from equation (2.3) is $\Omega_I \to \Omega_I' = \Omega_I \frac{h_I}{h_I'}$, which can be related to entropy

and thus heat by the second law of thermodynamics. A scale transformation in log space is a translation [28], naturally leading to

$$Q_I \rightarrow Q_I' = Q_I + \triangle Q_I' \tag{4.2}$$

where $Q_I = N_I k_b T \ln(\frac{V}{h_I^D})$, $Q_I' = N_I k_b T \ln(\frac{V}{h_I'^D})$ and $\triangle Q_I' = N_I k_b T D \ln(\frac{h_I}{h_I'})$, for a constant number of particles analogous to the partitioning of entropies from earlier and a system in steady state equilibrium. The measurement scale transformation is thus $\triangle Q_I'$. Here, $\triangle Q_I'$ is interpreted as a change in the measured composition of the heat by changing measurement accuracy, where the composition requires distinguishability of the heat. For instance, heat may be composed of photons of different frequencies being emitted from different locations on system. Because systems in steady state equilibrium (giving off as much heat as it is receiving), heat is measured over a unit time interval dt, giving $\triangle Q_I' = \int (\dot{Q}_I' - \dot{Q}_I)\, dt$. In this case $\triangle Q_I' = -\triangle Q_r'$ stating an increase in the measured heat from a system is equal to a decrease in remaining heat and vice-versa. Dividing equation (4.2) by T gives an equation for measurement scale transformation in terms of S_I.

If the observed number of particles is a function of h_I, a more general measurement scale transformation should be used to include transformations in $N_I \rightarrow N_I' = \lambda N_I$ where λ is unitless and $1 \le N_I' \le N_{max}$. This yields

$$Q_I \rightarrow Q_I' = \lambda Q_I + \lambda \triangle Q_I', \tag{4.3}$$

and postulates that the amount of measured heat depends on both the number of particles fluxing though the measuring apparatus and the accuracy of the measurement h_I. Because $N_I = N_I(h_I)$, changes in measurement accuracy h_I may not be independent of changes in N_I. An example of observing more heat by increasing

measurement accuracy is first only measuring the number of photons by the number of "clicks" a photodetector makes and then increasing spacial and/or momentum detector accuracy by distinguishing photon frequencies meaning that observer must have physically measured frequencies or the device has output more information (and thus more heat) to the observer. In this case the device would originally output a single number describing the total number of photons but if frequencies are measured, it will have to put out multiple numbers for the number of photons measured of particular frequency. A personal macroscopic semblance of this discussion is that the amount of heat observed depends on my location in the room, the size of my pupil, the brightness of the room, and the precision of my eyesight.

Examples of measurement scale transformation are discussed below:

4.1.1   Relation to Divergence Theorem - λQ_I

Consider a number of photons N_{max} emanating uniformly from a source with spherical symmetry and a constant $\rho = N_{max}/A$ in steady state equilibrium. The number of photons fluxing through a surface is given by,

$$N_I = \int_a \rho da' = \rho a \tag{4.4}$$

where a is the surface area that was integrated over. Consider the maximum measurable amount of heat of this system to be $Q_{max} = N_{max}\hbar\omega$ for some given time interval. We may write

$$Q_{max} = \frac{a}{A}N_{max}\hbar\omega + \frac{A-a}{A}N_{max}\hbar\omega = Q_I + Q_r \tag{4.5}$$

stating that the observer is measuring only a fraction, $\frac{a}{A}$ of the total heat coming from the source and hence $N_I = \frac{a}{A} N_{max}$ and $N_r = 1$. The value for a can vary as the observer changes distance from the source or decrease, or for instance, if the observer blocks light from entering the measurement apparatus. A fraction $\frac{A-a}{A}$ of photons go undetected or uncounted by the observer. Here the scale transformation is

$$Q_I \to Q_I' = \lambda N_I \hbar \omega = \frac{a'}{A} N_{max} \hbar \omega \tag{4.6}$$

such that $\lambda = \frac{a}{a'}$ and $\lambda \triangle Q_I' = 0$ from (4.3) because $h_I = h_I'$.

An example ensemble is solved below where N photons with angular frequency ω are pinned to the surface of a sphere of radius R. This system is chosen because it is a microcannonical semblance of a system with constant number of photons having angular frequency ω fluxing through the surface of a sphere over a given time interval:

A photon has phase-space volume $V_1 = V_q V_p$ where $V_q = A = 4\pi R^2$, the surface area of a sphere, and the total energy of the system is $E = Npc = N\hbar\omega$, so $V_p = \frac{E}{c}$. Since p is the magnitude of momentum, no integration is needed. Keep in mind that this example had different number of dimensions for V_q and V_p so the units of h_{min} so should match V giving $h_{min} = (\triangle x)^2 \triangle p$. The entropy for N_{max} photons is $S_{max} = N_{max} k_b \ln(\frac{V_1}{h_{min}})$ and terms like $N_{max}!$ do not contribute to the entropy as is shown later. The temperature of the system is $\frac{1}{T} = \frac{dS_{max}}{dE} = \frac{k_b N_{max}}{E}$, so $T = \frac{E}{N_{max} k_b}$. The maximum measurable heat is equal to $Q_{max} = TS_{max} = E\ln(\Omega) = N\hbar\omega \ln(\Omega_1)$, hence $\ln(\Omega_1) = 1$ from (4.5), and thus $h_{min} = \frac{V_1}{e^{A/A}}$ from equations (2.5) and (2.6) because $\Omega_I = e^{a/A} = \Omega_1^{\cos^2(\theta)}$. The value h_{min} is the value of h_I needed to measure every photon on the surface of the sphere. Here, Q_{max} can be broken up into Q_I

and Q_r by using $h_I = \frac{V_1}{e^{a/A}}$ such that,

$$Q_I = \frac{a}{A} N_{max} \hbar \omega = N_I \hbar \omega = N_{max} \hbar \omega \ln(V_1/h_I) \tag{4.7}$$

and

$$Q_r = \left(\frac{A-a}{A}\right) N_{max} \hbar \omega = (N_{max} - N_I) \hbar \omega = N_{max} \hbar \omega \ln(h_I/h_{min}) \tag{4.8}$$

implies that the observed heat from measuring photons in an area a out of the total area A with perfect measurement is equivalent to measuring the entire sphere with a larger uncertainty $h_I = \frac{V_1}{e^{a/A}} > h_{min}$ for this case. This shows that N_I has some dependence on h_I and vice-versa.

4.1.2 Blurry Object - $\triangle Q'_I$

The motivation of this section is to describe the blurriness of objects by varying $h_I > h_{min}$ and show that $\triangle Q'_I$ is the change in the measured composition of the observed heat.

An object appears blurry when photons from different points on an object occupy the same apparent spacial state. This can be seen directly by unfocusing your eyes on an object with multiple colors or removing prescription glasses. Note that an object of uniform color and light intensity will not appear blurry because the apparent displacement of colors does not change the image (due to symmetry), and the shape of an image is only apparent if the object has different color or light intensity than the background. The edge of an object will lose contrast as measurement uncertainty increases. The state of maximum blurriness occurs when all photons of a static system appear to be distributed everywhere on the object, and

no physical boundaries between color areas can be distinguished. A measurement using maximum blurriness gives zero internal information, making the system appear to be completely noise and/or uniform. This occurs when $h_I = V$ giving $I = 0$ and $S_r = S_{max}$ because all of the photons occupy the same h_I. Here S_r can be considered a signal noise notional to Shannon's paper [11].

The following discussion will be about the combinatorics of N photons in b bins. Photons are bosons and as such can occupy the same state, or in this case the same bin. To properly define the maximum number of countings for a system with Ω_{max} having multiple different colors, one cannot simply use the multiset coefficients [29]

$$w(N, b) = \frac{(N + b - 1)!}{N!(b - 1)!},$$

$$(4.9)$$

where N is the number of photons and b the number of bins, because it assumes bosons of the same stack height are indistinguishable. This would be the case if every stack of the same height was the same frequency photon. Therefore, consider the following functions that have N photons of distinguishable frequency:

The integer number partition function, $p(n)$ is a function that gives the number of ways an integer n can be partitioned; for example the number 4 has the five following partitions: $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, which is depicted in Figure 4.1 [2]. Appendix 6.1 lists the values for $p(n)$ up to $p(10)$ and only approximations are available for large n. Consider mapping the number partitions into bins, where each integer separated by a $+$ sign occupies a bin. The number of occupied bins is denoted o_i. The physical number of bins is $b = V/h_I^D$, where σ_p need only be smaller than the smallest difference in the momentum of the photons and greater than zero. Because N bosons can occupy the same bin, the number of distinguishable boson configurations Ω_{dB}, where the order of bosons in
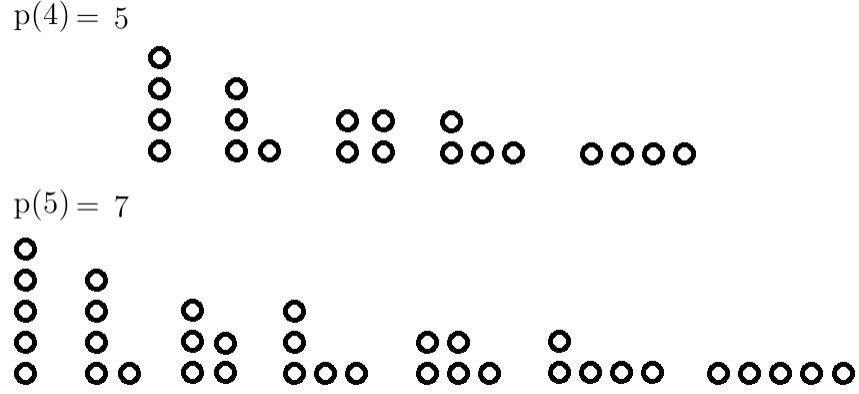
Figure 4.1: These are Ferrers diagrams for the number partition function $p(n)$ for $n = 4, 5$ [1]. This shows the number partition function gives the number of ways n bosons can be stacked among bins so permutations of such stacks can be accounted for later in the total number of configurations.

each stack is irrelevant but permutations of the stacks and color are counted, is

$$\Omega_{dB} = \sum_{i=1}^{p(N)} \Omega_i = \sum_{i=1}^{p(N)} \frac{b!N!}{(b-o_i)!f(k_{ij})\prod_j(k_{ij})!}, \tag{4.10}$$

where k_{ij} is the number of bosons occupying the jth boson stack in the ith number of occupied bins case, and $N!/\prod_j k_{ij}!$ is a multinomial coefficient. The function $f(k_{ij})$ is a product of factorials where the product runs over the number of groups of stacks g with the same height and the number in the factorial is the number of same height stacks in that group so,

$$f(k_{ij}) = \prod_{g=1}^{G}\left(\sum_{m=1}\delta_{k_{ig},k_{im}}\right)! \tag{4.11}$$

where stack heights of zero are not regarded, $\delta_{k_{ig},k_{im}}$ is a the Kronecker delta and is equal to 1 for stacks with equal heights, and G is the number of possible stack heights; i.e. the partition $2 + 1 + 1 + 1$ has $G = 2$. The about function $f(k_{ij})$ is used so than double permuting of the photons does not occur in equation (4.10) by

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permuting photons internally by frequency and then again by bin.

For $N \geq 4$ it is possible to have $o_n = o_m$ for $n \neq m$ because, for example, two of the number partitions of 4 occupy two bins, namely $3 + 1$ and $2 + 2$. Each permutation among bins including those with equal stack height are distinguishable and o_i is the number of occupied bins set by ith member if $p(n_i)$. Because physically $o_i \leq b$, if $o_i > b$ in the above relation, set $\Omega_i = 0$ or realize that $(b - o_i)!$ with $o_i > b$ is complex infinity and $1/\tilde{\infty} = 0$ as defined by wolframalpha. Because it is possible to have $o_n = o_m$ for $n \neq m$ in equation (4.10) due to the nature of the number partition function, a density of occupied bins function $g_i = g_i(N, o_i)$ is found for a given N and occupied number of bins o_i such that the reader knows how many times the same bin number is used in the sum from equation (4.10). The density of occupied bins function is partially recursive and is,

$$
g_i(N, o_i) = \begin{cases} p(N - o_i) & \text{if } 0 \geq (N - 2o_i); \\ p(N - o_i) - \sum_{j=o_i+1}^{N-o_i} g_j(N - o_i, j) & \text{if } 0 < (N - 2o_i). \end{cases} \tag{4.12}
$$

while remembering $o_i \leq N$. It can be show that $g(N, 1) = 1$, which is intuitive because there is only one partition for N particles occupying $o_i = 1$ bin. Figure 4.2 is a spreadsheet for the first few values of $g_i(N, o_i)$.

As it turns out, the number of permutations for distinguishable bosons is simply

$$
\Omega_{dB} = b^N \tag{4.13}
$$

for all tried values of N and b in (4.10). This is because both equation (4.10) and (4.13) do not consider stack order and count each distinguishable permutation. Equation (4.10) will likely not be used to calculate the total number of configurations due to it being much more cumbersome than equation (4.13) for N distinguishable

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o_i \ N	1	2	3	4	5	6	7	8	9	10	11
					Values of g(N,o_i)						
1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	2	2	3	3	4	4	5	5
3	0	0	1	1	2	3	4	5	7	8	10
4	0	0	0	1	1	2	3	5	6	9	11
5	0	0	0	0	1	1	2	3	5	7	10
6	0	0	0	0	0	1	1	2	3	5	7
7	0	0	0	0	0	0	1	1	2	3	5
8	0	0	0	0	0	0	0	1	1	2	3
9	0	0	0	0	0	0	0	0	1	1	2
10	0	0	0	0	0	0	0	0	0	1	1
											1
sum	1	2	3	5	7	11	15	22	30	42	56

Figure 4.2: Spreadsheet for the values of $g_i(N, o_i)$ up to $N = 11$ particles and the sum over i of $g_i(N, o_i)$ is computed in the bottom row to show that $\sum_i^N g_i(N, o_i) = p(N)$. The sum is equal to the integer partition function because $g_i(N, o_i)$ is summed over for all possible occipied bins configurations. This is most easily seen in $\sum_i^4 g_i(4, o_i) = p(4)$.

bosons.

Equation (4.10) might be easier to modulate to find remaining possible configurations after an observation has been made on the system because equation (4.10) "spells out" each configuration. For instance if the bins are measured to have equal number of photons in them then $\sum_i \Omega_i = \Omega_n$ because there is only 1 number partition for this instance, $b = o_n$, $f(k_{nj}) = b!$, $\prod_j (k_{nj})! = (m)!^b$ where m is the stack height, and $N = mb$, giving the remaining number of possible configurations from (4.10) to be $\Omega_r = \frac{N!}{(m)!^b}$, something that might not have been obvious from the form of (4.13). The remaining number of configurations in this case is a multinomial coefficient.

Removing $f(k_{ij})$ and the multinomial coefficient $N!/\prod_j k_{ij}!$ from equation (4.10) gives the number of configurations for N indistinguishable photons because the photons are no longer permuted among themselves. Dividing both equation (4.10) and (4.13) by $N!$ gives the equation for the approximate number of configurations

for indistinguishable particles from (3.9), in the limit $N << b$.

Entropy of a blurry object with variable contrast h_I

The explicit formula for the maximum entropy of a blurry object with N photons of different frequency is

$$S_{max} = k_b \ln(\Omega_{dB}) = k_b \ln \left(N_{max}! \Omega_{max}! \sum_{i=1}^{p(N_{max})} \frac{1}{(\Omega_{max} - o_i)! f(k_{ij}) \prod_j k_{ij}} \right). \quad (4.14)$$

where $\Omega_{max} = b = V \sigma_p^D / h^D$ are the spacial bins in D dimensions because momentum possibilities are permuted among the spacial bins in equation (4.14). Using $\Omega_{max} = \Omega_I \Omega_r$, log rules, and Stirling's approximation, I break up (4.14) into the measurable and relative entropy components

$$\begin{aligned}
S_{max} = &k_b \ln \left(\Omega_I^{\Omega_I} N_I^{N_I} \right) + k_b \ln \left(\Omega_r^{\Omega_r} N_r^{N_r} \right) \\
&+ k_b \ln \left(\sum_{i=1}^{p(N_{max})} \frac{1}{(\Omega_{max} - o_i)! f(k_{ij}) \prod_j k_{ij}} \right)
\end{aligned} \quad (4.15)$$

It is nontrivial to break up the last term into S_I and S_r and this term is removing permutations from the previous two terms. Now S_I and S_r are seemingly super extensive because Ω_I and Ω_r are multiplying factors inside and outside S_I and S_r. For simplicity consider using equation (3.5) and (4.13) and rather than equation (4.10) to give

$$S_{max} = k_b \ln \left(\Omega_I^{N_I} \right) + k_b \ln \left(\Omega_r^{N_I} \Omega_{max}^{N_{max} - N_I} \right) \quad (4.16)$$

but if every photon is observed, $N_I = N_{max}$ and

$$S_{max} = S_I + S_r = k_b \ln \left(\Omega_I^{N_{max}} \right) + k_b \ln \left(\Omega_r^{N_{max}} \right) \qquad (4.17)$$

thus transformations of the h_I from $\triangle Q_I'$ are required to change the amount of information observed.

The following discussion for photons pinned the the surface of a sphere. The energy of this system with N photons of different frequency is $E = \hbar \sum_{i=1}^{N_{max}} \omega_i = c \sum_i^{N_{max}} |p_i| = c|P|$. The size of momentum space per photon is $|P| = \frac{E}{c}$ and is 1 dimensional because at each position, the number of possible frequencies is N_{max}. The max entropy is,

$$S_{max} = N_{max} k_b \ln \left(\frac{V}{h_{min}} \right) = N_{max} k_b \ln \left(\frac{V_q E}{\triangle_{qmin}^2 \triangle_{pmin} c} \right), \qquad (4.18)$$

where $\triangle_{qmin}$ is the minimum uncertainty of position, $\triangle_{pmin}$ is the minimum uncertainty in momentum, $V_q = A = 4\pi r^2$, and the temperature is $T = \frac{E}{k_b N_{max}}$. The maximum heat is $Q_{max} = E \ln \left(\frac{V_q E}{\triangle_{qmin}^3 \triangle_{pmin} c} \right)$ thus $\triangle_{qmin}^3 \triangle_{pmin} = \left(\frac{V}{e} \right)$ so that $Q_{max} = E$. For constant E, changes in V_q change the required value of h_{min}. For $N_I = N_{max}$ and homogeneous transformations of h_I, the amount of blur is the the relative entropy term $S_r = N_{max} k_b \ln \left(\frac{\triangle_{qI}^2 \triangle_{pI}}{\triangle_{qmin}^2 \triangle_{pmin}} \right)$, which describes the number of possible position and momentum/frequency states for each inaccurately measured photon if all photons are measured.

4.1.3 Time Dependent $h_I(t)$

The transformation $h_I \rightarrow h_I(t)$ can be implemented if the measuring apparatus is changing measurement accuracy uniformly over space as a function of time. An

example of this is blinking your eyes. When your eyes are shut much less light is observable than with them open and in-fact the amount of light fluxing through your eyes varies as a function of $h(t)$. A crude approximation of this is $h_I(t) = V/4 \sin(\omega t) + V/2$, a sinusoidal change in h about $h_0 = V/2$. The amount of entropy measured as a function of time is,

$$S_I(t) = k_b N_I \ln\left(\frac{V}{h_I^D(t)}\right) \tag{4.19}$$

and $Q_I(t) = TS_I(t)$ but S_{max} will remain constant given the system is in steady state equilibrium. This idea could be combined with the notions of (??) to obtain $S_I(x, y, z, t)$.

4.1.4 Unitary Operator and Quantum Mechanics

For quantum systems consider the unitary operator $U = \left(\frac{n\hbar}{mh_I}\right)^i = \left(\frac{\alpha\hbar}{h_I}\right)^i = 1^i = 1$ and $U^\dagger = \left(\frac{mh_I}{n\hbar}\right)^i$ where $\alpha = \frac{n'}{m'} = \frac{h_I}{\hbar}$ describing the number of n per m. Equation (2.33) in 1D is derived using the unitary operator $(U^\dagger)^2$, which commutes with $\hat{H}$, and $|\psi\rangle = \sum_{n=1}^{\infty} c_n |n\rangle$ giving,

$$\hat{H}(U^\dagger)^2 |\psi\rangle = \frac{(mh_I\pi)^2}{2mL} |\psi\rangle = \frac{(mh_I\pi)^2}{2mL} \sum_{n=1}^{\infty} c_n |n\rangle \tag{4.20}$$

$$= E_m \sum_{m=1}^{\infty} c_{\alpha m} |\alpha m\rangle \tag{4.21}$$

where

$$c_{\alpha m} = \begin{cases} c_n \text{ for } \alpha m = n \\ 0 \text{ for } \alpha m \neq n \end{cases} \tag{4.22}$$

In as sense, the above relations represent "scaling" degeneracies for energy eigenvalues having $n = \alpha m$. The process of raising U or $U^\dagger$ to the N and multiplying it into $|\psi\rangle$ seems to be a way to arbitrarily scale any discrete quantum system. When $n \neq \alpha m$, the probability of observing E_m is zero. The sum of all zero probability energy from $n \neq \alpha m$ corresponds to Q_u the infinite unobservable energy, and energy states skipped over when scaling are considered to be a part of Q_r the remaining energy. Probabilities should be renormalized after scale transformation. It is possible that this could make a continuous energy appear discrete if positions are measured along with the kinetic energy which is a function of momentum.

Consider the quantum harmonic oscillator with $\hat{H}|\psi\rangle = \hbar\omega(a^\dagger a + 1/2)|\psi\rangle = \hbar\omega(n + 1/2)|\psi\rangle$. Applying the unitary operator $U^\dagger = \left(\frac{mh_I}{n\hbar}\right) = \left(\frac{h_I b^\dagger b}{\hbar a^\dagger a}\right)$ having the constraint $\alpha = \frac{n'}{m'} = \frac{h_I}{\hbar}$ gives,

$$U^\dagger \hat{H}|\psi\rangle = \hat{H}U^\dagger|\psi\rangle = h_I\omega(m + m/2n)|\psi\rangle = h_I\omega(m + 1/2\alpha)|\psi\rangle$$
$$= h_I\omega(m + 1/2\alpha)\sum_{n=0}^{\infty} c_n|n\rangle = E_m \sum_{m=0}^{\infty} c_{\alpha m}|\alpha m\rangle \qquad (4.23)$$

showing that $[H, U^\dagger] = 0$ and the c-numbers having the same relation as equation (4.22), hence scale transformation will preserve the value of c_0 before renormalization and the ground state energy. Probabilities should be renormalized after scale transformation. This is a postulate based on previous notions of this paper and may require experimentation to prove true or false. It states essentially that if we do not know precisely the frequency of a photon emitted, we do not know precisely the energy level of the system. The ground state energy remains constant in this case.

4.1.5 Macroscopic Environment

A probabilistic event requires a number of possible measurable outcomes. For a macroscopic event, like darts, rules are used to interpret the results of the game. Here, one could decide to make unconventional rules such that the dart board is partitioned into the left and right side. Now, as compared to a standard game of darts, the probabilities of the game have been rescaled. In this case $\triangle Q'_I$ may describe the difference in energy needed to distinguish outcomes between the two games, i.e. the difference in energy needed to determine dart scoring based on dart vicinities.

4.2 Phase-Space Volume V as a function of h_I

Above sections have considered $N_I = N(h_I)$, but it is possible that the system's measured value of E and/or V_q could be functions of h_I because V_q and E are never measured with infinite precision The discussion of $E(h)$ and $V_q(h)$ can most easily be discussed as the phase-space volume V being a function of h because $V = V(E, V_q)$. An observer measures V (although momentum space volume may not be directly measurable) and obtains an average value $\pm$ the uncertainty or standard deviation. Twice the magnitude of the uncertainty is the value of h_I, which will break up phase-space into the number of possible measurable configurations, Ω_I. The actual value of V is thus a random number within $\pm h_I/2$ about the value V. As h_I decreases to h'_I, the average measured value of V may shift to some V' within $\pm h_I$ of V. This will give an apparent shift in V, where the true value of V' has not changed. The V dependence on h_I is equivalent to changing the number of significant digits of V in accordance with changes in the significant digits of the uncertainty or standard deviation.

4.3 Unobservable Chemical Equilibrium with S_u

A system is in thermodynamic equilibrium if temperatures, chemical potentials, and pressures are equal respectively [5]. The zeroth law of thermodynamics reads, "If two bodies are in thermal equilibrium with a third body, they are in thermal equilibrium with each other" [5]. This being said, if only the unobservable entropy per system is considered, $S_u = k_b N \ln(\frac{h_{min}}{h_u})$, and summed across all systems, the described entropy/freedom is the location of N_{tot} particles within their respective h_{min} values. The Helmholtz free energy is $F_u = Q_u - TS_u = 0$ and using $\mu = \frac{\partial F}{\partial N}$ gives unobservable chemical potential $\mu_u = 0$. Because $\mu_u = 0$ for particles in a system, they are in unobservable chemical equilibrium with each other. Assuming that the system particles are in unobservable chemical equilibrium with their boundaries, because most boundaries are made of particles, each particle would be in unobservable chemical equilibrium with every other particle in the universe, given the zeroth law can be expanded to chemical equilibrium instead of restricted to thermal. This implies that no change in the unobservable entropy would occur when mixing particle systems. The temperature and pressures of particles within their h_{min} are not required to be the same because systems can vary in average kinetic energies, unless one takes the view that the temperature of the universe system is a constant in which case the universe is in thermal equilibrium with itself. To sum up the argument, all observers share that there are infinite unmeasurable coordinates about particles, but that does not take away from what is measurable, in fact it adds. Reference [20] derived an infinite equilibrium which may have connections to unobservable chemical equilibrium.

4.4   Granular and Non-Granular Universe

Reference [30] states, "... indiscernible objects of the universe form clusters of indistinguishable objects (granules, atoms, etc.). Thus from the rough set view the granularity of knowledge is due to the indiscernibility of objects caused by lack of sufficient information about them. Consequently granularity and indiscernibility are strictly connected". Hence the uncertainty principle leads to a minimum *measurable* value of h, namely $h_{min} = \hbar/2$, which gives a granular appearance to the measurable universe.

Many theories including relativity and classical mechanics treat space and momentum to be continuous and taking $h \to 0^+$ bridges quantum and classical systems [3] in some aspects. From this standpoint the universe is, in a way, both granular and non-granular (in measurement and in intuition respectively).

4.5   Second Law of Thermodynamics

The second law of thermodynamics states that the entropy of the universe is always increasing. In light of this and the notions of variable h, it seems that one could fix the phase-space hyper-volume of the universe while taking its average $h(t)$ to be a decreasing function in time, as a way of increasing the entropy of the universe over time. As an observer it seems that one could not tell the difference between the universe expanding or ones instruments getting more fine-tuned because both lead to an increase in the measurable phase-space states. This way light could be red shifted by either a local decrease in h_{min} or an acceleration of the space as indicated by relativity. An interesting discussion is the bound $h(t)$ approaches; is it $\hbar/2^+$ or 0^+? It may be that $h(t) \to \hbar/2^+$ in which case the entropy of the measurable universe approaches a maximum value; however, if one considers consequences of the

previous section, $h(t)$ could approach 0^+ leading to an infinite entropy and thereby an infinite time line because entropy will continue to increase indefinitely. Could Planck's constant $\hbar/2$ be decreasing as a function of time ever so slightly (far below the measured precision of $\hbar$) in a way that is inversely proportional to the expansion of space $V(t) \propto \frac{1}{\hbar(t)^3}$? Because $\hbar$ is so small and there are many units of $\hbar$ on the inter-solar system scale, a very small change in $\hbar$ could lead to apparent expansions of space. Either way; space is expanding or measurements on space are more fine-tuned as space gets older. If the universe began at a single point, this would be equivalent entropy-wise to $\hbar = V$, saying that initially the universe was in a single state measured by $\hbar$ equally the size of the universe or the initial phase-space volume of the universe before expansion was $\hbar$. Because scale transformations have strong ties with fractals [28, 31, 32], future research about the fractal nature of entropy can be investigated. An open ended question is, "As $h(t) \to 0^+$, could the macroscopic universe have self similar properties to unobservable entropy systems?".

Chapter 5

Conclusions

This paper expands the notion of h from being treated as an arbitrary constant to a variable defined by measurement apparatus or personal semantics (in the case of a game) giving the observer a choice on how fine he/she wishes to partition a system. The classical definition for a change in entropy has been expanded adding $dS_I = -\frac{Nk_b D}{h}dh$ to dS_r and the definition of the total entropy of a system has been broken into measured, ignored, and unobservable entropy in Eq. (2.7). Also the bound $h \to 0^+$ is explored leading to the development of S_u, an infinite unobservable entropy existing in all things with phase-space admitting there are infinite unmeasurable phase-space coordinates due to the restrictions of the uncertainty principle. An expression for infinite unobservable energy is found by $Q_u = TS_u$ which is interpreted as an additional unobservable heat a particle has as it explores unobservable/indistinguishable coordinates. This leads to an unobservable chemical equilibrium between all particles because $\mu_u = 0$ for every particle. A measurement scale transformation of variable h_I is obtained and it is the change in heat $\triangle Q'_I$ fluxing through the measuring apparatus. This is interpreted as a change in the information carrying heat coming from the system. Variable h is explored in three

example systems that cover: classical gas in a box, quantum mechanical particles in an infinite potential box, and classical gas in a box measured with two different values of h- one value on the left side and one on the right side of the box. This paper discusses $N_I = N_I(h_I)$ and states that the number of observed particles is a function of the measurement uncertainty h_I. This paper also develops a model for the entropy of a blurry object.

This paper allows for the development of several new research topics all revolving around variable h_I. One possibility is to see if it has applications in renormalization theory in particle physics because there are infinite degrees of freedom being renormalized into finite degrees of freedom after measurement. Also at some time I would like to investigate if relativity can be interpreted by inhomogeneous transformations of h_{min} that would cause space or atleast probability distributions to distort. The Schrdinger equation can be used to find probability distributions with a given potential energy. It could be possible to relate S_I, by appropriate shifts in $h_I(x)$, to the potential energy of the system and thereby leaving $S_r \approx S(p)$ the von Neumann entropy. Naively, an infinite potential box has $\sin(x)$ solutions so it could be possible that $\Omega_I \approx 1/\sin(x)$ which goes to infinity at $x = 0, \pi$, the same places the potential goes to infinity. This implies that any potential added to a quantum mechanical system acts as a partial measurement because it changes the wave function and thus $S(p)$. The index discusses some philosophy related to infinite unobservable entropy S_u.

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Chapter 6

Appendix

6.1 The Number Partition Function $p(n)$

Below is a list of some values for the number partition function $p(n)$:

p(0)	1
p(1)	1
p(2)	2
p(3)	3
p(4)	5
p(5)	7
p(6)	11
p(7)	15
p(8)	22
p(9)	30
p(10)	42
p(n$\to \infty$)	$\approx \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}$

Table 6.1: The first few and limiting cases of the number partition function [2].

6.2 MATLAB Code for Sub-particles Constrained to a Cubic Formation

```
clear

clc

tic

% POSITIONAL CONSTRAINT- ADJACENT sub-particleS in IN CUBE FORMATION in

% 3D SPACE

%-H REDUCES BY FACTOR OF 2- AND CONSTRAINT NSP=1/8*NPHASE



% choose K=value to set up 3D volume space elements



K=10;



for r= 2:1:K

x = 1:2.^(r-1);
```

```
y = 1:2.^(r-1);

z = 1:2.^(r-1);


% find vector lengths

lenx = length(x);

leny = length(y);

lenz = length(z);


% initialize space

phasespace = zeros(lenx,leny,lenz);


%loops that systematically count number of configurations while breaking if

%sub-particle in corner position is too close to boundary

count =0;

for i = 1:lenx

    for k= 1:lenz

      for j = 1:leny

        if j+leny/2-1>leny

            break

        end

        if i+lenx./2-1>lenx

            break

        end

        if k+lenz./2-1>lenz

            break

        end
```

```matlab
            phasespace(i,j,k) =1;


    if phasespace(i,j,k) ==1
        count=count+1;
        phasespace(i,j,k) =0;
        else
        count=count
    end
        end
      end
end




% The value of omega for a particular r or K (omega)
omega(r)=count;
% The number of sub-particles (nsp) that occupy 1/8th of the phase-space
nsp(r)=lenx*leny*lenz./8;
% The number of phase-space elements (nphase)
nphase(r)=lenx*leny*lenz;


end



omega
nsp
```

```
nphase


%For this case omega follows the rule omega_i=(2*cbrt{omega_{i-1}}-1)^3

%where omega_1=2^3   (not counting the 0th column)


%OR

% omega_i=8*omega_{i-1} FOR LARGE nphase

%nphase_i=8*nsp_i


%plot(omega,nphase)




toc
```

6.2.1   Matlab Results

```
I =

    10


omega =

         0           8          27         125         729        4913       35937      274625     2146689    16974593


nsp =

         0           1           8          64         512        4096       32768      262144     2097152    16777216


nphase =

         0           8          64         512        4096       32768      262144     2097152    16777216   134217728

Elapsed time is 44.579730 seconds.
```

Figure 6.1: These are the results from the above MATLAB code where omega is the number of measurable configurations Ω_I, nsp is the number of particles or subparticles, and nphase is the number of phase-space coordinates.

6.3   Philosophical Thoughts on S_u and Q_u

by: Kevin Vanslette and Stephen Hoffer

A property that every faith shares is that it is a system of belief(s) that originates out of an amount of unobservable or unobserved information, whatever that belief(s) may be. This paper implies, by infinite unobservable entropy S_u, that there will always be an infinite amount of unobservable and unobserved information and energy (that can do no work). This infinite basin of unobservable information S_u is taken as a domain for faith. The fact that there will always be an infinite amount of unobservable and unobserved information to observers (people) has been always been apparent; what has not been clear is whether this amount of information could possibly be known and if science could eventually eliminate the gap that faith fills. There is only a finite amount of information science can measure and therefore experimentally justify where the amount of information and energy that will go unmeasured is infinite.

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The infinite unobservable information described in is explicitly the exact knowledge of the positions and momentums of particles which is a physical seemingly boring and mundane domain for faith. However, because particle and light interactions govern nearly all forms of measurable interaction, including life-form interactions, there are infinite random variables in the unobservable domain contributing to a measurable outcome. Questions such as "Could faith in attributes of the unobservable domain contribute to a different measurable outcome?" are currently subjective. Therefore, the important message that this thesis gives from a philosophical standpoint is that science cannot eliminate this domain for faith, but infinite unobservable information and energy (following from thermodynamical law) will always remain. While the spectrum of faith lies in unobservable quantities from the forms of theism to the forms of atheism, what remains to an observer is a belief derived from faith in the infinite unobservable information and energy domain. I personally believe that S_u acts subjectively with the observer in that the knowledge of it changes your perspective and life path. The knowledge of an infinite amount of unobservable entropy and energy has enforced my theistic views because the world of invisible things can be understood through the things that were made.