

# Pricing of Multi-Name Credit Derivatives Using Copulas

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## **Executive Summary**

The credit derivatives market has been growing rapidly over the past years. Credit risk, like market risk or interest risk, is part of the risk family in the financial industry. A number of credit derivatives have been developed to protect investors against credit risk. As more and more participants appeared, the market calls for more quantitative ways to measure risks, and eventually to price credit derivatives more properly. The credit derivative class can be divided into two basic categories according to the number of credit risk products being protected, single-name and multi-name credit derivatives. The product is from only one issuer for the single-name credit derivatives, while there would be several issuers for products in multi-name derivatives. This paper focuses on the copula approach to the pricing of multi-name credit derivative baskets.

A copula function is basically a multi-dimensional distribution function with uniform marginal distributions. It explains the connections between the marginals from a more fundamental level. Unlike the traditional use of linear correlation, a copula function is able to separate the specific marginal distributions from the underlying dependence structure of the random vector. Thus with the same linear correlation, there can exist a number of random vectors with different copula functions. The copula function of a random vector does not change when monotonic transformations are applied to the marginals, while linear correlation stays unchanged only when linear transformation is performed to the marginal distributions. Another point worth noticing is the concept of tail dependence, which stands for the close dependence of the marginals in extreme cases. Copula functions also take into account this phenomenon which is very common in the financial field.

The problem to be solved in this project is to price a basic form of a first-to-default credit derivative basket via Monte Carlo simulation for different copula functions. Several typical members of the copula family are studied. The simulated results are compared to see the properties of different copula functions, including tail dependence, etc. An interface is developed during the course of the project to automate the simulation process and create graph for simulated random vectors. The functions also include the generation of meta-distributions, in which case the marginals are arbitrary, which is more closely related to the market.

The product to be priced is basically an insurance policy that covers the loss of the first bond to default in a portfolio of two. The idea is to obtain the proper premium for the basket from equate the present value of the coverage and the string of premiums. The probabilities involved in the calculations are acquired from simulations of individual cases, as a number of copula functions have implicit forms. Different amounts for premiums resulted from investors entering at different points in time before maturity are also part of the calculation.

Several set of analysis were performed and the results were compared and discussed. The copula model proves to be a more precise and complex approach in evaluating risks and pricing multi-name credit derivative baskets.

## **Abstract**

The goal of this project is to price multi-name credit derivatives using a copula approach. The properties and advantage copula functions have to other traditional methods are carefully evaluated. Monte Carlo simulations are studied and performed to obtain numerical results for copula functions with explicit and implicit forms. A model was developed to price a basic form of a first-to-default basket using different copula functions. The outcomes are analyzed and comparisons are carried out.

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# 1. Introduction

More and more attention has been paid to risks financial products are exposed to other than market risk or interest risk. Credit risk, which consists of default risk and credit spread risk, is an important member of the risk family. Default risk is the risk that a party fails to carry out pre-promised payments. Credit spread risk is the risk “due to the possible widening of the credit spread or worsening in credit quality” (Arvanitis, Gregory 4) that exists even when a party does not default. Various credit derivative products have been developed to protect investors against potential credit risks. With the ongoing credit crisis in the financial market worldwide, properly quantifying credit risk has become extremely important.

This project would focus mainly on multi-name credit derivative baskets that protect against default risk of one or a combination of securities inside a portfolio, rather than single-name credit derivative, which only covers securities from a single issuer. A large part of the pricing process calls for analysis of default correlation, namely, the correlation between the survival times (the amount of time before default happens) of the securities in the basket. The relationship between the survival times of different securities can be very complicated. The application of linear correlation, which only retains its power over linear transformations, would be very limited. The need for a more complex model led to the introduction of copula functions into the evaluation of default correlation.

*A copula function* is an n-dimensional distribution function which contains a lot more information about its marginal distribution than linear correlation. It would not be affected by monotone transformations applied to the marginal distributions, as it “provides a way of isolating the description of the dependence structure”(Embrechts,

Frey, McNeil 184) from the behaviors of the marginals. Alternative dependence measures could also be derived from copulas, like rank correlations and coefficients of tail dependence. Rank correlations are “simple scalar measures of dependence that depend only on the copula of a bivariate distribution and not on the marginal distribution.” (Embrechts, Frey, McNeil 206) Coefficients of tail dependence are “measures of the strength of dependence in the tails of a bivariate distribution” (Embrechts, Frey, McNeil 206), which provide measures of extremal dependence. The latter of the two plays a big part in the financial world, as the occurrence of extreme events tend to cluster together.

In the course of the project, a software interface is developed in Excel VBA. The workbook automates the process of Monte Carlo Simulation to produce pairs of marginal distributions with certain copula functions, outputs scatter plots for one or a pair of chosen copulas, and does pricing for a basic form of first-to-default basket. Analysis of the impacts of different copula functions have on the premiums of the first-to-default baskets are also performed and followed by results.

## 2. Copula Functions

### 2.1 Definition

An  $n$ -dimensional copula is a function  $C : [0,1]^n \rightarrow [0,1]$ , a mapping of the unit hypercube into the unit interval, for which the following properties must hold

1.  $C(u_1, u_2, \dots, u_n)$  is increasing in each component  $u_i$ .
2.  $C(1, \dots, 1, u_i, \dots, 1) = u_i$  for all  $i \in \{1, \dots, n\}, u_i \in [0,1]$ .
3. For all  $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in [0,1]^n$  with  $a_i \leq b_i$  we have

$$\sum_{i_1=1}^2 \dots \sum_{i_n=1}^2 (-1)^{i_1 + \dots + i_n} C(u_{1i_1}, \dots, u_{ni_n}) \geq 0,$$

where  $u_{j1} = a_j$  and  $u_{j2} = b_j$  for all  $j \in \{1, \dots, n\}$ .

Basically,  $C$  is a multivariate distribution function with uniform marginal distributions.

### 2.2 Sklar's Theorem

Let  $F$  be a joint distribution function with marginals  $F_1, \dots, F_n$ . Then there exists a copula  $C : [0,1]^n \rightarrow [0,1]$  such that, for all  $x_1, \dots, x_n$  in  $\mathfrak{R} = [-\infty, \infty]$ ,

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

If the marginals are continuous, then  $C$  is unique.

There is an important Corollary to Sklar's Theorem, when we apply

$$x_i = F^{\leftarrow}(u_i) = \inf\{y : F(y) \geq u_i\}, \quad 0 \leq u_i \leq 1, \quad i = 1, \dots, n,$$

$$C(u_1, \dots, u_n) = F\left(F_1^{\leftarrow}(u_1), \dots, F_n^{\leftarrow}(u_n)\right).$$

Sklar's Theorem indicates that the marginals and the dependence structure of a multivariate distribution can be separated, and how a Copula function works in such



process to “couple together” the marginal distributions. The Corollary is used for the set up of the simulation of random numbers generated by a specific copula, as it states how copulas can be “extracted” from multivariate functions with continuous marginals. It also “shows how copulas express dependence on a quantile scale, since the value  $C(u_1, \dots, u_n)$  is the joint probability that  $X_i$  lies below its  $u_i$ -quantile.” (Embrechts, Frey, McNeil 187)

## 2.3 Examples of Copulas

### 2.3.1 Fundamental Copulas

The independence copula  $\Pi(u_1, \dots, u_n) = \prod_{i=1}^n u_i$  is the case where the marginal distributions are independent.

The comonotonicity copula  $M(u_1, \dots, u_n) = \min\{u_1, \dots, u_n\}$  has perfectly positively dependent marginals.

The countermonotonicity copula  $W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$  is the joint distribution function of the random vector  $(U, 1-U)$ , where  $U \sim U(0,1)$ , i.e., the marginals are perfectly negatively dependent.

The Fréchet bounds are bounds for every copula  $C(u_1, \dots, u_n)$ ,

$$\max\left\{\sum_{i=1}^n u_i + 1 - n, 0\right\} \leq C(u) \leq \min\{u_1, \dots, u_n\}.$$

Clearly the upper and lower bounds are the comonotonicity and the countermonotonicity copulas.

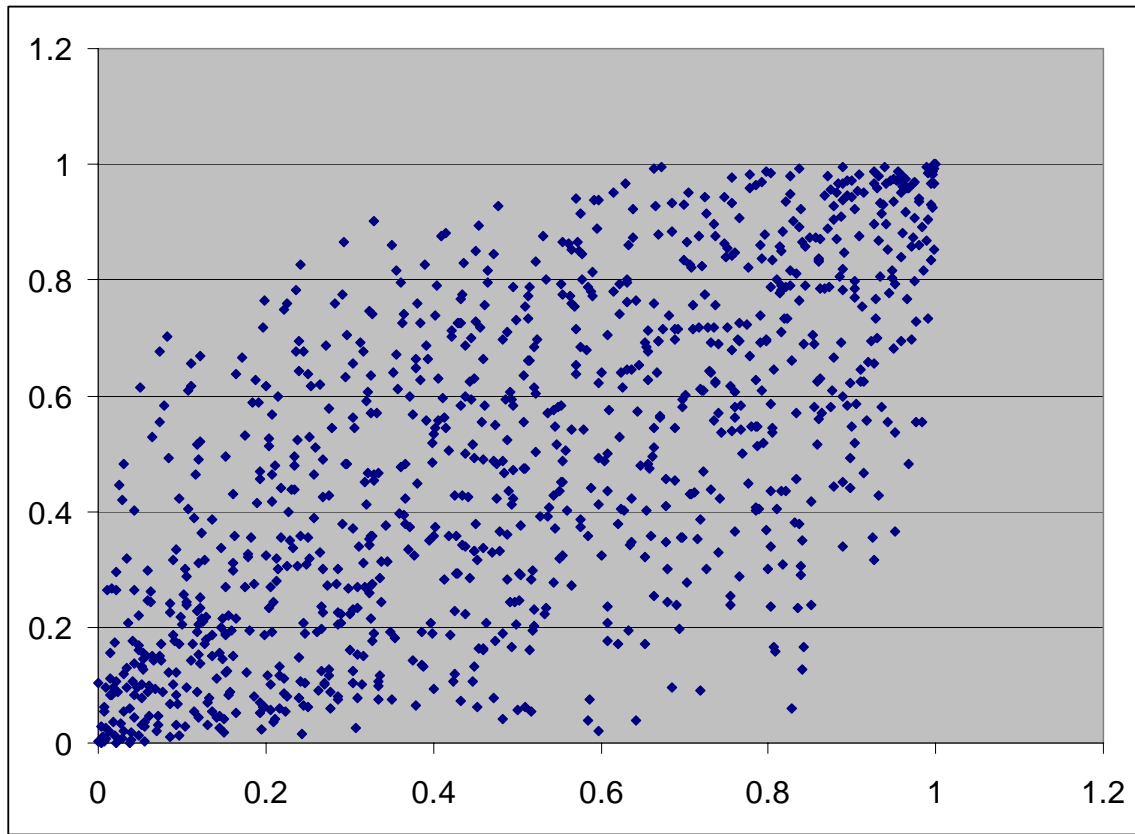
### 2.3.2 Gaussian and t Copulas

If  $Y \sim N_d(\mu, \Sigma)$  is a Gaussian random vector, then its copula is a Gaussian copula.

$$C_p^{Ga}(u) = P(\Phi(X_1) \leq u_1, \dots, \Phi(X_d) \leq u_d) = \Phi_p(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)),$$

where  $P = \rho(\Sigma)$  is the correlation matrix of  $Y$ ,  $\Phi$  is the standard univariate normal distribution function

and  $\Phi_p$  is the joint distribution function of  $X \sim N_d(0, P)$ .

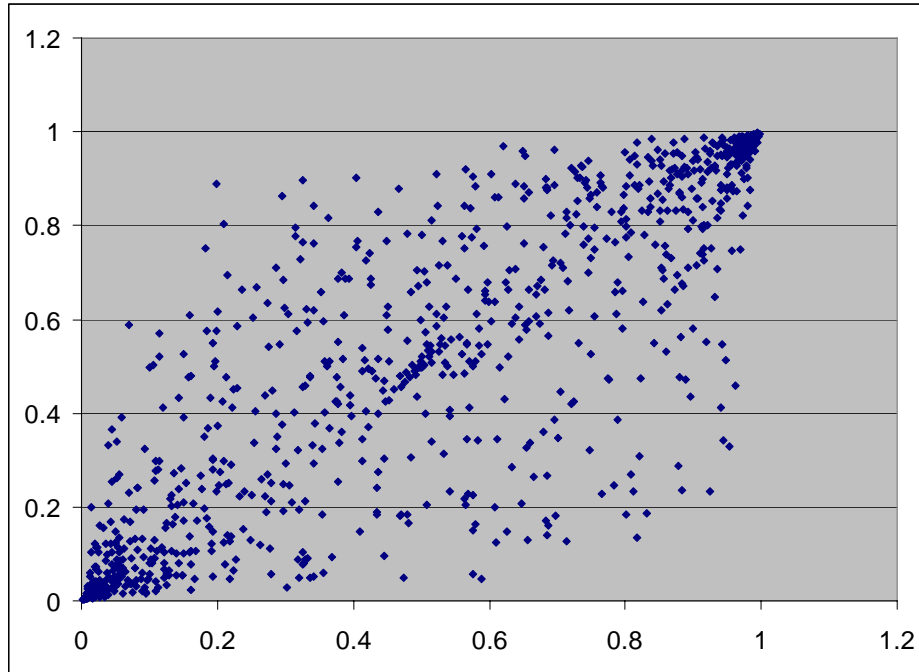


*Figure 1. 1000 simulated points for Gaussian copula with uniform marginals and correlation 0.7*

Similarly, the d-dimensional t copula

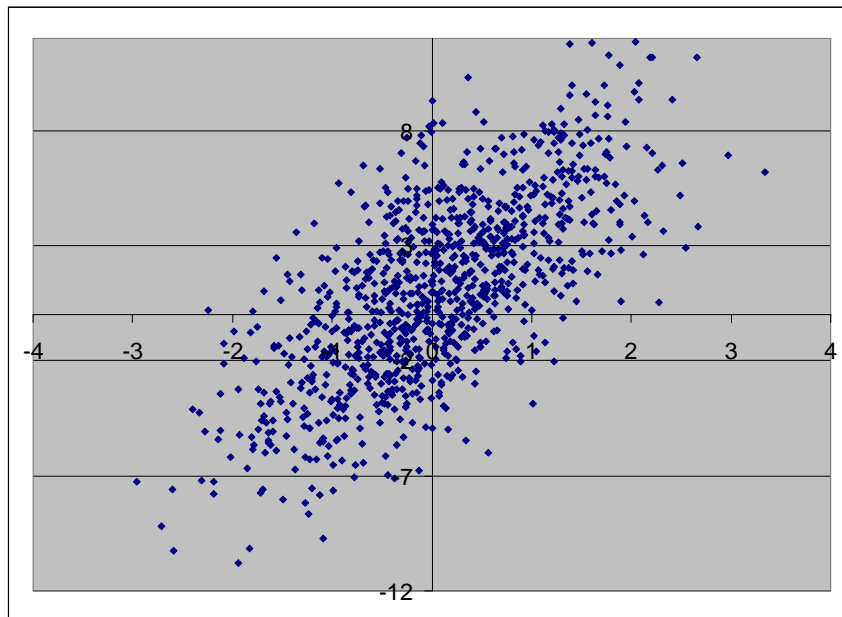
$$C_{\nu, P}^t(u) = t_{\nu, P}(t_{\nu}^{-1}(u_1), \dots, t_{\nu}^{-1}(u_d))$$

where  $t_{\nu}$  is the univariate  $t$  distribution function and  $t_{\nu, P}$  is the joint distribution function of  $X \sim t_d(\nu, 0, P)$ , and  $P$  is the correlation matrix.



**Figure 2.** 1000 simulated points for *t* copula with uniform marginals, correlation 0.9, and *df* 3

Meta distribution is the case where the marginal distributions are arbitrary distributions instead of just uniform distributions.



**Figure 3.** 1000 simulated points for Gaussian copula with correlation 0.7, the marginals are standard normal distribution, and normal distribution with mean 1, *sd* 4

### 2.3.3 Archimedean Copulas

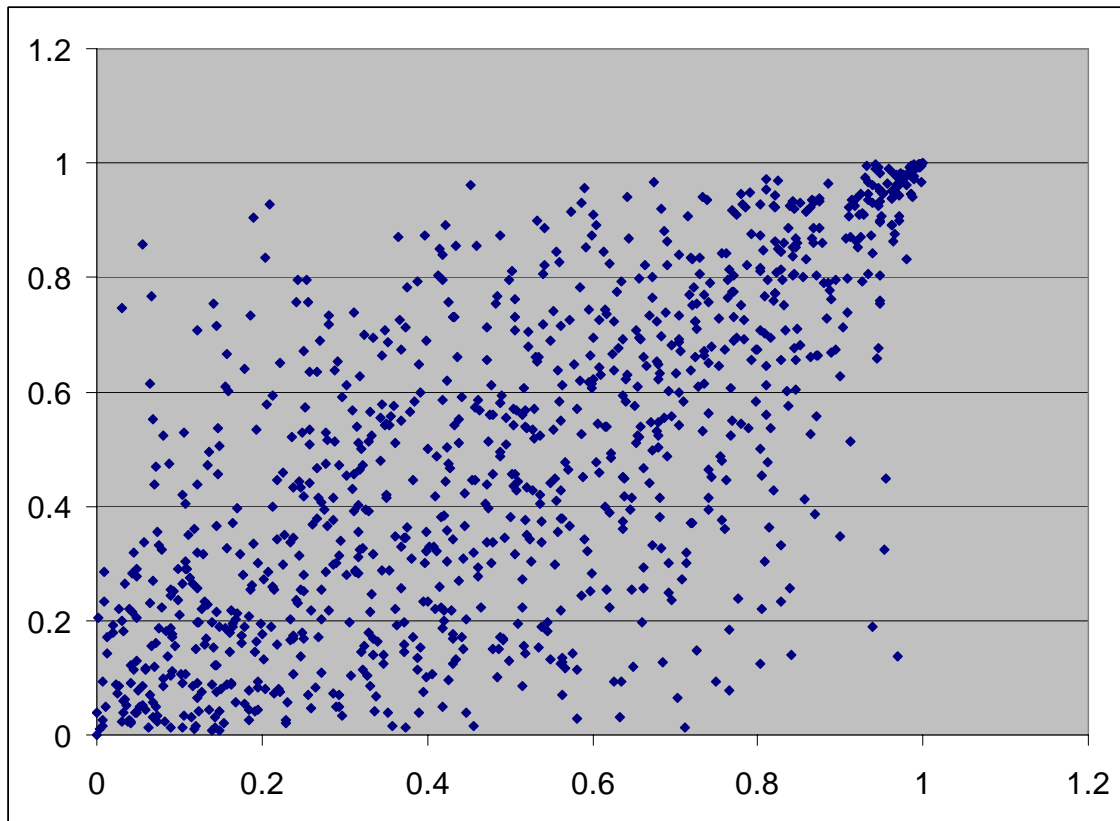
Archimedean copulas have the form  $C(u_1, u_2) = \phi^{-1}(\phi(u_1) + \phi(u_2))$ , where  $\phi$  is a decreasing function from  $[0,1]$  to  $[0, \infty)$ , with  $\phi(0) = \infty$ ,  $\phi(1) = 0$ , also known as the generator of the copula, and  $\phi^{-1}$  is its inverse. For such a function  $\phi$ , its pseudo-inverse

is defined as  $\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0) \\ 0, & \phi(0) < t \leq \infty \end{cases}$ . Hence  $C(u_1, u_2) = \phi^{[-1]}(\phi(u_1) + \phi(u_2))$  is a

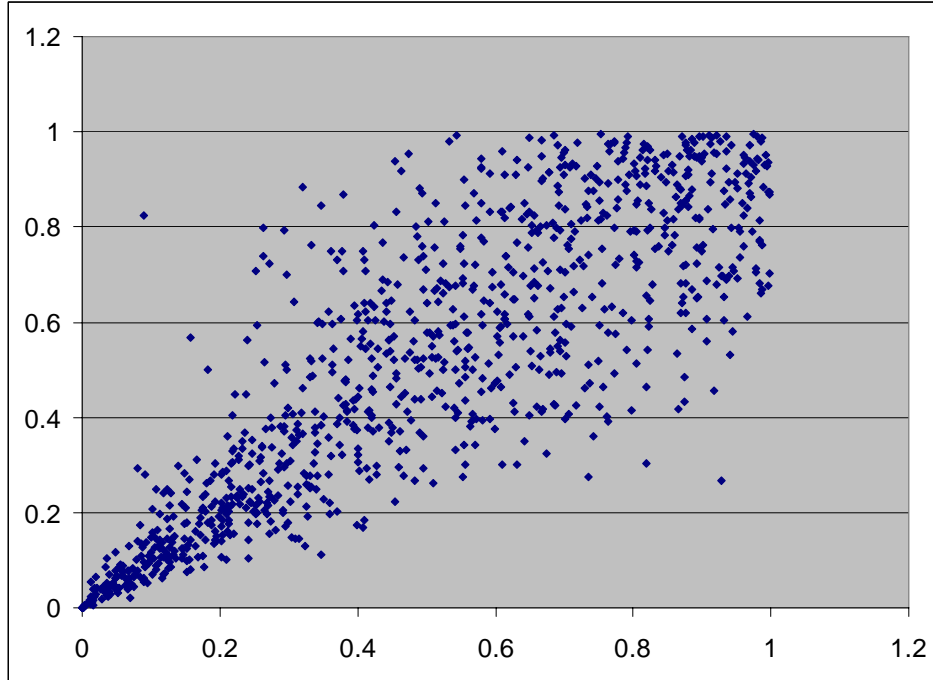
copula if and only if  $\phi$  is convex with the properties mentioned previously.

We would be talking mainly about three copulas in the Archimedean family: Gumbel copula, Clayton copula, and Frank copula.

$$\text{Gumbel: } C_{\theta}^{Gu}(u_1, u_2) = \exp\left\{-\left((-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right)^{1/\theta}\right\}, 1 \leq \theta < \infty.$$

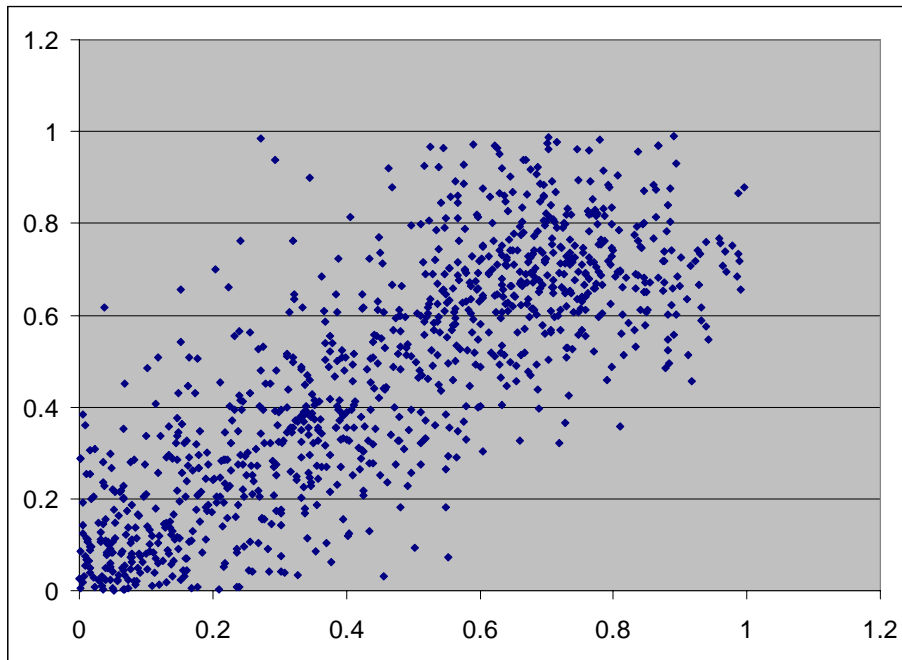


*Figure 4. 1000 simulated points for Gumbel copula with uniform marginals and parameter 2*



**Figure 5. 1000 simulated points for Clayton copula with uniform marginals and parameter 4**

$$\text{Frank: } C_{\theta}^{Fr}(u_1, u_2) = -\frac{1}{\theta} \ln \left( 1 + \frac{(\exp(-\theta u_1) - 1)(\exp(-\theta u_2) - 1)}{\exp(-\theta) - 1} \right), \theta \in \mathfrak{R}.$$



**Figure 6. 1000 simulated points for Frank copula with uniform marginals and parameter 10**

## 2.4 Measures of Dependence

### 2.4.1 Rank Correlation

“Rank correlations are simple scalar measures of dependence that depend only on the couple of a bivariate distribution and not on the marginal distributions, unlike linear correlation, which depends on both.” (Embrechts, Frey, McNeil 2006)

Kendall’s tau is a measure of concordance for bivariate random vectors. Two points  $(x_1, y_1)$  and  $(x_2, y_2)$  are concordant if  $(x_1 - x_2)(y_1 - y_2) > 0$ , discordant in  $(x_1 - x_2)(y_1 - y_2) < 0$ . Now for a random vector  $(X_1, X_2)$ ,  $(\tilde{X}_1, \tilde{X}_2)$  is an independent copy of  $(X_1, X_2)$ . Kendall’s tau can be defined as

$$\rho_\tau(X_1, X_2) = E(\text{sign}((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2))).$$

For random variables  $X_1$  and  $X_2$  with marginal distribution functions  $F_1$  and  $F_2$ , Spearman’s rho is given by  $\rho_s(X_1, X_2) = \rho(F_1(X_1), F_2(X_2))$ .

For example, Gaussian and t copulas, the rank correlations have

$$\rho_\tau(X_1, X_2) = \frac{2}{\pi} \arcsin \rho$$

$$\rho_s(X_1, X_2) = \frac{6}{\pi} \arcsin \frac{1}{2} \rho$$

## 2.4.2 Coefficients of Tail Dependence

Coefficients of tail dependence are also measures of pairwise dependence of two random variables that only depend on the copula functions. They provide information about extremal dependence, i.e., “measure of the strength of dependence in the tails of a bivariate distribution.” With tail dependence we would be able to tell how closely the two marginals are related in extreme cases, which is very important in the credit world.

Let  $X_1$  and  $X_2$  be random variables with marginal distribution functions  $F_1$  and  $F_2$ . The coefficient of upper tail dependence of  $X_1$  and  $X_2$  is

$$\lambda_u := \lambda_u(X_1, X_2) = \lim_{q \rightarrow 1^-} P(X_2 > F_2^{\leftarrow}(q) \mid X_1 > F_1^{\leftarrow}(q)), \text{ provided a limit } \lambda_u \in [0,1] \text{ exists,}$$

where  $G^{\leftarrow}$  denotes the generalized inverse of  $G$ , i.e.,  $G^{\leftarrow}(y) = \inf\{x : G(x) \geq y\}$ . If

$\lambda_u \in (0,1]$ , then  $X_1$  and  $X_2$  are said to show upper tail dependence of extremal

dependence in the upper tail; if  $\lambda_u = 0$ , they are asymptotically independent in the

uppertail. Similarly, the coefficients of lower tail dependence is

$$\lambda_l := \lambda_l(X_1, X_2) = \lim_{q \rightarrow 0^+} P(X_2 \leq F_2^{\leftarrow}(q) \mid X_1 \leq F_1^{\leftarrow}(q)), \text{ provided a limit } \lambda_l \in [0,1] \text{ exists.}$$

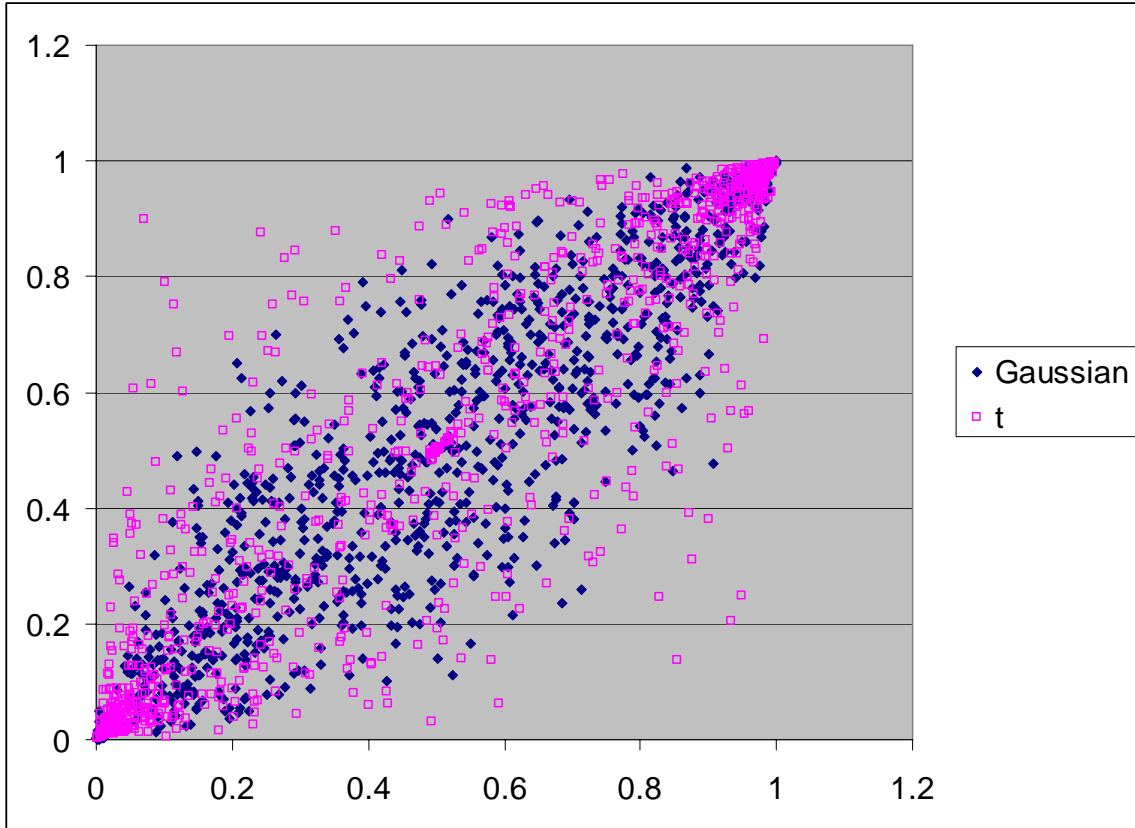
Gaussian copula and Frank copula do not have tail dependence. The coefficients of

t copula is  $\lambda = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right), \rho > -1$ . Gumbel copula does not have lower tail

dependence, and the coefficient of its upper tail dependence is  $\lambda_u = 2 - 2^{\frac{1}{\theta}}$ . Clayton

copula does not have upper tail dependence, and the coefficient its lower tail dependence

$$\text{is } \lambda_l = \begin{cases} 2^{-\frac{1}{\theta}}, & \theta > 0 \\ 0, & \theta \leq 0 \end{cases}.$$

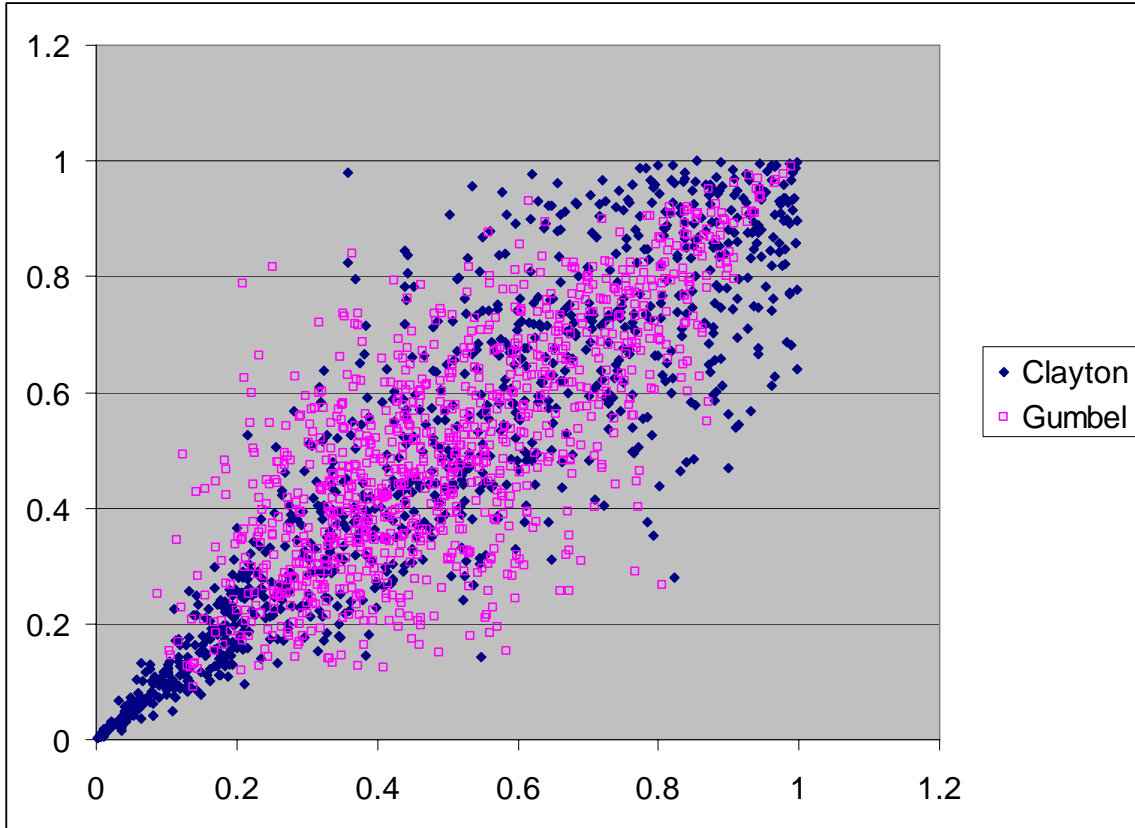


*Figure 7. 1000 simulated points for Gaussian & t copula with uniform marginals and correlation 0.9, t copula has df 3*

Type:	Kendall's tau:	Spearman's rho:	Upper Tail Dependence:	Lower Tail Dependence:
<b>Gaussian</b>	0.712867	0.89145613	0	0
<b>t</b>	0.712867	0.89145613	1.355111947	1.355111947

*Table 1. Measures of dependence for Gaussian & t copula with uniform marginals and correlation 0.9, t copula has df 3*





*Figure 8. 1000 simulated points for Clayton & Gumbel copula with uniform marginals and correlation 0.9*

Copula:	Kendall's tau:	Spearman's rho:	Upper Tail Dependence:	Lower Tail Dependence:	Theta:
Clayton	0.6666667	0.849203469	0	0.840896415	5.56309
Gumbel	0.5	0.70612141	0.585786438	0	3.7083596

*Table 2. Measures of dependence for Clayton & Gumbel copula with uniform marginals and correlation 0.9*

### 3. Simulation

One way to study the properties of copula functions is through simulation. Due to the implicit forms of certain copulas, like Gaussian and t, sampling appears to be a much simpler and straight forward solution. Monte Carlo simulation is performed for the five kinds of copula functions mentioned before, and the algorithms are as followed.

#### 3.1 Gaussian Copula

- Generate  $Z \sim N_2(0, P)$
- Obtain  $U = (\Phi(Z_1), \Phi(Z_2))'$  with  $C_P^{Ga}$ , where  $\Phi$  is the standard normal distribution function.

#### 3.2 t Copula

- Generate  $X \sim t_2(\nu, 0, P)$
- Get  $U = (t_\nu(X_1), t_\nu(X_2))'$  with  $C_{\nu, P}^t$ , where  $t_\nu$  is the distribution function of a standard univariate t distribution.

#### 3.3 Archimedean Copulas

- Generate  $V \sim G$  s.t.  $\hat{G} = \Phi^{-1}$ , where  $\hat{G}$  is the Laplace-Stieltjes transform of G, which is also the inverse of the generator  $\phi$  of the copula.

- For Clayton copula,  $V \sim Ga(\frac{1}{\theta}, 1), \theta > 0$
- For Gumbel copula,  $V \sim St(\frac{1}{\theta}, 1, \gamma, 0), \gamma = (\cos(\pi/(2\theta)))^\theta, \theta > 1$
- For Frank copula, the probability mass function of  $V$  is

$$p(k) = P(V = k) = (1 - \exp(-\theta))^k / (k\theta), k = 1, 2, \dots, \theta > 0.$$

- Generate  $X_1, X_2 \stackrel{iid}{\sim} U(0,1)$

- Return  $U = (\hat{G}(-\ln(X_1)/V), \hat{G}(-\ln(X_2)/V))'$

### 3.4 Convex Sums

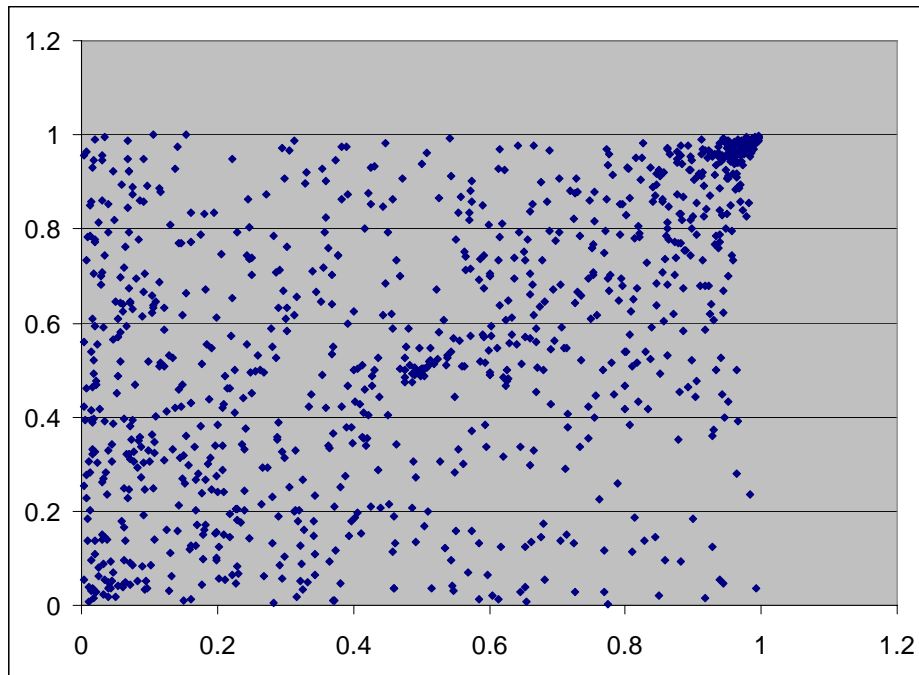
To obtain  $U$  with copula  $\lambda C_1 + (1 - \lambda)C_2$

- Generate  $W, S$  which have copulas  $C_1, C_2$
- Generate  $V \sim U(0,1)$
- Get  $u_i = \begin{cases} w_i, v_i \leq \lambda \\ s_i, v_i > \lambda \end{cases}$

### 3.5 Meta Distributions

Apply the inverse functions of the desired marginals to the uniform marginals obtained from previous steps.

- Return  $X = (F_1^{-1}(U_1), F_2^{-1}(U_2))$



**Figure 9. 1000 simulated points for 0.3\*Gaussian + 0.7\*t copula with correlation 0.9, t copula has df 3**

## **4. Pricing**

### ***4.1 Products***

There are two kinds of credit derivatives in the market, based on the number of securities it is protecting, single-name credit derivatives and multi-name credit derivative baskets. Default swap is a typical representative of single-name credit derivatives. It is “a bilateral contract that enables an investor to buy protection against the risk of default of an asset issued by a specified reference entity. Following a defined credit event, the buyer of protection receives a payment intended to compensate against the loss on the investment.”(O’Kane 25) However, in the current study, the studies would be mainly on the multi-name products. A basket default swap is a good example of this type of products. It is “similar to a default swap in which the credit event is the default of some combination of the credits in a specified basket of reference credits whose default triggers a payment to the protection buyer.” (O’Kane 25) A lot of analysis that would appear later in the paper is on first-to-default basket, it is “the first credit in a basket of reference credits whose default triggers a payment to the protection buyer.” (O’Kane 25)

## 4.2 Formula

The problem calls for a basic version of a first-to-default basket. The contract covers the loss of the first bond to default in a portfolio of two bonds. The contract starts at time 0, the maturity of the contract is  $T$ , force of interest spread is  $\delta$ , the weight of Bond 1 is  $\lambda$ , and the default times of the bonds are  $\tau_1, \tau_2$ . The premium for the basket is  $\Pi$ .

$$\begin{aligned} \sum_{t=0}^T (\Pi * \sum_{s=0}^t e^{-\delta s} P(\tau_1 > s, \tau_2 > s)) &= \sum_{t=0}^T e^{-\delta t} (\lambda * P(\tau_1 = t, \tau_2 > t) + (1 - \lambda) * P(\tau_1 > t, \tau_2 = t)) \\ \Pi &= \frac{\sum_{t=0}^T e^{-\delta t} (\lambda * P(\tau_1 = t, \tau_2 > t) + (1 - \lambda) * P(\tau_1 > t, \tau_2 = t))}{\sum_{s=0}^t e^{-\delta s} P(\tau_1 > s, \tau_2 > s)} \\ &\approx \frac{\sum_{t=0}^T e^{-\delta t} (\lambda * P(t < \tau_1 < t + \frac{1}{12}, \tau_2 > t) + (1 - \lambda) * P(\tau_1 > t, t < \tau_2 < t + \frac{1}{12}))}{\sum_{s=0}^t e^{-\delta s} P(\tau_1 > s, \tau_2 > s)} \end{aligned}$$

A two-dimensional random vector of a certain copula with certain marginals would be generated first, representing the default time random variables of the two bonds. Then the probability estimated in the formula would be obtained using the random sample that was generated.

## 5. Results

Simulations are performed for certain first-to-default basket contracts of two zero-coupon bonds. For each contract, 1000 pairs of default times would be simulated for the two bonds, and the premium would be calculated using the formula developed in the previous section. The force of interest spread is set at 2%, and the weight of Bond 1 is set at 0.3, and the value of the portfolio at time zero is 1. The first three cases have maturity of 5, and the t copula in all three cases have 3 degrees of freedom.

Case 1: The marginals are exponential distributions with mean 5 and 2.5, and the linear correlation between the two marginals is 0.9.

Case 2: The marginals are exponential distributions with mean 2.5, and the linear correlation between the two marginals is 0.9.

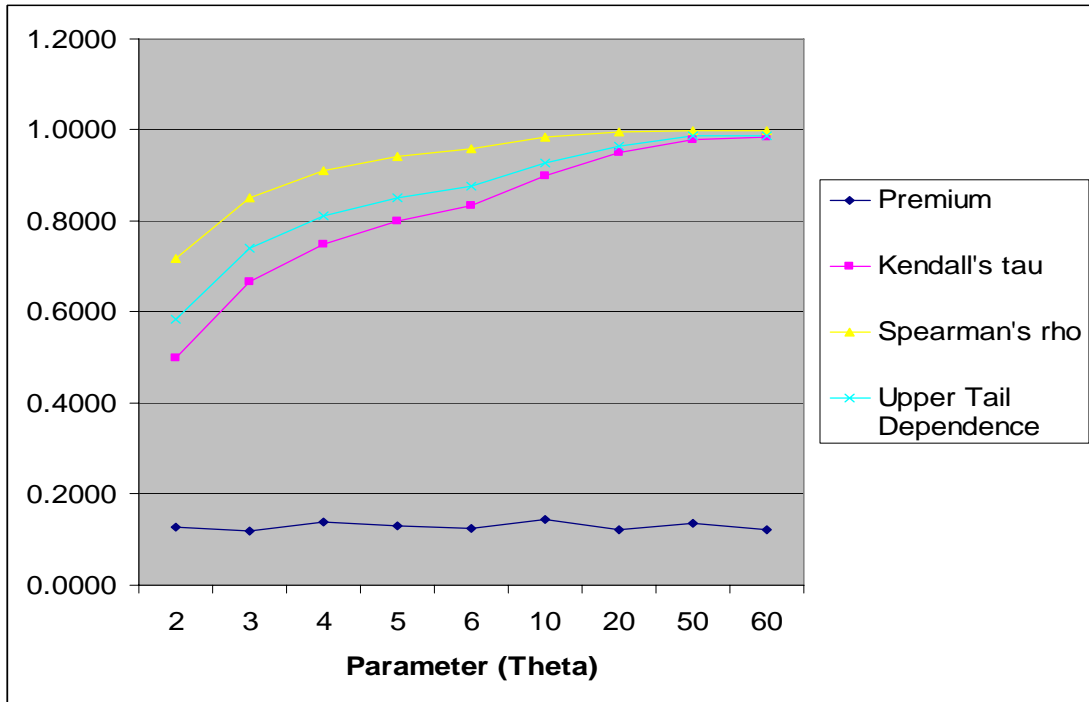
Case 3: The marginals are exponential distributions with mean 5 and 2.5, and the linear correlation between the two marginals is 0.7.

<b>Copula</b>	<b>Case 1</b>	<b>Case 2</b>	<b>Case 3</b>
<b>Clayton</b>	0.12567	0.09977	0.11313
<b>Gumbel</b>	0.13012	0.11206	0.11206
<b>Gaussian</b>	0.12026	0.10955	0.14861
<b>t</b>	0.07151	0.06401	0.07108

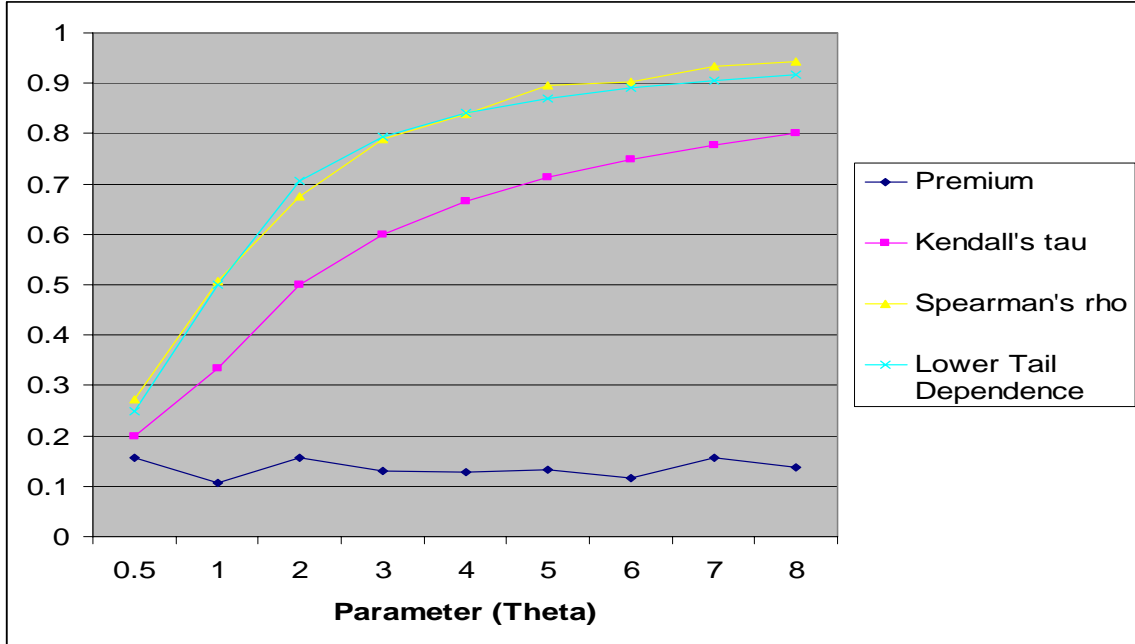
*Table 3. Premiums for the three cases*

From the tables above it could be read that for copulas that have lower tail dependence, like Clayton copula and t copula, the premiums tend to be lower. In all three cases, t copula produces the lowest premiums, and in Case 2 & 3, Clayton copula brings the second lowest price. On the other hand, copula with only upper tail dependence, e.g., Gumbel copula, requires the highest or second highest premiums in all three cases. The reason for this is at the beginning of the contract, although lower tail dependence would cause a higher default rate, the probability of neither bond defaulting is relatively high for

all copulas, which causes the coverage to payments ratio to stay low for all copulas including those with a higher defaulting rate. Upper tail dependence comes into play towards the end of the contract. At this moment a relatively high defaulting rate would easily cause a significant change in the probability of both bonds surviving, which leads to a high coverage to payments ration that brings up the premiums.



*Figure 10. Premiums & Measures of Dependence for Gumbel copula with exponential marginals with mean 5 and 2.5, weight of Bond 1 is 0.3*



**Figure 11. Premiums & Measures of Dependence for Clayton copula with exponential marginals with mean 5 and 2.5, weight of Bond 1 is 0.3**

Now take a look at the premiums and measures of dependence for Gumbel and Clayton copulas with the same marginals, weight and maturity as Case 1. From the graphs it is clear that there is no strong connection regarding the relationship between the premiums and the measures of dependence.

Take exponential marginals with mean 5 and 2.5, and apply different linear correlation each time, the following table of premiums can be obtained.

Corr	Clayton	Gumbel	Gaussian	t
<b>0.5</b>	0.157959	0.128466	0.13774749	0.070888
<b>0.6</b>	0.129699	0.182917	0.13948831	0.06838
<b>0.7</b>	0.105142	0.140107	0.11964756	0.08716
<b>0.8</b>	0.115772	0.12582	0.14580253	0.060166
<b>0.9</b>	0.131051	0.158769	0.10523178	0.071512

**Table 4. Premiums for copulas with different linear correlation coefficients**

The table can be looked at in two directions. First, it is obvious that even with the same linear correlation coefficients, if the copulas differ, one would get premiums with



considerable differences. Second, even for the same copula function, changes in linear correlation do not lead to a consistent pattern in the values of the premiums.

Next the marginals remain the same, and the correlation would be set to 0.9. The variable here would be the maturity.

<b>Maturity</b>	<b>Clayton</b>	<b>Gumbel</b>	<b>Gaussian</b>	<b>t</b>
<b>5</b>	0.131051	0.158769	0.105232	0.071512
<b>4</b>	0.108946	0.128355	0.126638	0.066733
<b>3</b>	0.105618	0.113207	0.122147	0.081431
<b>2</b>	0.095063	0.09555	0.084521	0.102989
<b>1</b>	0.069967	0.074427	0.079501	0.065859

*Table 5. Premiums for contracts with different maturities*

There is a general upward trend in the amounts of the premiums as the maturity increases. The explanation is straight forward: as the maturity goes up, the portfolio is exposed to higher risk of defaulting, which would certainly cause the raise of the premiums.

Another aspect of such analysis is to evaluate the premiums of different entering dates for the contract. The following two charts are for a Gaussian copula and a t copula with exponential marginals whose means are 5 and 2.5, the linear correlation is 0.9, and the maturity is 5. The first chart is the default time distributions, and the second chart is the premiums when entering the contract any time before maturity. Notice the jump around time 2 for the premiums of t copula resulted in the cluster around time 2 on the y axis in the default time graph. At this point Bond 2 has a really high default rate, and investors who enter the contract after this point are exposed to a lower level of default risk. The last chart is a comparison of all four copulas under the same condition. The overall downward trend in the premiums are the result of the fact that once the bond survives after a certain amount of time, it is less likely for it to default.

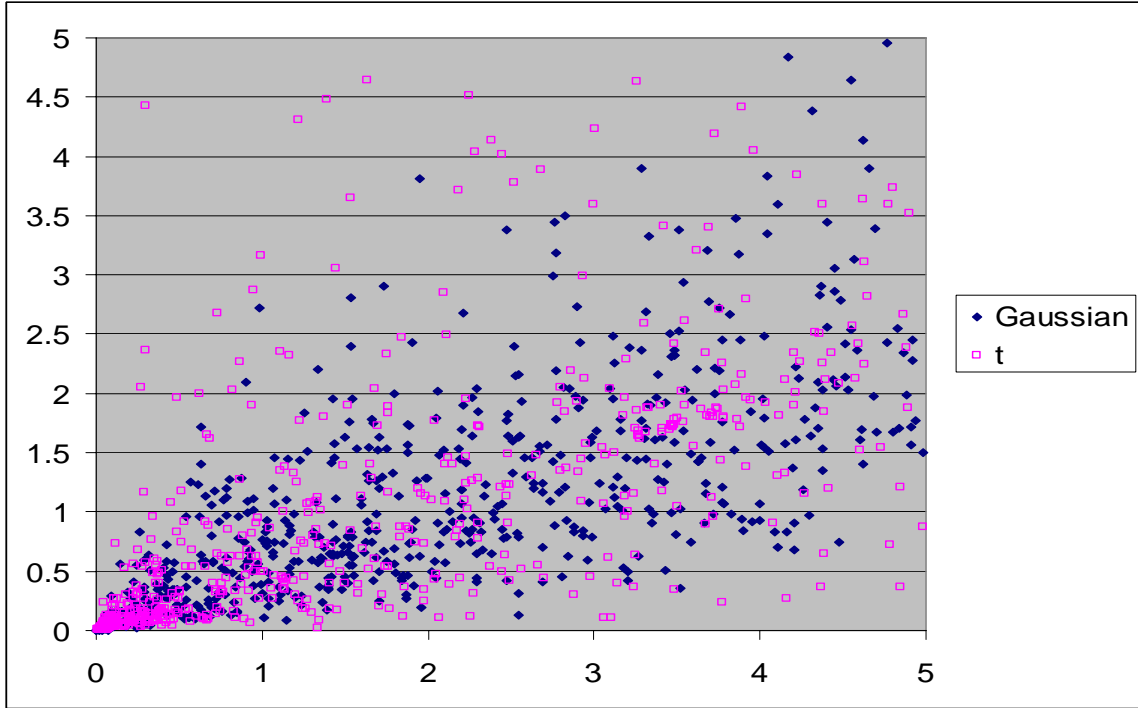


Figure 12. Default times for Case 1 with Gaussian and  $t$  copula

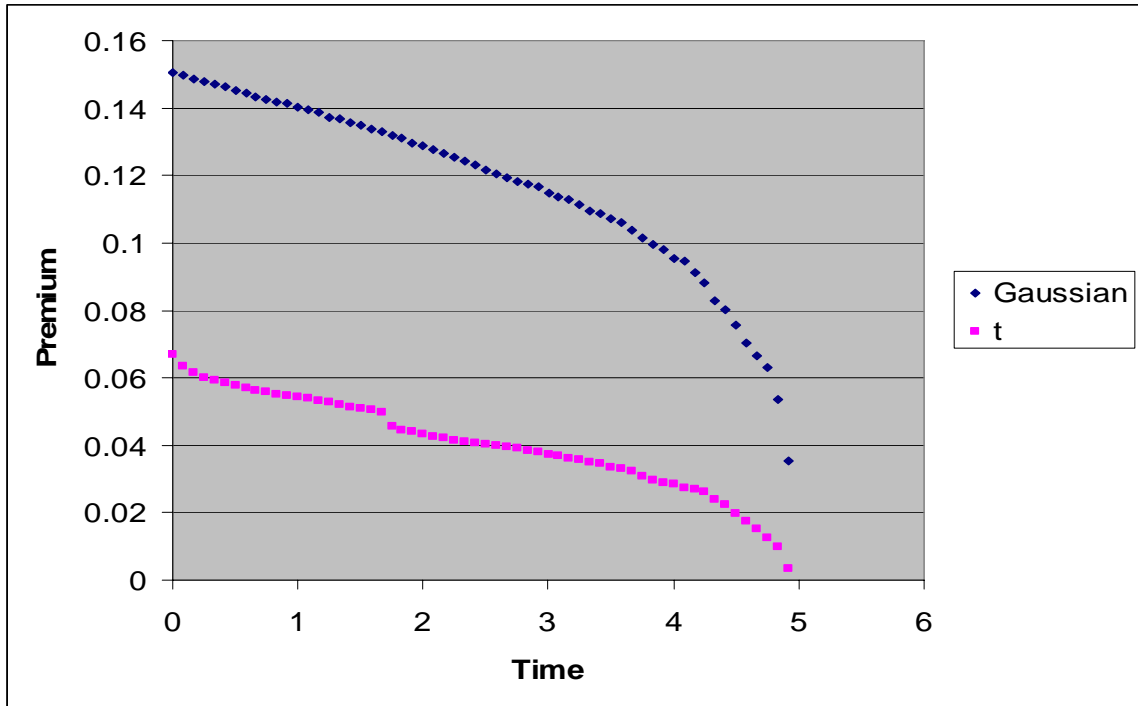
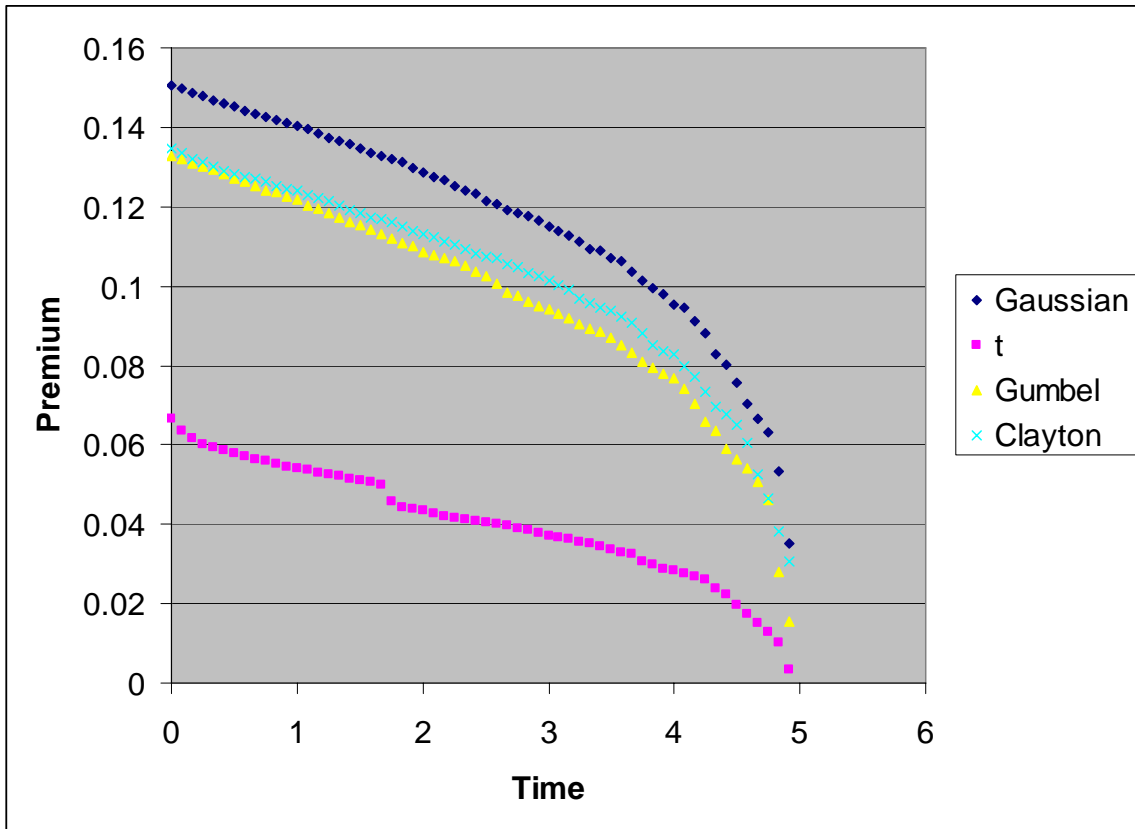


Figure 13. Premiums for different entering dates for Case 1 with Gaussian and  $t$  copulas



*Figure 14. Premiums for different entering dates for Case 1*

## **6. Conclusion**

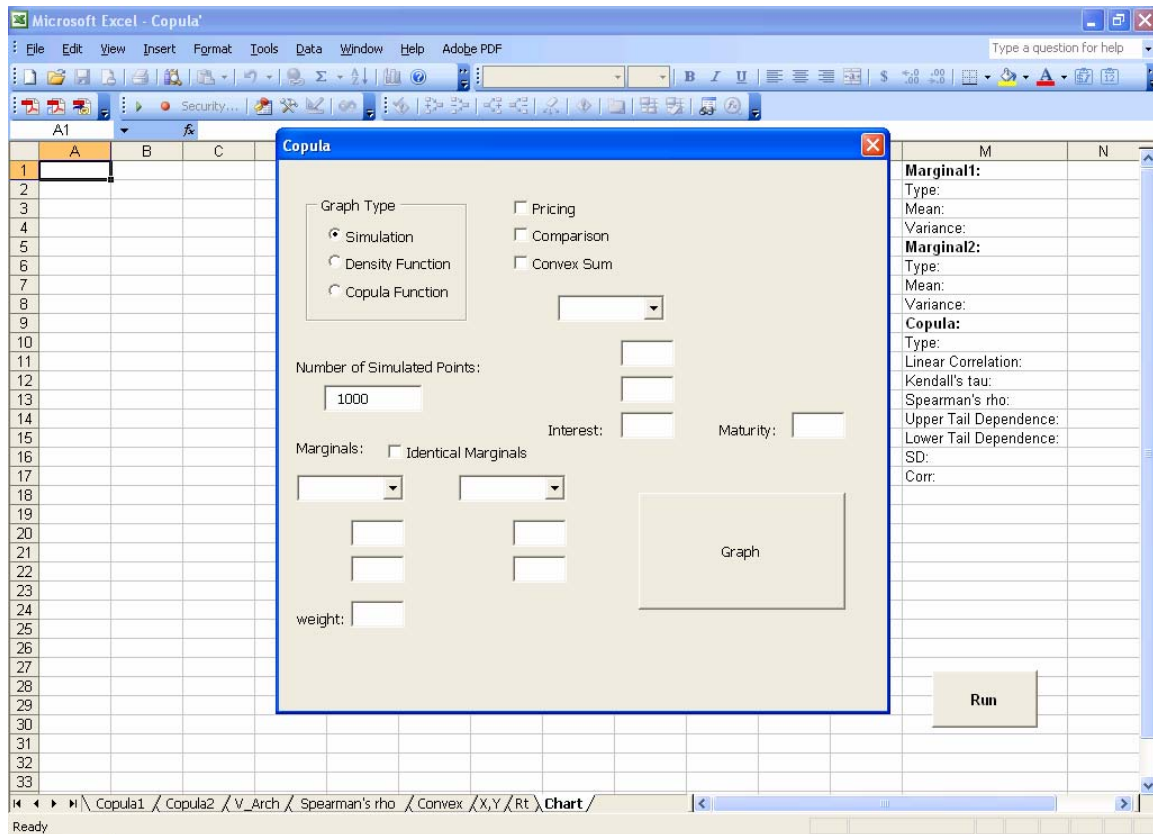
The credit risk market has been constantly growing. The evaluation of credit risk, especially the relations between different credit products has proven to be very complicated as well as important. The traditional way of linear correlation failed to separate the dependence structure and the marginal distributions. Copula functions provide a more precise approach that could set apart the influence of different marginal distribution functions. Several copula functions were studied. Monte Carlo simulations were performed for general understanding of the functions and for the later pricing part of the project. A real life example of a basic first-to-default basket of two zero-coupon bonds was introduced. The effects of different dependence structures on the premiums of the contract and the influence of entering time, etc. are also discussed. Using copula functions proved to be a good way to measure dependence between default times of credit risk products.

## 7. References

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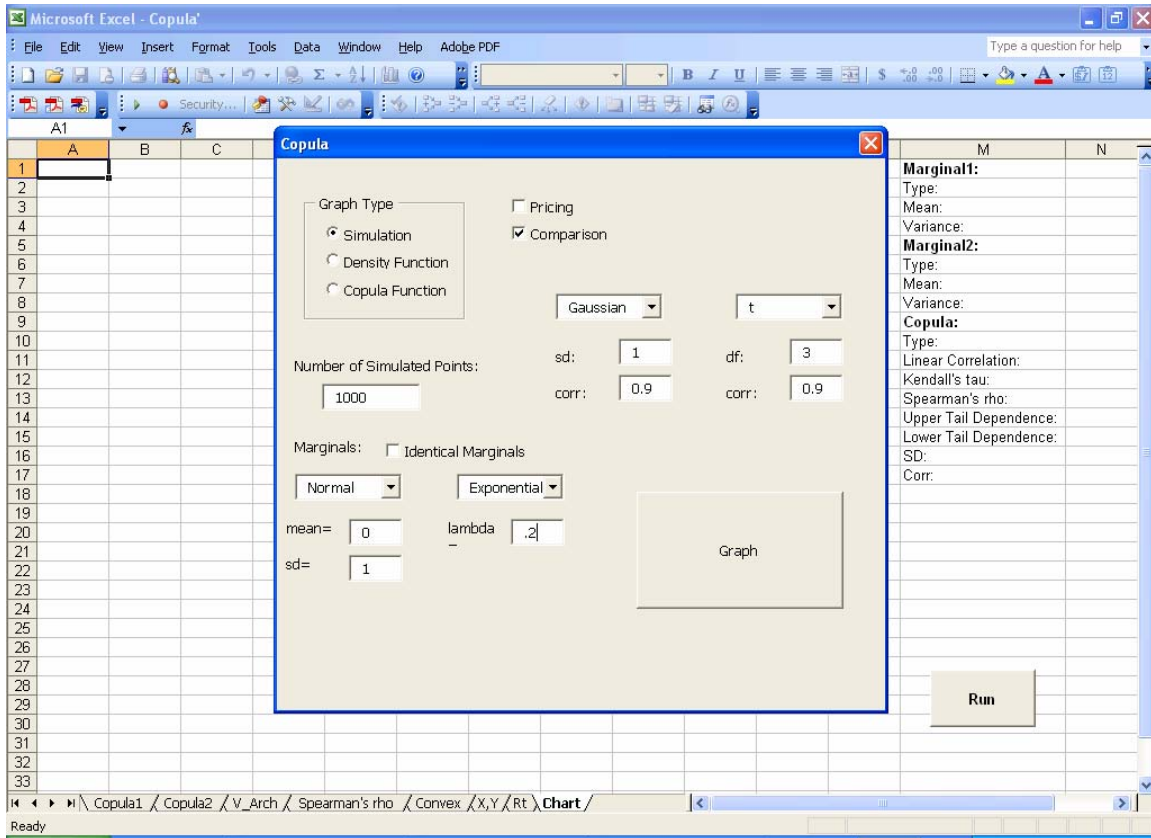
## 8. Appendix

A GUI interface was developed in Excel VBA to simulate random vectors with different copula functions and to price a basic form of first-to-default basket of two zero-coupon bonds.

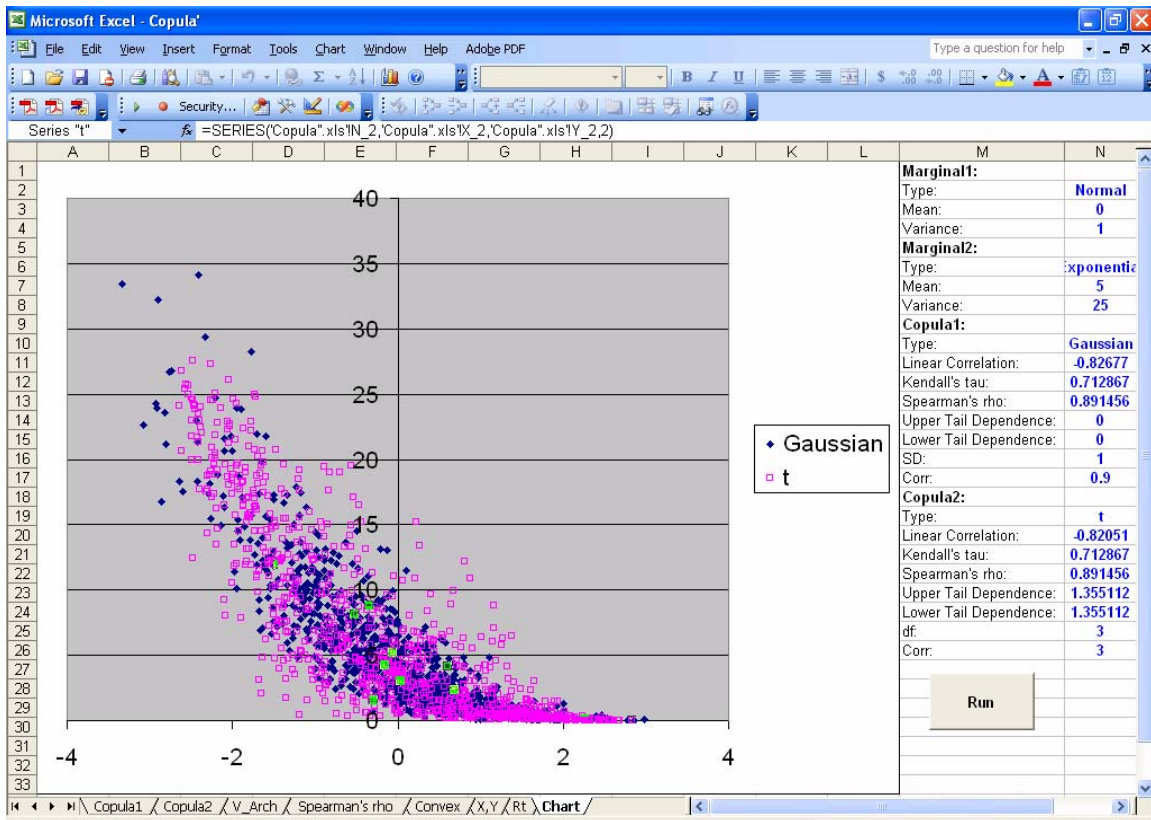


This image is how the interface would look like when the workbook is opened. Make sure to select “Simulation” for “Graph Type”. There are three options under “Simulation”, which are “Pricing”, “Comparison”, and “Convex Sum”. To simulate a single copula function and get graph, none of the three options needs to be checked. The default number of simulated points is 1000. The type and parameters of marginals can be selected, and there is also an option for identical marginals. To compare two copula functions, the option “Comparison” should be selected, and the types of copulas and

parameters can be inputted. When “Convex Sum” is selected, there is one more item to input, “lambda”, which is the weight of the first copula entered. The following is an example where Gaussian and t copulas with same correlation are compared, with normal and exponential marginals.



After “Graph” button is clicked, a series of random number generation and calculations would be carried out, and the “Chart” tab would be shown at the end of the process. A graph of the simulated points and key parameters would appear on the worksheet.



For the “Pricing” function, three more inputs are added. Force of interest spread, maturity, and the weight of the first bond. The default inputs are as in the next image. After the clicking of the “Graph” button under the “Pricing” option, the workbook would end on the “Rt” tab. A graph of premium amounts when entering at different points in time would be provided together with some basic information, as shown in the example to follow.

To perform another simulation/calculation, delete all the graphs currently on the workbook, and click the “Run” button on the “Chart” sheet. At this point inputs from the last run would still be stored on the GUI. To reset all the values to default, simply close the interface and click “Run” again.



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**Copula**

Graph Type

- Simulation
- Density Function
- Copula Function

Pricing

Convex Sum

Number of Simulated Points: 1000

Interest: 0.02 Maturity: 5

Marginals:  Identical Marginals

weight: 0.3

Graph

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Mean:  
Variance:

Marginal2:  
Type:  
Mean:  
Variance:

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Type:  
Linear Correlation:  
Kendall's tau:  
Spearman's rho:  
Upper Tail Dependence:  
Lower Tail Dependence:  
SD:  
Corr:

Chart

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**Copula**

Graph Type

- Simulation
- Density Function
- Copula Function

Pricing

Convex Sum

Number of Simulated Points: 1000

sd: 1 corr: 0.7 Interest: 0.02 Maturity: 5

Marginals:  Identical Marginals

Exponential Exponential

lambda .2 lambda .4

weight: 0.3

Graph

Run

Marginal1:  
Type:  
Mean:  
Variance:

Marginal2:  
Type:  
Mean:  
Variance:

Copula:  
Type:  
Linear Correlation:  
Kendall's tau:  
Spearman's rho:  
Upper Tail Dependence:  
Lower Tail Dependence:  
SD:  
Corr:

Chart

Ready

