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Qualitative Behavior of Solutions For Shallow Water Wave Equations with Relaxation

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Eyuel D. Abebe

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APPROVED:

Professor Tau Luo

MQP Advisor

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Abstract

This MQP report concerns the finding of the convergence rates to the equilibrium constant states of shallow water wave equations with relaxation as time goes to infinity and establishing the existence of global in time solutions. We study the initial value problem, under a certain set of initial conditions, by considering the system in Lagrangian coordinates using different transformations. We transform the hyperbolic conservation laws for flood wave motion from Eulerian coordinates to Lagrangian coordinates to a system of equations that is suitable for our analysis. We develop two essential lemmas that we employ to prove the global in time existence of solutions and show, moreover, how these lemmas yield the convergence rates, given the initial conditions. We also look at the general case by strengthening the initial conditions and arrive at another system of equations that is similar to the previous one, the analysis of which follows naturally.

Contents

1 Introduction

In this section we introduce the system for shallow water waves modeled by the hyperbolic conservation laws. We study this system under Lagrangian coordinates and we introduce transformations to reexpress it into a form suitable for our analysis in section 2. We also outline the initial conditions that need to be satisfied and subcharactersic conditions that we would require to be maintained through out this paper.

1.1 System of Hyperbolic Conservation Laws

The motion of shallow water wave on an inclined surface can be modeled by the following system of hyperbolic[1] conservation laws (1):

$$\begin{aligned} &h_\tau + (hu)_y = 0 \\ &(hu)_\tau + (hu^2 + \tfrac{1}{2}g'h^2)_y = g'hS - C_f u^2 \end{aligned} \tag{1}$$

where $g' = gcos\ \alpha$ is the gravitational acceleration, α also realized by $S = tan\ \alpha$ is a constant representing the angle of inclination of the river with $0 < \alpha < \frac{\pi}{2}$, $C_f > 0$ is the constant frictional coefficient, $h > 0$ and $u > 0$ are the depth and velocity of the water, and τ and y are time and space variables, respectively. Figure1 shows the physical interpretations of the variables h, α and u.

[1] A second order PDE is called hyperbolic if it is of the form $Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$ with det $\begin{pmatrix} A & B \\ B & C \end{pmatrix} = AC - B^2 < 0$.

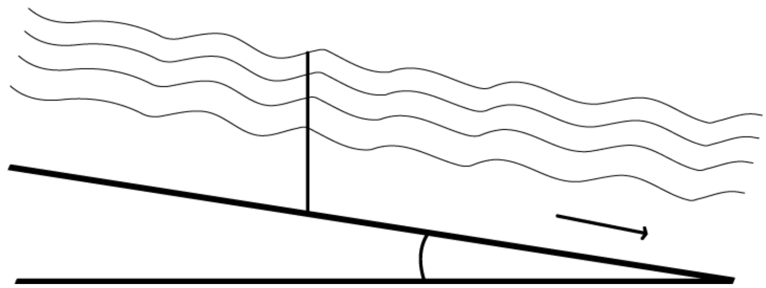

Figure 1: Model of water wave on an inclined surface.

1.2 Water Wave Equations in Lagrangian Coordinates

If we consider system (1) by switching to Lagrangian coordinates under the transformation,

$$x = \int_{\beta(\tau)}^{y} h(z,\tau)dz, \tag{2}$$

with $t = \tau$ and $\beta(\tau)$ is a trajectory followed by particles satisfying $\dot{\beta}(\tau) = u(\beta(\tau), \tau)$, we can represent the system more conveniently.

However, before applying this transformation we first make the following observations on the derivatives with respect to τ and y,

$$\begin{aligned}
\frac{\partial}{\partial\tau} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial\tau} + \frac{\partial}{\partial t}\frac{\partial t}{\partial\tau} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\tau} + \frac{\partial}{\partial t}\\
\frac{\partial}{\partial\tau} &= \frac{\partial}{\partial x}\frac{\partial}{\partial\tau}\left(\int_{\beta(\tau)}^{y} h(z,\tau)dz\right) + \frac{\partial}{\partial t}\\
\frac{\partial}{\partial\tau} &= \frac{\partial}{\partial x}\left(\int_{\beta(\tau)}^{y} h(z,\tau)dz - h(\beta(\tau),\tau)\beta'(\tau)\right) + \frac{\partial}{\partial t}\\
\frac{\partial}{\partial\tau} &= \frac{\partial}{\partial x}\left(\int_{\beta(\tau)}^{y} h(z,\tau)dz - h(\beta(\tau),\tau)u(\beta(\tau),\tau)\right) + \frac{\partial}{\partial t}\\
\frac{\partial}{\partial\tau} &= \frac{\partial}{\partial x}\left(-hu(y,t) + hu(\beta(\tau),\tau) - hu(\beta(\tau),\tau)\right) + \frac{\partial}{\partial t}\\
(*)\quad \frac{\partial}{\partial\tau} &= -hu\frac{\partial}{\partial x} + \frac{\partial}{\partial t}
\end{aligned}$$

In a similar fashion,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial y}$$
$$(**) \quad \frac{\partial}{\partial y} = h\frac{\partial}{\partial x}$$

Now, using transformation (2) and with the knowledge on the derivatives $(*)$ and $(**)$, we apply this to system (1). The first equation of system(1) becomes:

$$h_\tau + (hu)_y \quad = 0$$
$$-huh_x + h_t + h(hu)_x = 0$$
$$-huh_x + h_t + h(h_x u + hu_x) = 0$$
$$h_t + h^2 u_x = 0$$
$$\frac{h_t}{h^2} + u_x = 0$$
$$-(\frac{1}{h})_t + u_x = 0$$
$$v_t - u_x = 0$$

where $v = \frac{1}{h}$. The second equation of system (1) becomes:

$$(hu)_\tau + \ (hu^2 + \frac{1}{2}g'h^2)_y = g'hS - C_f u^2$$
$$-hu(hu)_x + (hu)_t + (hu^2 + \frac{1}{2}g'h^2)_y = g'hS - C_f u^2$$

Applying the product rule, we arrive at

$$-hh_x u^2 - h^2 uu_x + h_t u + u_t h + h(h_x u^2 + 2uu_x h) + \frac{h}{2}(g'_x h^2 + 2g'hh_x) = g'hS - C_f u^2$$
$$-hh_x u^2 - h^2 uu_x + h_t u + u_t h + hh_x u^2 + 2uu_x h^2 + \frac{h^3}{2}g'_x + g'h^2 h_x = g'hS - C_f u^2.$$

Here the first term cancels with the fifth term and dividing by h

$$
\begin{aligned}
u_t h + u h_t + h^2 u u_x + \frac{h^3}{2} g'_x + g' h^2 h_x &= g' h S - C_f u^2 \\
u_t + u \frac{h_t}{h} + h u u_x + \frac{h^2}{2} g'_x + g' h h_x &= g' S - C_f u^2 \frac{1}{h} \\
u_t + (\ln(h))_t u + \frac{h}{2} (u^2)_x + \frac{h^2}{2} g'_x + \frac{g'}{2} (h^2)_x &= g' S - C_f u^2 \frac{1}{h} \\
u_t + (\ln(h))_t u + \frac{h}{2} \left((u^2)_x + g'_x\right) + \frac{g'}{2} (h^2)_x &= g' S - C_f u^2 \frac{1}{h}
\end{aligned}
$$

We will take the gravitational acceleration g' to be a constant, hence $g'_x = 0$.

$$
\begin{aligned}
u_t + (\ln(h))_t u + \frac{h}{2} (u^2)_x) + \frac{g'}{2} (h^2)_x &= g' S - C_f u^2 \frac{1}{h} \\
u_t + \frac{h_t}{h} u + h u u_x + (\frac{g'}{2} v^{-2})_x &= g' S - C_f u^2 v \\
u_t - \frac{u_x}{v} u + h u u_x + (\frac{g'}{2} v^{-2})_x &= g' S - C_f u^2 \frac{1}{h} \\
u_t + p(v)_x &= g' S - C_f u^2 v
\end{aligned}
$$

where $h_t = (\frac{1}{v})_t = -(\frac{v_t}{v^2}) = -(\frac{u_x}{v^2})$ and $(\frac{g'}{2} v^{-2})_x = p(v)$. In order to simplify the system even more, we will set the gravitational acceleration to $g' = 1$. Following this simplifications, system (1) is transformed to:

$$
\begin{aligned}
&v_t - u_x = 0, \\
&u_t + p(v)_x = S - C_f u^2 v.
\end{aligned} \tag{3}
$$

This system is strictly hyperbolic when $0 < v < \infty$ with two distinct characteristic speeds $\lambda_1(v) = -\sqrt{-p'(v)} = -\sqrt{g'} v^{\frac{3}{2}}$, $\lambda_2(v) = \sqrt{-p'(v)} = \sqrt{g'} v^{\frac{3}{2}}$. When the relaxation term $S - C_f u^2 v$ vanishes, the system is in equilibrium and the equilibrium equation corresponding to system (3)is given by

$$
v_t - f(v)_x = 0,
$$

where $f(v) = \pm\sqrt{\frac{S}{C_f v}}$ satisfying $S - C_f (f(v))^2 v = 0$, assuming the case in a small neighborhood of (v,u) the equilibrium curve $u = \pm\sqrt{\frac{S}{C_f v}} = f(v)$ It is expected that system (3), as $t \to \infty$ is well approximated by the equilibrium equation given above, provided the subcharacteristic condition $|f'(v)| < \sqrt{-p'(v)}$ holds. This subcharacteristic condition serves as a stability condition and for our model it translates to the inclination angle being small, i.e $tan\ \alpha = S = 4C_f$ [1]. We notice we can even further simplify system(3) if we use the following transformation $w = v(x,t) - \bar{v}, z(x,t) = u - f(\bar{v})$. with given initial data $v(x,0) = v_0(x), u(x,0) = u_0(x)$, satisfying the conditions

$$\lim_{x\to\pm\infty} v_0(x) = \bar{v}, \lim_{x\to\pm\infty} u_0(x) = f(\bar{v})$$

where $\bar{v} > 0$ is a constant and

$$\int_{-\infty}^{+\infty} (v_0(x) - \bar{v})dx = 0.$$

Hence, system(3) becomes:

$$\begin{aligned} &w_t - z_x = 0 \\ &z_t + p(\bar{v} + w)_x = S - C_f(f(\bar{v}) + z)^2(\bar{v} + w) \end{aligned} \tag{4}$$

2 Analysis on Shallow Water Wave Equation

In this section we will investigate the trajectories taken by particles in water to understand the behavior of the motion of the wave based on initial data. Furthermore, we predict how this wave motion will behave through time given initial values in terms of the speed of particles and height of the water.

Before we begin the analysis on system(3), we introducing a new function $\varphi(x,t)$ given by,

$$\varphi(x,t) = \int_{-\infty}^{x} w(y,t)dy \tag{5}$$

which has a constant value in time as $x \to \infty$, as demonstrated below.

$$\begin{aligned} \frac{d}{dt}\varphi(+\infty,t) &= \frac{d}{dt}\int_{-\infty}^{+\infty} w(x,t)dx \\ &= \int_{-\infty}^{+\infty} w_t(x,t)dx = \int_{-\infty}^{+\infty} z_x(x,t)dx \\ &= z(+\infty,t) - z(-\infty,t) = 0. \end{aligned}$$

Now lets assume that,

$$z^+(t) = z(+\infty,t), w^+(t) = w(+\infty,t)$$

$$z^-(t) = z(-\infty,t), w^-(t) = w(-\infty,t)$$

and

$$w^+(0) = \lim_{x\to+\infty} w(x,0) = \lim_{x\to\infty}(v_0(x) - \bar{v}) = 0$$

$$z^+(0) = \lim_{x\to+\infty} z(x,0) = \lim_{x\to\infty}(u_0(x) - f(\bar{v})) = 0$$

$$w^-(0) = \lim_{x\to-\infty} w(x,0) = \lim_{x\to\infty}(v_0(x) - \bar{v}) = 0$$

$$z^-(0) = \lim_{x\to-\infty} z(x,0) = \lim_{x\to\infty}(u_0(x) - f(\bar{v})) = 0$$

then taking the time derivative of system(3), we find:

$$\frac{d}{dt}w^+(t) = 0$$
$$\frac{d}{dt}z^+(t) = S - C_f(f(\bar{v}) + z^+(t))^2(\bar{v} + w^+(t))$$

similarly,

$$\frac{d}{dt}w^-(t) = 0$$
$$\frac{d}{dt}z^-(t) = S - C_f(f(\bar{v}) + z^-(t))^2(\bar{v} + w^-(t))$$

The solutions to the O.D.E above are given by $w^+(t) = w^-(t) = 0$ and $z^+(t) = z^-(t) = 0$(this result implies $f(\bar{v}) = \sqrt{\frac{S}{C_f\bar{v}}})^2$. Hence $w(\pm\infty, t) = 0$ and $z(\pm\infty, t) = 0$ for all time $t \geq 0$.

Using the transformation(5) and the information on the time and position derivatives, we re-express system(4) as:

$$\begin{aligned}\varphi_{tt} + (p(\bar{v} + \varphi_x) - p(\bar{v}))_x = &-C_f(f(\bar{v}))^2\varphi_x - 2C_f f(\bar{v})\bar{v}\varphi_t - 2C_f f(\bar{v})\varphi_x\varphi_t \\ &-C_f\varphi_x\varphi_t^2\end{aligned} \quad (6)$$

From here on we will use system(6)[3] to study the motion of shallow water waves with respect to the initial given data.[4]

[2] It is clear from the definition of $\varphi(x,t)$ and analysis done; $\varphi_t(x,t) = z(x,t)$ and $\varphi_x(x,t) = w(x,t)$

[3] The left hand side from transformation(4)$S - C_f(f(\bar{v}) + \varphi_t)^2(\bar{v} + \varphi_x)$ is simplified to $-C_f(f(\bar{v}))^2\varphi_x - 2C_f f(\bar{v})\bar{v}\varphi_t - 2C_f f(\bar{v})\varphi_x\varphi_t - C_f\varphi_x\varphi_t^2$ by using the subcharacteristics condition $f(\bar{v}) = \sqrt{\frac{S}{C_f\bar{v}}}$.

[4] $\varphi_{tx} - \varphi_{xt} = 0$

Lemma 2.1. *Let $\varphi(x,t)$ be a solution of the initial value problem(6) satisfying $\varphi(.,t) \in H^3(\mathbb{R})$ and $\varphi_t(.,t) \in H^2(\mathbb{R})$ for $t \in [0,T]$. Then there exists a number $\delta > 0$, such that if*

$$\sup_{\substack{x \in (-\infty,+\infty) \\ t \in [0,T]}} \left(|\varphi_{xt}| + |\varphi_t| + |\varphi_x| \ + |\varphi|\right)(x,t) \leq \delta$$

then

$$\int_{-\infty}^{+\infty} (\varphi^2 + \varphi_t^2 + \varphi_x^2)(x,\tau) \ dx + \int_0^\tau \int_{-\infty}^{+\infty} (\varphi_t^2 + \varphi_x^2)(x,t) \ dxdt$$

$$\leq O(1) \int_{-\infty}^{+\infty} (\varphi^2 + \varphi_t^2 + \varphi_x^2)(x,0) \ dx$$

Remark 2.2. *For $d \geq 1, \Omega$ an open subset of $\mathbb{R}^d$ and $m \in \mathbb{N}$, the Sobolev Space $H^m(\Omega)$ is defined by*

$$H^m(\Omega) := f \in L^2(\Omega) : \forall |\alpha| \leq m, \partial_x^\alpha f \in L^2(\Omega)$$

where

$$L^2(\Omega) := f : \Omega \to \mathbb{R} : \int_\Omega |f|^2 dx \leq \infty.$$

Proof. We will apply the energy method used in [2] in order to prove this lemma. We start with multiplying system(6) by φ and integrating over the entire real line. We look at the left hand side(L.H.S) and the right hand side(R.H.S) separately.

L.H.S:

The first term becomes

$$\varphi_{tt}\varphi = (\varphi_t\varphi)_t - \varphi_t^2$$

$$\int_{-\infty}^{+\infty} \varphi_{tt}\varphi \ dx = \frac{d}{dt}\int_{-\infty}^{+\infty} \varphi_t\varphi \ dx - \int_{-\infty}^{+\infty} \varphi_t^2 \ dx.$$

For the second term we use the fact that $\varphi(\pm\infty,t) = 0$ and after integration by parts, we get

$$\int_{-\infty}^{+\infty}(p(\bar v+\varphi_x)-p(\bar v))_x\varphi\ dx\ = p(\bar v+\varphi_x)-p(\bar v))\varphi|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}(p(\bar v+\varphi_x)-p(\bar v))\varphi_x\ dx$$
$$\int_{-\infty}^{+\infty}(p(\bar v+\varphi_x)-p(\bar v))_x\varphi\ dx\ = -\int_{-\infty}^{+\infty}(p(\bar v+\varphi_x)-p(\bar v))\varphi_x\ dx$$

then the L.H.S becomes

$$\int_{-\infty}^{+\infty}[\varphi_{tt}\varphi\ +\ (p(\bar v+\varphi_x)-p(\bar v))_x\varphi]\ dx = \int_{-\infty}^{+\infty}[\frac{d}{dt}(\varphi_t\varphi)\ -\varphi_t^2\ +\ (p(\bar v)-p(\bar v+\varphi_x))\varphi_x]\ dx$$

We move on to examine the R.H.S in a similar approach, term by term;

R.H.S:

The first term is given by

$$\int_{-\infty}^{+\infty}-C_f(f(\bar v))^2\varphi\varphi_x\ dx = \int_{-\infty}^{+\infty}-C_f(f(\bar v))^2(\frac{1}{2}\varphi)_x\ dx$$
$$\int_{-\infty}^{+\infty}-C_f(f(\bar v))^2\varphi\varphi_x\ dx = -C_f(f(\bar v))^2(\frac{1}{2}\varphi)|_{-\infty}^{+\infty}=0.$$

The second term is simplified to

$$-2C_f(f(\bar v))\bar v\int_{-\infty}^{+\infty}\varphi\varphi_t\ dx = -2C_f(f(\bar v))\bar v\int_{-\infty}^{+\infty}(\frac{1}{2}\varphi)_t\ dx$$
$$-2C_f(f(\bar v))\bar v\int_{-\infty}^{+\infty}\varphi\varphi_t\ dx = -\frac{d}{dt}\int_{-\infty}^{+\infty}(C_f(f(\bar v))\bar v)\varphi_t\ dx$$

putting the terms together, the R.H.S becomes

$$-\frac{d}{dt}\int_{-\infty}^{+\infty}(C_f f(\bar v)\bar v)\varphi_t\ dx - 2C_f f(\bar v)\int_{-\infty}^{+\infty}\varphi_t\varphi_x\varphi\ dx\ -\ C_f\int_{-\infty}^{+\infty}\varphi_t^2\varphi_x\varphi\ dx$$

Putting the L.H.S and R.H.S together, System(6) is simplified to:

$$\int_{-\infty}^{+\infty}\frac{d}{dt}[\varphi_t\varphi\ +(p(\bar v)-p(\bar v+\varphi_x))\varphi_x-\varphi_t^2]dx = -\frac{d}{dt}\int_{-\infty}^{+\infty}(c_f(f(\bar v)\bar v)\varphi^2)dx$$
$$-2C_f f(\bar v)\int_{-\infty}^{+\infty}\varphi_t\varphi_x\varphi\ dx\ -C_f\int_{-\infty}^{+\infty}\varphi_t^2\varphi_x\varphi\ dx$$

Rearranging terms and based on the assumptions from Lemma(2.1), we get the following inequality;

$$\int_{-\infty}^{+\infty} \frac{d}{dt}[\varphi_t\varphi \ + (c_f(f(\bar{v})\bar{v})\varphi^2)]dx + \int_{-\infty}^{+\infty} (p(\bar{v}) - p(\bar{v}+\varphi_x))\varphi_x dx - \int_{-\infty}^{+\infty} \varphi_t^2 dx \\ \leq C\delta \int_{-\infty}^{+\infty} |\varphi_t\varphi_x| dx$$

At this point we use the AM-GM Inequality[5]on the integrand to get a better estimate;

$$\int_{-\infty}^{+\infty} \frac{d}{dt}[\varphi_t\varphi \ + (c_f(f(\bar{v})\bar{v})\varphi^2)]dx + \int_{-\infty}^{+\infty} (p(\bar{v}) - p(\bar{v}+\varphi_x))\varphi_x dx - \int_{-\infty}^{+\infty} \varphi_t^2 dx \\ \leq C\delta \int_{-\infty}^{+\infty} (\varphi_t^2 + \varphi_x^2) dx$$

We can further reduce this by working on the second term of the inequality above. We make use the Taylor's expansion of the integrand on the second term, and apply the $O(1)$ notation to represent the error terms (terms with order $\geqslant 2$).

$$-\int_{-\infty}^{+\infty} (p(\bar{v}+\varphi_x) - p(\bar{v}))\varphi_x \ dx \ = \int_{-\infty}^{+\infty} -p'(\bar{v})\varphi_x^2 \ dx + \\ \sum_{n=2}^{\infty} \frac{d^n}{dx}[\frac{(p(\bar{v}+\varphi_x) - p(\bar{v}))\varphi_x}{n!}]$$

$$-\int_{-\infty}^{+\infty} (p(\bar{v}+\varphi_x) - p(\bar{v}))\varphi_x \ dx = -p'(\bar{v}) \int_{-\infty}^{+\infty} \varphi_x^2 \ dx \ + \ O(1) \int_{-\infty}^{+\infty} |\varphi_x|^3 \ dx$$

$$\int_{-\infty}^{+\infty} \frac{d}{dt}[\varphi_t\varphi \ + (c_f(f(\bar{v})\bar{v})\varphi^2)] \ dx + \int_{-\infty}^{+\infty} (p(\bar{v}) - p(\bar{v}+\varphi_x))\varphi_x \ dx - \int_{-\infty}^{+\infty} \varphi_t^2 \ dx \\ \leq C\delta \int_{-\infty}^{+\infty} (\varphi_t^2 + \varphi_x^2) \ dx$$

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[5] AM-GM(Arithmetic Mean - Geometric Mean) is given by $\frac{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n}{\alpha} \geq \sqrt[\alpha]{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}$ where$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\forall \ x, \alpha_k, \alpha > 0$.

Using this information, the inequality further reduces to:

$$\int_{-\infty}^{+\infty} \frac{d}{dt}[\varphi_t\varphi \ dx + (c_f(f(\bar{v})\bar{v})\varphi^2)] \ dx + \int_{-\infty}^{+\infty} -p'(\bar{v})\varphi_x^2 \ dx - \int_{-\infty}^{+\infty} \varphi_t^2 \ dx$$
$$\leq C\delta \int_{-\infty}^{+\infty} (\varphi_t^2 + \varphi_x^2) \ dx + O(1) \int_{-\infty}^{+\infty} |\varphi_x|^3 \ dx$$
$$\int_{-\infty}^{+\infty} \frac{d}{dt}[\varphi_t\varphi \ + (c_f(f(\bar{v})\bar{v})\varphi^2)] \ dx + \int_{-\infty}^{+\infty} -p'(\bar{v})\varphi_x^2 \ dx \ - \int_{-\infty}^{+\infty} \varphi_t^2 \ dx$$
$$\leq C\delta \int_{-\infty}^{+\infty} (\varphi_t^2 + \varphi_x^2) \ dx \qquad (7)$$

for some constant C. Now that we have found this estimate, we look for another estimate by employing the same methods again except this time we multiply System(6) by φ_t.Again we look at the L.H.S and R.H.S separately.

L.H.S:

First term of the given by:

$$\int_{-\infty}^{+\infty} (\varphi_{tt}\varphi_t) \ dx = \int_{-\infty}^{+\infty} (\frac{1}{2}\varphi_t^2)_t \ dx = \frac{d}{dt}\int_{-\infty}^{+\infty} (\frac{1}{2}\varphi_t^2)(x,t) \ dx$$

The second term of the L.H.S becomes:

$$\int_{-\infty}^{+\infty} (p(\bar{v}+\varphi_x) - p(\bar{v}))\varphi_t \ dx = -\int_{-\infty}^{+\infty} (p(\bar{v}+\varphi_x) - p(\bar{v}))\varphi_{xt} \ dx$$
$$= \frac{d}{dt}\left(\int_{-\infty}^{+\infty} (p(\bar{v}+\varphi_x) - p(\bar{v}))\varphi_x \ dx\right) + \int_{-\infty}^{+\infty} (p(\bar{v}) - p(\bar{v}+\varphi_x))_t\varphi_x \ dx$$
$$= \frac{d}{dt}\left(\int_{-\infty}^{+\infty} (p(\bar{v}) - p(\bar{v}+\varphi_x))\varphi_x \ dx\right) + \int_{-\infty}^{+\infty} p'(\bar{v}+\varphi_x)_t\varphi_{xt}\varphi_x \ dx$$
$$= \frac{d}{dt}\left(\int_{-\infty}^{+\infty} (p(\bar{v}) - p(\bar{v}+\varphi_x))\varphi_x \ dx\right) + \int_{-\infty}^{+\infty} p'(\bar{v}+\varphi_x)_t(\frac{1}{2}\varphi_x^2)_t \ dx$$
$$= \frac{d}{dt}\left(\int_{-\infty}^{+\infty} (p(\bar{v}) - p(\bar{v}+\varphi_x))\varphi_x \ dx\right) + \frac{d}{dt}\left(\int_{-\infty}^{+\infty} \frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_x^2 \ dx\right) -$$
$$- \int_{-\infty}^{+\infty} \frac{1}{2}\varphi_x^2 p'(\bar{v}+\varphi_x)\varphi_{xt}dx$$
$$= \frac{d}{dt}\left(\int_{-\infty}^{+\infty} (p(\bar{v}) \ - p(\bar{v}+\varphi_x))\varphi_x + \frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_x^2 \ dx \ \right) -$$
$$- \int_{-\infty}^{+\infty} \frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_x^2\varphi_{xt} \ dx.$$

Putting all of the L.H.S together

$$\frac{d}{dt}\int_{-\infty}^{+\infty}[(\frac{1}{2}\varphi_t^2)+(p(\bar{v})-p(\bar{v}+\varphi_x))\varphi_x]\ \ dx+\frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_x^2\ dx\ \ -$$
$$-\ \int_{-\infty}^{+\infty}\frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_x^2\varphi_{xt}\ dx.$$

R.H.S:

Here we simply have

$$-C_f(f(\bar{v}))^2\int_{-\infty}^{+\infty}\varphi_x\varphi_t\ dx-2C_ff(\bar{v})\bar{v}\int_{-\infty}^{+\infty}\varphi_t^2\ dx-2C_ff(\bar{v})\int_{-\infty}^{+\infty}\varphi_x\varphi_t^2\ dx\ -$$
$$C_f\int_{-\infty}^{+\infty}\varphi_x\varphi_t^3\ dx.$$

For simplicity let q be given by $q := (p(\bar{v})-p(\bar{v}+\varphi_x))\varphi_x+\frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_x^2$, then we can put the R.H.S and L.H.S together and apply the condition that φ_{xt} and φ_x are less than some small δ to get the following inequality;

$$\frac{d}{dt}\int_{-\infty}^{+\infty}[(\frac{1}{2}\varphi_t^2)+q]\ dx+2C_ff(\bar{v})\bar{v}\int_{-\infty}^{+\infty}\varphi_t^2\ dx+C_f(f(\bar{v}))^2\int_{-\infty}^{+\infty}\varphi_x\varphi_t\ dx$$
$$=\int_{-\infty}^{+\infty}\frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_x^2\varphi_{xt}\ dx-2C_ff(\bar{v})\int_{-\infty}^{+\infty}\varphi_x\varphi_t^2\ dx$$
$$\frac{d}{dt}\int_{-\infty}^{+\infty}[(\frac{1}{2}\varphi_t^2)+q]\ dx+2C_ff(\bar{v})\bar{v}\int_{-\infty}^{+\infty}\varphi_t^2\ dx+C_f(f(\bar{v}))^2\int_{-\infty}^{+\infty}\varphi_x\varphi_t\ dx$$
$$\leq O(1)\delta\int_{-\infty}^{+\infty}\varphi_x^2\ dx+O(1)\delta\int_{-\infty}^{+\infty}\varphi_t^2\ dx$$
$$=O(1)\delta\int_{-\infty}^{+\infty}(\varphi_t^2+\varphi_x^2)\ dx \qquad (8)$$

At this point we can get an estimate based upon the information at initial time, by integrating the results from (7) and (8) up to a given time τ.This will give us an insight on how the motion of the water waves will behave through time(up to τ), given some initial constraints. So integrating (7) w.r.t time in $[0,\tau]$, we find:

$$\int_{-\infty}^{+\infty}[\varphi_t\varphi \ +(c_f(f(\bar{v})\bar{v})\varphi^2)](x,\tau)\ dx+\int_0^\tau\int_{-\infty}^{+\infty}-(p'(\bar{v})\varphi_x^2-\varphi_t^2)(x,t)\ dxdt$$
$$\leq\int_{-\infty}^{+\infty}[\varphi_t\varphi \ +(c_f(f(\bar{v})\bar{v})\varphi^2)](x,0)\ dx+$$
$$+\ O(1)\delta\int_0^\tau\int_{-\infty}^{+\infty}(\varphi_t^2+\varphi_x^2)(x,t)\ dxdt \qquad (9)$$

similarly integrating (8) w.r.t time in $[0,\tau]$, we have:

$$\int_{-\infty}^{+\infty}[(\frac{1}{2}\varphi_t^2)+q](x,\tau)\ dx+\int_0^\tau\int_{-\infty}^{+\infty}[2C_ff(\bar{v})\bar{v}\varphi_t^2+C_f(f(\bar{v}))^2\varphi_x\varphi_t](x,t)\ dxdt$$
$$\leq\int_{-\infty}^{+\infty}[(\frac{1}{2}\varphi_t^2)+q](x,0)\ dx+O(1)\delta\int_0^\tau\int_{-\infty}^{+\infty}(\varphi_t^2+\varphi_x^2)(x,t)\ dxdt \qquad (10)$$

Here we apply the Taylor expansion on q to get

$$q=(p(\bar{v})-p(\bar{v}+\varphi_x))\varphi_x+\frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_x^2=\frac{1}{2}p'(\bar{v})\varphi_x^2+O(1)|\varphi_x|^3.$$

and use a method introduced in [1] to get to our result. We add the result from (9) and (10) in the following fashion; (9) + k(10) for an appropriate positive constant in time k to be determined later. So adding (9) + k(10);[6]

$$\int_{-\infty}^{+\infty}[\varphi_t\varphi \ +(c_f(f(\bar{v})\bar{v})\varphi^2)+\frac{k}{2}\varphi_t^2+\frac{-kp'(\bar{v})}{2}\varphi_x^2+O(1)|\varphi_x|^3\ ](x,t)\ +$$
$$\int_0^\tau\int_{-\infty}^{+\infty}[-p'(\bar{v})\varphi_x^2+(2kC_ff(\bar{v})\bar{v}-1)\varphi_t^2+C_fk(f(\bar{v}))^2\varphi_t\varphi_x](x,t)\ dxdt$$
$$\leq\int_{-\infty}^{+\infty}[(\frac{1}{2}\varphi_t^2)+q+\varphi_t\varphi \ +(C_f(f(\bar{v})\bar{v})\varphi^2)](x,0)\ dx\ \ +$$
$$O(1)\delta\int_0^\tau\int_{-\infty}^{+\infty}(\varphi_t^2+\varphi_x^2)(x,t)\ dxdt$$
$$\leq O(1)\int_{-\infty}^{+\infty}(\varphi+\varphi_t^2+\varphi_x^2)(x,0)\ dx+O(1)\delta\int_0^\tau\int_{-\infty}^{+\infty}(\varphi_t^2+\varphi_x^2)(x,t)\ dxdt.$$

[6] Here we used the $O(1)$ notation to bound the following terms: $\frac{1}{2}\varphi_t^2\to\varphi_t^2, q\to\varphi_x^2$ and $(C_f(f(\bar{v})\bar{v})\varphi^2)\to\varphi^2$. Since $\varphi_t\varphi$ is a small positive term it can go away.

We choose k to guarantee each term on R.H.S is positive definite.Working with the first term, to guarantee the quadratic form $\left(\varphi_t\varphi\ + (c_f(f(\bar{v})\bar{v})\varphi^2) + \frac{k}{2}\varphi_t^2\right)$ is positive definite, we require

$$\begin{aligned}
&\varphi^2 - 4C_f f(\bar{v})\bar{v}\varphi^2\frac{k}{2} < 0\\
&\Rightarrow 1 - 4C_f f(\bar{v})\bar{v}\frac{k}{2}\ < 0\\
&\Rightarrow k > (2C_f f(\bar{v})\bar{v})^{-1} = (4C_f S\bar{v})^{-1/2}.
\end{aligned}$$

The quadratic form above is approximated by

$$\left(\varphi_t\varphi\ + (c_f(f(\bar{v})\bar{v})\varphi^2) + \frac{k}{2}\varphi_t^2\right) \geq O(1)\left(\varphi^2 + \varphi_t^2\right).^7$$

Likewise, looking at the second term, we require the quadratic form below to be positive definite.

$$-p'(\bar{v})\varphi_x^2 + (2kC_f f(\bar{v})\bar{v} - 1)\varphi_t^2 + C_f k(f(\bar{v}))^2\varphi_t\varphi_x$$

This is equivalent to[8]

$$\left(4\sqrt{C_f}S^{-3/2} - 2S^{-3/2}4\sqrt{C_f - S\bar{v}}\right)^{-1/2} < k < \left(4\sqrt{C_f}S^{-3/2} + 2S^{-3/2}4\sqrt{C_f - S\bar{v}}\right)^{-1/2}.$$

Similar to the first term, we can find a positive lower bound for the quadratic form in the second term. It can claim the following bound

$$-p'(\bar{v})\varphi_x^2 + (2kC_f f(\bar{v})\bar{v} - 1)\varphi_t^2 + C_f k(f(\bar{v}))^2\varphi_t\varphi_x > O(1)\left(\varphi^2 + \varphi_t^2\right)$$

Now that we have found two estimates for k from the first and second term, to find a bound that will be sufficient to both the terms of the R.H.S, we take the

[7] This inequality is a consequence from the fact that $\varphi_t\varphi > 0$ and $C_f f(\bar{v})\bar{v}, \frac{k}{2} > 0$.

[8] Here the subcharacterstic condition is used and the fact that $p'(\bar{v}) = -v^{-3}$ and $f(\bar{v}) = \sqrt{\frac{S}{C_f\bar{v}}}$.

maximum, i.e

$$
\begin{aligned}
max \quad & (4\sqrt{C_f}S^{-3/2} - 2S^{-3/2}4\sqrt{C_f - S}\bar{v})^{-1/2}, (4C_f S\bar{v})^{-1/2} \\
& < k \\
& < (4\sqrt{C_f}S^{-3/2} + 2S^{-3/2}4\sqrt{C_f - S}\bar{v})^{-1/2}. \qquad (14)
\end{aligned}
$$

Once we choose an appropriate k that guarantees the L.H.S to be positive, we integrate (8) over $[o, t]$ by virtue of the estimates due to k, we arrive at

$$
\int_{-\infty}^{+\infty} (\varphi^2 + \varphi_t^2 + \varphi_x^2)(x, \tau)\ dx + \int_0^{\tau} \int_{-\infty}^{+\infty} (\varphi_t^2 + \varphi_x^2)(x, t)\ dxdt
$$

$$
\leq O(1) \int_{-\infty}^{+\infty} (\varphi^2 + \varphi_t^2 + \varphi_x^2)(x, 0)\ dx.
$$

Lemma 2.3. *Let $\varphi(x, t)$ be a solution of the initial value problem(6) satisfying $\varphi(., t) \in H^3(\mathbb{R})$ and $\varphi_t(., t) \in H^2(\mathbb{R})$ for $t \in [0, T]$. Then there exists a number $\delta > 0$, such that if*

$$
\sup_{\substack{x \in (-\infty, +\infty) \\ t \in [0, T]}} \left(|\varphi_{xx}| + |\varphi_{xt}| + |\varphi_t| + |\varphi_x|\right)(x, t) \leq \delta
$$

then

$$
\begin{aligned}
\int_{-\infty}^{+\infty} (\varphi_{xt}^2 + \varphi_{xx}^2\ dx)(x, t)\ dx + \int_0^t \int_{-\infty}^{+\infty} (\varphi_{xt}^2 + \varphi_{xx}^2\ dx)(x, \tau)\ dxd\tau \\
< O(1)\delta \int_{-\infty}^{+\infty} (\varphi_{xx}^2 + \varphi_{xt}^2)(x, 0)\ dx.
\end{aligned}
$$

Proof. We start by differentiating System(6) w.r.t x and multiplying by φ_{xt} and then integrate over the whole real line to get an estimate that involves the derivatives of φ_t and φ_t.

L.H.S

the first term,

$$\int_{-\infty}^{+\infty} \varphi_{ttx}\varphi_{xt}\ dx = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\frac{\partial}{\partial t})(\varphi_{xx}^2)^2\ dx = \frac{d}{dt}\frac{1}{2}\int_{-\infty}^{+\infty}\varphi_{xt}^2\ dx.$$

and the second term, using the method of integration by parts;

$$\begin{aligned}\int_{-\infty}^{+\infty} p(\bar{v}+\varphi_x)\varphi_{xt}\ dx &= -\int_{-\infty}^{+\infty} p(\bar{v}+\varphi_x)_x\varphi_{xxt}\ dx\\ &= -\int_{-\infty}^{+\infty} p'(\bar{v}+\varphi_x)\varphi_{xx}\varphi_{xxt}\ dx\\ &= -\int_{-\infty}^{+\infty} p'(\bar{v}+\varphi_x)(\frac{1}{2}\varphi_{xx}^2)_t\ dx\\ &= -\int_{-\infty}^{+\infty} (\frac{1}{2}p'(\bar{v}+\varphi_x)\varphi_{xx}^2)_t\ dx + \frac{1}{2}\int_{-\infty}^{+\infty} p''(\bar{v}+\varphi_{xt})\varphi_{xt}\varphi_{xx}^2\ dx.\end{aligned}$$

All together the L.H.S becomes:

$$= \frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}(\varphi_{xt}^2 - p'(\bar{v}+\varphi_x)\varphi_{xx}^2)(x,t)\ dx + \frac{1}{2}\int_{-\infty}^{+\infty} p''(\bar{v}+\varphi_{xt})varphi_{xt}\varphi_{xx}^2\ dx.$$

R.H.S

$$\begin{aligned}= \int_{-\infty}^{+\infty} -2C_f f(\bar{v})\bar{v}\varphi_{xt}^2 - C_f(f(\bar{v}))^2\varphi_{xt}\varphi_{xx}\ dx\ -&\\ \int_{-\infty}^{+\infty}(2C_f f(\bar{v})[\varphi_{xx}\varphi_t + \varphi_x\varphi_{xt}\ dx + C_f[\varphi_{xx}\varphi_t^2 + 2\varphi_x\varphi_t\varphi_{xt}]\varphi_{xt}\ dx.&\end{aligned}$$

to simplify to following calculations we will assign I to be

$$I := \int_{-\infty}^{+\infty}(2C_f f(\bar{v})\varphi_x\varphi_{xt}^2 + 2C_f\varphi_t\varphi_{xt}\varphi_x + C_f\varphi_t^2\varphi_{xx})\varphi_{xt}\ dx$$

Putting the L.H.S and R.H.S together, we have

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}(\varphi_{xt}^2 - p'(\bar{v}+\varphi_x)\varphi_{xx}^2)(x,t)\ dx +$$
$$\int_{-\infty}^{+\infty} 2C_f f(\bar{v})\bar{v}\varphi_{xt}^2 + C_f(f(\bar{v})^2\varphi_{xt}\varphi_{xx}\ dx = -\frac{1}{2}\int_{-\infty}^{+\infty} p''(\bar{v}+\varphi_{xt})varphi_{xt}\varphi_{xx}^2\ dx - I.$$

Integrating the result in time from $[0,t]$ and in view of the lemma on sup of $|\varphi_{xt}|, |\varphi_t|$ and $|\varphi_x|$ we can get the estimate below by reducing I,

$$\int_{-\infty}^{+\infty}(\frac{1}{2}\varphi_{xt}^2 - p'(\bar{v}+\varphi_x)\varphi_{xx}^2)(x,t)\quad dx +$$
$$\int_0^t\int_{-\infty}^{+\infty}(2C_f f(\bar{v})\bar{v}\varphi_{xt}^2 + C_f(f(\bar{v}))^2\varphi_{xt}\varphi_{xx})(x,\tau)\ dxd\tau$$
$$= -\frac{1}{2}\int_0^t\int_{-\infty}^{+\infty}(p''(\bar{v}+\varphi_{xt})\varphi_{xx}^2)(x,t)\ dxd\tau\ + \int_0^t I\ d\tau$$
$$+\int_{-\infty}^{+\infty}(\frac{1}{2}\varphi_{xt}^2 - p'(\bar{v}+\varphi_x)\varphi_{xx}^2)(x,0)\ dx$$
$$\le \int_{-\infty}^{+\infty}\frac{1}{2}(\varphi_{xt}^2 - p'(\bar{v}+\varphi_x)\varphi_{xx}^2)(x,0)\ dx\ + O(1)\delta\int_0^t\int_{-\infty}^{+\infty}(\varphi_{xx}^2+\varphi_{xt}^2)(x,\tau)\ dxd\tau. \quad (\bigstar)$$

To get another estimate we multiply the derivative of System(6) w.r.t x by φ_{xx} and integrate of the the whole real line.

L.H.S

the first term

$$\int_{-\infty}^{+\infty}\varphi_{ttx}\varphi_{xx}\ dx = \frac{d}{dt}\int_{-\infty}^{+\infty}\varphi_{tx}\varphi_{xx}\ dx - \int_{-\infty}^{+\infty}\varphi_{tx}\varphi_{txx}\ dx$$
$$= \frac{d}{dt}\int_{-\infty}^{+\infty}\varphi_{tx}\varphi_{xx}\ dx.$$

for the second term

$$\int_{-\infty}^{+\infty} p(\bar v+\varphi_x)_{xx}\varphi_{xx}\ dx = -\int_{-\infty}^{+\infty} p(\bar v+\varphi_x)_x\varphi_{xxx}\ dx$$
$$= -\int_{-\infty}^{+\infty} p'(\bar v+\varphi_x)\varphi_{xx}\varphi_{xxx}\ dx$$
$$= -\int_{-\infty}^{+\infty} p'(\bar v+\varphi_x)(\frac{1}{2}\varphi_{xx}^2)_x\ dx$$
$$= \frac{1}{2}\int_{-\infty}^{+\infty} p''(\bar v+\varphi_x)\varphi_{xx}^3\ dx.$$

So the L.H.S becomes[9]:

$$= \frac{d}{dt}\int_{-\infty}^{+\infty}\varphi_{tx}\varphi_{xx}\ dx + \frac{1}{2}\int_{-\infty}^{+\infty} p''(\bar v+\varphi_x)\varphi_{xx}^3\ dx.$$

R.H.S

$$\int_{-\infty}^{+\infty} -C_f[f(\bar v)\varphi_{xx}\varphi_t + f(\bar v)\varphi_x\varphi_{xt} + \varphi_{xx}\varphi_t^2 + 2\varphi_x\varphi_{xt}\varphi_t]\varphi_{xx}\ dx.$$

We put the L.H.S and R.H.S together and using the using the fact $sup\ (|\varphi_t| + |\varphi_x|) = \delta$ [10], we get the inequality

$$\frac{d}{dt}\int_{-\infty}^{+\infty}(\varphi_t\varphi +\ c_f(f(\bar v)\bar v)\varphi^2)\ dx\ +\ \int_{-\infty}^{+\infty}(-p(\bar v)\varphi_x^2)dx\ -\ \int_{-\infty}^{+\infty}\varphi_t^2\ dx$$
$$+\ 2c_f f(\bar v)\int_{-\infty}^{+\infty}\varphi_t\varphi_x\varphi\ dx -\ c_f\int_{-\infty}^{+\infty}\varphi_t^2\varphi_x\varphi\ dx$$
$$\leq O(1)\delta\int_{-\infty}^{+\infty}(\varphi_{xx}^2+\varphi_{xt}^2)(x,t)\ dx.(\bigstar\bigstar)$$

□

To complete the proof, we make use of the method employed in lemma 2.1, we

[9] The term $\int_{-\infty}^{+\infty}\varphi_{tx}\varphi_{txx}\ dx \to 0$ and $[-\frac{1}{2}\int_{-\infty}^{+\infty}p'(\bar v+\varphi_x)\varphi_{xx}^2]_x = 0$.
[10] Notice the inequality $|\varphi_{xt}\varphi_{xx}| \leq \frac{1}{2}(\varphi_t^2+\varphi_x^2)$ also notice $\varphi_t^2\varphi_{xx}^2 \leq \delta^2\varphi_{xx}^2$

find a constant in time k such that $\bigstar + k\bigstar\bigstar$ is greater than or equal to

$$\geq c_1 \int_{-\infty}^{+\infty} (\varphi_{xx}^2 + \varphi_{xt}^2)(x,t)\ dx + C_2 \int_0^t \int_{-\infty}^{+\infty} (\varphi_{xx}^2 + \varphi_{xt}^2)(x,t)\ dx.$$

So

$$\begin{aligned}
&c_1 \int_{-\infty}^{+\infty} (\varphi_{xx}^2 + \varphi_{xt}^2)(x,t)\ dx + C_2 \int_0^t \int_{-\infty}^{+\infty} (\varphi_{xx}^2 + \varphi_{xt}^2)(x,t)\ dx \\
&\leq O(1) \int_{-\infty}^{+\infty} (\varphi_{xx}^2 + \varphi_{xt}^2)(x,0)\ dx + C_2 \int_0^t \int_{-\infty}^{+\infty} (\varphi_{xx}^2 + \varphi_{xt}^2)(x,0)\ dx \\
&\leq O(1) \int_{-\infty}^{+\infty} (\varphi_{xx}^2 + \varphi_{xt}^2)(x,0)\ dx.
\end{aligned}$$

□

Theorem 2.4. *There exists a positive number $\delta > 0$, such that if $||\varphi(.,0)||_{H^3(\mathbb{R})} + ||\varphi_t(.,0)||_{H^2(\mathbb{R})} < \delta$, then the initial value problem System(6) with the initial data $\varphi(x,0)$ and $\varphi_t(x,0)$, has a global in time solution. Moreover,*

$$\int_{-\infty}^{+\infty} (\varphi_x^2 + \varphi_t^2)(x,t)\ dx \leq \frac{C}{t} \qquad (12)$$

$$\int_{-\infty}^{+\infty} (u - f(\bar{v}))^2 + (v - \bar{v})^2(x,t) dx \leq \frac{C}{t}$$

and

$$\sup_{x\, \in\, (-\infty, +\infty)} (|\varphi_t(x,t)| + |\varphi_x(x,t)|) \leq \frac{C}{t^{\frac{1}{4}}}$$

$$\sup_{x\, \in\, (-\infty, +\infty)} (|u - f(\bar{v})|\ + |v - \bar{v})|) \leq \frac{C}{t^{\frac{1}{4}}} \qquad (13)$$

Proof. The first inequality comes from (8)by integrating over (t,T):

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (\ \frac{1}{2}\varphi_t^2\ + q)(t,T)\ dx\ +\ \int_t^T \int_{-\infty}^{+\infty} (\ 2c_f(f(\bar{v}))\ \bar{v}\ \varphi_t^2\ +\ c_f(f(\bar{v}))^2 \varphi_t\ \varphi_x)(x,s)\ dx\ ds \\
&\leq\ \int_{-\infty}^{+\infty} (\ \frac{1}{2}\varphi_t^2\ + q)(x,t)\ dx\ +\ O(1)\delta\ \int_t^T \int_{-\infty}^{+\infty} (\varphi_t^2\ +\ \varphi_x^2)(x,s)\ dx\ dt
\end{aligned}$$

then integrate the above result on $[0,T]$ for t:

$$T\int_{-\infty}^{+\infty}(\ \frac{1}{2}\varphi_t^2\ +q)(t,T)\ dx\ +\ \int_0^T\int_t^T\int_{-\infty}^{+\infty}(\ 2c_f(f(\bar{v}))\ \bar{v}\ \varphi_t^2\ +\ c_f(f(\bar{v}))^2\varphi_t\ \varphi_x)(x,s)\ dx\ ds\ dt$$
$$\le\ \int_0^T\int_{-\infty}^{+\infty}(\ \frac{1}{2}\varphi_t^2\ +q)(x,t)\ dx\ +\ O(1)\delta\ \int_0^T\int_t^T\int_{-\infty}^{+\infty}(\varphi_t^2\ +\ \varphi_x^2)(x,s)\ dx\ dt$$

because of the positive integrand, this reduces to[11]:

$$\Longrightarrow\ T\int_{-\infty}^{+\infty}(\ \frac{1}{2}\varphi_t^2\ +q)(x,T)\ dx\ \le\ O(1)\int_0^T\int_{-\infty}^{+\infty}(\varphi_t^2\ +\ \varphi_x^2)\ dx\ dt$$
$$\Longrightarrow\ T\int_{-\infty}^{+\infty}(\ \frac{1}{2}\varphi_t^2\ +q)(x,T)\ dx\ \le\ O(1)$$
$$\Longrightarrow T\int_{-\infty}^{+\infty}(\ \frac{1}{2}\varphi_t^2\ +\varphi_x^2)(x,T)\ dx$$
$$\le\frac{C}{T}.$$

Notice that (12) is a simple consequence derived from Schwarz's Inequality[12].

$$\begin{aligned}\varphi_t^2(x,t) &= \int_{-\infty}^{x}(\varphi_t^2(x,t))_x\ dx \\ &= \int_{-\infty}^{x} 2\varphi_t(y,t)\varphi_{xt}(y,t)\ dy \\ &\le 2\left(\int_{-\infty}^{x}\varphi_t^2(y,t)\ dy\right)^{\frac{1}{2}}\left(\int_{-\infty}^{x}\varphi_{xt}^2(y,t)\ dy\right)^{\frac{1}{2}} \\ &\le 2\left(\int_{-\infty}^{+\infty}\varphi_t^2(y,t)\ dy\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\varphi_{xt}^2(y,t)\ dy\right)^{\frac{1}{2}} \\ &\le \frac{C}{t^{\frac{1}{2}}} \\ &\Rightarrow |\varphi_t|\ \le\ \frac{C}{t^{\frac{1}{4}}}\end{aligned}$$

---

[11] We used integration by parts on the integrand $\int_0^T\int_t^T\int_{-\infty}^{+\infty}(\varphi_t^2\ +\ \varphi_x^2)(x,s)\ dx\ dt = t\int_t^T\int_{-\infty}^{+\infty}(\varphi_t^2\ +\ \varphi_x^2)(x,s)\ dx\ dt|_0^T + \int_0^T\int_{-\infty}^{+\infty}(\varphi_t^2\ +\ \varphi_x^2)(x,s)\ dx\ dt = \int_0^T\int_{-\infty}^{+\infty}(\varphi_t^2\ +\ \varphi_x^2)(x,s)\ dx\ dt$.

[12] Given two integrable functions $\varphi_t(x,t)$ and $\varphi_{xt}(x,t)$ in $[x,-\infty]$ then Cauchy's Inequality is given by $\int_{-\infty}^{x}\varphi_t(y,t)\varphi_{xt}(y,t)\ dy \le \left(\int_{-\infty}^{x}\varphi_t^2(y,t)\ dy\right)^{\frac{1}{2}}\left(\int_{-\infty}^{x}\varphi_{xt}^2(y,t)\ dy\right)^{\frac{1}{2}}$.

Similarly, we have $|\varphi_x| \leq \frac{C}{t^{\frac{1}{4}}}$. From here we get (12),

$$\sup_{x \in (-\infty, +\infty)} (|\varphi_t(x,t)| + |\varphi_x(x,t)|) \leq \frac{C}{t^{\frac{1}{4}}}.$$

The global in time existence follows by the standard local existence result of hyperbolic systems ([6]) and the a priori estimates(Lemma 2.1 and Lemma 2.2). This a priori estimates enable us to extend the local in time solution to global in time solution([7]).

□

3 More General Situation

In this section we analyze how the system of equations for shallow water wave would change if the initial conditions were changed. We look at the case where the velocity of the water at $\pm\infty$ is not the same. Instead we start with the assumptions listed below:

$$\lim_{x\to\pm\infty} v_0(x) = \bar{v}$$

$$\lim_{x\to+\infty} u_0(x) = u_+, \quad \lim_{x\to-\infty} u_0(x) = u_-$$

where u_+ and u_- are positive constants.

Now let

$$v^+(t) = \lim_{x\to+\infty} v(x,t), \quad u^+(t) = \lim_{x\to+\infty} u(x,t)$$

$$\bar{v} = \lim_{x\to-\infty} v(x,t), \quad u^-(t) = \lim_{x\to-\infty} u(x,t) \quad \textit{and } f(\bar{v}) = \sqrt{\frac{S}{C_f\bar{v}}}.$$

then rewriting system(3), we get[13];

$$v_t^+(t) = 0$$

$$u_t^+(t) = S - C_f(u^+)^2(v^+)c$$

At this point we introduce new transformations that can be used to express system(7) in a similar fashion as system(6). This approach is beneficial due to the fact that the lemmas developed to prove the existence of a global solution for system(6) could be directly applied here as well.

Set

$$U^+(t) = \frac{1+y^+(t)}{1-y^+(t)}\sqrt{\frac{S}{C_f\bar{v}}}, \quad U^-(t) = \frac{1+y^-(t)}{1-y^-(t)}\sqrt{\frac{S}{C_f\bar{v}}}$$

[13] Notice that $v^+(0) = \bar{v}, \quad u^+(0) = u_+, \quad v^+(t) = \bar{v}, \quad u_t^+ = S - C_f(u^+)^2\bar{v}.$

where,

$$y^+(t) = \frac{u_+\sqrt{C_f\bar{v}} - \sqrt{S}}{u_+\sqrt{C_f\bar{v}} + \sqrt{S}}\ exp\left(-\sqrt{SC_f\bar{v}t.}\right)$$
$$y^-(t) = \frac{u_-\sqrt{C_f\bar{v}} - \sqrt{S}}{u_-\sqrt{C_f\bar{v}} + \sqrt{S}}\ exp\left(-\sqrt{SC_f\bar{v}t.}\right)$$

We introduce new variables $\hat{v}(x,t)$ and $\hat{u}(x,t)$ defined as:

$$\hat{v}(x,t) = m_0(x)\int_t^{+\infty}\left(u^-(s) - u^+(s)\right)\ ds$$
$$\hat{u}(x,t) = \left(u^+(t) - u^-(t)\right)\int_x^{+\infty} m_0(y)\ dy$$
$$where \int_{-\infty}^{+\infty} m_0(y)\ dy = 1\ and\ supp\ m_0(x) = [-1,1].$$

Here we make a quick observation on the relation between $\hat{v}(x,t)$ and $\hat{u}(x,t)$;

$$\begin{aligned}
\frac{d}{dt}\hat{v}(x,t) &= \frac{d}{dt}\left(m_0(x)\int_t^{+\infty}\left(u^-(s) - u^+(s)\right)\ ds\right)\\
&= m_0(x)\frac{d}{dt}\left(\int_t^{+\infty}\left(u^-(s) - u^+(s)\right)\ ds\right)\\
&= m_0(x)\left(u^-(s) - u^+(s)\right)|_t^{+\infty}\\
&= m_0(x)\left(u^+(t) - u^-(t)\right).\\
\frac{d}{dx}\hat{u}(x,t) &= \frac{d}{dt}\left(\left(u^+(t) - u^-(t)\right)\int_x^{+\infty} m_0(y)\ dy\right)\\
&= \left(u^+(t) - u^-(t)\right)\frac{d}{dt}\left(\int_t^{+\infty} m_0(y)\ dy\right)\\
&= \left(u^+(t) - u^-(t)\right)m_0(y)\ |_{-\infty}^{x}\\
&= \left(u^+(t) - u^-(t)\right)m_0(x).
\end{aligned}$$

This shows that $\hat{u}_x(x,t) = \hat{v}_t(x,t)$.

Following similar steps we took in Section 2, we define the variables $w(x,t)$ and

$z(x,t)$ in the following way.

$$w(x,t) := v(x,t) - \bar{v} - \hat{v}$$
$$z(x,t) := u(x,t) - f(\bar{v}) - \hat{u}.$$

From the relation between $\hat{v}$ and $\hat{u}$, it follows that $w_t(x,t) = z_x(x,t)$ and if we look at the time derivative of $z(x,t)$, we find[14]

$$\begin{aligned}
&z_t = u_t - \hat{u}_t \\
&z_t = S - C_f u^2 v - p(v)_x - \bar{u}_t \\
&z_t = S - C_f u^2 v - p(\bar{v} + \hat{v} + w)_x - \bar{u}_t \\
&z_t + p(\bar{v} + \hat{v} + w)_x = S - C_f u^2 v - \bar{u}_t \\
&z_t + p\left((\bar{v} + \hat{v} + w) - p(\bar{v})\right)_x = S - C_f u^2 v - \bar{u}_t.
\end{aligned}$$

Combining these information we get a system of equations that is alike system(4)

$$\begin{aligned}
&w_t - z_x = 0 \\
&z_t + p\left((\bar{v} + \hat{v} + w) - p(\bar{v})\right)_x = S - C_f u^2 v - \bar{u}_t \qquad (15)
\end{aligned}$$

Now let;

$$\begin{aligned}
\varphi(x,t) &= \int_{-\infty}^{x} w(y,t)\ dy \\
&= \int_{-\infty}^{x} \left(v(y,t) - \bar{v} - \hat{v}\right)(y,t)\ dy
\end{aligned}$$

We want $\varphi(\pm\infty, t) = 0$ for all $t > 0$ and we want to show that $\varphi(\pm\infty, 0) = 0$ and

[14] From system(3), we have $u_t = S - C_f u^2 v - p(v)_x$ and $p\left((\bar{v} + \hat{v} + w) - p(\bar{v})\right)_x = p(\bar{v} + \hat{v} + w)_x$ because $p(\bar{v})_x = 0$, also $f(\bar{v})_x = \bar{v}_t = 0$

$\varphi_t(\pm\infty, t) = 0.$[15]

$$\begin{aligned}
\varphi(+\infty, 0) &= \int_{-\infty}^{+\infty} [v_0(x) - \bar{v} - \bar{v}] \ dx = 0 \\
&= \int_{-\infty}^{+\infty} \left[v_0(x) - \bar{v} - m_0(x) \left(\int_{t=0}^{+\infty} (u^-(s) - u^+(s)) \ ds \right) \right] = 0 \\
&\Rightarrow \int_{-\infty}^{+\infty} (v_0(x) - \bar{v}) \ dx = \int_{-\infty}^{+\infty} m_0(x) \left(\int_{t=0}^{+\infty} (u^-(s) - u^+(s)) \ ds \right) \ dx \\
&\Rightarrow \int_{-\infty}^{+\infty} (v_0(x) - \bar{v}) \ dx = \int_{t=0}^{+\infty} (u^-(s) - u^+(s)) \ ds
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\varphi(+\infty, t) &= \int_{-\infty}^{+\infty} (v_t - \bar{v}_t(x,t)) \ dx \\
&= \int_{-\infty}^{+\infty} (u_x - \bar{u}_x(x,t)) \ dx \\
&= (u(+\infty, t) - u(-\infty, t)) - (\hat{u}(+\infty, t) - \hat{u}(-\infty, t)) \\
&= \big(u^+(t) - u^-(t)\big) - \big(u^+(t) - u^-(t)\big) = 0.
\end{aligned}$$

Once we have the conditions listed above, we are ready to express the system of equations for the general case in terms of $\varphi(x, t)$.

$$\begin{aligned}
&\varphi_x = v - \bar{v} - \hat{v} = w, \quad \varphi_t = \int_{-\infty}^{x} z_x \ dx \\
&\varphi_t = z = u - f(\bar{v}) - \hat{u} \\
&\varphi_{tt} = z_t
\end{aligned}$$

Hence the equation for the general case becomes,

$$\varphi_{tt} + p\left((\bar{v} + \hat{v} + \varphi_x) - p(\bar{v})\right)_x = S - C_f(f(\bar{v}) + \hat{u} + \varphi_t)^2(\bar{v} + \hat{v} + \varphi_x) - \hat{u}_t. \tag{16}$$

Once we have found this equation, we apply all the arguments presented in section 2 to arrive at the global solution and find the convergence rate.

[15] It is clear that $\varphi(-\infty, t) = \varphi_t(-\infty, t) = \varphi(-\infty, 0) = 0$

4 Conclusion

This MQP project looks at the partial differential equations describing Shallow Water Wave(SWW). The motivation for the project comes from the journal papers [2] and [2], where the local existence of the solutions of SWW were proved. However, the convergence rate for the solution and the existence of the global in time solutions were out of reach.

In this project we study SWW equations with certain given initial data. We start by assuming the height of the surface and the inclination of the surface are small with respect to the horizontal. Using this assumption and applying multiple transformations we convert the SWW equations to a single equation that we found suitable. From there on we use concepts from real analysis to arrive at convergence rate for the Shallow Water Wave. Furthermore, we introduce two new lemmas that we consider as a-priori conditions to prove the existence of the global solution.

In conclusion, we try to generalize by looking ahead at general situations, by minimizing the restriction on the initial conditions. Here, we introduce transformations that would carry the system of SWW equations to a single equation similar to what we have found and studied on previous section. We leave the details from showing the existence and convergence rate of the global solution, as the argument follows naturally from the previous section.

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