Distance Labelings of Möbius Ladders

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by

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Abstract

A distance-two labeling of a graph G is a function $f : V(G) \to \{0, 1, 2, \ldots, k\}$ such that $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$ and $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ for all $u, v \in V(G)$. A labeling is optimal if k is the least possible integer such that G admits a k-labeling. The $\lambda_{2,1}$ number is the largest integer assigned to some vertex in an optimally labeled network. In this paper, we examine the $\lambda_{2,1}$ number for Möbius ladders, a class of graphs originally defined by Richard Guy and Frank Harary [9]. We completely determine the $\lambda_{2,1}$ number for Möbius ladders of even order, and for a specific class of Möbius ladders with odd order. A general upper bound for $\lambda_{2,1}(G)$ is known [6], and for the remaining cases of Möbius ladders we improve this bound from 18 to 7. We also provide some results for radio labelings and extensions to other labelings of these graphs.

Executive Summary

A graph is a pair $G = (V, E)$, such that $V(G)$ is the vertex set, and $E(G)$ is the set of edges. For simple graphs (i.e., undirected, loopless, and finite), the concept of a radio labeling was first introduced in 1980 by Hale [8]. A radio labeling is formally defined as a function:

$$f : V(G) \to \{0, 1, 2, \ldots, k\}$$

such that

$$|f(u) - f(v)| + d(u, v) \geq diam(G) + 1$$

for all u, v in $V(G)$. Here, $d(u, v)$ denotes the length of the shortest uv-path, and $diam(G)$ is the diameter of G, i.e., the maximum distance among all vertex pairs in G. This definition puts $diam(G)$ constraints on the labeling assignment of vertices. The radio number of G, denoted $rn(G)$, is the smallest k such that G can be radio labeled with $k + 1$ labels (including zero). Any radio labeling that achieves this bound is said to be optimal.

Investigations of optimal radio labelings of graphs are better managed in smaller cases and with labeling schemes that present fewer constraints. A natural starting point, then, is the $L(2, 1)$ labeling problem, which only imposes restrictions on vertices within a distance-two neighborhood of any vertex. Formally, adjacent vertices must receive labels that differ by at least two, and vertices which are distance two may receive labels that differ by at least one. Note that this problem is exactly that of radio labeling for graphs of diameter two. An optimal $L(2, 1)$ labeling uses $\lambda_{2,1}(G) + 1$ labels (including zero).

An $L'(2, 1)$ labeling has the same distance contraints as an $L(2, 1)$ labeling, with the additional imposition that the function must also be one-to-one. The parameter $\lambda'_{2,1}(G)$ is the minimum value k such that G admits a k-labeling, corresponding to the $L'(2, 1)$ labeling constraints.

Our objective for this project is to determine the $\lambda_{2,1}$ number for a previously unresolved class of graphs called Möbius ladders. Möbius ladders, denoted M_n, are formed by taking a cycle of vertices C_n and connecting all vertex pairs (u, v) such that $d(u, v) = diam(C_n)$. We refer to M_n as an *even* or *odd* Möbius ladder depending on whether n is even or odd. Though the construction and some basic properties of Möbius ladders are the same, the even and odd classes exhibit several fundamentally different characteristics. For instance, even Möbius ladders are regular of degree three, while odd ladders are regular of degree four. We consider these cases separately due to the structural differences.

The main theorems of the paper are:

- $\lambda_{2,1}(M_n) = 6$ for n even and $n \neq 8$.
- For n odd and $n \neq 11, 17$, $\lambda_{2,1}(M_n)$ is either 6 or 7.

- $\lambda'_{2,1}(M_n) = n - 1$ for all n.
- $rn(M_n) \geq \frac{3n}{2} - 2$, for $n \geq 17$ and $n \in \{11, 14, 15\}$.

In the cases where equality is proved, algorithms are presented which attain the optimal value. There is also a discussion and an upperbound on the $L(h, 1)$ labelings of Möbius ladders which follows naturally from the $L(2, 1)$ labeling results presented. We conclude with some suggestions for future research and possible extensions to the concept of the Möbius ladders.

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Chapter 1

Introduction

It is important to establish some basic graph theoretic concepts and their relevance to the frequency labeling problem before we begin establishing results on graphs. What follows are definitions and concepts that arise in examples, results, and proofs throughout this paper.

Most basically, we must first provide a precise definition of a graph. A *graph* *G=(V,E)* consists of a vertex set $V(G)$ and an edge set $E(G)$. When the referenced graph is obvious, these shall be shortened to V and E. In this paper, we will only consider simple, finite, undirected graphs.

A *finite graph* is a graph in which both the vertex and edge sets have a finite number of elements. A *simple* graph is a graph that has no loops or multiple edges. A *loop* is an edge which connects a vertex to itself. At most one edge $e = (u, v)$ may join two distinct vertices $u, v \in V(G)$.

All labeling schemes rely heavily upon the concept of distance. The *distance* between any two vertices u and v is the length of the shortest uv-path in G, and is denoted $d(u, v)$. The diameter of a graph, denoted $diam(G)$, is the maximum distance between vertices in G. In other words, $diam(G) = max_{u,v \in V} d(u, v)$.

A *labeling* is a function which maps the vertex set of G to another set S. In this paper, we only consider labelings that map the vertices to the nonnegative integers,

$$f : V \to S = \mathbb{Z}^+ \cup \{0\}.$$

In some cases, it is only necessary to consider the neighborhood of a vertex. The *neighborhood* about a vertex u is the set $\{v \in V \mid d(u, v) = 1\}$. The *distance-two neighborhood* of a vertex u, denoted $\mathcal{N}_2(u)$ is defined to be the set of vertices $\{v \in V \mid 1 \leq d(u, v) \leq 2\}$. The closed neighborhood of u is the set $\{u\} \cup \mathcal{N}(u)$, and the closed distance-two neighborhood is defined analogously. It is natural for frequency strength to dissipate the farther a signal is broadcast.

This interpretation lends itself to the idea of neighborhoods. Another relevant notion which plays an important role in our analysis is that of a dominating set. A *dominating set* of a graph G is any set of vertices whose closed neighborhoods contain every vertex in G.

For a specific labeling f, the *span* of f, denoted $\sigma(f)$ is defined to be $max(f) - min(f)$. In the cases presented here, $\sigma(f) = max(f)$, as these labelings begin at zero. This convention is widely used for $L(h, k)$ labelings. The span of a frequency network is better known as the bandwidth. Since bandwidth is finite, it is important to reduce the span as much as possible to conserve it while retaining sufficient signal propagation.

Chapter 2

Background

2.1 Radio Channel Assignment Problem

An interesting problem that arises in application is radio channel frequency assignment. The closer two towers are within a network of radio towers, the further apart their frequencies must be in order to minimize interference. Towers that are far apart do not run the chance of frequency interference and may broadcast on closer frequencies. The ability to broadcast on a closer range of frequencies is especially helpful in conservation of bandwidth. Minimizing the total span of frequencies subject to a configuration of radio towers is naturally formulated as an optimization problem.

Given a graph $G = (V, E)$, the radio transmitters can be represented as the set of vertices V and the overall closeness between the transmitters described by the edge set E. This optimization problem is thus reduced to one of graph labeling. This simple model works under the assumptions that frequencies are emitted in all directions equally without reverberation and regardless of wavelength, allowing for relaxation of otherwise impeding restrictions such as directedness [10].

To solve this problem, the graph labeling must reflect the physical constraints of frequency assignment. This manifests itself in distance-dependent labeling schemes. Intuitively, vertices that are closer in the graph must receive labels that are farther apart, and vertices that are farther apart may receive labels that are closer. Perhaps the most natural of these labeling schemes is the radio labeling.

Given a graph $G = (V, E)$, a *radio labeling* is a labeling f, such that

$$|f(u) - f(v)| \geq diam(G) + 1 - d(u, v).$$

This constraint is referred to as the *radio condition.* This creates $diam(G)$

separate constraints for each vertex. The *radio number* of a graph, denoted $rn(G)$, is the minimum span of any radio labeling on that graph. Formally,

$$rn(G) = min(\sigma(f)).$$

A radio labeling that achieves this bound is an optimal radio labeling.

Lemma 2.1.1. $rn(G) \geq |V| - 1$.

Proof. Let $G = (u, v)$ and $u, v \in V$. By the definition of a radio labeling, $|f(u) - f(v)| \geq diam(G) + 1 - d(u, v)$. Because $d(u, v) \leq diam(G)$ by definition, then $|f(u) - f(v)| \geq 1$. So any two arbitrary vertices must receive different labels. Therefore, the labeling must be one-to-one, and $\sigma(f) \geq |V| - 1$. Since σ is bounded below by $|V| - 1$, then $rn(G)$ must be also. □

2.1.1 Basic Results on Radio Labelings

Some known results are discussed on complete graphs, paths, and cycles. The latter two results appeared in [13].

Complete Graphs A complete graph on n vertices, K_n, has an edge between every two vertices. This implies that $diam(G) = 1$, and therefore, the radio condition reduces to $|f(u) - f(v)| \geq 1$. It is evident that $rn(K_n) = n - 1$.

Cycles A cycle on n vertices, denoted by C_n, is a connected, two-regular graph.

$$rn(C_n) = \begin{cases} \frac{n-2}{2}\phi(n) + 1, & \text{if } n \equiv 0, 2 \pmod 4; \\ \frac{n-1}{2}\phi(n), & \text{if } n \equiv 1, 3 \pmod 4. \end{cases}$$

With $n = 4k + r$, ($r \in \mathbb{Z}_4$), and

$$\phi(n) = \begin{cases} k + 1, & \text{if } r = 1; \\ k + 2, & \text{else.} \end{cases}$$

Paths The path on n vertices, referred to as P_n, is given by the vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$, and an edge e is in the edge set if and only if $e = (v_{i-1}, v_i)$ for $i = 0, 1, \ldots n$. Liu and Zhu proved the following:

$$rn(P_n) = \begin{cases} 2k^2 + 2, & \text{if } n = 2k + 1; \\ 2k(k - 1) + 1, & \text{if } n = 2k. \end{cases}$$

Refer to Figures 2.1 and 2.2 for optimal radio labelings of a cycle and a complete graph, respectively.

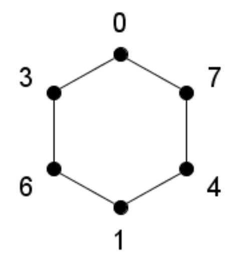

Figure 2.1: An optimal radio labeling for a cycle on 6 vertices. Observe that $6 \equiv 2 \bmod 4$ so $\phi = 3$.

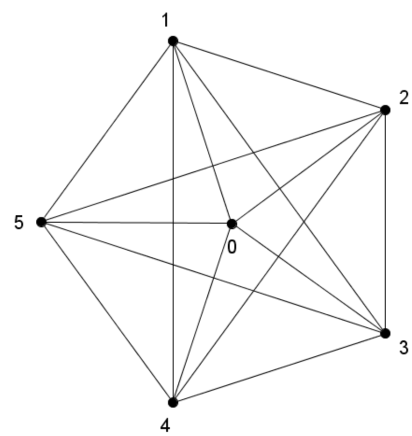

Figure 2.2: An optimal radio labeling of K_6 with maximum label $n - 1$.

2.1.2 $L(2,1)$ Labelings

A simple analog to the standard radio channel assignment problem is the $L(2,1)$ labeling. If f is an $L(2,1)$ labeling, then:

$$|f(u) - f(v)| \geq \begin{cases} 2, & \text{if } u \text{ and } v \text{ are adjacent;} \\ 1, & \text{if } d(u,v)\text{=2.} \end{cases}$$

There are no restrictions on the labels of u and v for $d(u,v) \geq 3$ as there would be for a radio labeling. This labeling scheme was first introduced in [7]. For a given graph G, the minimum possible span of any $L(2,1)$ labeling, $\lambda_{2,1}(G)$, is defined as follows.

$$\lambda_{2,1}(G) = min(\sigma(f))$$

If a graph can admit an $L(2,1)$ labeling of span k, then the graph is said to be *k-labelable*. Note that G is k-labelable for all $k \geq \lambda_{2,1}(G)$. It is important to keep in mind that a k-labeling of G is not necessarily an optimal labeling.

Lemma 2.1.2. *If $diam(G) = 2$, then $\lambda_{2,1}(G) = rn(G)$.*

Proof. Recall the radio condition of a graph such that $diam(G) = 2$:

$$|f(u) - f(v)| \geq 3 - d(u,v).$$

If u and v are adjacent, then $d(u,v) = 1$ and $|f(u) - f(v)| \geq 2$. If u and v are distance two, then $d(u,v) = 2$ and $|f(u) - f(v)| \geq 1$. These statements plainly define the necessary constraints of an $L(2,1)$ labeling. □

Previous results

We once again consider the classes of complete graphs, cycles and paths. Determining $\lambda_{2,1}(K_n)$, is a relatively simple feat. All vertices are adjacent to each other and must therefore receive labels that differ by two. This leads to the elementary property:

$$\lambda_{2,1}(K_n) = 2(n-1).$$

Paths Consider the first four paths: P_1, P_2, P_3, and P_4. It is not difficult to determine that $\lambda_{2,1}(P_1) = 0$, $\lambda_{2,1}(P_2) = 2$ and $\lambda_{2,1}(P_3) = \lambda_{2,1}(P_4) = 3$. In [7], it was shown that for $n \geq 5$,

$$\lambda_{2,1}(P_n) = 4.$$

The proof for this relies heavily on the following lemma.

Lemma 2.1.3. *For a given graph G and a subgraph H of G, $\lambda(G) \geq \lambda(H)$.*

Proof. Assume that there exists a graph G and a subgraph H such that $\lambda(G) < \lambda(H)$. Let $\lambda(G) = k$. Then there exists a proper k-labeling of G. Deleting any vertices in G preserves the feasibility of this labeling. If we delete all the vertices in $(G \cap H)^C$, then what remains is a feasible k-labeling of H, contradicting the initial assumption. □

Note that this result holds regardless of which labeling scheme is used.

Cycles First observe that the cycle C_n has P_n as a subgraph, so as a consequence of Lemma 2.1.3, $\lambda_{2,1}(C_n) \geq 4$. A construction that attains this bound is presented in [7] and is given below:

Let $V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\}$, and $E = \{(v_i, v_j) | j = i + 1 \pmod{n}\}$.

If $n \equiv 0 \bmod 3$,

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 0 \bmod 3; \\ 2, & \text{if } i \equiv 1 \bmod 3; \\ 4, & \text{if } i \equiv 2 \bmod 3. \end{cases}$$

If $n \equiv 1 \bmod 3$ then this same labeling scheme applies, with the exception of vertices in the set $\{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\}$, which are redefined as follows

$$f(v_i) = \begin{cases} 0, & \text{if } i = n-3; \\ 3, & \text{if } i = n-3; \\ 1, & \text{if } i = n-2; \\ 4, & \text{if } i = n-1. \end{cases}$$

If $n \equiv 2 \bmod 3$ then the only vertices that need new labels are v_{n-2} and v_{n-1}. Redefine these as follows:

$$f(v_i) = \begin{cases} 1, & \text{if } i = n-2; \\ 3, & \text{if } i = n-1. \end{cases}$$

To see an example of this labeling scheme, refer to Figure 2.3 below.

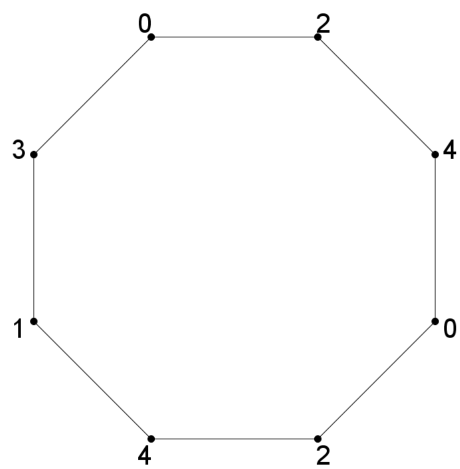

Figure 2.3: An optimal $L(2,1)$ labeling of C_8 with maximum label 4.

There are also results on many different families of graphs, some of which are mentioned below [2].

- Wheels, W_n
- Infinite lattices with $\Delta = 3, 4, 6, 8$
- Cartesian products of paths, cycles, complete graphs
- Planar graphs
- K_4-minor free graphs

- Hypercubes
- Trees

Note that the $L(2,1)$ labeling problem is not completely solved for all the above families of graphs. Likewise, there are many more specific families with established results that are not on this list. The reader is referred to [2] for a more complete list of results.

2.1.3 $L(h,k)$ Labelings

A generalization of the $L(2,1)$ labeling problem is the $L(h,k)$ labeling problem. It is defined as follows:

$$|f(u) - f(v)| \geq \begin{cases} h, & \text{if } u \text{ and } v \text{ are adjacent;} \\ k, & \text{if } d(u,v)\text{=2.} \end{cases}$$

Likewise, the minimum span of an $L(h,k)$ labeling is $\lambda_{h,k}(G)$. For an extensive overview on results regarding the $L(h,k)$ labeling problem, refer to [2], the highlights of which are presented here.

$L(h,0)$

The $L(1,0)$ problem has been extensively studied, as this problem is analogous to graph coloring. For a more rigorous definition of a coloring, refer to section 2.2.1. The only difference between coloring a graph and applying an $L(1,0)$ labeling is that the minimum label in a coloring is one, whereas $L(1,0)$ labelings permit zero as a label. For this reason, $\lambda_{1,0}(G) = \chi(G) - 1$, where $\chi(G)$ is the chromatic number of G. In fact, $\lambda_{h,0}(G) = h \cdot \chi(G) - 1$, as this problem is isomorphic to vertex coloring with a required difference of h between adjacent vertices.

$L(1,1)$

The *square* of a graph G, denoted G^2, is formed by adding an edge $e = (u,v)$ between all vertices u and v if $d(u,v) = 2$. With this in mind, it is not hard to see that $\lambda_{1,1}(G) = \chi(G^2) - 1$.

2.1.4 $L'(2,1)$ Labelings

An $L'(2,1)$ labeling is an $L(2,1)$ labeling with the additional constraint that the function must be one-to-one. Thus, $\lambda'_{2,1}(G)$ is the minimum one-to-one $(2,1)$-labeling admitted by G. Since the labeling begins at zero, we may say that $\lambda'_{2,1}(G) \geq n-1$. A graph that admits an $L'(2,1)$ labeling with $\sigma(f) = n-1$ is said to be *perfectly labelable*.

2.2 Möbius Ladders

Originally introduced in 1966 by Richard Guy and Frank Harary [9], the Möbius ladder on n vertices, denoted M_n, is constructed by connecting vertices u and v in the cycle C_n if $d(u,v) = diam(C_n)$. Vertices that satisfy this condition are *antipodal*. Looking at the left representation of M_{16} in Figure 2.4, it is easy to see why this family is called the Möbius ladders.

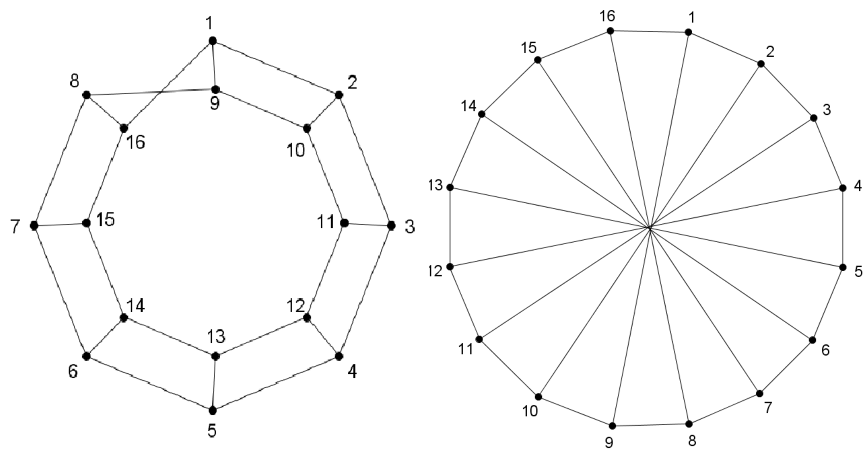

Figure 2.4: An example of the two representations of M_{16} with vertex labels indicating the associated isomorphism.

Although the original paper by Guy and Harary defined Möbius Ladders for all natural numbers $n \geq 5$, many authors only consider n even [1, 12, 14]. The two cases (n even or odd) change the structure of these graphs quite radically. For this reason, the main result of the paper shall deal with them separately. Unless otherwise stated, assume that the vertex set of M_n is labeled clockwise about the outer cycle, as in Figure 2.4.

2.2.1 Basic Properties

As evidenced by the number of publications, there are many more established results for even Möbius ladders than for odd. They are three-regular and Hamiltonian. Thus the edge set consists of a spanning cycle and a one-factor. Clearly, for n even, $|E(M_n)| = \frac{3n}{2}$ by the handshaking lemma. The ladder shape consists of $\frac{n}{2}$ contiguous 4 cycles in a prism-like arrangement. In one of these 4-cycles, a "twist" is introduced to create the topological Möbius strip.

When n is odd, M_n is four-regular and therefore two-factorable. The graph is composed of n contiguous triangles rather than the contiguous 4-cycles used to create the rungs of the even ladder. Likewise, there is one triangle that is "twisted". An example of an odd order Möbius ladder is given in Figure 2.5. In this figure, the twist is introduced at the bottom, between vertex 1 and vertex 17. By the handshaking lemma, $|E(M_n)| = 2n$ for n odd.

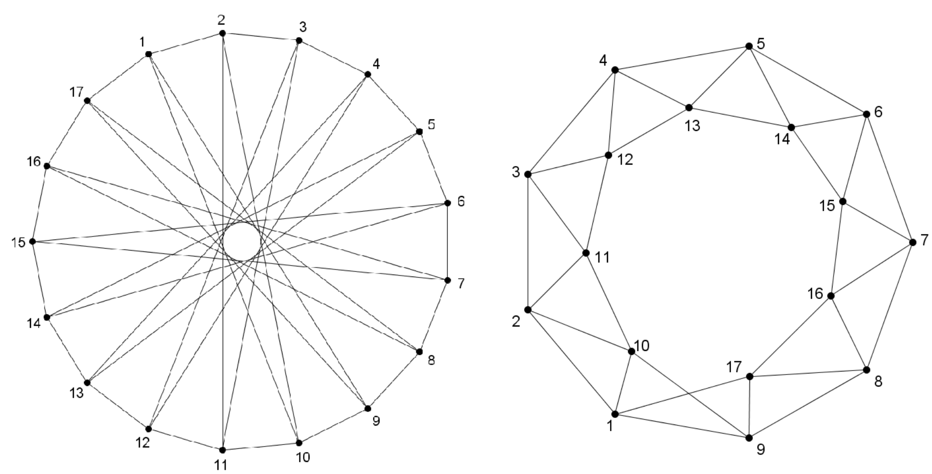

Figure 2.5: The two main representations of an odd Möbius ladder, M_{17}. The vertex labels indicate an isomorphism from one to the other.

Crossing Number

The *crossing number* of a graph, denoted *cr(G)*, is the minimum number of edge crossings in any plane drawing of G. The main result of [9] is that

$$cr(M_n) = 1$$

for all $n \geq 5$. In this sense, the Möbius ladders are minimally nonplanar. Guy and Harary even considered these graphs as a possible extension to Kuratowski Graphs, since K_5 and $K_{3,3}$ are isomorphic to M_5 and M_6, respectively.

Diameter and Distance

Radio labelings rely heavily on the notion of a graph's diameter. If $n = 2r$ then diam(M_n)=$\lceil \frac{r}{2} \rceil$. That is,

$$diam(M_n) = \begin{cases} \frac{r}{2}, & \text{if } r \text{ is even;} \\ \frac{r+1}{2}, & \text{if } r \text{ is odd;} \end{cases}$$

This same formula applies to the odd case, or for $n = 2r + 1$.

Chromatic Number

A *vertex coloring* of a graph is a function which assigns colors to the vertices such that no adjacent vertices receive the same color. In practice, the colors are simply the integers $\{1, 2, \ldots, k\}$. Any vertex coloring that uses k distinct colors is also referred to as a k-coloring. Given a graph G, the *chromatic number*, $\chi(G)$, is the minimum number of colors k such that G admits a k-coloring. In order to determine the chromatic number of a Möbius ladder, we require the following theorems:

Theorem (Brooks'): For a graph G with maximum degree Δ, $\chi(G) \leq \Delta$, unless G is an odd cycle or a complete graph.

Theorem: G is 2-colorable if and only if G contains no odd cycles.

With these preliminaries established we can now determine the chromatic number for Möbius ladders.

For an even Möbius ladder, M_n (with $n = 2r$)

$$\chi(M_n) = \begin{cases} 2, & \text{if } r \text{ is odd;} \\ 3, & \text{if } r \text{ is even.} \end{cases}$$

Proof. Consider the cycles in an even Möbius ladder. All the rungs create contiguous 4-cycles, and the outside (Hamiltonian) cycle has size $2r$. Note that the construction of a Möbius ladder begins with an n-cycle (which is the Hamiltonian cycle that we just mentioned), and joins antipodal vertices with an edge. This edge cuts the original Hamiltonian cycle in half, creating an interior cycle of size $\frac{n}{2} + 1 = r + 1$. Therefore, if r is odd, then there are no odd cycles, and M_n is 2-colorable. However, if r is even then there is an odd cycle and a third color must be introduced.

Note that it is possible to 3-color M_n as a result of Brook's Theorem, and thus, this coloring is optimal for r even. □

When dealing with odd Möbius ladders, the chromatic number is slightly higher.

$$\chi(M_n) = \begin{cases} 3, & \text{if } n \equiv 3 \bmod 6; \\ 4, & \text{otherwise.} \end{cases}$$

Proof. Note that any odd Möbius Ladder can be 4-colored as a result of Brook's Theorem. It suffices to show that M_n can admit a 3-coloring only when n is congruent to 3 modulo 6.

We begin by assuming that odd Möbius graphs are tripartite and show for which classes this is true. Thus we assume that we may label the nodes of any odd Möbius graph with three distinct labels.

By the structure of the odd Möbius ladder, it is clear that labeling any two adjacent vertices determines the labels of the vertices that are within the same 3 cycle.

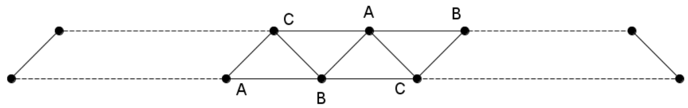

Figure 2.6: Arbitrary labeling of two vertices, A and B, which determine the immediate neighbors of both A and B.

Without loss of generality, label the two left-most adjacent vertices and the bottom right-most vertex of the ladder as in the following figure.

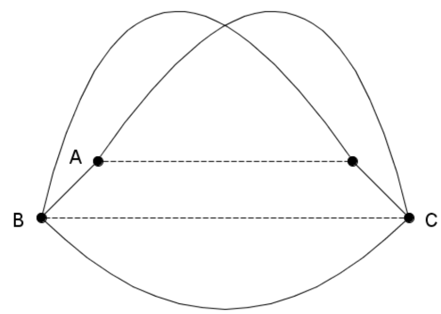

Figure 2.7: Determination of the arbitrary end vertices of a Möbius ladder which influence the number of allowable intermediary vertices.

Now, because of the deterministic nature of the labeling forced by the structure of the ladder, the only way to maintain feasibility is by introducing $3h$ vertices between the vertices which we initially labeled, where h is any nonnegative integer. This brings our total number of vertices to $3 + 3h = 3(h + 1)$.

Since we are working with odd ladders, we require that the above product also be odd. That is, $3(h + 1)$ must contain no even factors. Since 3 is odd, we require that

$$h + 1 = 2m + 1 \implies h = 2m.$$

Looking back at our original equation for the number of vertices in this set, we have $3 + 6m$. In other words, we have obtained uniquely the class of odd Möbius ladders with $n = 3 \mod 6$. $\square$

Chromatic Polynomials

Consider the following question: Given a graph G, and k colors, how many different ways are there to k-color G? This question can be answered by the *chromatic polynomial* of G. Let p_k be the number of ways to k-color G. Then the chromatic polynomial, $P_G(k)$ is defined to be the n^{th} degree interpolating polynomial of the points (k, p_k), for $k = 0, 1, \ldots, n$.

Note that if $k < \chi(G)$, then $P_G(k) = p_k = 0$.

For an even Möbius ladder, Biggs et al [1] showed that,

$$P_{M_n}(k) = (k^2 - 3k + 3)^n + (k-1)[(3-k)^n + (1-k)^n] - 1.$$

Circulant Matrices

For a graph of order n, the adjacency matrix A is an nxn matrix satisfying the following:

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E(G); \\ 0, & \text{otherwise} \end{cases}$$

A circulant matrix is a matrix such that $a_{ij} = a_{i-1,j-1}$, where subtraction is done modulo n. In this sense, a circulant matrix is entirely defined by its first column. A circulant graph, is simply a graph where the adjacency matrix is circulant. The Möbius ladders are circulant graphs. The first column will always be given by the vector $[b_1, b_2, \ldots, b_n]^T$, with

$$b_i = \begin{cases} 1, & \text{if } i \in \{2, \frac{n}{2}+1, n\} \text{ and } n \text{ even}; \\ 1, & \text{if } i \in \{2, \frac{n+1}{2}, \frac{n+3}{2}, n\} \text{ and } n \text{ odd}; \\ 0, & \text{otherwise.} \end{cases}$$

One important feature of circulant graphs, is their rotational symmetry. A Möbius ladder can be drawn with vertices on a regular n-gon, and the symmetry group of the n-gon also describes M_n.

Chapter 3

Results

As previously mentioned, the parity of n greatly affects the structure of M_n. Thus, the even and odd cases must be dealt with separately.

3.1 Even Case

It has been shown that 3-regular, Hamiltonian graphs have a $\lambda_{(2,1)}$ labeling between 5 and 9, inclusive [5, 11]. In the case of the Möbius ladders, these bounds can be improved from above and below.

Lemma 3.1.1. *The lower bound for an $L(2,1)$ labeling on any even Möbius graph, M_n, is 6.*

Proof. If it can be shown that a distance-two neighborhood about a given vertex a of M_n has a minimal $L(2,1)$ labeling 6, then $\lambda_{2,1}(M_n)$ is at least 6, by Lemma 2.1.3.

Any given vertex in M_n has three vertices distance one away and four vertices distance two away. For simplicity, label a zero. Note that we are referring to our vertices in the same manner as described in Section 2.2, with the vertices labeled clockwise about the outer cycle. This implies that vertex a will be adjacent to vertices $a+1$, $a-1$ and $a+\frac{n}{2}$.

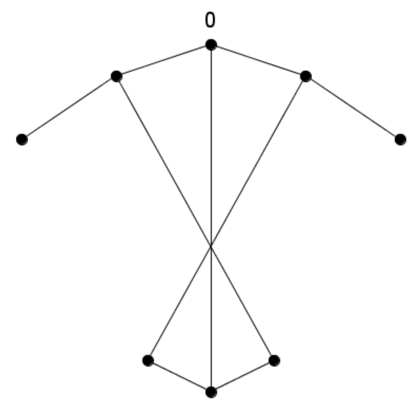

Figure 3.1: Closed distance-two neighborhood of an arbitrary vertex (labeled zero) in an even Möbius ladder.

There cannot be four vertices with the same label among this set, as no four labels are all distance three from each other. So suppose three of these vertices acquire the same label. Then there must exist a one-to-one mapping from the remaining vertices onto some four-element subset of the set {1,2,3,4,5}. It is also true that each of these three points must be at least distance three from each other. The only three points that satisfy this constraint are the vertices $a+2, a-2$, and $a+\frac{n}{2}$.

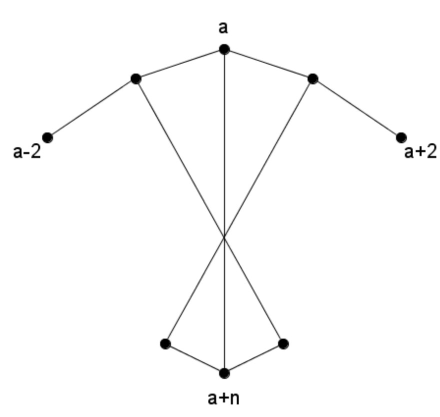

Figure 3.2: The only three vertices of the distance-two neighborhood in Lemma 3.1.1 which are all distance-three from each other.

This collection forms a dominating set of the deleted neighborhood about 0.

Then there is at least one element of the set {1,2,3,4,5} not admitted by M_n. In this case, $\lambda_{(2,1)}(M_n) \geq 6$.

Now suppose M_n admits one pair of vertices with a given label and another distinct pair of vertices with a different label. The nine possible pairs of vertices are $\{a+\frac{n}{2}+1, a-1\}$, $\{a+\frac{n}{2}+1, a-2\}$, $\{a-2, a+2\}$, $\{a-2, a+1\}$, $\{a-1, a+2\}$, $\{a+\frac{n}{2}, a-2\}$, $\{a+\frac{n}{2}, a+2\}$, $\{a+\frac{n}{2}-1, a+1\}$, and$\{a+\frac{n}{2}-1, a+2\}$. Any two pairs that together form a dominating set of the deleted neighborhood about 0 will result in the previous argument, forcing $\lambda(M_n) \geq 6$. The only sets of pairs that don't form this dominating set are $\{a+\frac{n}{2}, a-2\}$ and $\{a+\frac{n}{2}+1, a-1\}$, $\{a+\frac{n}{2}-1, a+1\}$ and $\{a+\frac{n}{2}, a-2\}$, and $\{a-1, a+2\}$ and $\{a-2, a+1\}$. Since $\{(a+\frac{n}{2}, a-2), (a+\frac{n}{2}+1, a-1)\}$ and $\{(a+\frac{n}{2}-1, a+1), (a+\frac{n}{2}, a-2)\}$ are symmetrically equivalent, we need only consider one of these pairs.

We first examine the vertex pairs $\{a+\frac{n}{-}1, a+1\}$ and $\{a+\frac{n}{2}, a-2\}$ and show that $\lambda_(2,1) \geq 6$ if $\{a+\frac{n}{2}-1, a+1\}$ and $\{a+\frac{n}{2}, a-2\}$ receive the same labels.

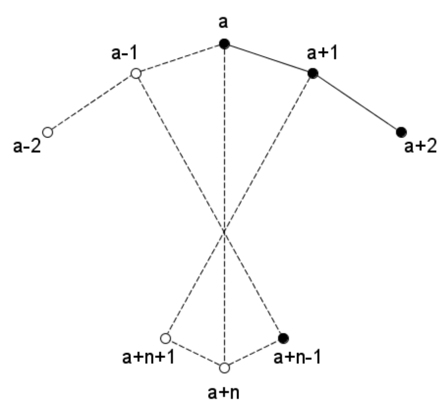

Figure 3.3: The vertex pairs $a+\frac{n}{2}, a-2$ and $a+\frac{n}{2}-1, a+1$ and neighboring vertices. Dashed lines indicate edges incident to at least one of the vertices in the vertex pairs. White nodes are the chosen vertex pairs.

We begin by looking at the relationship between the vertices $a+\frac{n}{2}, a+\frac{n}{2}-1$, and $a-1$.

$$|f(a+\frac{n}{2}) - 1 - f(a+\frac{n}{2})| \geq 2;$$

$$|f(a+\frac{n}{2}-1) - f(a-1)| \geq 2;$$

and because $f(a+\frac{n}{2}+1) = f(a-1))$,

$$|f(a+\frac{n}{2}) - f(a-1)| \geq 2.$$

Thus, the difference between the greatest and least values of these vertices is at least 4. Since $f(a) = 0$ and a is the only vertex to receive that label, it must be true that the lowest possible value of any one of the remaining vertices is 1, raising the lower bound here to 5. Since we require that this neighborhood receive a labeling of no greater than 5, one of the vertices $a + \frac{n}{2}$, $a - 1$, and $a + \frac{n}{-}1$ must receive a label of 1. The only vertex that may receive this label while maintaining a feasible labeling is $a + \frac{n}{2} - 1$. The remaining vertices $a + \frac{n}{2}$ and $a - 1$ must receive labels 3 and 5. That leaves labels 2 and 4 to be split among $a + 1$ and $a + 2$. Figure 3.4 illustrates the possible labelings.

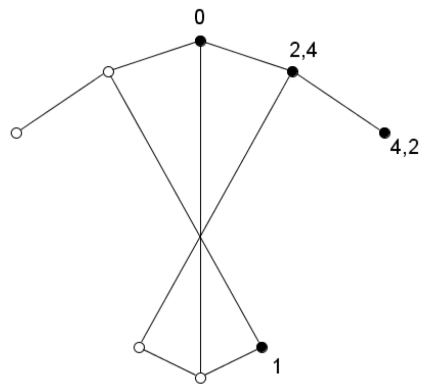

Figure 3.4: Partial feasible labelings of a distance-two neighborhood of an arbitrary vertex of M_n.

Regardless of the label $a + 1$ receives, $a + \frac{n}{2} + 1$ may not receive the label 3. Thus, the labeling is determined as in Figure 3.5:

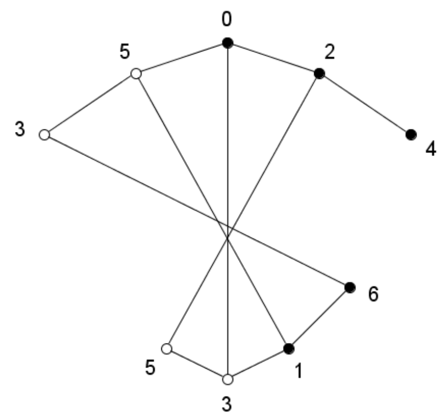

Figure 3.5: Contradiction forced by the construction of the labeling.

The labeling produced on the original distance-two neighborhood of a is infeasible if vertices $a-2$ and $a+2$ are joined to form M_8. Thus, we introduce the vertex antipodal to $a-2$ and see that the smallest possible label it may receive is 6. Thus, this case does not admit a 5-labeling. As previously stated, this precludes the symmetric case from being 5-labeled.

The only remaining case in which the distance-two neighborhood about a may be 5-labeled is when the vertex pairs $a-1, a+2$ and $a-2, a+1$ are chosen. First, note that the only possible vertices that may receive the label 1 are $a+\frac{n}{+}1$ and $a+\frac{n}{-}1$. Without loss of generality, suppose $f(a+\frac{n}{+}1)=1$. Figure 3.6 depicts the labeling scheme determined by this choice of labels.

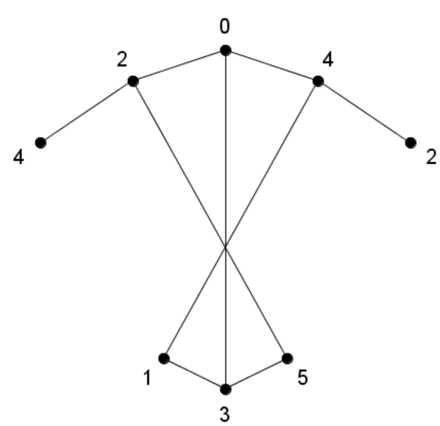

Figure 3.6: Optimal labeling of a distance-two neighborhood of a Möbius ladder.

Figure 3.6 cannot be connected while maintaining a feasible labeling, so we must introduce two more vertices and label them accordingly. It is important to note that the vertex labeled $(0, 1)$ may only receive the label 1. This is true because we have thus far determined antipodal vertex labels. Put simply, the label of the vertex opposite the vertex labeled 0 is 3, the label opposite 1 is 4, and the label opposite 2 is 5. The following figure demonstrates the inclusion of a single pair of vertices that retain feasibiity of the labeling. The objective is to be able to connect the graph while never using a label above 5.

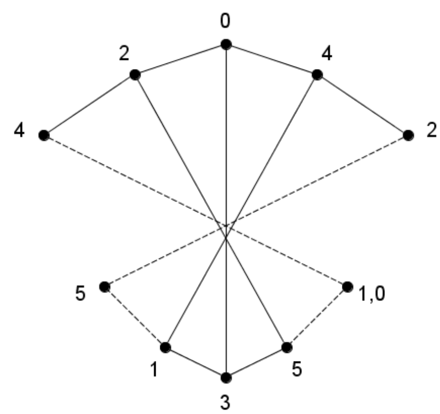

Figure 3.7: The extended neighborhood about a, where even labels must be on the top, and odd labels on the bottom.

The restriction posed by the five labeling happens to fully determine the labels of newly introduced vertices. A pattern develops after enough new vertices have been added, and it is left to the reader to confirm that the top half of the constructed graph contains only even labels, while the bottom contains only odd. Note that no choice of new vertices has the potential to ever produce a subgraph that may be connected while retaining feasibility. $\square$

Lemma 3.1.2. *Given a Mobius ladder M_n, if $n \equiv 4 \bmod 8$ then $\lambda(M_n) = 6$.*

Proof. Note that the lower bound of six was established in Lemma 3.1.1, so any labeling that achieves this bound must be optimal. This can be done using the following algorithm.
Starting at any vertex v_i with $f(v_i) = 0$, then label along the outer cycle such that $f(v_{i+1}) = (f(v_i)+2) \bmod 8$. This recursive formula generates the sequence (0,2,4,6).
If $n \equiv 4 \bmod 8$, then clearly, $n = 4k$ for some k odd. Therefore, the sequence (0,2,4,6) which consists of four elements, is repeated exactly k times. Because k is odd, the adjacent vertices v_i and $v_{i+\frac{n}{2}}$ which lie across from each other on the cycle receive labels that always differ by four. An alternative way to

see this, is given $n \equiv 4 \bmod 8$ then $\frac{n}{2} \equiv 2 \bmod 4$. And since this scheme essentially determines labels according to the vertex indices modulo 4, then $|f(v(i)) - f(v(i + \frac{n}{2})| = 4$.
In order for this to be a proper $L(2,1)$ labeling, then both constraints must be satisfied. Note that the adjacency constraint must be satisfied, as
$|f(v(i)) - f(v_(i \pm 1))| = 2$, and $|f(v(i)) - f(v(i + \frac{n}{2}))| = 4$.
For the distance two constraint, since the difference between all the labels is at least two, it suffices to show that no vertices within distance two of $v(i)$ receive the same label $f(v(i))$. The four vertices that are distance two from $v(i)$ are $v(i \pm 2)$ and $v(i + \frac{n}{2} \pm 1)$. It is clear to see that the two vertices $v(i \pm 2)$ receive the values $f(v(i)) \pm 4$ modulo 8. Further, because $f(v(i + \frac{n}{2})) = f(v(i)) + 4 \bmod 8$, then it implies that $f(v(i + \frac{n}{2} \pm 1)) = f(v(i)) + 4 \pm 2 \bmod 8$. Therefore, no label that is distance two from $v(i)$ receives the same label $f(v(i))$, so the distance two constraint is also satisfied. □

This construction leads to the following theorem.

Theorem 3.1.3. $\lambda(M_n) = 6$, *for all even* n $\neq 8$.

Proof. Given a Möbius ladder, M_n with $n \equiv 4$ mod 8, an L(2,1) labeling of six or less is possible (as shown in the previous Lemma). This construction enables us to insert a "rung" of the ladder, consisting of two new vertices, such that the 6-labeling is preserved.
Between each pair of adjacent vertices u, v such that f(u)=0 and f(v)=2, a new vertex w can be inserted such that f(w)=5. Likewise, in between each pair of vertices u' and v' such that f(u')=4 and f(v')=6, a new vertex w' can be inserted such that f(w')=1, and w' is adjacent to w. For a concrete example, refer to Figure 3.8.
There will be exactly $\frac{n}{4}$ such pairs of vertices that can be inserted, that preserve the L(2,1) labeling. This follows from the fact that n is a multiple of four, and there is exactly one "rung" that can be inserted in the ladder for every four cycle of (0,2,6,4).

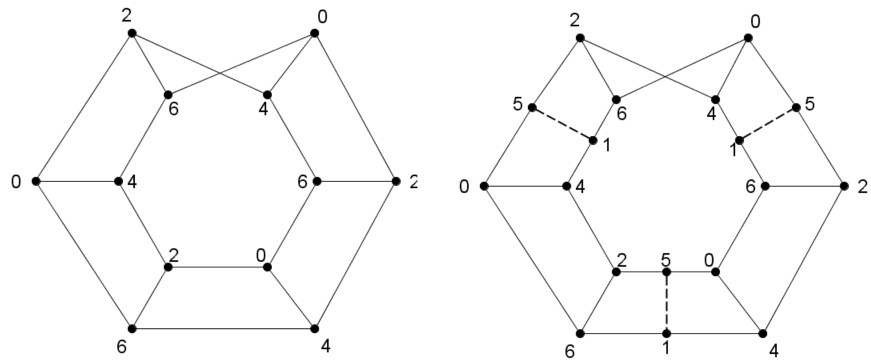

Figure 3.8: M_{12} labeled by the algorithm in Lemma 3.1.2, and then the additional rungs (dotted edges) which can be inserted to get to M_{14}, M_{16}, and M_{18}.

Therefore, because M_n with $n \equiv 4 \bmod 8$ has a proper $L(2,1)$ labeling equal to six, so does $M_{n+\frac{n}{4}}$, and all even Möbius ladders in between. Note that this does not cover the three smallest cases, as the first defined Möbius ladder with $n \equiv 4 \bmod 8$ is M_{12}. So it remains to be shown the 6-labelings exist for M_6, and M_{10}. The reader is referred to Figure 3.9 to see an example of such a labeling. Note that because $diam(M_8) = 2$, it cannot be properly labeled with fewer than $n - 1 = 7$ labels as a consequence of Lemmas 2.1.1 and 2.1.2. M_8 is the only even Möbius ladder that requires more than six labels. We later prove that M_8 can be 7-labeled, as a consequence of Theorem 3.4.1. Therefore, $\lambda(M_n) \leq 6$, for all $n \neq 8$. $\square$

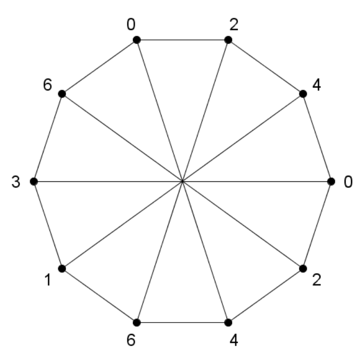

Figure 3.9: An optimal six labeling of M_{10}

3.2 Odd Case

3.2.1 Preliminaries

When trying to determine $\lambda_{2,1}(G)$, one potentially useful parameter is the total number of vertices in G that may receive any single label in a proper $L(2,1)$ labeling. This parameter will be denoted $\alpha(G)$, and is formally defined by:

$$\alpha(G) = max|S_j|$$

Where S_j is a label set with cardinality maximized over all $L(2,1)$ labelings.

Theorem 3.2.1. *For n odd, $\alpha(M_n) = \lfloor \frac{n}{5} \rfloor$.*

Proof. Each vertex $v \in V(M_n)$ has a closed neighborhood of size five. If $n \equiv 5 \bmod 10$ then $n = 5k$. In this case, we can decompose M_n into sets $V_1, V_2, \ldots, V_k$ such that $\bigcup_{j=1}^{k} V_j = V(M_n)$ and $|V_j| = 5$ for all $j \in \{1, 2, \ldots, k\}$ and such that each set V_j can be expressed as the closed neighborhood of a vertex v_j. Note that each v_j must be distance three from each other, because their closed neighborhoods are all disjoint. Therefore, each v_j may be assigned the same label, and $\alpha(M_n) = k = \lfloor \frac{n}{5} \rfloor$.
If there are additional vertices, (say $n = 5k+r$, with $r < 5$), we may decompose M_n into k sets of five, and one set of r. The k sets of five may once again be expressed as the closed neighborhoods of specific vertices within those sets. However, the set containing r elements is not large enough to be a closed neighborhood on its own, and therefore each vertex must be distance two from one of the surrounding v_j. Due to this proximity, no vertex in the r-set can receive the label l. Therefore, $\alpha(M_n) = \frac{n-r}{5} = \lfloor \frac{n}{5} \rfloor$. □

This concept is a useful tool for providing initial lower bounds. If no more than $\alpha(G)$ vertices can receive a single label, then $\lambda_{2,1}(G) \geq \frac{n}{\alpha(G)}$. In the case of Möbius Ladders, we can construct this lower bound as follows:

$$\alpha(M_n) = \lfloor \frac{n}{5} \rfloor \leq \frac{n}{5}$$

And thus,

$$\lambda_{2,1}(M_n) \geq \frac{n}{(\frac{n}{5})} = 5.$$

This bound is not very helpful, as [5] shows that an r-regular graph requires at least $r+2$ labels for a proper L(2,1) labeling. The four-regularity of odd Möbius Ladders implies that $\lambda_{2,1}(M_n) \geq 6$. As the following theorem demonstrates, this bound is achievable.

Theorem 3.2.2. *If $n \equiv 7 \bmod 14$, then $\lambda(M_n) = 6$.*

Proof. It suffices to show that a six labeling exists. Consider the following construction. The graph H on seven vertices, and it's corresponding labeling is given in Figure 3.10.

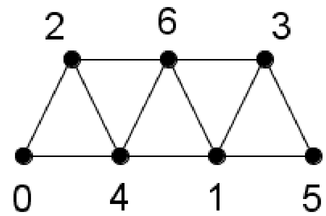

Figure 3.10: The base graph H on 7 vertices which the construction in Theorem 3.2.2 is based upon.

Note that with the introduction of three edges [(0,5), (2,5), (0,3)], this would become the Möbius ladder M_7. Further, after the introduction of these edges the L(2,1) labeling is still feasible. This implies that H can be catenated with itself k times (where k is odd) to create a graph on $7k$ vertices. Also note that the endpoints will be the same, and by introducting the same three edges, you have a 6 labeling of M_{7k}. Refer to Figure 3.11 to see how this construction yields a 6 labeling of M_{21}.

The aforementioned construction can be simply expressed as follows: Given the standard clockwise labeling of the Möbius ladder's vertices, $V = \{v_1, v_2, \ldots, v_n\}$, the label function is given by:

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1 \bmod 7; \\ 4, & \text{if } i \equiv 2 \bmod 7; \\ 1, & \text{if } i \equiv 3 \bmod 7; \\ 5, & \text{if } i \equiv 4 \bmod 7; \\ 2, & \text{if } i \equiv 5 \bmod 7; \\ 6, & \text{if } i \equiv 6 \bmod 7; \\ 3, & \text{if } i \equiv 0 \bmod 7 \end{cases}$$

□

This construction contains a nice property. In Section 2.2.1 it was noted that odd Möbius ladders are 2-factorable. Looking at the construction of M_{21} in Figure 3.11, we notice that this graph can be decomposed into two disjoint spanning cycles that reflect the two constraints of an L(2,1) labeling. That is, along one cycle C_1 the distance one constraint is held tight. For any two adjacent vertices u and v, $f(u) = (f(v) + 2) \bmod 7$.
Along the other cycle C_2, the distance two constraint is held tight. Any vertices u and v that are distance two in this cycle receive labels such that
$f(u) = (f(v) + 1) \bmod 7$. For instance, a vertex with the label of 6 will always be distance two from a vertex labeled 0. For a visual aid, refer to Figure 3.12.

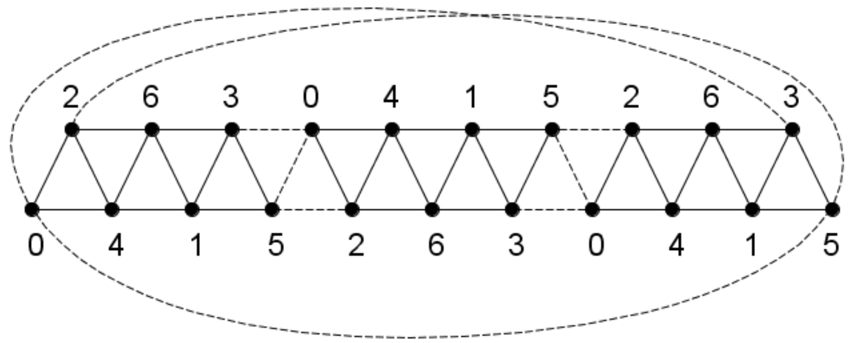

Figure 3.11: An optimal L(2,1) labeling of M_{21} according to the construction in Theorem 3.2.2. The solid edges denote the 3 multiples of H. The dotted edges are those that must be introduced to catenate.

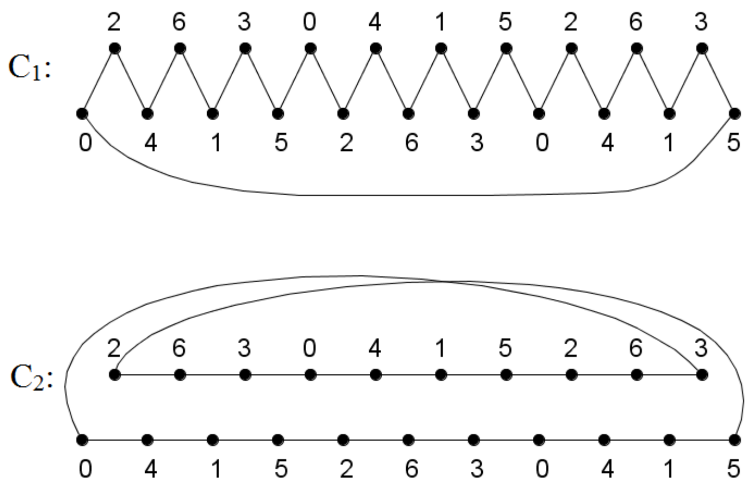

Figure 3.12: The two spanning cycles of M_{21}. C_1 corresponds to the distance one constraint and C_2 to the distance two constraint.

3.2.2 Upper Bounds

The best established upper bound for a graph with maximum degree Δ is given by Gonçalves in [6]. That is, $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$. For the odd Möbius ladders this implies that $\lambda_{2,1}(M_n) \leq 18$. The intent of the following is to improve this upper bound to seven.

Lemma 3.2.3. *The least upper bound for an $L(2,1)$ labeling on any Möbius graph, M_n, for n odd and greater than or equal to 19, is 7.*

Proof. We separate the graphs into three distinct classes, namely their order modulo six. It is important to note that each case uses the same base labeling of 7 vertices, which is given in Figure 3.10. The first case to be considered is $n \equiv 1 \bmod 6$. The idea was to find a set of 6 vertices whose labeling may be held constant and which, by indefinite appending to the base labeling, retains feasibility for odd Möbius ladders. The following figure illustrates the suggested labeling and how this set of six vertices may be affixed to the base set indefinitely.

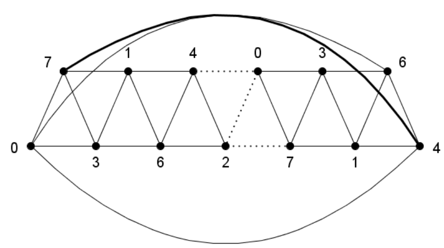

Figure 3.13: The base labeling used to establish the 1 mod 6 case.

The second natural case is $n \equiv 3 \bmod 6$, and the approach is nearly identical. It begins by introducing a slight modification of the same base labeling adjoined to the first by a single vertex. The following figure illustrates this idea.

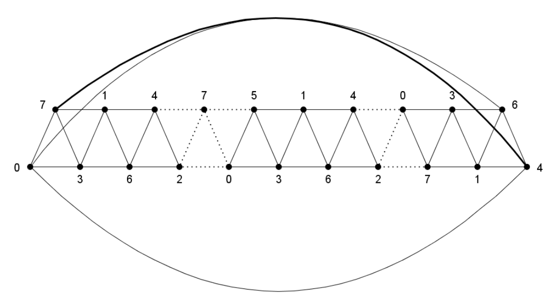

Figure 3.14: The base labeling used to establish the 3 mod 6 case.

The same set of 6 vertices established in the initial case is used in the same manner here, thus establishing the upper bound for this case.

The final case is $n \equiv 5 \mod 6$. The beginning base is increased to 23, and the method of demonstration is identical to the second case.

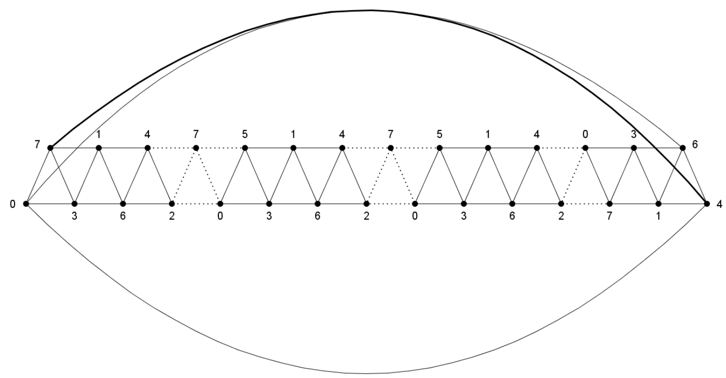

Figure 3.15: The base labeling used to establish the 5 mod 6 case.

By broadening the base for each case, some smaller Möbius graphs are inevitably overlooked. These cases are $n = 5$, 9, 11, and 17. The arguments for the optimal labelings of these graphs are laid out in the following section. $\square$

As a corollary of Lemma 3.2.3, $\lambda(M_n)$ with n odd must either be six or seven.

The only case we have found where an odd Möbius ladder can be six labeled, is when $n \equiv 7 \bmod 14$. It is conjectured that for n odd, $\lambda(M_n) = 6$ if and only if $n \equiv 7 \bmod 14$.

3.2.3 Remaining Cases

There are small cases that the proof of Lemma 3.2.3 does not resolve. These are the three smallest cases when $n \equiv 5 \bmod 6$. Determining $\lambda_{2,1}(M_5)$ is trivial, as it is isomorphic to the complete graph K_5.

Lemma 3.2.4. *When* n*=11,* $\lambda(M_n) \geq 8$.

Proof. We proceed by contradiction. First, assume that M_{11} admits a 7 labeling. Namely, there exist 8 disjoint 'label' sets S_0, S_1, … ,S_7 such that $\bigcup_{j=0}^{7} S_j = V(M_{11})$.

It is clear that for any vertex v_i, there exists uniquely (up to symmetry) a vertex $v_{\bar{i}}$, such that $d(v_i, v_{\bar{i}}) = 3$. That is to say, there is only one vertex (up to symmetry) that can receive the same label as v_i in M_{11}.

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ denote the closed neighborhoods of vertices v_i and $v_{\bar{i}}$, respectively. Note that $\mathcal{N}_1$ and $\mathcal{N}_2$ are disjoint sets with $|\mathcal{N}_1| = |\mathcal{N}_2| = 5$. So clearly, the subgraph induced by the vertices $V \backslash (\mathcal{N}_1 \cup \mathcal{N}_2)$ is the graph consisting of a single vertex. In other terms, there is only one vertex that is distance 2 away from both v_i and $v_{\bar{i}}$. This implies that if S_j contains two vertices, then there can be a maximum of one vertex in the union of the sets S_{j+1} and S_{j-1}.

Now by the pigeonhole principle, if 8 labels are to be applied to 11 vertices, there must be at least 3 sets that contain 2 vertices. This follows directly from the fact that $\alpha(M_{11}) = \lfloor 11/5 \rfloor = 2$, and thus any set can have at most 2 vertices, by definition of $\alpha(M_n)$.

Let F be a valid 7 labeling of M_{11}, in which exactly 3 label sets contain 2 vertices. It is clear to see that there must be at least one label set that is empty. This occurs at either S_{j+1} or S_{j-1}, where $|S_j| = 2$ and $j \neq 0, 7$. Because there are 11 vertices and 8 sets, exactly 3 of which have cardinality 2, there cannot be an empty set among the remaining five sets of cardinality one. Thus, the labeling F is not valid, contradicting our initial assumption.

The only other possibility for a labeling is if exactly 4 sets have cardinality 2. This follows from a simple observation. Assume that $|S_j| = 2$ for some $j \in \{1, \ldots, 6\}$. Since $|S_{j-1} \cup S_{j+1}| \leq 1$, Then clearly neither S_{j-1} nor S_{j+1} can have cardinality 2. That is, there can be no consecutive sets both containing 2 vertices. Therefore, the maximum number of sets with cardinality 2 in M_{11} is 4.

Let G be a 7-labeling of M_{11} with exactly 4 label sets which contain 2 vertices apiece. The reader may note that if $|S_j| = |S_{j+2}| = 2$, then $S_{j+1} = \emptyset$. This is due to the fact that $S_j \cup S_{j+2}$ forms a dominating set, and therefore, no vertex can feasibly obtain the label *j+1*.

Because G has exactly four label sets with cardinality 2, then there must be at least 2 label sets which are empty. Out of 8 sets of vertices, exactly 4 have cardinality 2, and 2 have cardinality 0. This implies that 1 of the 2 remaining sets must have 2 vertices, thus arriving at our contradiction.

Therefore, it is not possible to 7 label M_{11}. □

Refer to Figure 3.16 for a proper 8-labeling of M_{11}, implying $\lambda_{2,1}(M_{11}) = 8$.

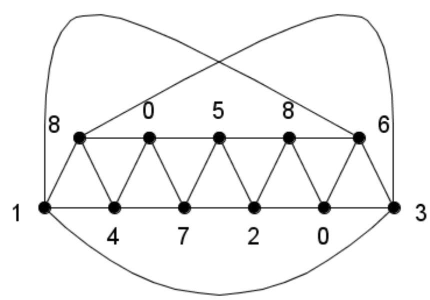

Figure 3.16: An optimal 8-labeling of M_{11}.

The aforementioned construction also neglects the case when n=17, which leads us to the following claim:

Claim: $\lambda_{2,1}(M_{17}) \geq 8$

Before proving this, some simple observations must be made about the structure of this graph.

Lemma 3.2.5. *In M_{17}, if $|S_j| = 3$, then $|S_{j-1} \cup S_{j+1}| \leq 2$, where j+1 and j-1 are defined.*

Proof. Consider any three vertices u, v, and w in the vertex set of M_{17} that can receive the same label j. Let $\mathcal{N}(u)$ denote the closed neighborhood of u (defined analogously for v and w). Each of these sets contain five vertices, and since u, v, and w are all at least distance three from each other, then $\mathcal{N}(u)$, $\mathcal{N}(v)$, and $\mathcal{N}(w)$ are all disjoint.
Observe that the subgraph induced by $V \backslash (\mathcal{N}(u) \cup \mathcal{N}(v) \cup \mathcal{N}(w))$ contains only two vertices. These are the only two vertices in the graph that are distance two from every vertex with label j. Therefore, these are the only two vertices that can receive consecutive labels j+1 or j-1. $\square$

Lemma 3.2.6. *In M_{17}, if $|S_j| = 3$, and $|S_{j+1}| = 2$, then $|S_{j+2}| \leq 1$.*

Proof. Let u, v, w be defined as in Lemma 3.2.5. Now consider all the possibilities of where these three vertices can be placed in M_{17}. To start, let v be any vertex (note that this graph is vertex transitive, so without loss of generality we shall refer to the v labeled in Figure 3.17).

The only other vertices that can receive the same label as v are those labeled

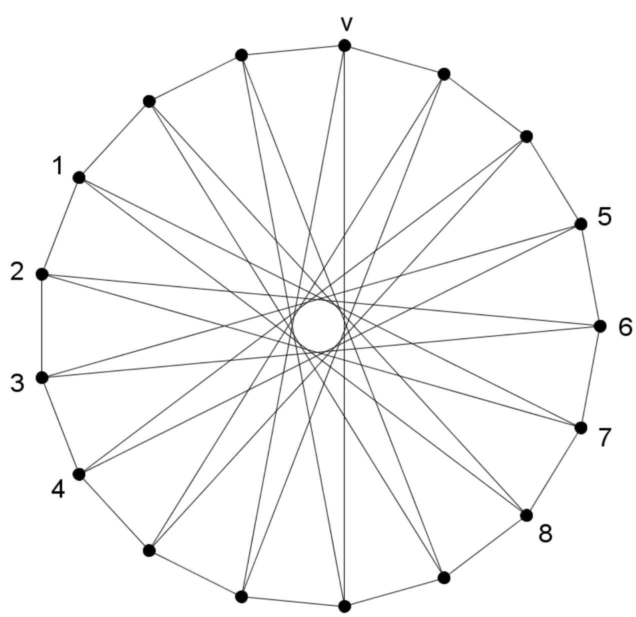

Figure 3.17: An example of M_{17} with v and vertices 1-8 which are eligible to receive the same label as v.

one through eight in Figure 3.17. Two of these vertices must be designated as

u and w. To take a better look at the possibilities, refer to Figure 3.18.

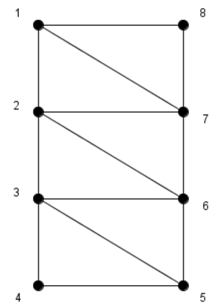

Figure 3.18: The subgraph H induced by vertices 1 through 8 in M_{17}

Clearly, the vertices labeled 2 and 6 cannot be considered, because they are not distance three from any other vertex in this subgraph H. There are six possibilities for choosing two vertices in H that can receive the same label, and these six possibilities can be sorted into two cases, due to the rotational symmetry of Möbius Ladders.
Case 1: The vertices (1,5), (5,8), or (1,4) are chosen to be u and w. This case entails three vertices that are each distance three from each other in M_{17}. The subgraph induced by $V\backslash(\mathcal{N}(u) \cup \mathcal{N}(v) \cup \mathcal{N}(w))$ leaves two vertices that are distance three from each other, and can thus receive the same label.
Case 2: The vertices (7,4), (3,8), or (4,8) are chosen for u and w. This case entails one vertex that is distance three to both the others, and the remaining two are distance four. In this case, the subgraph induced by
$V\backslash(\mathcal{N}(u) \cup \mathcal{N}(v) \cup \mathcal{N}(w))$ leaves two vertices that are adjacent, and therefore must receive different labels.
It is obvious that the only event when $|S_{j+1}|$ can equal 2 occurs in the first of the previous cases. Refer to Figure 3.19 for an example of this.

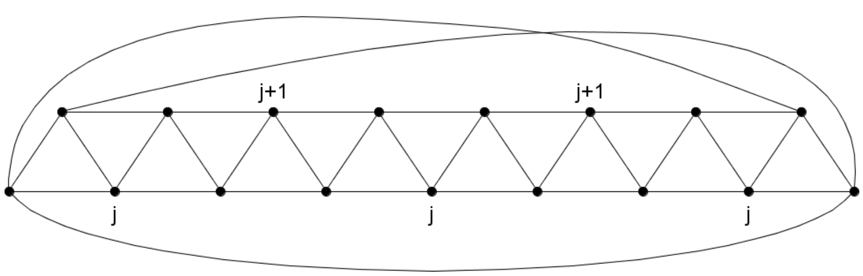

Figure 3.19: An example where $|S_j| = 3$ and $|S_{j+1}| = 2$ in M_{17}.

Let the two vertices that receive label $j+1$ be denoted by x and y. The only vertices in the graph that are available to receive the label *j+2* are those that are not in the closed neighborhoods of x or y, and also those which are not previously labeled. Therefore, we can look at the subgraph induced by

$$V \backslash (\mathcal{N}(x) \cup \mathcal{N}(y) \cup \{u, v, w\}),$$

which excludes all the aforementioned vertices (See Figure 3.20). This subgraph has diameter two, and therefore at most one vertex in this subgraph can receive the label $j+2$. □

It is clear that this principle also holds for subtraction instead of addition. That is,

$$|S_j| = 3, \text{ and } |S_{j-1}| = 2 \Longrightarrow |S_{j-2}| \leq 1.$$

The proof for this is identical, and therefore omitted.

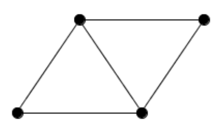

Figure 3.20: The subgraph of M_{17} which can receive label $j+2$, assuming $|S_j| = 3$ and $|S_{j+1}| = 2$.

Lemma 3.2.7. *In* M_{17}*,*
(i:) If $|S_j| = |S_{j+2}| = 3$*, for some label* j*, then* $|S_{j+1}| \leq 1$*.*
(ii:) Further, if $|S_{j+1}| = 1$*, then* M_{17} *does not admit a* 7*-labeling.*

Proof. Let $S_j = \{u_1, u_2, u_3\}$, and $S_{j+2} = \{v_1, v_2, v_3\}$. Let vertex a receive the label of $j+1$. Note that a cannot be adjacent to any element of $S_j \cup S_{j+2}$. So $\mathcal{N}(a) \cap S_j = \mathcal{N}(a) \cap S_{j+2} = \emptyset$, where $\mathcal{N}(a)$ denotes the closed neighborhood of a. If this neighborhood is removed from M_{17}, we are left with a triangular ladder on twelve vertices. We refer to this subgraph as T. T must contain three vertices with label j, and three vertices with label $j+2$. Note that the only possible arrangement such that there is distance three between all elements of S_j and S_{j-2} is given in Figure 3.21.

Observe that either set S_j or S_{j+2} dominates the unlabeled vertices in T. Therefore, no vertex in T can be labeled $j+1$ or $j+3$. This concludes the proof of part (i).

Now note that the vertices in T that are yet to be labeled form two disjoint 3 cycles.

Recall that $\lambda_{2,1}(C_3) = 4$, so $\lambda_{2,1}(T) \geq 4$ by Lemma 2.1.3. This implies that T will need an additional five labels labels (because '0' is included in an optimal

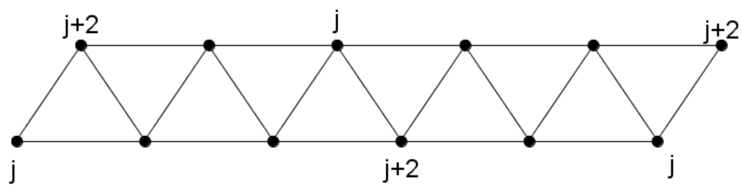

Figure 3.21: The subgraph T, and the only possible assignment of S_j and S_{j+2}.

labeling of C_3) to properly label one of these 3 cycles. However, we have already introduced four labels, which belong to the set $\{j, j+1, j+2, j+3\}$. If five additional labels are required to feasibly label T, this brings the total number of labels to nine. Which implies that it is impossible to label 7-label M_{17}, provided $|S_{j+1}| = 1$. □

With these preliminaries established, we are ready to prove the initial claim that $\lambda_{2,1}(M_{17}) \geq 8$.

Theorem 3.2.8. *M_{17} does not admit a 7 labeling.*

Proof. As was the case with M_{11}, we proceed by contradiction. Assume that M_{17} admits a 7-labeling. That is to say, the vertices can be partitioned into eight disjoint sets $S_0, S_1, \ldots, S_7$, where the vertices in set S_j receive the label j.

The key observation to be made is that

$$|S_{j-1} \cup S_j \cup S_{j+1}| \leq 6 \text{ for } j \in \{1, 2, \ldots, 6\}.$$

This is a direct consequence of the three previously established lemmas. If $|S_{j-1}| = 3$, then $|S_j| \leq 2$, by Lemma 3.2.5. Further, if this inequality is tight, then $|S_{j+1}| \leq 1$ by Lemma 3.2.6.

Because we assume that M_{17} admits a 7-labeling, the contrapositive of Lemma 3.2.7 (ii) implies that if $|S_j| = |S_{j+2}| = 3$, then $S_{j+1} = \emptyset$. In any case, the sum $|S_{j-1}| + |S_j| + |S_{j+1}| \leq 6$. This will be referred to as *the 3 set condition* for M_{17}.

By the pigeonhole principle, the set with maximum cardinality must have at least $\lceil \frac{17}{8} \rceil = 3$ vertices.

There can also be at most three vertices designated to the same label, because $\alpha(M_{17}) = \lfloor \frac{17}{5} \rfloor = 3$.

With this information, it is possible to categorize all possibilities for the labeling of M_{17}. The four cases are analyzed below.

Case I: There is exactly one set that has cardinality three. It is easy to see that the other seven sets must all have cardinality two.
However, this directly violates the 3 set condition, as there would be three consecutive label sets with cardinality that sums to seven.

Case II: There are exactly two sets with cardinality three. This leaves eleven vertices and only six sets in which to put them, implying that there are five sets with cardinality two, and one set with cardinality one. Let $S_a = S_b = 3$, for $a, b \in \{0, 1, \ldots, 7\}$. Observe that there are no label sets which are empty, and thus at least one of S_a or S_b must be in a sequence of 3 consecutive sets which sum to 7, thereby violating the 3 set condition.

Case III: There are exactly three sets with cardinality three. This possibility leads to two cases in which the remaining eight vertices are partitioned differently. Observe that there must exist $\hat{j} \in \{1, 2, \ldots, 6\}$ such that $|S_{\hat{j}}| = 3$.

III(a): 3 sets of 3, 4 sets of 2, and 1 set which is empty. Consider $S_{\hat{j}}$. Either $S_{\hat{j}-1}$ or $S_{\hat{j}+1}$ must be empty in order to satisfy Lemma 3.2.5, because there are no sets of cardinality one. Assume $S_{\hat{j}-1}$ is empty. If $\hat{j} \in \{1, 2, \ldots, 5\}$, then $|S_{\hat{j}} \cup S_{\hat{j}+1} \cup S_{\hat{j}+2}| \geq 7$, because the remaining six sets all have at least two elements. This violates the 3 set condition. If $\hat{j} = 6$ then S_7 must have cardinality two, in order to satisfy the 3 set condition. This leaves the remaining five sets $S_0, \ldots, S_4$ two of which have cardinality three, and three of which have cardinality two. It is impossible for this to remain feasible with respect to the three set condition.

III(b): 3 sets of 3, 3 sets of 2, and 2 sets of 1. This case is simple, because there are no empty label sets, then none of the sets of cardinality three can be within two of each other. More formally, there cannot exist a j such that two of $\{S_{j-1}, S_j, S_{j+1}\}$ have cardinality three. The only case when this happens, is if $|S_0| = |S_7| = |S_3|$ (note that the last set can also be S_4, but we may assume it is S_3 without loss of generality). In order to satisfy Lemma 3.2.5, then S_2 and S_4 must both have cardinality one. However, this implies that $|S_5| = |S_6| = 2$, which violates the three set condition.

Case IV: There are exactly four sets with cardinality three. There are an additional two cases in which the remaining five vertices are partitioned differently.

IV(a): 4 sets of 3, 2 sets of 2, 1 set of 1, and 1 set which is empty.
If four of the eight sets are full, then there are at least two sets which must lie directly between two full sets. So by Lemma 3.2.7, these two sets must be empty. However, this labeling only admits one empty set, and is thus infeasible.

IV(b): 4 sets of 3, 1 set of 2, 3 sets of 1.
The same argument that applied to case IV(a) applies here. There must be at least two empty sets for this to be feasible, and there are none. So this labeling is not a valid $L(2, 1)$ labeling.

Note that it is impossible for M_{17}, to have five sets with cardinality three. This follows directly from Lemma 3.2.6.

□

3.3 Extension to $L(h,1)$ Labelings

In [6], Gonçalves provides the upperbound of $\lambda_{h,1}(G) = \Delta^2 + (h-1)\Delta - 2$, for any graph G. We present a labeling scheme which improves this bound for Möbius ladders. This construction is the direct extension of Theorems 3.1.2, and 3.2.3.

For an even Möbius ladder, $\lambda_{h,1}(M_n) \le 3h$. Note that this is an improvement from the general upper bound of $3h+4$. Beginning with the case of $n \equiv 4 \bmod 8$, we label the vertices

$$f(v_i) = \begin{cases} 0; & \text{if } i \equiv 0 \bmod 4; \\ h; & \text{if } i \equiv 1 \bmod 4; \\ 2h; & \text{if } i \equiv 2 \bmod 4; \\ 3h; & \text{otherwise.} \end{cases}$$

This will create $\frac{n}{4}$ 4-cycles which are labeled $(0, h, 3h, 2h)$. A 'rung' can be inserted into each of these 4-cycles with vertices labeled $2h+1$ and 1. To see this labeling scheme, refer to Figure 3.22. Because this construction is directly derived from the construction in Theorem 3.1.2, it also does not cover the small cases of $n = 8, 10$.

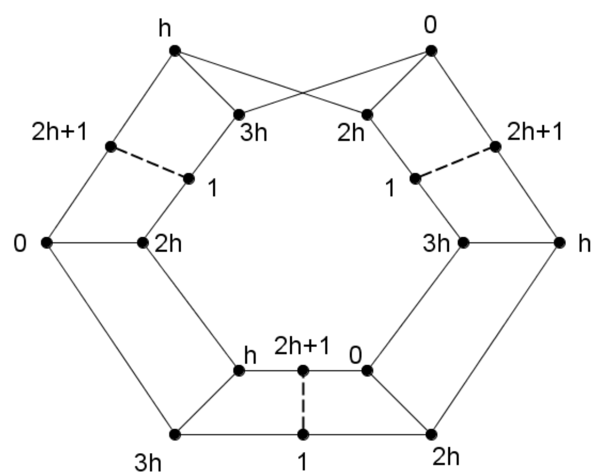

Figure 3.22: The addition of rungs on M_{12}, corresponding to an $L(h,1)$ labeling scheme. The dotted edges can be added to feasibly obtain M_{14}, M_{16}, and M_{18}.

When dealing with odd Möbius ladders, the upper bound of $3h$ has only been found when $n \equiv 7 \bmod 14$. This is achieved by catenating the base set of seven vertices given in Figure 3.23. Note that this base set becomes a feasible Möbius ladder with the addition of three edges: $(0, 2h+1)$, $(0, h+1)$, and $(h, 2h+1)$.

This graph can be indefinitely appended to itself in order to obtain a feasible $L(h, 1)$ labeling of M_n, where $n \equiv 7 \bmod 14$.

The odd Möbius ladders are split into three cases in order to prove the more basic bound of $3h + 1$. These are the odd integers modulo six. The same construction presented in Theorem 3.2.3 can be extended analogously. This does not cover the smaller cases of $n = 17, 11, 5$.

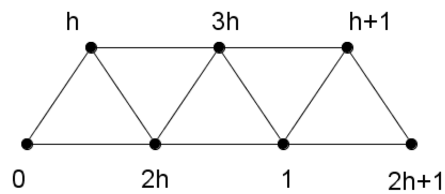

Figure 3.23: $L(h, 1)$ labeling scheme for the Möbius ladder on 7 vertices.

3.4 $L'(2, 1)$ Labelings

Recall that any $L'(2, 1)$ labeling requires at least $n - 1$ labels. A graph that can achieve this lower bound is said to be *perfectly labelable.*

Theorem 3.4.1. *All Möbius Ladders of order $n \geq 6$ are perfectly labelable. That is, $\lambda'_{2,1}(M_n) = n - 1$.*

Proof. Note that M_5 is a complete graph with diameter one. Thus, it requires 10 labels, as shown in Section 2.1.2.

We present a labeling algorithm that adequately labels Möbius graphs. It is important to note that this scheme is not the only such scheme, but it is a simple and convenient mechanism to describe.

Since an $L'(2, 1)$ labeling is one-to-one by definition, all vertices must receive different labels, thus satisfying the distance-two constraint. By then imposing the restriction that no two adjacent vertices be labeled consecutively, we receive an optimal labeling of $n - 1$.

M_n exhibits cyclic properties that may be described by the group $\mathbb{Z}_n$. The binary operation for $\mathbb{Z}_n$ is addition modulo n. An element $x \in \mathbb{Z}_n$ is a *generator* of $\mathbb{Z}_n$ if and only if

$$gcd(x, n) = 1.$$

Thus, by starting at an intial vertex, $v(0)$ and assigning it label 0, and proceeding clockwise around the perimeter of M_n in successive steps of length

x, every vertex will be labeled in exactly $n-1$ steps. So,

$$f(v(i+x)) = f(v(i)) + 1,$$

where the addition of $(i+x)$ is performed modulo n. Further, x must satisfy $gcd(n, x) = 1$, in order to ensure that each vertex is labeled exactly once. For even Möbius ladders, the distance one constraint suggests that x may not be an element of $\{1, \frac{n}{2}, n-1\}$.

In the case of odd Möbius ladders, the same labeling scheme holds. However, in order to verify that this construction does not violate the distance-one constraint, x may not be in the set $\{1, \frac{n+1}{2}, \frac{n-1}{2}, n-1\}$. □

3.5 Radio Labelings

3.5.1 Even Case

We begin with a discussion on the distances between particular vertices in even Möbius ladders. For $\frac{n}{2}$ odd, M_n has precisely two distinct vertices which are distance $diam(M_n)$ from any given vertex v as demonstrated in Figure 3.24.

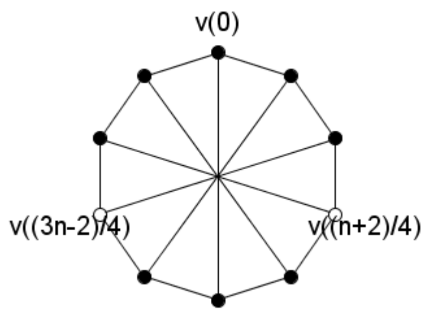

Figure 3.24: An example on M_{10} for which $\frac{n}{2}$ is odd. The white nodes represent those which are $diam(M_{10})$ from $v(0)$.

Without loss of generality, call $v(0)$ the initial vertex. Traveling clockwise around the outer cycle of the Möbius ladder gives the two vertices of interest as $v(\frac{n+2}{4})$ and $v(\frac{3n-2}{4})$. It remains to be seen exactly how far apart these vertices are from each other. The length of the path along the outer cycle, beginning on vertex $v(\frac{n+2}{4})$ to vertex $v(\frac{3n-2}{4})$, is exactly $\frac{2n-4}{4}$. This is because

$$\frac{3n-2}{4} - \frac{n+2}{4} = \frac{2n-4}{4}.$$

On the other hand, the same vertex may be reached by taking the path beginning with edge $(v(\frac{n+2}{4}), v(\frac{3n+2}{4}))$, and then continuing along the outside cycle until $v(\frac{3n-2}{4})$ is reached. Note that this secondary path, (excluding the edge which crosses to antipodal partner) has length

$$\frac{n+2}{4} + \frac{n}{2} - \frac{3n-2}{4} = 1.$$

With the initial edge included, this path has length two. Since the distance between two vertices is determined by the shortest path between them,

$$d(v(\frac{n+2}{4}), v(\frac{3n-2}{4})) = min\{\frac{2n-4}{4}; 2\}.$$

The only time $\frac{2n-4}{4}$ is chosen is when

$$\frac{2n-4}{4} < 2,$$

which is only true when $n < 4$. Observe that there is no Möbius ladder on less than 4 vertices. So for this case to be nontrivial, $d(v(\frac{n+2}{4}), v(\frac{3n-2}{4})) = 2$.

For $\frac{n}{2}$ even, M_n has precisely four distinct vertices which are distance $diam(M_n)$ from any given vertex $v(0)$. Since exactly two of them are to be examined at any given time, there are $\binom{4}{2} = 6$ possible vertex pairs to be studied. The four vertices of interest are $v(\frac{n}{4})$, $v(\frac{n+4}{4})$, $v(\frac{3n-4}{4})$, and $v(\frac{3n}{4})$. Again, the distances between these vertices must be considered.

Beginning with the vertex $v(\frac{n}{4})$, $d(v(\frac{n}{4}), v(\frac{n+4}{4})) = d(v(\frac{n}{4}), v(\frac{3n}{4})) = 1$. Now

$$d(v(\frac{n}{4}), v(\frac{3n-4}{4})) = min\{2, \frac{2n-4}{4}\}.$$

Once again, we may conclude that $d(v(\frac{n}{4}), v(\frac{3n-4}{4})) = 2$, as no Möbius ladders are defined for $n < 4$.

For vertex $v(\frac{n+4}{4})$,

$$d(v(\frac{n+4}{4}), v(\frac{3n-4}{4})) = min\{\frac{n-4}{2}; 3\}.$$

The only case in which $\frac{n-4}{2} < 3$ is when $n < 10$. The following figures demonstrate optimal labelings for M_6 and M_8.

Otherwise, it may be inferred that $d(v(\frac{n+4}{4}), v(\frac{3n-4}{4})) = 3$.

$$d(v(\frac{n+4}{4}), v(\frac{3n}{4})) = min\{\frac{n-2}{2}; 2\}$$

A similar analysis on the traversal of the perimeter for $d(v(\frac{n+4}{4}), v(\frac{3n}{4}))$ reveals that $d(v(\frac{n+4}{4}), v(\frac{3n}{4})) = 2$ is nontrivial.

It only remains to check $d(v(\frac{3n-4}{4}), v(\frac{3n}{4}))$. $d(v(\frac{3n-4}{4}), v(\frac{3n}{4})) = 1$.

Now an analysis can begin on consecutive labelings of vertices in a radio labeling of a Möbius ladder.

Theorem 3.5.1. *An optimal radio labeling for an even Möbius ladder M_n admits no three consecutive labels for* $\{n \geq 18\} \cup \{14\}$.

Proof. Suppose u and v are arbitrary vertices of M_n and that $|f(u) - f(v)| = 1$. Then

$$|f(u) - f(v)| = 1 \geq diam(M_n) + 1 - d(u, v),$$

where $d(u, v) \geq diam(M_n)$.

But by definition of the diameter of a graph, the distance between any two points of M_n may not exceed $diam(M_n)$. Thus,

$$d(u, v) = diam(M_n).$$

Now suppose that there exists a third vertex x in $V(M_n)$ such that $f(x) = f(v) + 1 = f(u) + 2$. Then

$$|f(x) - f(v)| = 1 \geq diam(M_n) + 1 - d(x, v)$$

$$d(x, v) \geq diam(M_n)$$

and

$$|f(x) - f(u)| = 2 \geq diam(M_n) + 1 - d(x, u)$$

$$d(x, u) \geq diam(M_n) - 1.$$

From these previous equations, it can be seen that there must exist two different vertices each $diam(M_n)$ away from v. Incorporating results from the above analysis on the classification of these points, $\frac{n}{2}$ odd and $\frac{n}{2}$ even must be treated separately.

Suppose $\frac{n}{2}$ is odd. Then for the nontrivial case, $d(u, x) = 2$. Then

$$diam(M_n) = \frac{n+2}{4} \leq 3 \implies n \leq 10$$

Thus, the only even Möbius graphs which may receive three consecutive labels on n vertices with $\frac{n}{2}$ odd are M_6 and M_{10}.

Now suppose $\frac{n}{2}$ is even. For the nontrivial cases, $d(u, v) = 1, 2,$ or 3.

Suppose $d(u, v) = 1$. Then

$$diam(M_n) = \frac{n}{4} \leq 2 \implies n \leq 8.$$

For $d(u, v) = 2$,

$$diam(M_n) = \frac{n}{4} \leq 3 \implies n \leq 12.$$

Finally, consider $d(u, v) = 3$. Then

$$diam(M_n) = \frac{n}{4} \leq 4 \implies n \leq 16.$$

Thus, the only Möbius ladders with $\frac{n}{2}$ even that admit three consecutive labels are M_4,M_8, M_{12}, and M_{16}. Concisely, M_n admits no three consecutive labels for $\{n \geq 18\} \cup \{14\}$. □

3.5.2 Odd Case

Theorem 3.5.2. *An optimal radio labeling for an odd Möbius ladder, M_n admits no three consecutive labels for $\{n \geq 15\} \cup \{11\}$.*

Proof. Let $v(0)$ be any vertex in M_n. First, consider the case when r is odd. This implies that the diameter is $\frac{r+1}{2}$. The two vertices which are distance $diam(M_n)$ from $v(0)$ are $\frac{r+1}{2}$ and $\frac{3r+1}{2}$. By the cyclic nature of our labeling scheme $v(i)$ is adjacent to $v(i+r)$ and $v(i+r+1)$, where addition is done modulo n. This implies that vertices $v(\frac{r+1}{2})$ and $v(\frac{3r+1}{2})$ must be adjacent. In other words, $d(v(\frac{r+1}{2}), v(\frac{3r+1}{2})) = 1$.

If $v(0)$, $v(\frac{r+1}{2})$, and $v(\frac{3r+1}{2})$ are to receive consecutive labels i.e.

$$f(v(0)) = f(v(\frac{r+1}{2})) + 1 = f(v(\frac{3r+1}{2}) - 1,$$

then

$$|f(v(0)) - f(v(\frac{r+1}{2}))| = |f(v(0)) - f(v(\frac{3r+1}{2}))| = 1,$$

and

$$|f(v(\frac{r+1}{2})) - f(v(\frac{3r+1}{2}))| = 2.$$

However, by the radio condition, this implies

$$2 \geq diam(M_n) + 1 - 1 = diam(M_n),$$

substituting in the expression for a Möbius ladder's diameter (with r odd):

$$\frac{r+1}{2} \leq 2 \implies r \leq 3.$$

It is clear to see that this labeling can only hold for $n \leq 7$.

Now consider the case when r is even. Let $v(0)$ be any initial vertex in $V(M_n)$. Once again, there are exactly four vertices which are distance $diam(M_n)$ away from $v(0)$. This implies that there are $\binom{4}{2} = 6$ pairs of vertices to consider. However, odd Möbius ladders are 4-regular, and this makes the cases simpler. The four vertices which are $diam(M_n)$ away from $v(0)$ are: $v(\frac{r}{2})$, $v(\frac{r+2}{2})$, $v(\frac{3r}{2})$, and $v(\frac{3r+2}{2})$. Observe that these vertices form the same subgraph shown in Figure 3.20, which has diameter two. In fact, the only two vertices which are not adjacent to each other are $v(\frac{r+2}{2})$ and $v(\frac{3r}{2})$.

It is not hard to see that $d(v(\frac{r+2}{2}), v(\frac{3r+2}{2}) = 2$. This knowledge allows us to solve explicitly the values of n for which M_n may admit 3 consecutive labelings. If $v(0)$, $v(\frac{r}{2})$ and $v(\frac{3r+2}{2})$ are to admit three consecutive labels, then

$$|f(v(\frac{r+2}{2})) - f(v(\frac{3r}{2}))| = 2.$$

By the definition of a radio labeling:

$$2 \geq diam(M_n) + 1 - 2 = diam(M_n) - 1,$$

which reduces to

$$\frac{r}{2} \leq 3 \implies r \leq 6.$$

Thus, n must be less than or equal to 13 in order for M_n to admit 3 consecutive labels. $\square$

As a corollary to these theorems, a lower bound is established for the radio number of Möbius ladders. If M_n admits no three consecutive labels, then the sequence of n labels which contains the smallest span is given by the (ordered) set: $\{0, 1, 3, 4, 6, \ldots, \frac{3n}{2} - 2\}$. Which provides the lower bound:

$$rn(M_n) \geq \frac{3n}{2} - 2.$$

This inequality holds for all $\{n \geq 17\} \cup \{11, 14, 15\}$. No labelings have been found that achieve this bound.

Chapter 4

Suggestions for Future Work

4.1 $L(2,1)$ Labelings of Odd Möbius Ladders

This paper presents a range of two values for optimally labeling an odd Möbius ladder. That is, for n odd:

$$6 \leq \lambda_{2,1}(M_n) \leq 7.$$

The only case we have discovered that achieves this bound is when n is congruent to 7 modulo 14, covered in section 3.2.2. For this reason, we put forth the following conjecture: For odd Möbius ladders, $\lambda_{2,1}(M_n) = 6$ if and only if $n \equiv 7 \bmod 14$.

This may be related to the structure of the 7 mod 14 case. A 6-labeling of any graph requires 7 labels, and it is easy to see that this is the only case when all 7 labels appear with the same multiplicity. However, this is not known to be a sufficient condition for optimal labelings on Möbius ladders.

4.2 Radio Numbers of Möbius Ladders

A working lower bound of $\frac{3n}{2} - 2$ has been established for $rn(M_n)$, with n even. We believe that no labeling can achieve this bound, but that has yet to be proven. One might also devote time to establishing an upperbound for $rn(M_n)$.

4.3 Generalizations of Möbius Ladders

One possible generalization of a Möbius ladder, is the *Möbius lattice*. M_n is constructed by taking the ladder graph on n vertices, and connecting the edges with a 'twist'. This same construction may be applied to a grid graph.

The *grid graph* $G_{m,n}$ is the cartesian product $P_m \Box P_n$ of paths on m and n vertices. This graph has order $m \cdot n$. Note that the ladder graph on n vertices is a subclass of grid graphs, $G_{\frac{n}{2},2}$. One possible construction for a generalized Möbius ladder, denoted $M_{m,n}$, is introducing a twist, and connecting the ends of the grid graph $G_{m,n}$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a subset of $V(G_{m,n})$ such that $(a_i, a_{i-1}) \in E(G_{m,n})$ for all $2 \leq i \leq n$. Further, let $deg(a_i) = 3$ for $i = 2, \ldots, n-1$, and $deg(a_1) = deg(a_n) = 2$. This clearly describes one of the paths of length n which compose an 'edge' of the grid graph. Let $B = \{b_1, b_2, \ldots, b_n\}$ be defined the same way.

With these sets A and B, we can formally describe the twist in the Möbius ladder $M_{m,n}$. Add an edge between each pair of vertices (a_i, b_{n+1-i}) for $i = 1, \ldots, n$. Observe that trying to 'untwist' the ladder will simply introduce the same twist in an adjacent set of rungs, preserving the topological Möbius strip. Refer to Figure 4.1 for a depiction of $M_{10,3}$.

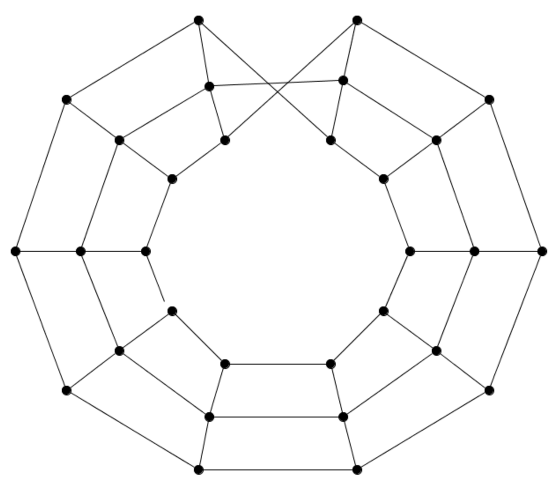

Figure 4.1: An example of a generalized Möbius ladder, $M_{10,3}$.

4.4 $L(h,k)$ Labelings

An upperbound for the $\lambda_{h,1}$ number of Möbius ladders was provided. In the case of $h = 2$, this provides a good bound, and equality in half the cases. However, for the general $L(h,1)$ labeling problem, not as many general results have been established, and there is not a decent lower bound for $\lambda_{h,1}(M_n)$. It remains to be seen if the upper bound established is tight.

A more general $L(h,k)$ labeling, with $k > 1$ was not looked at, and is sure to be a fruitful topic of investigation.

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