Image Deblurring

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Abstract

This project was carried out in order to improve our understanding of the mechanisms behind image deblurring. In order to gain a more thorough understanding of the basic processes that are employed in image deblurring, a number of references and textbooks were used to build up a background of understanding. Experiments were performed to explore the effectiveness of the basic methods. Through this investigation, we were able to develop an understanding of the sensitivity towards error that is experienced by some of the naïve deblurring solutions that were investigated. Additionally, we gained a better appreciation of the ability of spectral filtering techniques to mitigate the influence of noise on deblurring techniques.

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1 Introduction

Image deblurring is an old problem in the realm of image processing, one that continues to garner attention from academics and businesses alike. It has applications in many different real-world problems and serves as an easy way to visualize examples of a larger range of inverse problems in many fields [8]. Image deblurring seeks to take a blurry image and restore it to its original form algorithmically. When deblurring images, a mathematical description of how it was blurred is very important to maximizing the effectiveness of the deblurring process. With real-world photos, we do not have the luxury of knowing the mathematical function by which the image was blurred. However, there exist methods to approximate how blur occurred.

There are many different sources of blur in a photograph, such as motion blur, camera shake and long exposure times. Even a relatively small amount of any of these effects can be enough to ruin an otherwise good photograph. Image deblurring tools and research seek to solve this problem by taking the blurred images and attempting to restore or alter them to a sharper, clearer state. Because every picture taken comes out blurry to some degree, image deblurring is fundamental in making pictures sharp and useful [1].

Image deblurring and similar techniques are also applicable in commercial settings. Image enhancement techniques were used to enhance the original Star Wars trilogy for its DVD release [2]. Image deblurring techniques can also be applied in fields ranging from the analysis of astronomical data to barcode readers at supermarkets or shipping companies [3]. Because of image deblurring techniques' ability to be used for valuable purposes such as enhancing videos for higher definition releases, there is a lot of interest in developing algorithms and models that can be used to improve their effectiveness.

Most modern cameras come with some form of hardware-based image stabilization feature. This allows the camera to reduce the noise caused by blur in the image without needing to depend on a software level solution. The most common form used in camera is called Optical Image Stabilization (OIS) [9]. Different companies use slightly different methods to achieve this result, but it allows the same likelihood of a sharp image for shutter speeds 8-16 times slower than without any OIS [10]. Other hardware-based stabilization techniques include gyroscopically stabilized gimbals, which are both heavy and expensive, and boosting the film speed, which can result in a noisy (grainy) photograph. Due to technological limitations, digital cameras are unable to perform software level deblurring on pictures.

Through this investigation, we seek to improve our understanding of the techniques and limitations surrounding image-deblurring technologies, and to explore their effectiveness in simple, controlled cases. We also seek to use this investigation to develop a more sophisticated understanding and appreciation for some of the linear algebra techniques used in these methods.

In the following section, some background information on image deblurring and associated techniques will be presented. Following that, we will discuss results that were obtained, as well as discussing some of the weaknesses and limitations in the investigation.

2 Background

In order to be able to better understand the methods of image deblurring that were applied during this project, research was done into the mechanism behind some of the naïve approaches to image deblurring. The investigation went through three stages of deblurring techniques of increasing sophistication in order to build up an understanding of the basic methods and weaknesses of image deblurring. For the rest of this report, unless otherwise stated, all images will be assumed to be in a gray scale.

2.1 General Method of Image Deblurring

The general method of image deblurring is a direct method for obtaining a blurred or deblurred image from the original or blurred image, respectively. The most basic method is simple matrix multiplication of row and column blurring matrices with the original image. We start with a very simple model for the blurring effects on an image.

$$\mathbf{A}_{\mathrm{c}}\mathbf{X}\mathbf{A}_{\mathrm{r}}^{\mathrm{T}} = \mathbf{B}$$

Equation 1: Linear Model for Blurring

Where $\mathbf{A}_{\mathrm{c}}$ and $\mathbf{A}_{\mathrm{r}}$ are the column and row blurring matrices, $\mathbf{A}_{r}^{\mathrm{T}}$ is the transpose, **B** is the blurred image, and **X** is the original image. **B** and **X** are matrices in $\mathbb{R}^{m\times n}$ (The set of m-by-n matrices of real numbers), $\mathbf{A}_{\mathrm{c}}$ is in $\mathbb{R}^{m\times m}$ and $\mathbf{A}_{\mathrm{r}}$ is in $\mathbb{R}^{n\times n}$. We would expect $\mathbf{X}_{naïve} = \mathbf{A}_{\mathrm{c}}^{-1}\mathbf{B}\mathbf{A}_{\mathrm{r}}^{-\mathrm{T}}$ to be the case, but a fairly simple example shows that the resulting output bears no resemblance to the original image. Because of factors like noise and imprecision, knowing the exact blurring matrices for **X** is not sufficient to restore the image. A simple example is provided below [1].

Figure 1: Original Image of Pumpkins

Figure 2: Blurred Image of Pumpkins

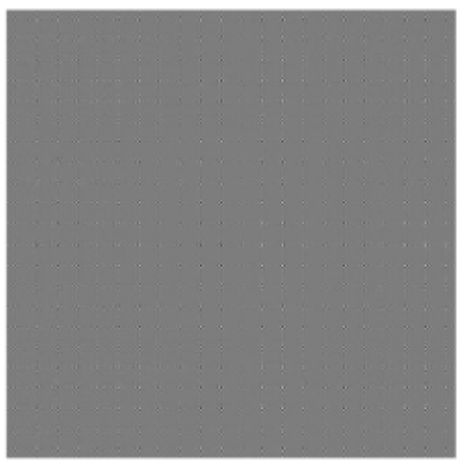

Figure 3: Deblurred with Gaussian Elimination

Because of this, an error term needs to be included in the equation.

$$\mathbf{B} = \mathbf{E} + \mathbf{A}_c \mathbf{X} \mathbf{A}_r^{\mathrm{T}}$$

Equation 2: Error Term in Blurring Equation

Unfortunately, the value of the noise term is unknown, and so the goal of more sophisticated methods is to minimize the influence of the inverted noise $\mathbf{A}_c^{-1} \mathbf{E} \mathbf{A}_r^{-\mathrm{T}}$. The reason that the noise causes the deblurred image to be unrecognizable is because its highfrequency components are amplified when the small singular values of the blurring matrices are inverted [1]. The simplest way to reduce this effect is by creating rank-k versions of the row and column matrices from their singular value decompositions.

$$\mathbf{A}_{\mathrm{k}}^{\dagger} = \begin{bmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_k \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{u}_1^T \\ \vdots \\ \mathrm{u}_k^T \end{bmatrix}$$

Equation 3: Rank-k Truncated Matrix

While this method provides a better restoration than the previous solution, the noise still has a large effect on the resulting image using $\mathrm{vec}(\mathbf{X}) = \mathbf{A}_k^{-1}\mathrm{vec}(\mathbf{B})$.

Figure 4: Rank-k Deblurring with k=800 (Out of 169744)

Despite the improvement, this simple of an approach is insufficient to appropriately reconstruct a blurred image.

2.2 Point Spread Function and Boundary Conditions

In order to increase the accuracy of deblurring functions, a more in-depth look at the blurring matrix **A** must be taken. Understanding the method by which the blurring matrix is generated is the first step to take. Since **A** will blur any original image **X** in the same way, we can imagine applying **A** to a black image with a single pixel of light, known as a point source. The resulting 'image' that describes how **A** acts on individual pixels is called the Point Spread Function (PSF) [1]. Depending upon the type of blurring that is being described, we have different functions that generate the PSF around a central point of (k, l). A few example PSFs are included below [1].

$$p_{i,j} = \begin{cases} 1/(\pi r^2) & \text{if } (i-k)^2 + (j-l)^2 \le r^2 \\ 0 & \text{elsewhere} \end{cases}$$

Equation 4: Out-of-Focus Blur

$$\boldsymbol{p_{i,j}} = \exp\left(-\frac{\mathbf{1}}{\mathbf{2}} \begin{bmatrix} \boldsymbol{i-k} \\ \boldsymbol{j-l} \end{bmatrix}^{\boldsymbol{T}} \begin{bmatrix} \boldsymbol{s_1^2} & \boldsymbol{\rho^2} \\ \boldsymbol{\rho^2} & \boldsymbol{s_2^2} \end{bmatrix}^{-\mathbf{1}} \begin{bmatrix} \boldsymbol{i-k} \\ \boldsymbol{j-l} \end{bmatrix} \right)$$

Equation 5: Atmospheric Blur

$$\boldsymbol{p_{i,j}} = \left(\mathbf{1} + \left(\frac{(\boldsymbol{i-k})}{\boldsymbol{s_1}} \right)^2 + \left(\frac{(\boldsymbol{j-l})}{\boldsymbol{s_2}} \right)^2 \right)^{-\boldsymbol{\beta}}$$

Equation 6: Astronomical Telescope Blur

The function for atmospheric blur is commonly known as the two-dimensional Gaussian function [4], and the function used for astronomical telescopes is called the Moffat function [5]. By convoluting the generated PSF and the original image **X**, we can generate the blurred image without needing to construct the larger blurring matrix. Convolution can also be used to smooth noise or enhance edges in a picture with an appropriate PSF.

Alone, the PSF only allows us to increase the speed at which we reach a poorly deblurred image. In order to improve the accuracy of image reconstructions, we need to take into account that the image we are looking at does not exist in a vacuum, but rather, extends infinitely beyond the captured area. As such, noise can be generated from outside the boundary of the image. In order to counteract this effect, the use of boundary conditions allows us to make a simple assumption about the state of the world outside of the image's frame.

There are three types of boundary conditions commonly used for this purpose, with different situations where they are used. For each boundary condition, the matrix **X** is placed in the center of a matrix three times as large as it. In the zero boundary condition, all other entries are left as zero. Zero boundary conditions assume that all space beyond the image is empty, which is a good assumption to make when working with astronomical data. A periodic boundary condition assumes that the image repeats infinitely, and so tiles the original image in each of the blocks of the larger matrix. The reflexive boundary condition assumes that the image is mirrored outside of the border, and is created by flipping the original image depending on what block it is in, vertically if it is above or below, horizontally if it is to the left or right, and both if it is diagonal of the middle block where the original image was placed.

For certain states of the PSF and assumptions of the boundary condition, we can use fast algorithms in order to calculate the blurred image and a naïve solution from it. When the PSF is separable[1], under any boundary condition, we can interpret the blurring matrix **A** as a Kronecker product of two smaller matrices. Using the PSF, we are able to calculate the $\mathbf{A}_c$ and $\mathbf{A}_r$ whose Kronecker product form the blurring matrix **A** without ever having to construct **A**. We can then apply singular value decomposition to these

[1] A PSF $\mathbf{P} \in \mathbb{R}^{m \times n}$ is separable if there exists $\boldsymbol{c} \in \mathbb{R}^m$ and $\boldsymbol{r} \in \mathbb{R}^n$ such that $\mathbf{P} = \boldsymbol{c}\boldsymbol{r}^T$

matrices as in earlier techniques to generate the naïve reconstruction of an image. Because the SVD of a Kronecker product can be represented in terms of the SVDs of the matrices that form it, we can use these singular value decompositions in order to more efficiently compute the naïve solution than if we were to decompose the matrix itself [1].

$$\mathbf{P} = \mathbf{c}\mathbf{r}^T$$
$$\mathbf{A} = \mathbf{A}_\mathrm{r} \otimes \mathbf{A}_\mathrm{c}$$
$$\mathbf{A}_\mathrm{r} = \mathbf{U}_\mathrm{r}\Sigma_\mathrm{r}\mathbf{V}_\mathrm{r}^\mathrm{T}$$
$$\mathbf{A}_\mathrm{c} = \mathbf{U}_\mathrm{c}\Sigma_\mathrm{c}\mathbf{V}_\mathrm{c}^\mathrm{T}$$
$$\mathbf{X}_{na\ddot{\imath}ve} = \mathbf{V}_\mathrm{c}\Sigma_\mathrm{c}^{-1}\mathbf{U}_\mathrm{c}^\mathrm{T}\mathbf{B}\mathbf{U}_\mathrm{r}\Sigma_\mathrm{r}^{-1}\mathbf{V}_\mathrm{r}^\mathrm{T}$$

Equation 7: Deblurring with Kronecker Decomposition and SVD

For PSFs that are doubly symmetric, under the reflexive boundary condition we can apply the two-dimensional Discrete Cosine Transform (DCT) algorithm, which computes the eigenvalues of the blurring matrix **A** from the PSF and blurs/deblurs the image without constructing **A** [1].

$$\hat{x}_k = \omega_k \sum_{j=1}^{n} x_j \cos\frac{\pi(2j-1)(k-1)}{2n}$$

Equation 8: Discrete Cosine Transform

The two-dimensional Fast Fourier Transform (FFT) algorithm can be applied whenever there is a periodic boundary condition and works in a similar manner to the DCT algorithm.

$$\hat{x}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j e^{-2\pi i(j-1)(k-1)/n}$$

Equation 9: Discrete Fourier Transform

For both the FFT and DCT, we compose their results over each dimension to obtain the resulting matrix [4][6][7]. These algorithms allow us to compute efficiently the blur and naïve restoration of large images efficiently, but are still incredibly vulnerable to noise in the data corrupting the image.

2.3 Spectral Analysis

In each of these efficient algorithms for deblurring images, small eigenvalues of the blurring matrix cause a large buildup of inverted noise that corrupts the restoration of the original image. In order to damp noise from the blurred image, a process called spectral filtering in introduced to the deblurring process [1]. The method used here involves generating an m-by-n matrix **Φ**, and element-wise dividing it by the eigenvalue matrix **S**. Truncated Singular Value Decomposition (Section 2.1) is an example of spectral filtering using the following equation. An implementation method of this is to only keep singular values that are greater than or equal to a specified tolerance value.

$$\phi_i \equiv \begin{cases} 1 & i = 1, \dots, k \\ 0 & i = k+1, \dots, N \end{cases}$$

Equation 10: TSVD Method for Spectral Filtering

Another method for generating the filter factors is the Tikhonov Method.

$$\phi_i \equiv \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2}$$

Equation 11: Tikhonov Method for Spectral Filtering

The α in the Tikhonov Method equation is called the regularization parameter; this parameter behaves in much the same way as the choice of k does in TSVD in damping small eigenvalues that would increase the inverted noise.

Figure 5: Examples of Tikhonov and TSVD Filtering

In the above example [1], Tikhonov regularization (bottom left) and TSVD (bottom right) were selected for the most visually pleasing result.

While these methods allow a much better reconstruction of the original image, more sophisticated methods of image deblurring are required in order to generate more accurate deblurred images.

3 Experimental Results

The results reported below were obtained under a single set of conditions. All PSFs were generated using Gaussian (atmospheric) blur with $\rho = 0$ and $s_1 = s_2$, and all images were manipulated under periodic boundary conditions with the Fast Fourier Transform method. This was done both to prevent getting bogged down in the differences between different algorithms on a given image and because the FFT method had the fasted clock time for images of any size. Furthermore, we will be only applying deblurring methods to single, blurred images to attempt to retrieve restored images. Unless otherwise noted, the images worked with are all based off of a sharp original image that is blurred through a given method so that the PSF of the blur is known.

The following picture was taken at the primary researcher's home, using a tripod for stabilization. The focus of the image was on the bowl of fruit and the picture was taken with an aperture of 5.5. For comparison purposes, some of the deblurred images will focus on just the small section containing the tabletop and fruit bowl itself in order to convey differences that may not be easily apparent with the full image at this scale.

Figure 6: Original Image

3.1 Blurring and Deblurring Without Noise

The first thing that was done to manipulate the above image was to test how it handled being blurred and deblurred under a number of different PSF s-values. In the psfGauss function for MATLAB provided as supplemental material to [1][2], the default value of s is 2, so that will be the starting point for provided examples of blurring and deblurring.

Figure 7: Gaussian Blur with s=2

Figure 8: Applying the FFT Algorithm

[2] The MATLAB functions can be found at http://www2.imm.dtu.dk/~pcha/HNO/

It was determined that as the radius of the PSF increased, the noise that was observed in the restored image would increase as well.

Figure 9: Center Area of Original Image

Figure 10: Gaussian Blur with s=10

Figure 11: Central Part of Deblurred Image, to Showcase Graininess (s=10)

We observe that even with a Gaussian blur using s=10, which leaves the blurred image such that the fruit bowl is simply an unidentifiable blur, there is very little graininess in the restored image. What noise is noticeable from this deblurring is concentrated around edges in the photo (such as the edges of the table). When the blur is increased to use s=15, the blurred image becomes even less recognizable, and in the full-sized image we are able to see a noticeable amount of grain.

Figure 12: Gaussian Blur with s=15

Figure 13: Central Part of Deblurred Image, to Showcase Graininess (s=15)

Simply by convoluting the grainy image with a low pass filter[3], we can smooth the graininess from the restored image; however, we receive a slightly blurry image in exchange. The low pass filter used and resultant image are shown below.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} / 10$$

Equation 12: An Example Low Pass Filter

Figure 14: Applying a Low Pass Filter to the Grainy Deblurred Image

As can be seen, an image that has been blurred without any random noise added in can be blurred to a high degree without significant loss or noise in the restored image.

3.2 Sensitivity of the Eigenvalue Matrix

Now we look at the sensitivity of this method of image deblurring to the introduction of random error to the blurred image. We will be using the same approach as used in the previous section to create the blurred image and then deblur the image with noise. The

[3] A low pass filter averages each pixel with its neighbors using some weighting, and the sum of its elements is 1

following images are based on the fruit bowl image which has been blurred by a PSF with s=2. Below is the Gaussian PSF in a 20-by-20 grid centered on the blurring point.

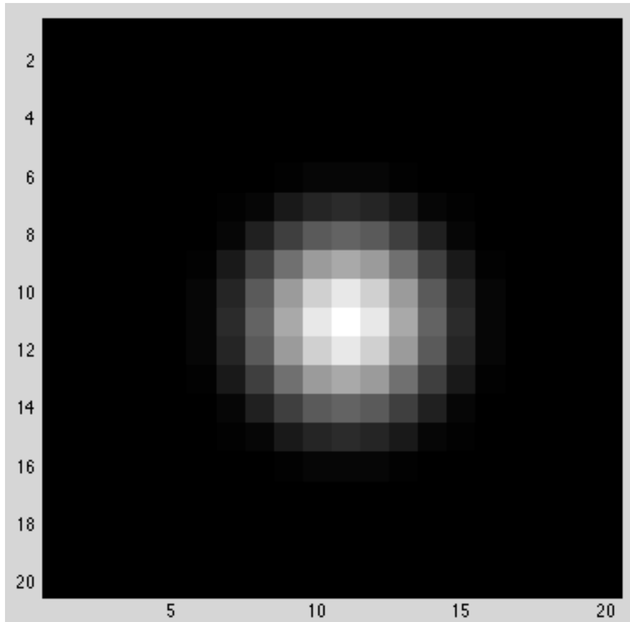

Figure 15: A PSF Using the Gaussian Method and s=2

$$\mathbf{B} = \mathbf{B} + err\|\mathbf{B}\|_{\mathrm{F}}\mathbf{E}$$

Equation 13: Adding Random Noise to the Blurred Image

In Equations 13, **E** is a matrix containing normalized random noise of the same size as **B**, and $\|\mathbf{B}\|_{\mathrm{F}}$ is the Frobenius norm[4] of **B**. We can see that the amount of error noise that is added to the image must be incredibly small in order to not cause the deblurred image to be overcome with inverted noise to the point of there being nothing but noise left in the image.

[4] The Frobenius norm of a matrix is $\|\mathbf{A}\|_{\mathrm{F}} = trace(\mathbf{A}^{*}\mathbf{A})$

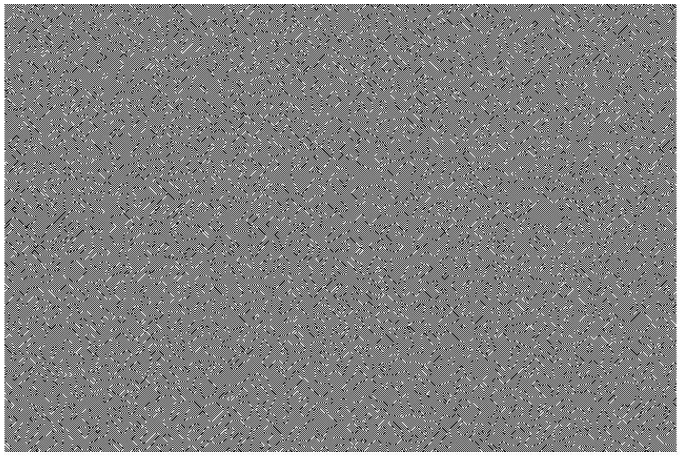

Figure 16: Deblurred Image with 1% Noise

We can observe that the condition number of the eigenvalue matrix **S** is 2.68e+18. This means that even a very small amount of noise is magnified to massive degrees when inverted in the deblurring process. When **S** element-wise divides the Fourier transform of the noisy blurred image, the noise is magnified by an enormous factor, dominating the image. By testing this out with different degrees of error, we find that we require reducing the amount of error to 1e-12 in order for less than half of the image to be dominated by noisy patches.

Figure 17: Deblurred Image with 1e-12 Noise

Stepping beyond that, we observe that reducing the noise by a factor of ten reduces the effect of inverted noise to a level where we must look closer at any section of the restored image to notice the corruption due to inverted noise. Going one step further to an error factor of 1e-14 we are able to more or less purge the inverted noise from the image.

Figure 18: Deblurred Image with 1e-13 Noise

Figure 19: Deblurred Image with 1e-14 Noise

As we can see, this naïve of a deblurring method is insufficient to successfully remove any amount of noise that would occur in a real world situation, so we need to advance to a technique that allows us to cancel out these small singular values that are exploding the inverted noise.

3.3 Spectral Filtering Techniques

The method by which we remove the small singular (spectral) values is known as spectral filtering, as described in the background section. At this stage, we implemented spectral filtering algorithms in order to eliminate the noise. Primarily, the method that was used was the Truncated Singular Value Decomposition (TSVD) based method, where all singular values below a certain tolerance were removed from the matrix **S**. Some operations in the algorithm are adjusted in order to account for this change.

The tolerance or regularization parameters must be carefully chosen to maximize the sharpness of the resulting image. If the parameter is too low, noisy perturbations dominate the image. However, if it is too high, the deblurring will fail to be effective in restoring the image to its original state.

Figure 20: TSVD Filtering with tol = 0.5*err

Figure 21: TSVD Filtering with tol = 0.5

For PSFs with s=2, we found that with the TSVD filtering approach the tolerance needed to be between 0.5 and 0.5 times the error in order to produce a reasonable image. The range is smaller as the amount of blur induced in the image increases;for s=4, a range of 0.05 to the error amount produces a good reconstruction. This implies that the range of tolerances that will allow for the deblurred image to be good quality will shrink as the blurred image gets blurrier.

Despite the limitations on how restrictive the tolerance parameter can be in order to reconstruct an image into something that is both recognizable and of reasonable quality, we are able to apply this filtering technique and obtain good results with error noise as high as 0.1%.

Figure 22: TSVD Filtering With err=0.1% and tol=0.01

The tolerance here is ten times the error parameter, and the resultant image is of good quality, though the image is made choppy by the noise that is not canceled. Reducing the error by one order of magnitude while keeping the tolerance at 0.01 reduces the amount of inverted noise even more.

Figure 23: TSVD Filtering with err=0.01% and tol=0.01

Applying the Tikhonov method to filter the singular values generates results that have less noise than their TSVD counterparts, but are blurrier than them as well. This is due to the different way the singular values are trimmed by the different filters.

Figure 24: Tikhonov Filtering with alpha=0.1 and err=0.01 (top) and 0.001 (bottom)

As can be seen with the above examples, this approach is insufficient to create a truly sharp restored image from the blurry, noise added image.

3.4 Extension to General Blurry Images

Below is a test case for extending this approach to real world noise and blur. A picture with sharp foreground is provided for reference of what the scene ought to look like. Then, a naturally blurry image is deblurred using spectral filtering. The result shown is chosen for its low noise. The tolerance and PSF parameters chosen in the deblurring attempt were 0.1 and 12, respectively.

Figure 25: Photo of Plant, Sharp Foreground, Blurry Background

Figure 26: Photo of Plant, Blurry

Figure 27: Deblurring the Photo of a Plant Using TSVD Filtering

As we can see, the deblurring leaves much to be desired when compared to a sharper image, though it is sharper than the starting image. However, given that the PSF parameter that best represented the image was so high, the reconstruction did fairly well.

4 Weaknesses and Limitations

The method for dealing with single blurred images used in this report suffers from a number of weaknesses, including needing strong knowledge of the blurring 'function' that the image went through [11]. There exist a number of techniques for restoring images both from blurry images, and from grainy (underexposed) images. Working with several underexposed images, techniques can be used to average them together and reduce the noise variance. A multiple blurred image approach reduces the need for a strong knowledge of the original image and blurring function, and can even be applied to video, but takes a significantly longer time to achieve results. A fourth possible technique takes a single blurry image and a single grainy image and combines them in order to restore the original image, and functions as a fast and reliable deblurring method [9].

Additionally, the primary resource, *Deblurring Images*, does not provide many details on some of the more advanced approaches that can be used for image deblurring, such as using multiple blurry images for a single restoration of combining a single blurry image with a single grainy image to create a sharper restored image. Because of this and time constraints in the project, we were limited to using the naïve approaches from the resource in order to meet our schedule.

5 Conclusions

Advances in image deblurring and similar techniques are important both to the development of modern photography and to the restoration of images and videos that are not as sharp as they can be. The ability to remove noise from images captured in highly technical fields such as astronomy and medicine is critically important to increasing the ability of related professionals to fulfill their jobs in the most effective manner possible. This provides a strong reason for research into improving the accuracy and sophistication of image deblurring techniques in both academic and corporate settings.

Though our investigation was limited to some of the more basic and ill-conditioned methods of deblurring, the results demonstrate the value in image deblurring techniques and in looking into study of more sophisticated techniques. Investigation into the more advanced and effective techniques of image deblurring would certainly be worth doing in order to obtain a stronger understanding of the problems and algorithms in image deblurring.

Through this study, we were able to gain a better understanding of both the implementation and the concepts behind how images are handled in the deblurring process. Additionally, through learning about these methods, a stronger understanding of linear algebra concepts such as Fourier transforms was developed. Further, through this investigation, we learned more about the importance of techniques such as singular value decomposition and gained an appreciation for the value of spectral filtering techniques for eliminating the small singular values that are inverted in the reconstruction. Additionally, we gained an understanding about some of the limitations that are present in image deblurring due to the weakness of naïve methods.

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