Symmetric Informationally Complete Positive Operator Valued Measures

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Abstract

We consider the question of the existence of d^2 equiangular lines in d-dimensional complex space $\mathbb{C}^d$. In physics, such a set of equiangular lines is called a symmetric, informationally complete positive operator valued measure (SIC-POVM or SIC). The question of existence of SIC-POVMs has been studied for several decades now with very little progress. While numerical solutions have been found in dimensions up to $d = 121$ and a few dozen exact solutions are known, the 1999 conjecture of Zauner that SIC-POVMs exist in dimension d for all integers $d \geq 2$ is still open.

The purpose of this project is first to survey all known simple constructions of SIC-POVMs and second to explore these solutions for additional symmetries. We focus on three main constructions: the Weyl-Heisenberg construction; constructions using Hadamard matrices; and Hoggar's construction. In the Weyl-Heisenberg approach, a set of d^2 unit vectors spanning our lines is found as an orbit of the Weyl-Heisenberg matrix group. When a vector $v \in \mathbb{C}^d$ has the property that it belongs to an orbit of size d^2 with this equiangular property, we say that v is a fiducial vector. We then explore fiducial vectors in dimensions 7 and 19 which have additional attractive number-theoretic properties. Finally, we generate geometric configurations of the Majorana states of some SIC-POVMs in dimensions 3 and 4. We then plot the Majorana vectors of those SIC-POVMs in $\mathbb{R}^3$; these are then found to have special symmetry. In the process of generating the Majorana states, two potentially new SIC-POVMs in dimension three have been found.

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Chapter 1

Introduction

Equiangular lines have been studied for over 65 years, and their construction remains "one of the most challenging problems in algebraic combinatorics" [8] . Do there exist d^2 equiangular lines, the maximum possible, in all finite complex dimensions d? Zauner conjectured 15 years ago that the answer is yes [14]. However, the problem still remains unsolved. This question has attracted increased attention from the quantum physics community recently. In quantum theory, such a set of d^2 equiangular lines in d-dimensional complex space is equivalent to a symmetric informationally complete positive-operator-valued measure (SIC-POVM), which is a special case of a generalized measurement on a Hilbert space.

In quantum theory, a mixed quantum state is represented by a density matrix, which is a d by d Hermitian matrix. For example, a 3 by 3 density matrix can be described as

$$\rho = \begin{pmatrix} a & b_1 + b_2 i & c_1 + c_2 i \\ b_1 - b_2 i & d & e_1 + e_2 i \\ c_1 - c_2 i & e_1 - e_2 i & f \end{pmatrix}. \tag{1.1}$$

We want to find out the minimum number of measurements required to determine the density matrix. In fact, the real vector space of d by d Hermitian matrices has dimension d^2. In the example of dimension 3, the density matrix ρ has 9 real parameters. Therefore, to estimate a density matrix we need d^2 values, which implies that we need d^2 projectors. SIC-POVMs provide a good scheme of measurements for estimating the density matrix, and they have been shown to lead to the minimum statistical errors in quantum tomography [11]. For physicists, it is very desirable that SIC-POVMs exist in all finite complex dimensions.

The mathematical definition of a SIC-POVM is the following: let $S = \{v_1, v_2 \cdots v_{d^2}\}$ be a set of d^2 vectors in $\mathbb{C}^d$ which is said to be a SIC-POVM or a set of equiangular lines if it satisfies that

$$|\langle v_j, v_k \rangle|^2 = \begin{cases} 1 , & \text{if } j = k; \\ \frac{1}{d+1}, & \text{if } j \neq k. \end{cases}$$

for all $v_j, v_k \in S$, where $\langle v_j, v_k \rangle = v_j^* v_k$ denotes the Hermitian inner product and $*$ denotes the conjugate transpose.

Despite a number of exact solutions, as well as a longer list of numerical solutions, the problem whether SIC-POVMs exist in every dimension remains open. The exact solutions, found by hand in a few cases and by computer algebra software in the others, are known in the following dimensions: $d = 2\text{-}24, 28, 30, 31, 35, 37, 39, 43, 48.$ [9] The numerical solutions are known in all dimensions up to and including $d = 151$, as well as a handful of other dimensions up to $d = 323$ [9, 10].

Chapter 2

Weyl-Heisenberg Construction of SIC-POVMs

Almost every known SIC-POVM has been constructed as an orbit of a simple family of matrix groups called Weyl-Heisenberg groups. The Weyl-Heisenberg group is a discrete group first brought into quantum mechanics by Weyl [12], and is defined as follows. For any dimension d, let $\omega = e^{2\pi i/d}$ be a primitive d^{th} root of unity. Let $\{e_0, e_1, \cdots e_{d-1}\}$ be an orthonormal basis for $\mathbb{C}^d$. Then, we construct the shift operator X and the phase operator Z as follows:

$$Xe_j = e_{j+1} \qquad Ze_j = \omega^j e_j$$

where the shift is modulo d: $Xe_{d-1} = e_0$. These two operators may also be represented as two unitary matrices. For example, in dimension 3,

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

We easily find that

$$XZX^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \omega^{-1}Z,$$

and therefore $ZX = \omega XZ$ for this example. In fact, these operators satisfy the Weyl commutation relation, $X^r Z^t = \omega^{-rt} Z^t X^r$. The operators X and Z are very "close" to commuting except for an additional phase term.

The Weyl-Heisenberg displacement operators in dimension d are defined by

$$D_{rt} = -\omega^{rt/2} X^r Z^t. \tag{2.1}$$

The product of two displacement operators is another displacement operator with different phase:

$$\begin{aligned} D_{rt}D_{sm} &= (-\omega^{rt/2} X^r Z^t)(-\omega^{sm/2} X^s Z^m) \\ &= \omega^{(rt/2+sm/2)} X^r Z^t X^s Z^m \\ &= \omega^{(rt/2+sm/2)} X^r \omega^{-st} X^s Z^t Z^m \\ &= \omega^{(rt/2+sm/2-st)} X^{r+s} Z^{t+m} \\ &= -\omega^{-3st/2-rm/2} D_{r+s,t+m} \end{aligned}$$

where every sum is modulo d.

Therefore, by allowing the generators to be multiplied by phase factors, we can define a group, known as the Weyl-Heisenberg group in dimension d. Notice in $\mathbb{C}^d$, there are a total of d^2 Weyl-Heisenberg displacement operators. We call a vector ϕ a fiducial vector if the set $\{D_{rt}\phi\}_{r,t=0}^{d-1}$ is a representation of vectors in a SIC-POVM. Zauner [14] conjectured that such fiducial vectors exist in all finite complex dimensions. This conjecture is far from being proved but is well supported by examples in small dimensions discussed in the introduction chapter. Some examples of fiducial vectors will be further discussed later in Chapters 5 and 6.

Chapter 3

Construction of Complex Equiangular Lines from Hadamard Matrices

3.1 Complex Equiangular Lines From Hadamard Matrix

There are just a few known cases where researchers have generated SIC-POVMs without using the Weyl-Heisenberg group. Jonathan Jedwab and Amy Wiebe found a simple construction of complex equiangular lines using Hadamard matrices [7]. This construction method is simple and easy to verify by hand, as compared to other constructions. However, this method only applies in dimension 2, 3 and 8 and is not applicable in its current form for other dimensions. We shall now present the construction.

An order d complex Hadamard matrix is a d by d matrix, all of whose entries are in $\mathbb{C}$ and are of magnitude 1, for which

$$HH^* = dI_d, \tag{3.1}$$

where H^* denotes the conjugate transpose of H. Observe that all row vectors of H are pairwise orthogonal so Hadamard matrices are in one-to-one correspondence with ordered unitary bases. If, additionally, the entries of H are all in $\{1, -1\}$, then H is called a real Hadamard matrix.

If H is a Hadamard matrix and we swap any two rows, then the resulting matrix is also Hadamard. This works for columns as well. One may also check that multiplying any row or column of a Hadamard matrix by a unit scalar preserves the Hadamard property. Two Hadamard matrices are considered to be *equivalent* if we may obtain one from the other by permuting rows and/or columns and possibly multiplying some rows or columns by phases of magnitude 1.

For convenience, we denote any row vector as $[u_1, u_2, \ldots]$ and denote any column vector as $(u_1, u_2, \ldots)$ so that $(u_1, u_2, \ldots) = [u_1, u_2, \ldots]^T$.

Let H be an order d complex Hadamard matrix. Define h_j as the jth row of H and let $\{h_j\}_{j=1}^d$ denote the set of all rows. For a complex number v and $1 \leq k \leq d$, let $H_k(v)$ denotes the set of d vectors in $\mathbb{C}^d$ obtained by scaling the kth entry of each h_j by v. Thus, given v, we obtained d sets $H_1(v), H_2(v), \ldots, H_d(v)$ and we consider $H(v) = \bigcup_{k=1}^d H_k(v)$.

Examples 3.1.1. *Let H be the following order 2 complex Hadamard matrix:*

$$H = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

Then $H(v)$ consists of the following 4 vectors:

$$[v, i], [v, -i] \in H_1(v)$$
$$[1, vi], [1, -vi] \in H_2(v).$$

For example, $H(2) = \{[2, i], [2, -i], [1, 2i], [1, -2i]\}$.

Notice that in the Weyl-Heisenberg construction, the elements of a SIC-POVM are represented by unit column vectors. In the construction using the Hadamard matrix, we represent the elements of a SIC-POVM as unnormalized row vectors.

3.2 Allowable Construction Parameters

In the set $H(v)$, the inner product of any vector with itself is $d - 1 + |v|^2$. Aside from these, there are only three types of inner products that can arise between distinct vectors of $H(v)$:

    (i) the inner product of two distinct vectors within a set $H_j(v)$;

    (ii) the inner product of two vectors of distinct sets $H_j(v), H_k(v)$ which are derived from the same row of H;

    (iii) the inner product of two vectors of distinct sets $H_j(v), H_k(v)$ which are derived from distinct rows of H.

Then $H(v)$ forms a set of d^2 equiangular lines if and only if the equations obtained by equating the magnitudes of every inner product of Type (i), (ii) and (iii) have a solution. And $H(v)$ also forms a SIC-POVM if each vector in $H(v)$ is normalized. For example, in the set $H(2)$ constructed above, the vectors all have length $\sqrt{5}$ with Type (i),(ii), (iii) inner products equal to ± 3, 4 and 0, respectively. This example provided is neither a set of equiangular lines nor a SIC-POVM.

Lemma 3.2.1. *Let $v = a + ib$ for $a, b \in \mathbb{R}$. For all d, every inner product of Type (i) has magnitude $|a^2 + b^2 - 1|$ and every inner product of Type (ii) has magnitude $|2a + d - 2|$.*

Proof. Consider any two row vectors from H_d:

$$h_j = [h_{j1}, h_{j2}, ..., h_{jd}],$$
$$h_k = [h_{k1}, h_{k2}, ..., h_{kd}],$$

where h_{mn} denotes the entry of H in m^{th} row and n^{th} column. Since all row vectors of H_d are pairwise orthogonal, then $\langle h_j, h_k \rangle = 0$ for all $j, k \in \{1, ..., d\}$ and $j \neq k$. According to Equation (3.1), $\langle h_j, h_k \rangle = d$ for $j = k$.

For inner product of Type (i), consider any two distinct vectors $V_{jl}, V_{kl} \in H_l(v)$ such that v is multiplied by the lth coordinate of h_j, h_k (in this case, $j \neq k$):

$$V_{jl} = [h_{j1}, h_{j2}, ..vh_{jl}., h_{jd}],$$
$$V_{kl} = [h_{k1}, h_{k2}, ..vh_{kl}., h_{kd}].$$

Then the magnitude of the inner product of V_{jl} and V_{kl} is given by

$$|\langle V_{jl}, V_{kl} \rangle| = \left| \left(\sum_{i=1, i \neq l}^{d} \overline{h_{ji}} h_{ki} \right) + \overline{vh_{jl}} v h_{jl} \right|$$

$$= \left| \left(\sum_{i=1}^{d} \overline{h_{ji}} h_{ki} \right) + (|v|^2 - 1) \overline{h_{jl}} h_{jl} \right|$$

$$= |\langle h_j, h_k \rangle + (|v|^2 - 1)|$$

$$= ||v|^2 - 1|$$

$$= |a^2 + b^2 - 1|.$$

For inner product of Type (ii), consider the corresponding vectors in $H_l(v)$ and $H_m(v)$ $(m \neq l)$ obtained from the same row h_j :

$$V_{jl} = [h_{j1}, h_{j2}, ..vh_{jl}...h_{jm}, ..., h_{jd}],$$
$$V_{jm} = [h_{j1}, h_{j2}, ..h_{jl}...vh_{jm}, ..., h_{jd}].$$

The magnitude of the inner product between V_{jl} and V_{jm} is

$$|\langle V_{jl}, V_{jm} \rangle| = \left| \left(\sum_{i=1, i \neq l, m}^{d} \overline{h_{ji}} h_{ji} \right) + \overline{vh_{jl}} h_{jl} + \overline{h_{jm}} v h_{jm} \right|$$

$$= \left| \left(\sum_{i=1}^{d} \overline{h_{ji}} h_{ji} \right) + (\overline{v} - 1) \overline{h_{jl}} h_{jl} + (v - 1) \overline{h_{jm}} h_{jm} \right|$$

$$= |\langle h_j, h_j \rangle + (\overline{v} - 1) + (v - 1)|$$

$$= |d + (\overline{v} - 1) + (v - 1)|$$

$$= |2a + d - 2|.$$

$\square$

Proposition 3.2.2. *Let H be an order 2 complex Hadamard matrix. Then $H(v)$ is a set of 4 equiangular lines in $\mathbb{C}^2$ if and only if*

$$v \in \left\{ 1/2(1 \pm \sqrt{3})(1 + i), 1/2(1 \pm \sqrt{3})(1 - i), -1/2(1 \pm \sqrt{3})(1 - i), -1/2(1 \pm \sqrt{3})(1 - i) \right\}.$$

Proof. Up to equivalence, the only order 2 complex Hadamard matrix is

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{3.2}$$

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Then it is easy to check by hand that all inner products of Type (iii) that occur in $H(v)$ (where $v = a + bi$, $a, b \in \mathbb{R}$) have magnitude $|v - \overline{v}| = |2b|$. Therefore by Lemma 3.2.1, $H(v)$ is a set of equiangular lines if and only if $v = a + ib$ satisfies the equations

$$|a^2 + b^2 - 1| = |2a| = |2b|. \tag{3.3}$$

This can be done exactly when $a \in \{1/2(1 \pm \sqrt{3}), -1/2(1 \pm \sqrt{3})\}$ and $b = \pm a$. $\qquad \square$

Proposition 3.2.3. *Let H be an order 3 complex Hadamard matrix. Then $H(v)$ is a set of 9 equiangular lines in $\mathbb{C}^3$ if and only if $v \in \{0, -2, 1 \pm \sqrt{3}i\}$.*

Proof. Claim: Let $\omega = e^{2\pi i/3}$, up to equivalence, the only order 3 complex Hadamard matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}. \tag{3.4}$$

Recall that for any order 3 Hadamard matrix, we can use two methods to get an equivalent Hadamard matrix:

(i) multiplying any row or/and column by a unit scalar,

(ii) permuting any row or/and column.

Let M be any order 3 Hadamard matrix, and let m_{jk} denotes the entry of M in jth row and kth column. First we use (i) to make the entries of the first row and column to be all 1's. Then we obtained an Hadamard matrix M' that is equivalent to M.

$$M' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & m'_{22} & m'_{23} \\ 1 & m'_{32} & m'_{33} \end{bmatrix}.$$

Since M' is also Hadamard, row 1 is orthogonal to row 2. Then we have $1 + m'_{22} + m'_{23} = 0$. The only solution of this equation is $\{m'_{22}, m'_{23}\} = \{\omega, \omega^2\}$ or $\{\omega^2, \omega\}$. Similarly, $\{m'_{32}, m'_{33}\} = \{\omega, \omega^2\}$ or $\{\omega^2, \omega\}$. Furthermore, row 2 and row 3 cannot be the same since they are orthogonal to each other, then $M' = H$ or M' is equivalent to H by permuting row 2 and row 3. Thus, H is the only order 3 complex Hadamard matrix up to equivalence.

Here are some useful properties of any primitive cube root of unity ω:

$$|\omega^j| = 1 \quad \forall j = \in \mathbb{Z}, \tag{3.5}$$
$$1 + \omega + \omega^2 = 0, \tag{3.6}$$
$$\omega^3 = 1, \tag{3.7}$$
$$\{\omega, \omega^2\} = \{-1/2 + i\sqrt{3}/2, -1/2 - i\sqrt{3}/2\}, \tag{3.8}$$
$$\overline{\omega} = \omega^2, \tag{3.9}$$

where Equation (3.8) is determined by applying the Euler's formula on $\omega = e^{2\pi i/3}$.

All inner products of Type (iii) that occur in $H(v)$ are derived from two different row vectors of H, which have inner product $1 + \omega + \omega^2 = 0$. For any two vectors in $H(v)$, we can verify that their inner product takes the form $\omega^n(v + \bar{v}\omega + \omega^2)$ or $\omega^n(v + \omega + \bar{v}\omega^2)$ for some $n \in \{0,1,2\}$. If $v = a + ib$ with $a, b \in \mathbb{R}$, these inner products have magnitude $|a - 1 + b\sqrt{3}|$ and $|a - 1 - b\sqrt{3}|$, respectively (the results come from directly substituting Equation (3.8) and $v = a + ib$). Therefore by Lemma 3.2.1, $H(v)$ is a set of 9 equiangular lines if and only if v satisfies the equations

$$|a^2 + b^2 - 1| = |2a + 1| = |a - 1 + b\sqrt{3}| = |a - 1 - b\sqrt{3}| \qquad (3.10)$$

This can be done exactly when $(a, b) \in \{(0,0), (-2,0), (1, \pm\sqrt{3}/2)\}$. $\qquad\square$

Theorem 3.2.4. *Let $d > 3$ and let H be an order d complex Hadamard matrix. Then $H(v)$ is a set of d^2 equiangular lines if and only if $d = 8$, where H is equivalent to a real Hadamard matrix and $v \in \{-1 \pm 2i\}$.*

Proof. Let $H = (h_{jk})$ be an order d complex Hadamard matrix. We consider two cases.
Case 1 : Suppose for every pair of distinct row vectors of H, all summands of the inner product of the rows take values in a set $\{\xi, -\xi\}$ for some $\xi \in \mathbb{C}$ (depending on the row pair) of magnitude 1.

$$H = \begin{bmatrix} h_{11} & h_{12}, & \cdots & h_{1k} & \cdots & h_{1d} \\ h_{21} & h_{22} & \cdots & h_{2k} & \cdots & h_{2d} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{j1} & h_{j2} & \cdots & h_{jk} & \cdots & h_{jd} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{d1} & h_{d2} & \cdots & h_{dk} & \cdots & h_{dd} \end{bmatrix}, \qquad (3.11)$$

We may divide each column of H by the first entry to get a new Hadamard matrix H' that is equivalent to H:

$$H' = \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ \dfrac{h_{21}}{h_{11}} & \dfrac{h_{22}}{h_{12}} & \cdots & \dfrac{h_{2k}}{h_{1k}} & \cdots & \dfrac{h_{2d}}{h_{1d}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \dfrac{h_{j1}}{h_{11}} & \dfrac{h_{j2}}{h_{12}} & \cdots & \dfrac{h_{jk}}{h_{1k}} & \cdots & \dfrac{h_{jd}}{h_{1d}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \dfrac{h_{d1}}{h_{11}} & \dfrac{h_{d2}}{h_{12}} & \cdots & \dfrac{h_{dk}}{h_{1k}} & \cdots & \dfrac{h_{dd}}{h_{1d}} \end{bmatrix}. \qquad (3.12)$$

Suppose all summands of the inner product between row 1 and row j take values in a set $\{\xi_j, -\xi_j\}$ for some $\xi_j \in \mathbb{C}$ of magnitude 1; i.e. $h_{jk} \cdot \overline{h_{1k}} \in \{\xi_j, -\xi_j\}$ for all k. Now we multiply row j by the corresponding term ξ_j^{-1} so that every entry of H' (except for $j = 1$) is ± 1

$$\frac{h_{jk}}{h_{1k}} \cdot \frac{1}{\xi_j} = \frac{h_{jk} \cdot \overline{h_{1k}}}{h_{1k} \cdot \overline{h_{1k}}} \cdot \frac{1}{\xi_j} = \frac{h_{jk} \cdot \overline{h_{1k}}}{\xi_j} = \pm 1. \qquad (3.13)$$

Thus we obtain a real Hadamard matrix H'' that is also equivalent to H.

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Notice that for each row $j \neq 1$ in H'' , the number of 1's and -1's should be equal to $d/2$ so that row j is orthogonal to the first row. Then for inner product of Type (iii), the inner product would be either

$$(-1)^j(v - \bar{v} + (d/2 - 1)(1) + (d/2 - 1)(-1) = (-1)^j(v + \bar{v} - 2)$$

or

$$(-1)^j(v + \bar{v} + (d/2 - 2)(1) + (d/2)(-1) = (-1)^j(v - \bar{v})$$

for some $j \in \{0, 1\}$.

For $v = a + ib$ with $a, b \in \mathbb{R}$, these inner products have magnitude $|2a - 2|$ or $|2b|$ respectively, and both magnitudes occur. Using Lemma 3.2.1, $H(a + ib)$ is therefore a set of equiangular lines if and only if we can solve the equations

$$|a^2 + b^2 - 1| = |2a + d - 2| = |2a - 2| = |2b|. \tag{3.14}$$

This can be done exactly when $d = 8$ and $(a, b) = (-1, \pm 2)$. $\qquad \square$

3.3   Solving The Equations

The solutions of all the systems of equations discussed earlier are obtained using the following method.
For example, in Theorem 3.2.4, we need to solve

$$|a^2 + b^2 - 1| = |2a + d - 2| = |2a - 2| = |2b|. \tag{3.15}$$

Square both sides of each equation and turn it into a system of equations:

$$(a^2 + b^2 - 1)^2 = (2a - 2)^2,$$
$$(2a - 2)^2 = 4b^2,$$
$$(2a + d - 2)^2 = 4b^2.$$

The solution of first two equations is $a = -1, b = \pm 2$ and $a = 1, b = 0$. Substituting the solutions into the third equation, we get $a = -1, b = \pm 2, d = 8$, and $a = 1, b = 0, d = 0$. Since we require d to be greater or equal to 3, the only solution of the system of equation is $d = 8$ and $(a, b) = (-1, \pm 2)$. The solution of the other systems of equations are obtained in a similar way.

Chapter 4

Hoggar's Construction

Hoggar gave a construction of a SIC-POVM in dimension 8 in 1981 [5, 6] from the diameters of a polytope in quaternionic space. The quaternions are the number system that extends the complex numbers. Quaternions are generally represented in the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and i, j, k are the fundamental quaternion units satisfying

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ijk = -1.$$

It is shown that the set, when complexified to 64 lines in $\mathbb{C}^8$, becomes equiangular, with $\cos^2 \theta = 1/9$ for any pair of distinct lines. The angle θ between two lines is defined via representative vectors u, v, by $\cos^2 \theta = \dfrac{|\langle u, v \rangle|^2}{(|u||v|)^2}$, where $\langle u, v \rangle = \sum \bar{u}_i v_i$ denotes the Hermitian inner product, and $|u|^2 = |\langle u, u \rangle|$.

4.1 Hoggar's Construction

Let $r = \sqrt{2}$, $i = \sqrt{-1}$, $s = \dfrac{1+i}{r}$, $t = \dfrac{1-i}{r} = \bar{s}$. Let O, D, S and R be the columns of the matrix of vectors in $\mathbb{C}^2$ shown below, with rows numbered 1 to 4.

Row	O	D	S	R
1	(0,0)	(s,t)	$(s,-s)$	$(0,r)$
2	(0,0)	$(t,-s)$	(s,s)	$(r,0)$
3	(0,0)	(t,s)	$(s,-s)$	$(0,ri)$
4	(0,0)	$(s,-t)$	$(s,-s)$	$(ri,0)$

Table 4.1: Matrix of vectors in $\mathbb{C}^2$ giving the 64 lines

We use notation C_m to denote the entries of the matrix, where C denotes the column and m denotes the row. For example,

$$D_3 = (t, s), \quad S_1 = (s, -s), \quad R_4 = (ri, 0).$$

Now define the vectors in $\mathbb{C}^8$. Each row in the table above defines one vector. For example, for row 1 :

$$v_1 = (O_1, D_1, S_1, R_1) = (0, 0, s, t, s, -s, 0, r).$$

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Inserting "−" signs in front of any pair in $\{D, S, R\}$, then there are 4 different sign patterns denoted by p:

$$p = 1 : + + ++,$$
$$p = 2 : + - -+,$$
$$p = 3 : + - +-,$$
$$p = 4 : + + --,$$

where the signs are in front of O, D, S, R respectively. For example, 3 other vectors can be obtained from row 1.

$$v_2 = (O_1, -D_1, -S_1, R_1),$$
$$v_3 = (O_1, -D_1, S_1, -R_1),$$
$$v_4 = (O_1, D_1, -S_1, -R_1).$$

Doing the same operation for each row in the table, we obtain a total of 16 vectors spanning distinct lines through the origin in $\mathbb{C}^8$. We call these Type 1 lines.

Now, for each $n = 2, 3, 4$, we obtain 16 vectors said to be of Type n by interchanging columns 1 and n in the matrix, and similarly the remaining two columns.

$$\text{Type } 1 : ODSR,$$
$$\text{Type } 2 : DORS,$$
$$\text{Type } 3 : SROD,$$
$$\text{Type } 4 : RSDO.$$

This gives an action of the Klein 4-group on our set of vectors, with the types as blocks of imprimitivity.

For each type we obtain 16 vectors using the previous method. In this way, we construct a set of 64 vectors spanning distinct lines through the origin in $\mathbb{C}^8$. Now let's define our notation for the vectors. For any vector v_l, where $l = 16(n-1) + 4(m-1) + s$ ($1 \le s, n, m \le 4$), it is generated from row m, with sign pattern p and of Type n. For example, a vector from row 3 , with sign pattern 4 and of Type 2 would be denoted by

$$v_{28} = (D_3, O_3, -R_3, -S_3).$$

Notice that for each vector v_l ($l = 1, 2, ..., 64$), we have

$$|v_l|^2 = \langle D_j, D_k \rangle + \langle S_j, S_k \rangle + \langle R_j, R_k \rangle.$$

For example,

$$\begin{aligned}
|v_1|^2 &= \langle O_1, O_1 \rangle + \langle D_1, D_1 \rangle + \langle S_1, S_1 \rangle + \langle R_1, R_1 \rangle \\
&= |(0, 0)|^2 + |(s, t)|^2 + |(s, -s)|^2 + |(0, r)|^2 \\
&= 0 + 2 + 2 + 2 \\
&= 6
\end{aligned}$$

	1	2	3	4
1	2,2,2	2i,0,0	0,0,2i	0,2i,0
2	−2i,0,0	2,2,2	0,2i,0	0,0,2i
3	0,0,2i	0,−2i,0	2,2,2	−2i,0,0
4	0,2i,0	0,0,2i	2i,0,0	2,2,2

Table 4.2: Table of the triple $(\langle D_j, D_k \rangle, \langle S_j, S_k \rangle, \langle R_j, R_k \rangle)$

We then construct a table with entry of jth row and k^{th} column being the triple $(\langle D_j, D_k \rangle, \langle S_j, S_k \rangle,$ $\langle R_j, R_k \rangle)$. We use this table to help us check the inner product between any two vectors.

Notice that for each vector v_l $(l = 1, 2, ..., 64)$, we have

$$|v_l|^2 = \langle D_j, D_k \rangle + \langle S_j, S_k \rangle + \langle R_j, R_k \rangle.$$

Using Table 2, we find that independent of m,

$$|v_l|^2 = 2 + 2 + 2 = 6.$$

To verify that the angle θ between any two lines satisfies $\cos^2 \theta = 1/9$, we must show that $|\langle v_{l_1}, v_{l_2} \rangle|^2 = 4$ for any distinct l_1, l_2; i.e. all the inner products have modulus 2.

For inner product of the same type, observe that the inner products all take the form

$$|\langle D_{m_1}, D_{m_2} \rangle \pm \langle S_{m_1}, S_{m_2} \rangle \pm \langle R_{m_1}, R_{m_2} \rangle|.$$

Excluding the case when $m_1 = m_2$ and all signs are positive.

The $+$ sign occurs when two vectors have the same sign pattern; the $-$ sign occurs when two vectors have different sign patterns. Using Table 2 again, it is obvious that all inner products of the same type have modulus 2.

For inner product of vectors of different types, observe that the inner products all take the form

$$|\langle L_{m_1}, M_{m_2} \rangle \pm \langle M_{m_1}, L_{m_2} \rangle \pm \langle N_{m_1}, O_{m_2} \rangle \pm \langle O_{m_1}, N_{m_2} \rangle|.$$

where L, M, N are chosen from $\{S, R, D\}$. Since O have all 0 entries, then the inner product is just $|\langle L_{m_1}, M_{m_2} \rangle \pm \langle M_{m_1}, L_{m_2} \rangle|$, for each pair (L, M) chosen from $\{S, R, D\}$.

To check this inner product, we need to calculate $\langle S_j, R_k \rangle$, $\langle D_j, R_k \rangle$, and $\langle S_j, D_k \rangle$ for all j, k from 1 to 4. Using Table 1, we calculated that the inner products $\langle S_j, R_k \rangle$ and $\langle D_j, R_k \rangle$ take values in the set $\{\pm rs, \pm rt, \pm irs, \pm irt\}$ and $\langle S_j, D_k \rangle$ takes values in the set $\{r(s \pm t), ri(s \pm t), rs(1 \pm i), rt(1 \pm i)\}$ They all have modulus 2. Therefore we have verified that these 64 vectors represent a SIC-POVM in $\mathbb{C}^8$. It turns out that the SIC-POVM $H(v)$ we obtained in $\mathbb{C}^8$ in the previous chapter using the Hadamard matrix is equivalent to Hoggar's line up to a unitary transformation [13].

4.2 Group Construction Of Hoggar's Line

In 2011, Chris Godsil and Aidan Roy [3] found that Hoggar's line may also be constructed using a group of 64 unitary matrices acting on a single vector. Moreover, d^2 equiangular lines can be constructed using this particular class of matrices only in dimension 2 and 8.

Let X, Y, Z be the Pauli matrices,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Then $\langle X, Y, Z \rangle$ is the Pauli group. We consider the group modulo $-I$, that is, choose any three matrices from $\{X, Y, Z, I\}$, multiplicity allowed, and apply tensor product to these three matrices. Then we have a group of 64 elements, each of these elements is represented as an 8 by 8 matrix over $\mathbb{C}$.

Let $v = (0, 0, s, t, s, -s, 0, r)$ be the same as the first row of Table (4.1) in the Hoggar's construction. Then $\{Av \mid A \in G\}$ is a set of 64 equiangular lines, equivalent to Hoggar's lines constructed previously.

Lemma 4.2.1. *For any $d = 2^k$, consider $v = (v_1, v_2, ...v_d) \in \mathbb{C}^d$. Let $G_k = \{X, Y, Z, I\}^{\otimes k}$. The d^2 lines $\{Av | A \in G_k\}$ can only be equiangular for $k = 1$ or 3.*

Proof. Suppose $S = \{Av \mid A \in G_k\}$ is a SIC-POVM in $\mathbb{C}^d$, where $d = 2^k$. Let $v = (v_1, v_2...v_d) \in S$ and let $a = (a_1, a_2, ...a_d)$, where $a_i = \bar{v}_i v_i$ ($1 \leq i \leq d$). Then each a_i is real and since $\langle v, v \rangle = 1$,

$$a_1 + a_2 + ... + a_d = 1.$$

Let $Z_k = (I \otimes I... \otimes I \otimes Z)$, which is a k-fold tensor product

$$Z_k = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & -1 \end{pmatrix}.$$

Since $Z_k \in G_k$, $Z_k v \in S$. Furthermore, because S is a SIC-POVM,

$$\langle v, Z_k v \rangle = a_1 - a_2 + a_3 - a_4 + ... + a_{d-1} - a_d = \pm \frac{1}{\sqrt{d+1}}.$$

Let H be a d by d Hadamard matrix:

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes k}.$$

For a string s defined as $s = \{s_1, s_2, \cdots, s_k\} \in \{0, 1\}^k$, define $Z(s) = Z^{s_1} \otimes Z^{s_2} \cdots \otimes Z^{s_k}$. E.g. for $s = \{001\}, Z_s = I \otimes I \otimes Z$. The Hadamard matrix has its rows naturally indexed by binary strings from $\{0, 1\}^k$. More precisely, the rows of Hadamard matrix could be written in this notation,

$$h_s = \begin{bmatrix} 1 \\ (-1)^{s_1} \end{bmatrix}^T \otimes \begin{bmatrix} 1 \\ (-1)^{s_2} \end{bmatrix}^T \otimes \cdots \otimes \begin{bmatrix} 1 \\ (-1)^{s_k} \end{bmatrix}^T,$$

where each component is a row vector.

For example, in a Hadamard matrix of order 8, the fourth row could be written as

$$h_4 = (1, -1, -1, 1, 1, -1, -1, 1)$$

$$= \begin{bmatrix} 1 \\ (-1)^0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ (-1)^1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ (-1)^1 \end{bmatrix}$$

$$= h_{\{011\}}.$$

Claim: $Z(s)v = h_s^T \circ v$, where $\circ$ denotes the entry-wise product.

E.g.

$$h_{\{011\}}^T \circ v = h_4^T \circ v = [v_1, -v_2, -v_3, v_4, v_5, -v_6, -v_7, v_8]$$

$$= \begin{pmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & -1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & 1 \end{pmatrix} v$$

$$= (Z^0 \otimes Z^1 \otimes Z^1)v$$

$$= Z(\{001\})v.$$

Then

$$\langle v, Z(s)v \rangle = \langle v, h_s^T \circ v \rangle = \sum_j \overline{v}_j h_{s_j}^T v_j = \langle \overline{h}_s^T, \overline{v} \circ v \rangle = \langle \overline{h}_s^T, a \rangle = h_s a.$$

The result will be a single number since h_s is a row vector and a is a column vector.

For any row h_s of H except for row 1, indexed $\{000 \cdots 0\}$, we have $Z(s) \in \{I,Z\}^{\otimes k}$ satisfying $\langle v, Z(s)v \rangle = h_s a = \pm \dfrac{1}{\sqrt{d+1}}$. The first row of H is trivial since $Z(\{00 \cdots 0\}) = I_d$ and $\langle v, I_d v \rangle = 1$.

We now obtain a system of equations:

$$Ha = \frac{1}{\sqrt{d+1}} \begin{pmatrix} \sqrt{d+1} \\ \pm 1 \\ . \\ . \\ . \\ \pm 1 \end{pmatrix}.$$

Since $HH^T = dI$, then $H^{-1} = \dfrac{1}{d}H^T = \dfrac{1}{d}H$. We use this to solve our system for the column vector a :

$$a = \frac{1}{d\sqrt{d+1}} H \begin{pmatrix} \sqrt{d+1} \\ \pm 1 \\ . \\ . \\ . \\ \pm 1 \end{pmatrix}. \tag{4.1}$$

The jth entry of this column vector a can be described in the form :

$$a_j = \frac{\sqrt{d+1}+C_j}{d\sqrt{d+1}},\tag{4.2}$$

for some odd integer C_j, since d is even.

Next, consider the terms of the form $b_j = \bar{v}_j v_{j+1}$. Let $X_k = (I \otimes ... \otimes X)$, which is another k-fold tensor product

$$X_k = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Let $b = (b_1, \bar{b}_1, b_3, \bar{b}_3, \cdots, b_{d-1}, \bar{b}_{d-1})$. We have $X_k Z(s) \in G_k$, therefore $X_k Z(s)v \in S$. Then

$$\begin{aligned}
\frac{1}{\sqrt{d+1}} &= |\langle v, X_k Z(s)v \rangle| \\
&= |v^* X_k Z(s)v| \\
&= |(X_k v)^* Z(s)v| \\
&= |\langle X_k v, h_s^T \circ v \rangle| \\
&= |\sum_j (\overline{X_k v})_j (h_s^T)_j v_j| \\
&= |\sum_j (h_s^T)_j (\overline{X_k v})_j v_j| \\
&= |\langle h_s^T, b \rangle| \\
&= |h_s b|.
\end{aligned}$$

If $s_k = 0$, then

$$\begin{aligned}
h_s b &= \left(\begin{bmatrix} 1 \\ (-1)^{s_1} \end{bmatrix}^T \otimes \begin{bmatrix} 1 \\ (-1)^{s_2} \end{bmatrix}^T \otimes \cdots \otimes \begin{bmatrix} 1 \\ (-1)^0 \end{bmatrix}^T \right) b \\
&= \sum_j \pm(b_j + \bar{b}_j) \\
&= \pm \frac{1}{\sqrt{d+1}}
\end{aligned}$$

(Since $b_j + \bar{b}_j$ is always real, then $h_s b$ is also real).
If $s_k = 1$, then

$$\begin{aligned}
h_s b &= \left(\begin{bmatrix} 1 \\ (-1)^{s_1} \end{bmatrix}^T \otimes \begin{bmatrix} 1 \\ (-1)^{s_2} \end{bmatrix}^T \otimes \cdots \otimes \begin{bmatrix} 1 \\ (-1)^1 \end{bmatrix}^T \right) b \\
&= \sum_j \pm(b_j - \bar{b}_j) \\
&= \pm \frac{i}{\sqrt{d+1}}
\end{aligned}$$

(Since $b_j - \bar{b}_j$ is always pure imaginary, then $h_s b$ is also pure imaginary).

Since s is the binary string for the row number of h_s, then if s_k is 0, s is even, and if s_k is 1, s is odd. Now we could obtain another system of equation:

$$Hb = \frac{1}{\sqrt{d+1}} \begin{pmatrix} \pm 1 \\ \pm i \\ \vdots \\ \pm 1 \\ \pm i \end{pmatrix}.$$

Solve the system of equation for the column vector b,

$$b = \frac{1}{d\sqrt{d+1}} H \begin{pmatrix} \pm 1 \\ \pm i \\ \vdots \\ \pm 1 \\ \pm i \end{pmatrix}. \tag{4.3}$$

All entries of b take the form $b_j \in \dfrac{\pm l(1+i)}{d\sqrt{d+1}}$ for some integer l. Thus, $b_j \bar{b}_j = \dfrac{m}{d^2(d+1)}$ for some integer m; i.e. $b_j \bar{b}_j$ is rational.

Now, in a similar way, define $g_j = \bar{v}_j v_{j+2}$ and $h_j = \bar{v}_{j+1} v_{j+2}$. Let $M_1, M_2 \in \{I, X\}^{\otimes k}$ be permutation matrices , where M_1 takes coordinate j to $j+2$ on v (mod d), and M_2 takes coordinate $j+1$ to $j+2$ on v (mod d). Then by considering $\langle v, M_1 Z(s) v \rangle$ and $\langle v, M_2 Z(s) v \rangle$, we similarly get that $g\bar{g}$ and $h\bar{h}$ are rational.

Furthermore, notice that

$$g\bar{g} = \bar{v}_j v_{j+2} v_j \bar{v}_{j+2} = a_j a_{j+2}. \tag{4.4}$$

and

$$h\bar{h} = \bar{v}_{j+1} v_{j+2} v_{j+1} \bar{v}_{j+2} = a_{j+1} a_{j+2}. \tag{4.5}$$

From Equation (4.2), for any integer m and n,

$$a_m a_n = \frac{d+1+C_m C_n + (C_m + C_n)\sqrt{d+1}}{d^2(d+1)},$$

which is rational if and only if one of the following cases is satisfied:

case 1: $\sqrt{d+1}$ is rational ;

case 2: $(C_m + C_n) = 0$ for all integers m, n.

If case 2 is satisfied, from Equations (4.4) and (4.5), we will have $C_j = -C_{j+1}$; $C_j = -C_{j+2}$; $C_{j+1} = -C_{j+2}$. This system of equations has no solution since C_j is odd for all j. Then case 2 can only be satisfied when $d = 2$ and $C_1 = -C_2$.

If case 1 is satisfied, we claim that $\sqrt{2^k + 1}$ is rational only for $k = 3$ ($k > 0$).

Recall $d = 2^k$ and suppose that $\sqrt{2^k + 1}$ is rational, then $2^k + 1 = (x/y)^2$ where x, y are positive integers which, without loss of generality, have no common factor. We have $y^2(2^k + 1) = x^2$. Suppose y has a prime factor p, then $x^2 = m^2 p^2 (2^k + 1)$. Since $p|x^2$, we have $p|x$. This contradicts the assumption that x and y have no common factor. Therefore y cannot have any prime factors; i.e. $y = 1$. Therefore

$$2^k = x^2 - 1 = (x - 1)(x + 1),$$

so $x - 1$ and $x + 1$ are both powers of 2. The only powers of 2 differing by 2 are 2^1 and 2^2, so $x = 3$ and $k = 3$.

Now we have shown that $\{Av | A \in G_k\}$ is a SIC-POVM if and only if k is 1 or 3. $\qquad \square$

Chapter 5

SIC-POVMs In Dimension 7 And 19

In the analytical solution of fiducial vectors found by Grassl and Scott [9], most solutions are messy. However, there are a few neat solutions in some specific dimensions that are very short compared to other solutions. In addition, we found some interesting number theoretic properties from solutions in dimension 7 and 19.

5.1   Argument Legendre Fiducial Vectors

Let p be a prime number, recall the Legendre symbol

$$\left(\frac{j}{p}\right) = \begin{cases} 0 & \text{if } j \equiv 0 \bmod p \\ 1 & \text{if } j \text{ is a quadratic residue} \\ -1 & \text{if } j \text{ is a quadratic non-residue} \end{cases}.$$

Definition 5.2. *We say a fiducial vector $z \in \mathbb{C}^p$ is* Argument Legendre (AL) *if there exist $a, b, \theta \in \mathbb{R}$ such that*

$$z_j = \begin{cases} b & \text{if } j = 0 \\ ae^{i\left(\frac{j}{p}\right)\theta} & \text{if } j \neq 0 \end{cases}.$$

This definition is introduced in Khatirinejad's Ph.D. thesis in 2008 [8]. Notice that here z is indexed from 0 to $d-1$, therefore from the definition, z could be represented by three terms : 0th term, which is real; terms in the position of a quadratic residue (QR terms); and terms in the position of a quadratic non-residue (NQR terms). In addition, the entries of QR terms and NQR terms are complex conjugates of one another. For example, the solution 7b, found by Grassl and Scott [9], may be written as

$$\phi_7 = (\alpha, \beta, \beta, \gamma, \beta, \gamma, \gamma), \tag{5.1}$$

where $\alpha = -2 - \sqrt{2} + \sqrt{2}\sigma i$, $\beta = 2$, $\gamma = \sqrt{2} - 1 + \sigma i$ and $\sigma = \sqrt{\sqrt{8} + 1}$. Working mod 7, the quadratic residues (perfect squares in mod 7) are 1, 2 and 4; the quadratic non-residues are 3, 5 and 6. If we multiply the fiducial vector ϕ_7 by a complex phase such that the 0th term is real, we further discover that the QR terms and the NQR terms are complex conjugate with each other. Therefore ϕ_7 is an AL fiducial vector.

There is another solution in dimension 19 which satisfies such a property. The solution 19e, also found by Grassl and Scott [9], may be written as

$$\phi_{19} = (\alpha, \beta, \gamma, \gamma, \beta, \beta, \beta, \beta, \gamma, \beta, \gamma, \beta, \gamma, \gamma, \gamma, \gamma, \beta, \beta, \gamma), \tag{5.2}$$

where $\alpha = \sigma + 1$, $\beta = 1 - \sigma i$, $\gamma = 1 + \sigma i$ and $\sigma = \sqrt{\sqrt{20} + 1}$. The quadratic residues mod 19 are 1, 4, 5, 6, 7, 9, 11, 16 and 17; the quadratic non-residues are 2, 3, 8, 10, 12, 13, 14, 15 and 18. It is obvious that ϕ_{19} is also Argument Legendre.

Currently, we have found only two known analytical solutions that are AL: one in dimension 7 and one in dimension 19. It is proved in Khatirinejad's Ph.D thesis [8] that AL fiducial vectors only exist in dimension 7 and 19 and can not be generalized in higher dimensions. However, with such a special property, we can further classify all AL fiducial vectors in dimension 7.

5.3   Classification Of AL Fiducial Vectors In Dimension 7

First we write the AL fiducial vector in $\mathbb{C}^7$ as

$$v = (z_0, z_r, z_r, z_n, z_r, z_n, z_n), \tag{5.3}$$

where z_0 is the 0th term, z_r is the QR term and z_n is the NQR term. The phase and displacement operators in dimension 7 are

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^6 \end{pmatrix},$$

where $\omega = e^{2\pi i/7}$ and $\omega^7 = 1$. Also, X and Z are unitary matrices and satisfy that $X^7 = Z^7 = I$ and $X^r Z^t = \omega^{-rt} Z^t X^r$ for any r and t. According to the Weyl-Heisenberg construction discussed previously in Chapter 2, by defining $v_{jk} = D_{jk} v = -\omega^{jk/2} X^j Z^k v$, we have $S = \{v_{jk} \mid j, k \in \mathbb{Z}, 0 \leq j, k \leq 6\}$ is a SIC-POVM. e.g. $v_{00} = v$. Therefore, for any two vectors $v_{lm}, v_{st} \in S$, they must satisfy

$$|\langle v_{lm}, v_{st} \rangle| = \frac{1}{\sqrt{8}}. \tag{5.4}$$

Claim: For any two vectors $v_{lm}, v_{st} \in S$, they also satisfy

$$|\langle v_{lm}, v_{st} \rangle| = |\langle v_{00}, v_{nk} \rangle| = \frac{1}{\sqrt{8}}, \tag{5.5}$$

where $v_{nk} \in S$.

For example, using the property of a unitary matrix U, we have

$$\langle v_{26}, v_{35} \rangle = \langle U v_{26}, U v_{35} \rangle.$$

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If we let U be $Z^1 X^5$ (U is unitary since Z and X are unitary), then

$$U v_{26} = -\omega^6 Z X^5 X^2 Z^6 v = -\omega^{-1} v.$$

Therefore we have

$$
\begin{aligned}
\langle v_{26}, v_{35} \rangle &= \langle U v_{26}, U v_{35} \rangle \\
&= \langle -\omega^{-1} v, -\omega^{1/2} Z X^5 X^3 Z^5 v \rangle \\
&= \langle -\omega^{-1} v, -\omega^{1/2} Z X Z^5 v \rangle \\
&= \langle -\omega^{-1} v, -\omega^{1/2} \omega X Z Z^5 v \rangle \\
&= \langle -\omega^{-1} v, -\omega^{3/2} X Z^6 v \rangle \\
&= \langle -\omega^{-1} v, -\omega^{-3/2} \omega^3 X Z^6 v \rangle \\
&= \langle -\omega^{-1} v, \omega^{-3/2} v_{16} \rangle.
\end{aligned}
$$

Since ω has modulus 1, we further get that

$$|\langle v_{26}, v_{35} \rangle| = |\langle v_{00}, v_{16} \rangle|.$$

Using similar means, we can show that Equation (5.5) holds for any two vectors $v_{lm}, v_{st} \in S$. Now, to verify that S is a SIC-POVM, we only need to check that $|\langle v_{00}, v_{nk} \rangle| = \frac{1}{\sqrt{8}}$ for every $v_{nk} \in S$. There are only 49 calculations we need to do, which is much easier than checking all $\binom{49}{2}$ inner products. We use Maple to calculate the inner products between every vector in S and the fiducial vector v. Since v is AL, then we can further restrict that $z_0 = a$, $z_r = b e^{i\theta}$ and $z_n = b e^{-i\theta} = \bar{z}_r$ for a, b. The inner product between the fiducial vector and itself is 1, this gives the equation:

$$a^2 + 6b^2 = 1. \tag{5.6}$$

Furthermore, we calculate the rest of the equations from (5.5) using Maple. The remaining equations can further be reduced to 3 more equations:

$$5b^4 + 2a^2 b^2 = 1/8 \tag{5.7}$$
$$4b^4 \cos^2 \theta (4 \cos^2 \theta - 3) + a^2 b^2 + 2b^3 a \cos \theta = 1/8 \tag{5.8}$$
$$4b^4 \cos^2 \theta - b^4 + 4b^3 a \cos \theta (2 \cos^2 \theta - 1) = 1/8 \tag{5.9}$$

Solving Equations (5.6)-(5.9) we get that

$$a = \sqrt{\frac{2 + 3\sqrt{2}}{14}}, \qquad b = \sqrt{\frac{4 - \sqrt{2}}{28}}, \qquad \cos \theta = -\sqrt{\frac{1 + 1\sqrt{2}}{4}}.$$

We further calculated that this solution is equivalent to the solution 7b, found by Grassl and Scott [9] .

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Chapter 6

SIC-POVMs In 3D

We know that most known SIC-POVMs are group covariant, and they have some degrees of geometrical symmetry. For example, in $\mathbb{C}^2$, a SIC-POVM can be represented by 4 unit vectors on a Bloch sphere as shown in the figure below.
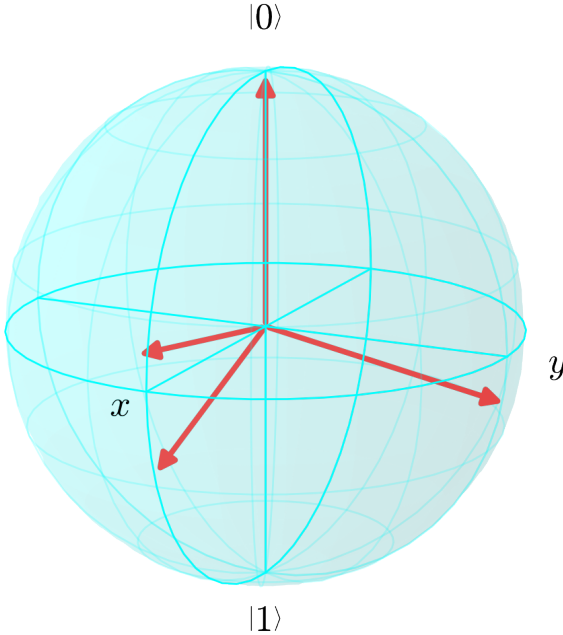

$|0\rangle$

y

x

$|1\rangle$

Figure 6.1: 4 unit vectors on a Bloch sphere represent a SIC-POVM in $\mathbb{C}^2$

We can visually observe that they are equiangular and form the shape of a regular tetrahedron. What would SIC-POVMs look like in higher dimensions? What geometrical symmetry do they have? Since plots in $\mathbb{R}^3$ are the most easily understood, we want to interpret SIC-POVMs in high dimensions using vectors in $\mathbb{R}^3$ and see what geometrical symmetries they might have in this setting.

To accomplish this, we need to introduce the Majorana representations that allow us to represent a vector in $\mathbb{C}^d$ by $d-1$ vectors in $\mathbb{R}^3$.

6.1 Majorana States In $\mathbb{C}^3$

In $\mathbb{C}^3$, up to normalization, a vector can be described in the form

$$v = (1, \frac{C_1}{C_0}, \frac{C_2}{C_0}), \tag{6.1}$$

where $C_0, C_1, C_2 \in \mathbb{C}$. In Majorana's approach, we represent this vector as:

$$v = (1, \frac{\alpha_1 + \alpha_2}{\sqrt{2}}, \alpha_1 \alpha_2) \tag{6.2}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ are called Majorana parameters (M-parameters). Then we can calculate the values of these two M-parameters by solving the quadratic equation:

$$z^2 - \sqrt{2}\frac{C_1}{C_0}z + \frac{C_2}{C_0} = 0. \tag{6.3}$$

For each M-parameter α_i, we define the corresponding Majorana vector (M-vector) a_i as a point on the Riemann Sphere [4] with the Cartesian coordinates:

$$a_i = \left(\frac{2Re(\alpha_i)}{1 + |\alpha_i|^2}, \frac{2Im(\alpha_i)}{1 + |\alpha_i|^2}, \frac{1 - |\alpha_i|^2}{1 + |\alpha_i|^2} \right). \tag{6.4}$$

The Riemann sphere and the extended complex plane (with the added point at infinity) are in one-to-one correspondence with each other under stereographic projection. If one takes the complex plane to be in the equatorial plane of the Riemann sphere, then the image of any point on the sphere is obtained by drawing the line from the south pole to it and seeing where it intersects the equatorial plane.

For each vector v in $\mathbb{C}^3$, we obtain two M-vectors in $\mathbb{R}^3$ using the method just introduced. Therefore, we have a total of 18 M-vectors for a SIC-POVM in dimension 3. We can look at the 3D plots of the M-vectors and pick out any special symmetries they may possess. In the following section, a few examples of SIC-POVMs are provided and their M-vectors are calculated and plotted. Furthermore, we can sometimes use M-vectors with special symmetry to construct new SIC-POVMs.

6.2 Examples

Examples 6.2.1. *Hesse SIC*

Taking the fiducial vector as $(0, 1, -1)$, we generate the entire SIC-POVM using the Weyl-Heisenberg construction. Then we obtain 9 unnormalized vectors in $\mathbb{C}^3$:

$$v_1 = (-\omega, 0, 1), \quad v_2 = (-\omega^2, 0, 1), \quad v_3 = (-1, 0, 1)$$
$$v_4 = (\omega^2, -1, 0), \quad v_5 = (\omega, -1, 0), \quad v_6 = (1, -1, 0)$$
$$v_7 = (0, 1, -\omega), \quad v_8 = (0, 1, -\omega^2), \quad v_9 = (0, 1, -1),$$

where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.

The M-vectors of these 9 vectors, calculated using the method previously discussed, can be divided into 3 groups:

$$\text{Group 1} \begin{cases} v_1 : a_1 = (-1/2, \sqrt{3}/2, 0), \quad a_2 = (1/2, -\sqrt{3}/2, 0), \\ v_2 : a_1 = (-1/2, -\sqrt{3}/2, 0), \quad a_2 = (1/2, \sqrt{3}/2, 0), \\ v_3 : a_1 = (1, 0, 0), \quad a_2 = (-1, 0, 0), \end{cases}$$

$$\text{Group 2} \begin{cases} v_4 : a_1 = (0, 0, 1), \quad a_2 = (\sqrt{2}/3, -\sqrt{2/3}, -1/3) \\ v_5 : a_1 = (0, 0, 1), \quad a_2 = (\sqrt{2}/3, \sqrt{2/3}, -1/3) \\ v_6 : a_1 = (0, 0, 1), \quad a_2 = (\dfrac{-2\sqrt{2}}{3}, 0, -1/3) \end{cases}$$

$$\text{Group 3} \begin{cases} v_7 : a_1 = (0, 0, -1), \quad a_2 = (\sqrt{2}/3, -\sqrt{2/3}, 1/3) \\ v_8 : a_1 = (0, 0, -1), \quad a_2 = (\sqrt{2}/3, \sqrt{2/3}, 1/3) \\ v_9 : a_1 = (0, 0, -1), \quad a_2 = (\dfrac{-2\sqrt{2}}{3}, 0, 1/3) \end{cases}$$

The 3-D plot of these three groups of M-vectors, shown from Figure 6.2 to Figure 6.4, point towards the vertices of a regular hexagon in the x-y plane, and two regular tetrahedra. Observe that the vectors in Group 3 are simply the reflection of those in Group 2 in the x-y plane as shown in Figure(6.4).
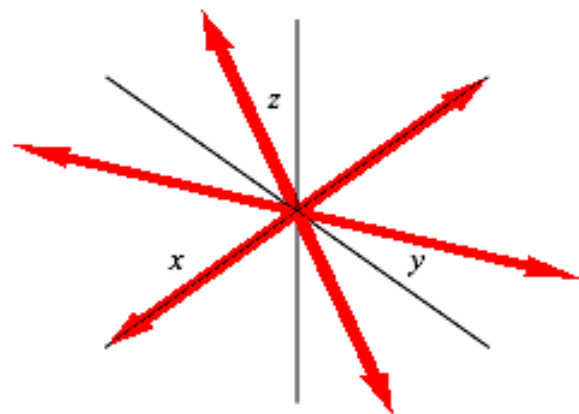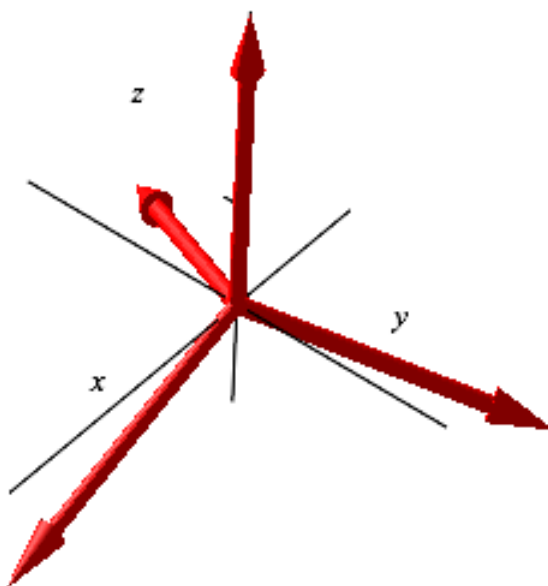
Figure 6.2: M-vectors in Group 1 of Hesse SIC

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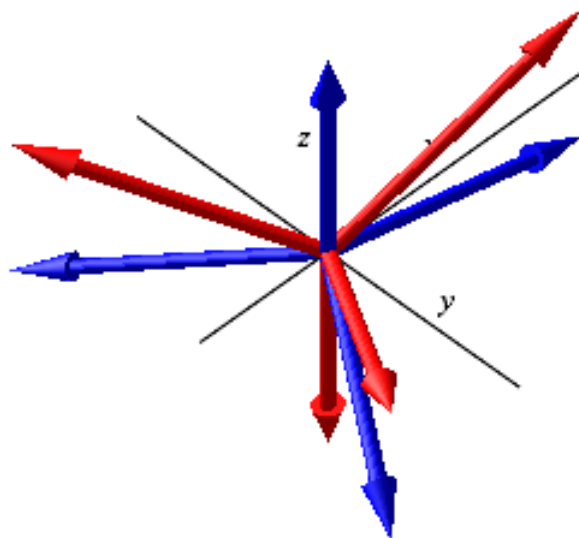Figure 6.3: M-vectors in Group 2 of Hesse SIC

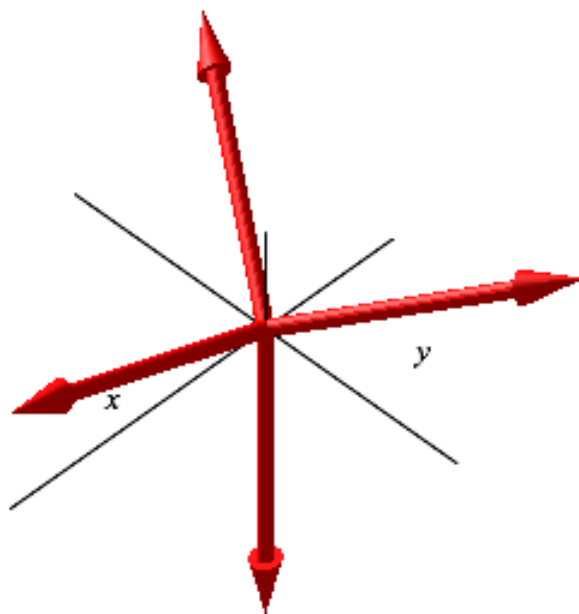Figure 6.5:   M-vectors in Group 2 (blue) and Group 3 (red) of Hesse SIC

Figure 6.4: M-vectors in Group 3 of Hesse SIC

30

Examples 6.2.2. *Appleby's SIC*

Taking the fiducial vector as $(0, e^{it}, e^{-it})$, where t is from 0 to $\pi/6$, we generate the entire SIC-POVM using the Weyl-Heisenberg group. The 9 unnormalized vectors in $\mathbb{C}^3$ are expressed with a parameter t from 0 to $\pi/6$:

$$v_1 = (\omega e^{-it}, 0, e^{it}), \quad v_2 = (\omega^2 e^{-it}, 0, e^{it}), \quad v_3 = (e^{-it}, 0, e^{it})$$
$$v_4 = (\omega e^{it}, e^{-it}, 0), \quad v_5 = (\omega^2 e^{it}, e^{-it}, 0), \quad v_6 = (e^{it}, e^{-it}, 0)$$
$$v_7 = (0, e^{it}, \omega e^{-it}), \quad v_8 = (0, e^{it}, \omega^2 e^{-it}), \quad v_9 = (0, e^{it}, e^{-it}).$$

The squared modulus of the Hermitian product of every pair is $1/4$ upon normalization, therefore the set is a SIC-POVM for any value of t between 0 and $\pi/6$. This SIC is first discovered by Appleby [1].

The 9 M-vectors, expressed in terms of the parameter t, can also be divided into 3 groups:

Group 1 $\begin{cases} v_1 : a_1 = (-\sin(t + \pi/3), \cos(t + \pi/3), 0), \quad a_2 = (\sin(t + \pi/3), -\cos(t + \pi/3), 0), \\ v_2 : a_1 = (-\sin(t - \pi/3), \cos(t - \pi/3), 0), \quad a_2 = (\sin(t - \pi/3), -\cos(t - \pi/3), 0), \\ v_3 : a_1 = (-\sin(t), \cos(t), 0), \quad a_2 = (\sin(t), -\cos(t), 0) \end{cases}$

Group 2 $\begin{cases} v_4 : a_1 = (\sin(\theta_0)\cos(2t + 2\pi/3), -\sin(\theta_0)\cos(2t + 2\pi/3), -\cos(\theta_0)), \quad a_2 = (0, 0, 1), \\ v_5 : a_1 = (\sin(\theta_0)\cos(2t + 4\pi/3), -\sin(\theta_0)\cos(2t + 4\pi/3), -\cos(\theta_0)), \quad a_2 = (0, 0, 1), \\ v_6 : a_1 = (\sin(\theta_0)\cos(2t + 2\pi/3), -\sin(\theta_0)\cos(2t + 2\pi/3), -\cos(\theta_0)), \quad a_2 = (0, 0, 1), \end{cases}$

Group 3 $\begin{cases} v_7 : a_1 = (\sin(\theta_0)\cos(2t + 2\pi/3), -\sin(\theta_0)\cos(2t + 2\pi/3), \cos(\theta_0)), \quad a_2 = (0, 0, -1), \\ v_8 : a_1 = (\sin(\theta_0)\cos(2t + 4\pi/3), -\sin(\theta_0)\cos(2t + 4\pi/3), \cos(\theta_0)), \quad a_2 = (0, 0, -1), \\ v_9 : a_1 = (\sin(\theta_0)\cos(2t + 2\pi/3), -\sin(\theta_0)\cos(2t + 2\pi/3), \cos(\theta_0)), \quad a_2 = (0, 0, -1), \end{cases}$

where $\sin(\theta_0) = \frac{2\sqrt{2}}{3}$ and $\cos(\theta_0) = 1/3$. The 3-D plots of these 3 groups of M-vectors are similar to those of the Hesse's SIC . But they differ by a rotation angle of t. In fact, when $t = \pi/6$, Appleby's SIC has exactly the same geometric configuration as Hesse's SIC .

Plotting the first group of M-vectors of the Hesse SIC and Appleby's SIC at $t = 0$ together (as shown in Figure 6.6), we observe that they differ by a rotation of $\pi/6$. Next, plotting the second group of M-vectors of the Hesse SIC and Appleby's SIC at $t = 0$ together (as shown in Figure 6.7), we observe that they differ by a rotation of $\pi/3$. Also, the the third group of M-vectors of Appleby's SIC are the reflection of the second group in the x-y plane.

We conclude that as the parameter t varies, the M-vectors of the states in the groups (v_4, v_5, v_6) and (v_7, v_8, v_9) get rotated relative to the vectors for $t = 0$ by the counter-clockwise angle $2t$ about the z-axis, while for the group (v_1, v_2, v_3) the rotation is counter-clockwise by the angle t . Since only the relative orientation of the vectors in the different groups is significant, we can take the vectors of v_1, v_2 and v_3 to be fixed and let those of the other states rotate relative to them by the counter-clockwise angle t. Then, as t varies between 0 to $\pi/6$, all the distinct configurations of this system are realized.
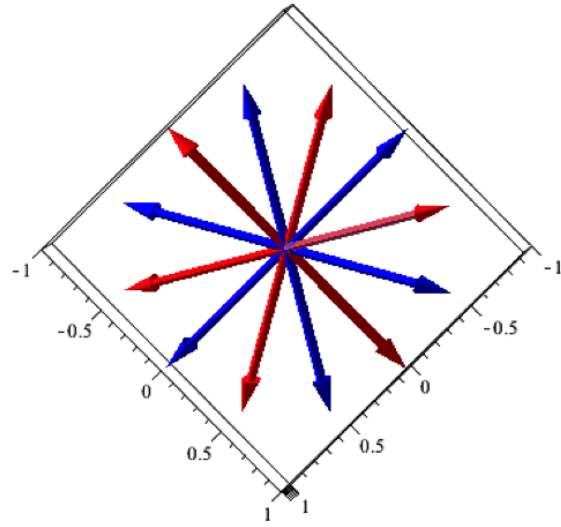
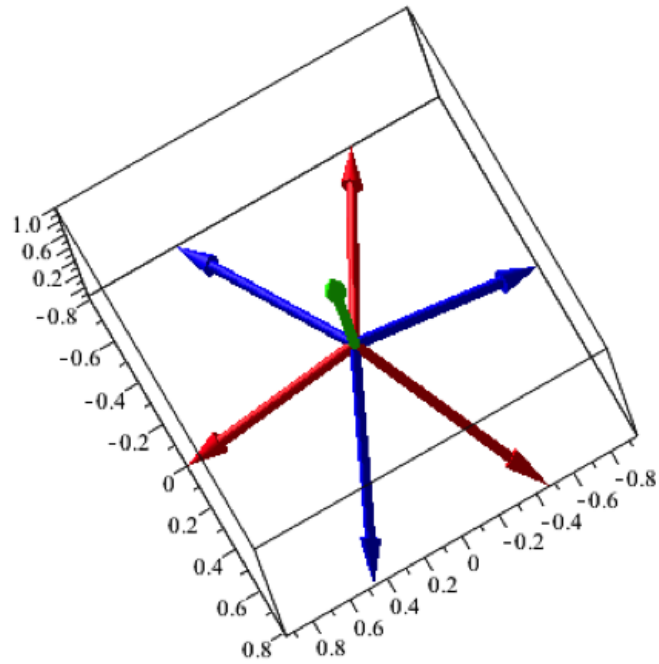Figure 6.6: Group 1, Hesse SIC (blue) and Appleby's SIC at $t = 0$ (red).

Figure 6.7: Group 2, Hesse SIC (blue) and Appleby's SIC at t=0 (red)

6.3 New Construction Of SIC-POVMs

From the examples studied so far, we see that the M-vectors of SIC-POVMs in dimension 3 have threefold symmetry about the z-axis. We can generate new SIC-POVMs based on this symmetry.

Examples 6.3.1. *Aravind-1 SIC*

In Appleby's SIC, the rotation angle of the three groups of M-vectors are related. What if they are rotated at arbitrary angles with respect to each other? Will one still have a SIC-POVM? Consider a SIC-POVM whose 18 M-vectors (denoted in spherical coordinate (θ, ϕ)) are

$$
\text{Group 1} \begin{cases} v_1: & a_1 = (\pi/2, 0), \quad a_2 = (\pi/2, \pi), \\ v_2: & a_1 = (\pi/2, \pi/3), \quad a_2 = (\pi/2, 4\pi/3), \\ v_3: & a_1 = (\pi/2, 2\pi/3), \quad a_2 = (\pi/2, 5\pi/3), \end{cases}
$$

$$
\text{Group 2} \begin{cases} v_4: & a_1 = (0,0), \quad a_2 = (\pi - \theta_0, \phi_1), \\ v_5: & a_1 = (0,0), \quad a_2 = (\pi - \theta_0, 2\pi/3 + \phi_1), \\ v_6: & a_1 = (0,0), \quad a_2 = (\pi - \theta_0, 4\pi/3 + \phi_1), \end{cases}
$$

$$
\text{Group 3} \begin{cases} v_7: & a_1 = (\pi, 0), \quad a_2 = (\theta_0, \phi_2), \\ v_8: & a_1 = (\pi, 0), \quad a_2 = (\theta_0, 2\pi/3 + \phi_2), \\ v_9: & a_1 = (\pi, 0), \quad a_2 = (\theta_0, 4\pi/3 + \phi_2), \end{cases}
$$

where $\theta_0 = \cos^{-1}(1/3)$ and ϕ_1 and ϕ_2 are arbitrary values between 0 and $\pi/6$. The geometric configuration of these 3 groups of vectors is similar to Figure 6.2, except for the difference in their relative rotation angles. In this case, the first group remains fixed, while the rotation angle of second group and third group relative to the first one is ϕ_1 and ϕ_2, respectively. These Majorana states are the same as those of Appleby's SIC when $\phi_1 = \phi_2$. However, these 18 M-vectors are more general than M-vectors of Appleby's SIC, because the M-vectors in the three groups can be rotated by arbitrary angles relative to each other. In Appleby's case, the two tetrahedrons are always the reflections of each other in the x-y plane and only the hexagon can rotate relative to them.

We can generate the vectors in $\mathbb{C}^3$ that correspond to these M-vectors. The 9 unnormalized vectors are:

$$
\begin{aligned}
v_1 &= (1, 0, -1), \quad v_2 = (1, 0, -\omega), \quad v_3 = (1, 0, -\omega^2), \\
v_4 &= (1, e^{i\phi_1}, 0), \quad v_5 = (1, \omega e^{i\phi_1}, 0), \quad v_6 = (1, \omega^2 e^{i\phi_1}, 0), \\
v_7 &= (0, 1, e^{i\phi_2}), \quad v_8 = (0, 1, \omega e^{i\phi_2}), \quad v_9 = (0, 1, \omega^2 e^{i\phi_2}).
\end{aligned}
$$

It can be verified that upon normalization, the squared modulus of the inner product of any two vectors is 1/4. Therefore, these 9 vectors form a SIC-POVM. This SIC was discovered by P.K.Aravind using geometrical arguments. We have shown here that it includes both the Appleby's and Hesse SICs as special cases.

Examples 6.3.2. *Aravind-2 SIC*

Consider a SIC whose M-vectors (in spherical coordinate (θ, ϕ)) are

$$\text{Group 1} \begin{cases} v_1 : a_1 = (\theta_1, 0), & a_2 = (\pi - \theta_1, \pi) \\ v_2 : a_1 = (\theta_1, 2\pi/3), & a_2 = (\pi - \theta_1, 5\pi/3) \\ v_3 : a_1 = (\theta_1, 4\pi/3), & a_2 = (\pi - \theta_1, \pi/3) \end{cases}$$

$$\text{Group 2} \begin{cases} v_4 : a_1 = (\theta_2, 0), & a_2 = (\theta_2, 2\pi/3) \\ v_5 : a_1 = (\theta_2, 2\pi/3), & a_2 = (\theta_2, 4\pi/3) \\ v_6 : a_1 = (\theta_2, 4\pi/3), & a_2 = (\theta_2, 0) \end{cases}$$

$$\text{Group 3} \begin{cases} v_7 : a_1 = (\pi - \theta_2, \pi), & a_2 = (\pi - \theta_2, 5\pi/3) \\ v_8 : a_1 = (\pi - \theta_2, 5\pi/3), & a_2 = (\pi - \theta_2, \pi/3) \\ v_9 : a_1 = (\pi - \theta_2, \pi/3), & a_2 = (\pi - \theta_2, \pi). \end{cases}$$

These 18 M-vectors were obtained using purely geometric argument by P.K. Aravind (private communication). The geometric configuration of these states (shown in Figure 6.9 - 6.11) is quite different from examples we previously discussed.
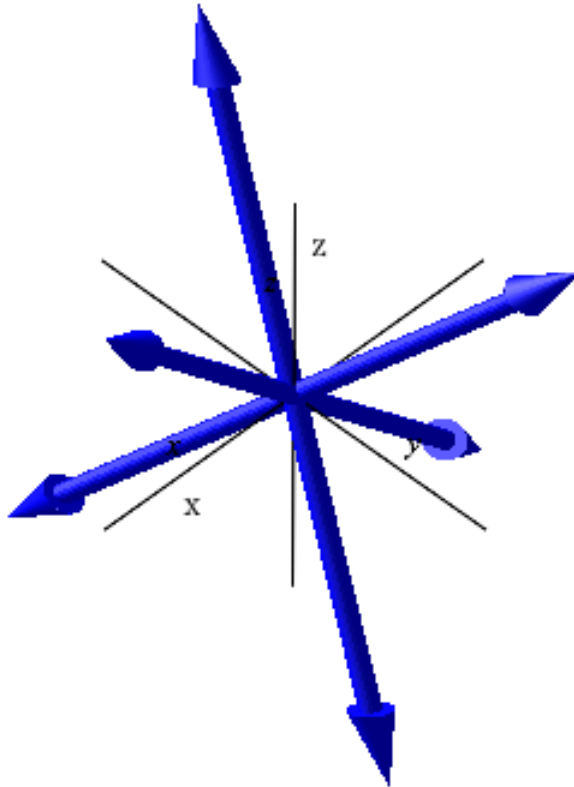

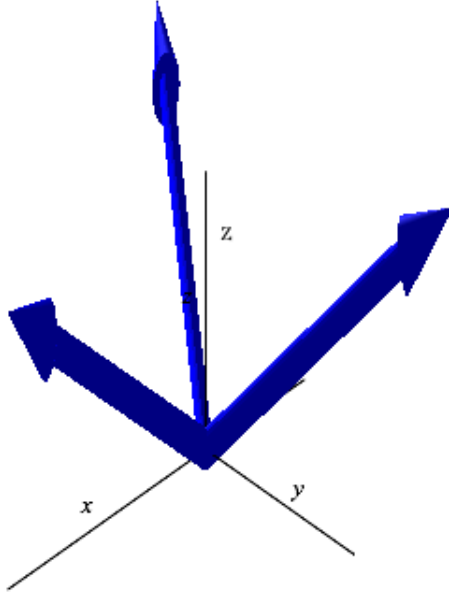Figure 6.8: M-vectors in Group 1 of Aravind-2 SIC

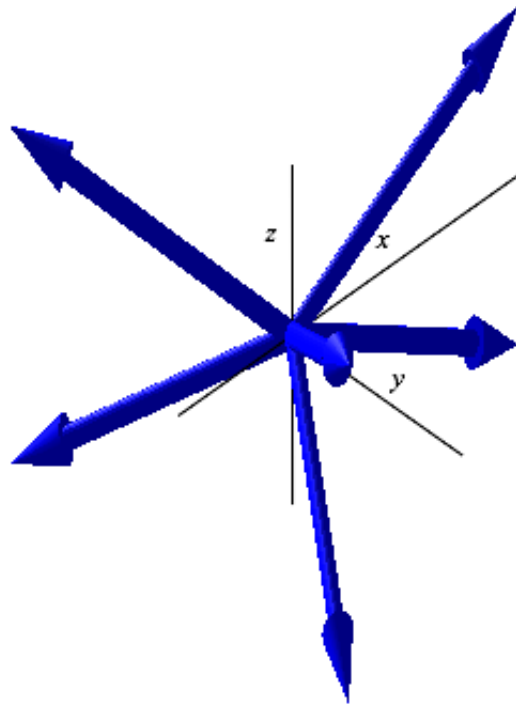Figure 6.9: M-vectors in Group 2 of Aravind-2 SIC

Figure 6.10: M-vectors in Group 3 of Aravind-2 SIC

The unnormalized vectors corresponding to these M-vectors are

$$v_1 = (1, -2, -1), \quad v_2 = (1, 2\omega^{-1/2}, \omega^{1/2}), \quad v_3 = (1, 2\omega^{1/2}, \omega^{-1/2})$$

$$v_4 = (1, \frac{1}{2}\omega^{1/2}, -\frac{1}{2}\omega^{-1/2}), \quad v_5 = (1, \frac{1}{2}, -\frac{1}{2}), \quad v_6 = (1, \frac{1}{2}\omega^{-1/2}, -\frac{1}{2}\omega^{1/2})$$

$$v_7 = (1, -\omega^{1/2}, -2\omega^{-1/2}), \quad v_8 = (1, 1, 2), \quad v_9 = (1, -\omega^{-1/2}, -2\omega^{1/2})$$

It is verified that upon normalization that the squared modulus of the inner product of any two vectors is also 1/4. Therefore these 9 vectors form a SIC-POVM. However, these 9 vectors are not in the same Weyl-Heisenberg orbit and none of these vectors can be used as a fiducial vector to generate a SIC-POVM.

To confirm that this is indeed a new SIC-POVM (i.e. that this SIC is inequivalent to other SICs), we need to find some invariants that can distinguish between them. According to Zhu's paper [13], if $\rho_i = v_i v_i^*$, where * denotes the conjugate transpose, the trace of the triple product $tr(\rho_j \rho_k \rho_l)$ is invariant under unitary transformations of the vectors v_1, v_j and v_k. We shall be concerned with these triple products when j, k and l independently take on all values from 1 to 9. Since the magnitude of the trace is the same, we shall only be concerned with the phase of the trace. If the phase $\phi = |\text{Arg}(tr(\rho_j \rho_k \rho_l))|$, we can obtain a total of $\binom{9}{3} = 84$ phases from a SIC-POVM. Then two SIC-POVMs are considered to be inequivalent if their phases have the distinct patterns. The phase ϕ is also known as the Bargmann invariant [2].

The phases distribution for Aravind-2 is shown in Table (6.1), and the phase distribution for Aravind-1 with arbitrary ϕ_1 and ϕ_2 is shown in Table (6.2).

Value of the phase	Number of multiplicity
0	45
$\pi/3$	27
$2\pi/3$	9
π	3

Table 6.1: Table of ϕ distribution for Aravind-2 SIC

Value of the phase	Number of multiplicity
0	10
$\pi/3$	36
$2\pi/3$	10
π	2
$-2\phi_1 + \phi_2$	10
$2\pi/3 - 2\phi_1 + \phi_2$	8
$-2\pi/3 - 2\phi_1 + \phi_2$	8

Table 6.2: Table of ϕ distribution for Aravind-1 SIC

From the distribution of the phases, we can tell that these two SICs are inequivalent for any value of ϕ_1 and ϕ_2. Similarly, we can show that Aravind-2 SIC is also inequivalent to Hesse's SIC

and Appleby's SIC. We anticipated this result because the Aravind-2 SIC is not a Weyl-Heisenberg orbit. However, we don't know whether it is equivalent to other SICs under unitary transformation. We mark this SIC-POVM as a potential new SIC and perhaps it should be studied further.

The most striking conclusion of the Majorana approach to studying SIC-POVMs in dimension 3 is that the states of the SIC come in three groups of three each, with the M-vectors of the states in each group being related to each other by a threefold rotation about the z-axis. It is possible to use this symmetry alone to deduce a large class of SIC-POVMs in dimension 3, as has been shown by P.K.Aravind (private communication).

Chapter 7

SIC-POVMs In Dimension 4

We have found that the M-vectors of SIC-POVMs in dimension 3 have a characteristic symmetry, namely, a threefold symmetry about the z-axis. We will now follow a similar approach to seek symmetries of the SIC-POVMs in dimension 4. We concentrate on a particular SIC-POVM in dimension 4 obtained by Appleby [1], whose vectors are

$$
\begin{aligned}
&v_1 = (x_0, 1, 1, 1), && v_2 = (x_0, 1, -1, -1), && v_3 = (x_0, -1, 1, -1), && v_4 = (x_0, -1, -1, 1), \\
&v_5 = (1, 1, x_0 i, -i), && v_6 = (1, 1, -x_0 i, i), && v_7 = (1, -1, x_0 i, -i), && v_8 = (1, -1, -x_0 i, -i), \\
&v_9 = (1, i, 1, -x_0 i), && v_{10} = (1, i, -1, x_0 i), && v_{11} = (1, -i, 1, x_0 i), && v_{12} = (1, -i, -1, -x_0 i), \\
&v_{13} = (1, x_0 i, i, -1), && v_{14} = (1, x_0 i, -i, 1), && v_{15} = (1, -x_0 i, i, 1), && v_{16} = (1, -x_0 i, -i, -1),
\end{aligned}
$$

where $x_0 = \sqrt{2 + \sqrt{5}}$. It can be verified that the squared modulus of the inner product of any two vectors is $1/5$ upon normalization. Therefore these 16 vectors form a SIC-POVM. Next, we want to work out their M-parameters.

Any vector in $\mathbb{C}^4$ can be parametrized by three M-parameters, which can be chosen as the complex numbers $\alpha_1, \alpha_2, \alpha_3$ or, alternatively, three unit vectors a_1, a_2, a_3 on the Riemann sphere.

$$
(1, c_1, c_2, c_3) = (1, \frac{\alpha_1 + \alpha_2 + \alpha_3}{\sqrt{3}}, \frac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1}{\sqrt{3}}, \alpha_1\alpha_2\alpha_3). \tag{7.1}
$$

The complex parameters $\alpha_1, \alpha_2, \alpha_3$ are the roots of the Majorana Polynomial:

$$
z^3 - \sqrt{3}c_1 z^2 + \sqrt{3}c_2 z - c_3 = 0 \tag{7.2}
$$

By solving this cubic equation, we can obtain three M-parameters for each of the vectors in the SIC-POVM. An investigation shows that the M-parameters of the 16 vectors can be expressed in terms of the following 15 quantities: 9 complex numbers $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9$, 4 real numbers $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ and 2 imaginary numbers γ_1, γ_2. Table (7.1) shows how the states can be expressed in terms of these parameters.

$v_1 = (\eta_1, \overline{\eta}_1, \beta_1)$	$v_5 = (\eta_2, \eta_3, \eta_4)$	$v_9 = (\eta_5, -\overline{\eta}_5, \gamma_1)$	$v_{13} = (\eta_7, \eta_8, \eta_9)$
$v_2 = (\beta_2, \beta_3, \beta_4)$	$v_6 = (\overline{\eta}_2, \overline{\eta}_3, \overline{\eta}_4)$	$v_{10} = (\eta_6, -\overline{\eta}_6, \gamma_2)$	$v_{14} = (-\overline{\eta}_7, -\overline{\eta}_8, -\overline{\eta}_9)$
$v_3 = (-\eta_1, -\overline{\eta}_1, -\beta_1)$	$v_7 = (-\eta_2, -\eta_3, -\eta_4)$	$v_{11} = (-\eta_5, \overline{\eta}_5, -\gamma_1)$	$v_{15} = (-\eta_7, -\eta_8, -\eta_9)$
$v_4 = (-\beta_2, -\beta_3, -\beta_4)$	$v_8 = (-\overline{\eta}_2, -\overline{\eta}_3, -\overline{\eta}_4)$	$v_{12} = (-\eta_6, \overline{\eta}_6, -\gamma_2)$	$v_{16} = (\overline{\eta}_7, \overline{\eta}_8, \overline{\eta}_9)$

Table 7.1: Table of the M-parameters of the 16 vectors of Appleby's SIC

From the M-parameters, we can calculate their corresponding M-vectors. Recall that a complex parameter α determines the M-vector $a = (a_x, a_y, a_z)$ via the relations

$$a_x = \frac{2Re(\alpha)}{1 + |\alpha|^2}, \qquad a_y = \frac{2Im(\alpha)}{1 + |\alpha|^2}, \qquad a_x = \frac{1 - |\alpha|^2}{1 + |\alpha|^2}. \tag{7.3}$$

We denote the M-vectors as h_i corresponding to M-parameter η_i, b_i corresponding to β_i and c_i corresponding to γ_i. Also note that replacing a complex parameter by its conjugate, or its negative, or its negative conjugate, causes the corresponding M-vector to change as follows:

$$\alpha \to \overline{\alpha} \qquad \Longrightarrow a = (a_x, a_y, a_z) \to (a_x, -a_y, a_z) \equiv \overline{a}$$
$$\alpha \to -\alpha \qquad \Longrightarrow a = (a_x, a_y, a_z) \to (-a_x, -a_y, a_z) \equiv \tilde{a}$$
$$\alpha \to -\overline{\alpha} \qquad \Longrightarrow a = (a_x, a_y, a_z) \to (-a_x, a_y, a_z) \equiv \check{a}$$

We have introduced a compact symbol for each transformed M-vector at the end of each of the above lines. With this notation, the M-vectors of the 16 SIC states are shown in Table (7.2).

Group 1	Group 2	Group 3	Group 4
$v_1 = (h_1, \overline{h}_1, b_1)$	$v_5 = (h_2, h_3, h_4)$	$v_9 = (h_5, \check{h}_5, c_1)$	$v_{13} = (h_7, h_8, h_9)$
$v_2 = (b_2, b_3, b_4)$	$v_6 = (\overline{h}_2, \overline{h}_3, \overline{h}_4)$	$v_{10} = (h_6, \check{h}_6, c_2)$	$v_{14} = (\check{h}_7, \check{h}_8, \check{h}_9)$
$v_3 = (\tilde{h}_1, \check{h}_1, \tilde{b}_1)$	$v_7 = (\tilde{h}_2, \tilde{h}_3, \tilde{h}_4)$	$v_{11} = (\tilde{h}_5, \overline{h}_5, \overline{c}_1)$	$v_{15} = (\tilde{h}_7, \tilde{h}_8, \tilde{h}_9)$
$v_4 = (\check{b}_2, \check{b}_3, \check{b}_4)$	$v_8 = (\check{h}_2, \check{h}_3, \check{h}_4)$	$v_{12} = (\check{h}_6, \overline{h}_6, c_2)$	$v_{16} = (\overline{h}_7, \overline{h}_8, \overline{h}_9)$

Table 7.2: M-vectors of SIC in Table (7.1)

The actual values of the 15 independent vectors occurring in the SIC were calculated using MAPLE and found to be:

$h_1 = (0.010, 0.982, 0.159)$ $\qquad$ $h_2 = (0.575, -0.293, -0.764)$ $\qquad$ $h_3 = (-0.399, -0.073, 0.863)$

$h_4 = (-0.307, 0.906, -0.293)$ $\qquad$ $h_5 = (0.838, -0.529, 0.129)$ $\qquad$ $h_6 = (0.639, 0.559, -0.528)$

$h_7 = (0.782, -0.520, 0.344)$ $\qquad$ $h_8 = (-0.313, 0.607, 0.730)$ $\qquad$ $h_9 = (-0.057, 0.510, -0.858)$

$b_1 = (0.924, 0, 0.382)$ $\qquad$ $b_2 = (0.983, 0, -0.185)$ $\qquad$ $b_3 = (0.779, 0, 0.627)$

$b_4 = (-0.985, 0, 0.170)$ $\qquad$ $c_1 = (0, 0.657, -0.754)$ $\qquad$ $c_2 = (0, -0.906, 0.424)$.

To visualize the result shown in Table 7.2, we plot the projection on the x-y plane of the M-vectors of the four states in each of the columns of Table 7.2. The results are shown from Figure 7.1 to Figure 7.4 . Red, yellow, blue and green dots denote the x, y coordinates of four different states. Dots are connected with thin lines if their z coordinates are positive and with thick lines if their z coordinates are negative. Since all vectors are unit vectors, all information about the vectors can be obtained from the plots. We find that the plots have C_{2v} symmetry in every case.
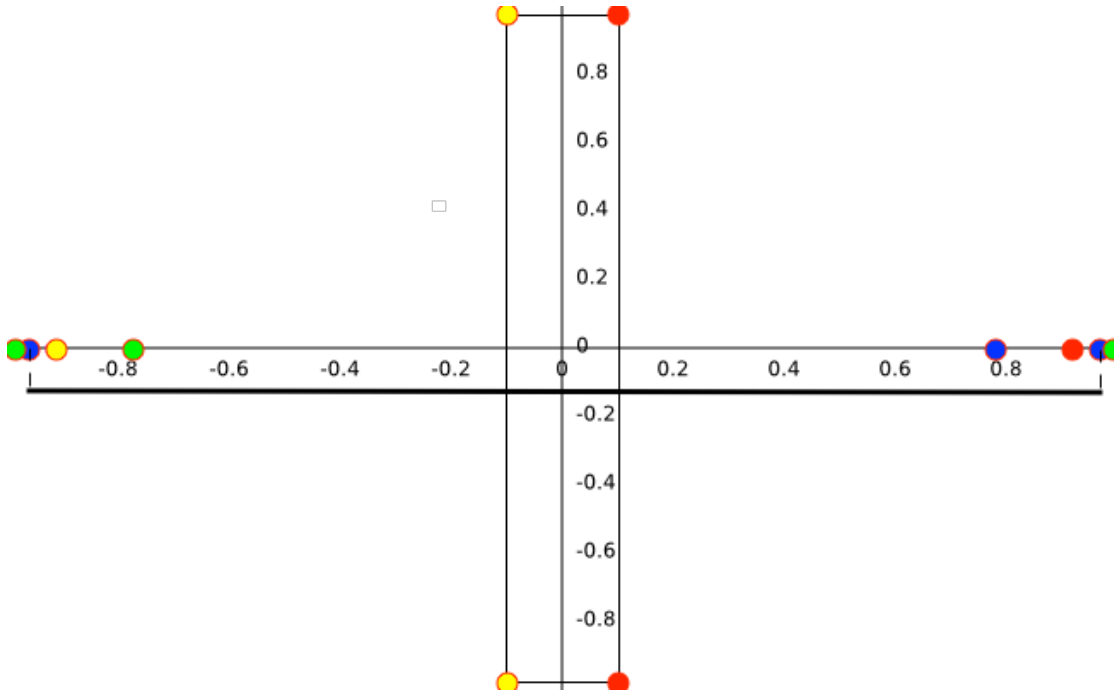
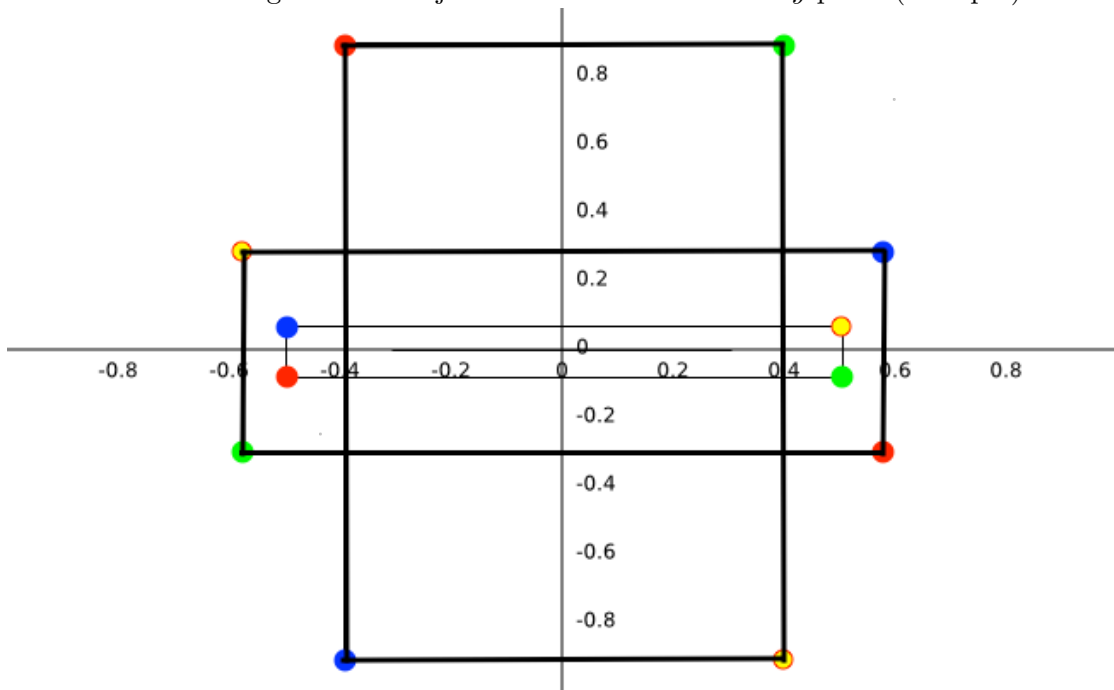Figure 7.1: Projection of M-vectors onto x-y plane (Group 1)

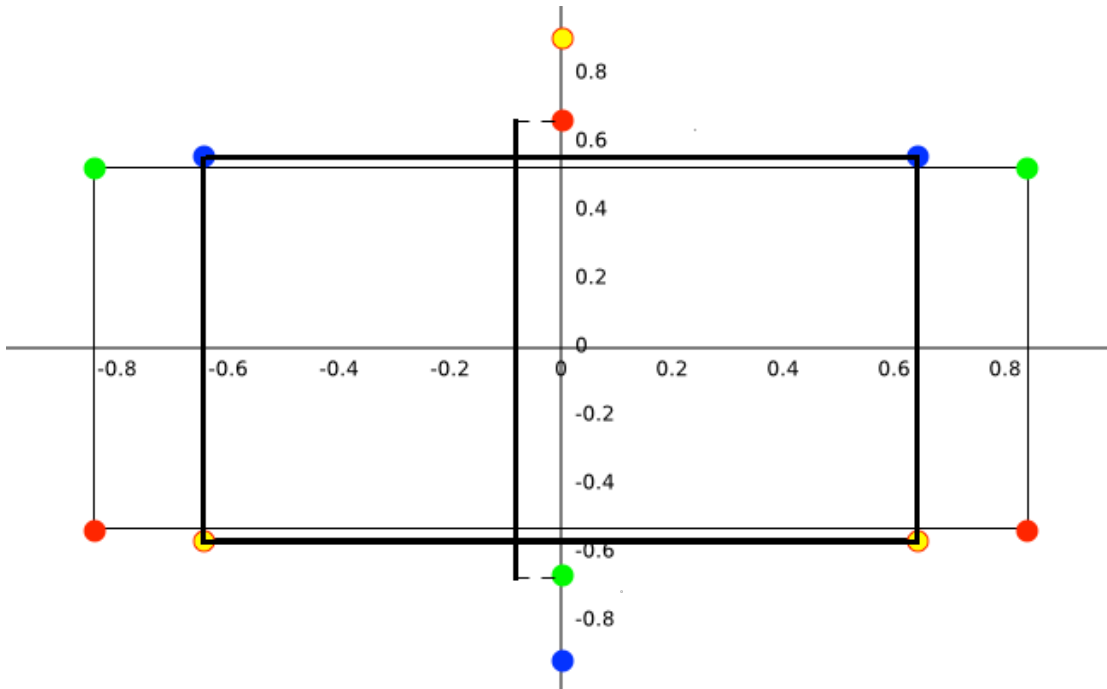Figure 7.2: Projection of M-vectors onto x-y plane (Group 2)

Figure 7.3: Projection of M-vectors onto x-y plane (Group 3)
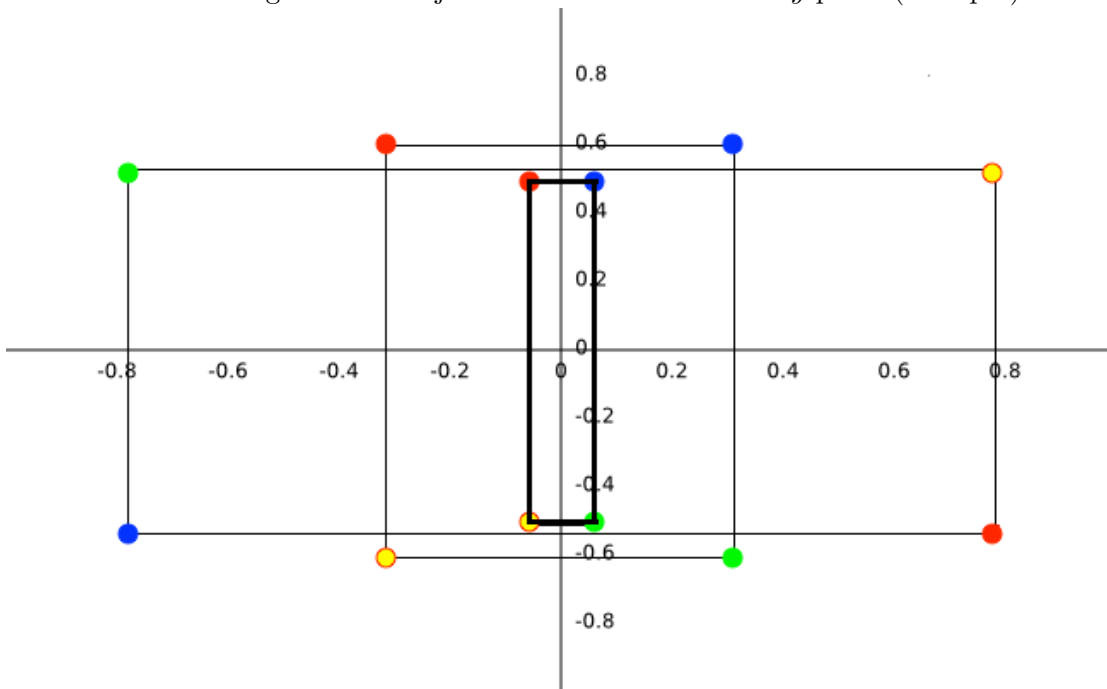

Figure 7.4: Projection of M-vectors onto x-y plane (Group 4)

There appear to be no further symmetries aside from C_{2v} .

This suggests that we can construct a SIC by looking for 4 equiangular states having C_{2v} symmetry and put four such groups together that form a SIC. It still remains to be seen if this approach will work.

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