A Holographic Approach to the Hawking Effect for a Kerr-Newman Black Hole

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May 5, 2022

Abstract

This work uses holographic physics to compute the Hawking temperature for a Kerr-Newman black hole. Kerr-Newman black holes, possess mass, angular momentum, and an electric charge. We start deriving the methodology by exploring the Unruh effect, we later extend this logic to consider black holes in the near horizon limit. Once the analytical models are derived, we simulate black hole evaporation for a variety of black holes.

Acknowledgements

We would like to thank our advisors for their assistance throughout the year, Dr. Sarah Olson of the Mathematical Sciences department, Dr. L. Ramdas Ram-Mohan of the Physics department, and Dr. Craig Wills of the Computer Science department. Their continued guidance and support enabled us to successfully complete this project. We would also like to thank Dr. Leo Rodriguez, Dr. Shanshan Rodriguez, and Raid Suleiman for their assistance in teaching us the relevant material throughout the year. They were essential in grounding our understanding of the material and without their guidance and support we would not have been able to accomplish this project. We also wanted to thank Kylar Coleman-Foley for the time she spent with us following our project. Her involvement and passion for the project motivated us throughout the year.

Finally, we would like to thank our many supportive peers, among them, Gaby Cano, Mara Decesare, Ryan Hanna, Morgan Kaler, Ben Lunden, Luciano Malavasi, Charlie Tribble, Daniil Volkov, as well as the faculty and staff in the physics department for their continued moral and emotional support throughout our entire time at WPI.

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Introduction

Black Holes as Thermodynamic Objects

Shortly after Einstein published his General Theory of Relativity in 1915, Karl Schwarzschild found an exact solution to the Einstein field equations describing a gravitational field outside a spherically symmetric mass [13]. For a sufficiently large mass, an object with a physical singularity and an apparent event horizon would form, these objects are what we refer to as black holes.

During the 1970's many researchers became interested in the thermodynamic properties of black holes. One particular problem was that the existence of black holes appeared to violate the second law of thermodynamics. Black holes, as described by classical general relativity have no microstates, hence they have no entropy. One could imagine, as John Wheeler did, a system in which there is a black hole and a hot teacup. The teacup, having a variety of microstates, has an associated entropy. If one imagined throwing the teacup into the black hole, they would find that the entropy of the system (the universe) has decreased. This is a violation of the second law of thermodynamics. Therefore, in order for the predictions of General Relativity to abide by known physics, black holes must somehow have an entropy.

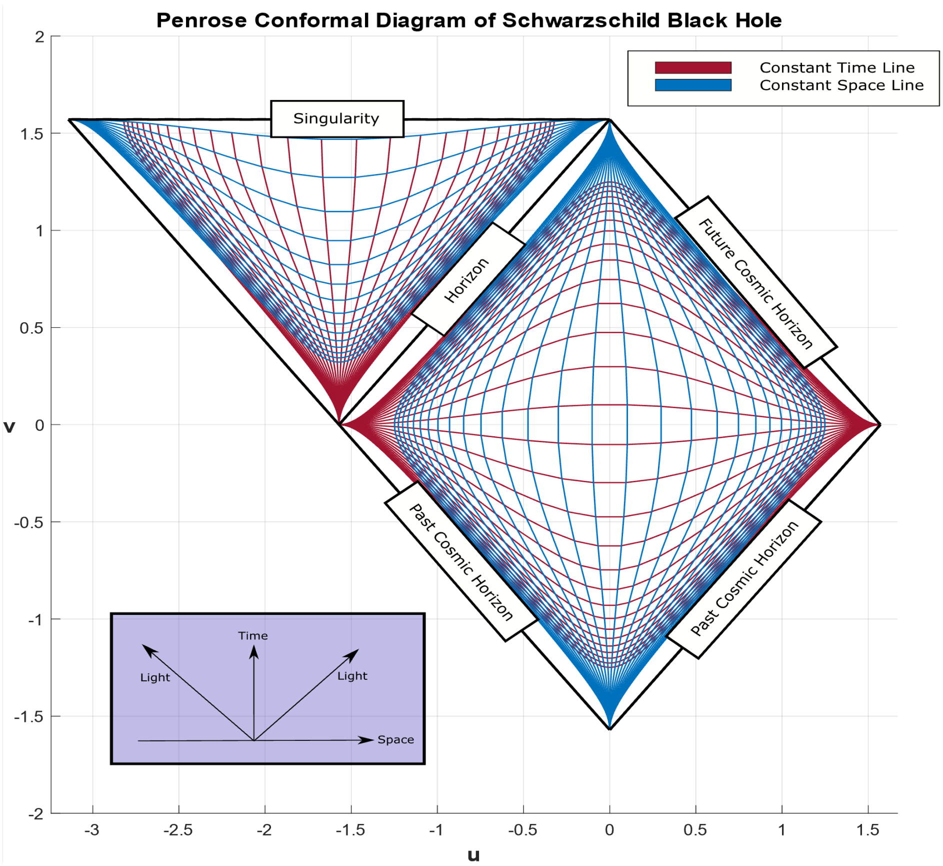


Figure 1: Conformal Penrose diagram of an eternal Schwarzschild black hole.

Jacob Bekenstein resolved this thought experiment from the perspective of quantum information theory by demonstrating that black holes are thermodynamic objects and have an associated entropy even though the information about the black hole interior remains inaccessible to an outside observer.

This entropy does not appear as a result of classical thermodynamics, but as the result of a deeper connection with quantum mechanics. He found that the entropy of a black hole to be proportional to its surface area [1].

$$S_{BH} \propto A$$

Stephen Hawking's work in 1974 showed that due to quantum mechanical effects in a classical general relativistic background, black holes must radiate particles as if they were a black body with a given temperature proportional to the surface gravity of the black hole κ.

$$T = \frac{\hbar\kappa}{2\pi k_B} = \frac{\hbar c^3}{8\pi G k_B M}$$

This result, as well as his later work in 1975 [6], confirmed that not only should black holes radiate, but they should also evaporate. Hawking's calculations also confirmed Bekenstein's result of black hole entropy, leading to the Bekenstein-Hawking Entropy formula.

$$S = \frac{c^3 A}{4G\hbar}$$

Present in this relation are terms from both classical and quantum mechanics, hinting towards a deeper theory of quantum gravity.

Computational Approach to Obtain Thermodynamic Properties of Black Holes

This project extends previous work developed by Dominic Chang [3] of applying methods from the derivation of the Unruh Effect to (1+1)-D dimensional holographic duals of (3+1)-D black holes to derive their Hawking temperature. In his work, he analyzed the properties of a Kerr black hole. We extend this method to consider a Kerr-Newman black hole, as it is the most general representation of a black hole, possessing mass, angular momentum, and an electric charge. We make use of computers to automate these calculations to obtain the Hawking temperature for a given black hole. All calculations in this paper use natural units,

$$c = G = \hbar = k_B = 1.$$

We also use the mostly positive metric signature $(- + ++)$.

The Unruh Effect

Overview

The Unruh effect is a coordinate dependent effect that arises due a change in reference frames between an inertial and accelerated observe. Quantum mechanical invariants are only preserved under transformations belonging to the same unitary group. The transformation defined by an observer undergoing constant acceleration is not unitary, hence causes disturbances in the quantum field theory. This results in a changing ground state energy level of the quantum fields from the perspective of the accelerating frame.

We begin this section by deriving the transformation between an inertial Minkowski observer and an accelerated observer in Rindler coordinates. We then define a classical field theory describing a massless scalar field on that spacetime background and quantize it. Finally, we calculate the Bogolyubov coefficients to compute the ground state energy in the accelerated frame. This allows us to make use of the number operator to determine the expected number of particles produced in the accelerated frame. These particles follow a Bose-Einstein distribution with an associated temperature parameter.

Inertial and Accelerated Frames in Flat Spacetime

We start this section by defining our metric for flat Minkowski spacetime as well as our line-element.

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

For simplicity, we will work in a (1+1)-D Minkowski spacetime and we will perform a rotation to work in light-cone coordinates. Our sign convention $(- + ++)$ dictates our light-cone coordinates to be,

$$x_+ = \frac{x+t}{\sqrt{2}} \quad \text{and} \quad x_- = \frac{x-t}{\sqrt{2}}. \tag{3}$$

We use this coordinate system to simplify future calculations involving our field modes in this background. From **Equation (3)**, we have the following relations,

$$dx_+ = \frac{dx+dt}{\sqrt{2}} \quad \text{and} \quad dx_- = \frac{dx-dt}{\sqrt{2}}. \tag{4}$$

From our relation in our line-element definition **Equation (2)**,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 = 2\left(\frac{1}{2}(-dt+dx)(dt+dx)\right) = 2dx_+dx_-. \tag{5}$$

From this relation, we can see that only the off diagonal terms survive, and since we have a coefficient of two, our metric in the light-cone coordinate basis is,

$$g^{(M)}_{\mu\nu} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{6}$$

We are interested in finding the hyperbolic path generated by an accelerating observer from the perspective of a Minkowski observer. To do this, we shall derive a few relations to better understand the geometry of Minkowski spacetime. Let us start by considering velocity. The definition of a four-velocity is given by,

$$U^\mu = \frac{dx^\mu}{d\tau}.$$

τ is the proper time parameter. From our line element definition and the fact that $ds^2 = -d\tau^2$,

$$-d\tau^2 = g^{(M)}_{\mu\nu} dx^\mu dx^\nu,$$

$$g^{(M)}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1,$$

$$g^{(M)}_{\mu\nu} U^\mu U^\nu = -1.$$

This means that the dot product between two four-velocities in light-cone coordinates is,

$$g^{(M)}_{\mu\nu} U^\mu U^\nu = 2\frac{dx_+}{d\tau}\frac{dx_-}{d\tau} = 2\dot{x}_+\dot{x}_- = -1.$$

$$\dot{x}_+\dot{x}_- = -\frac{1}{2}. \tag{7}$$

Taking the derivative of this relationship with respect to proper time informs us that acceleration must be perpendicular to velocity.

$$g^{(M)}_{\mu\nu} a^\mu U^\nu = \frac{d}{d\tau}(\dot{x}_+\dot{x}_-) = 0$$

$$\ddot{x}_+\dot{x}_- + \dot{x}_+\ddot{x}_- = 0$$

$$\ddot{x}_+\dot{x}_- = -\dot{x}_+\ddot{x}_-$$

This implies that acceleration is perpendicular to velocity and

$$\frac{\ddot{x}_+}{\dot{x}_+} = -\frac{\ddot{x}_-}{\dot{x}_-}. \tag{8}$$

Since velocity is perpendicular to acceleration, and from the observer traveling in the accelerated frame, velocity is in the direction of time, then acceleration must be spatial.

$$a^2 = g^{(M)}_{\mu\nu} \ddot{x}^\mu_+ \ddot{x}^\nu_- = 2\ddot{x}_+\ddot{x}_- \tag{9}$$

Combining our relations in **Equation (7)** and **Equation (8)** and multiplying by the acceleration in our x_- direction, we can form a differential equation with a solution describing the hyperbolic arc.

$$\ddot{x}_- = -\frac{\ddot{x}_+}{\dot{x}_+}\dot{x}_-$$

$$\ddot{x}_- = \frac{\ddot{x}_+}{2\dot{x}_+^2}$$

$$\ddot{x}_-\ddot{x}_+ = \frac{\ddot{x}_+^2}{2\dot{x}_+^2}$$

$$\frac{a^2}{2} = \ddot{x}_-\ddot{x}_+ = \frac{\ddot{x}_+^2}{2\dot{x}_+^2}$$

$$a^2 = 2\ddot{x}_+\ddot{x}_- = \frac{\ddot{x}_+^2}{\dot{x}_+^2}$$

From this, we can write our differential equation as,

$$\ddot{x}_+ = a\dot{x}_+. \tag{10}$$

This differential equation has a known solution of the form,

$$x_+ = \frac{A}{a}e^{a\tau} + B. \tag{11}$$

We can select a reference where $B = 0$ and using **Equation (7)** we can find a differential equation describing x_-,

$$\dot{x}_+ = Ae^{a\tau},$$

$$\dot{x}_+\dot{x}_- = -\frac{1}{2} \implies \dot{x}_- = -\frac{1}{2A}e^{-a\tau}$$

$$x_- = \frac{1}{2Aa}e^{-a\tau}.$$

Remember that $-t^2 + x^2 = 2x_+x_- = \frac{1}{a^2}$ and our light-cone coordinates are normalized by a factor of $1/\sqrt{2}$. We can set $A = 1/\sqrt{2}$ and arrive at our equations relating our light-cone coordinates to a hyperbolic curve in Minkowski space.

$$x_+ = \frac{1}{a\sqrt{2}}e^{a\tau} \quad \text{and} \quad x_- = \frac{1}{a\sqrt{2}}e^{-a\tau} \tag{12}$$

We can rewrite this curve in terms of our original Minkowski coordinates t, x,

$$x(\tau) = \frac{x_+ + x_-}{\sqrt{2}} = \frac{e^{a\tau} + e^{-a\tau}}{2a} = \frac{1}{a}\cosh a\tau,$$

$$t(\tau) = \frac{x_+ - x_-}{\sqrt{2}} = \frac{e^{a\tau} - e^{-a\tau}}{2a} = \frac{1}{a}\sinh a\tau,$$

$$t = \frac{1}{a}\sinh a\tau \quad \text{and} \quad x = \frac{1}{a}\cosh a\tau. \tag{13}$$

Our curves are expressed by the hyperbola,

$$x^2 - t^2 = \frac{1}{a^2}.$$

A diagram describing the accelerated trajectory can be seen below in **Figure (2)**.

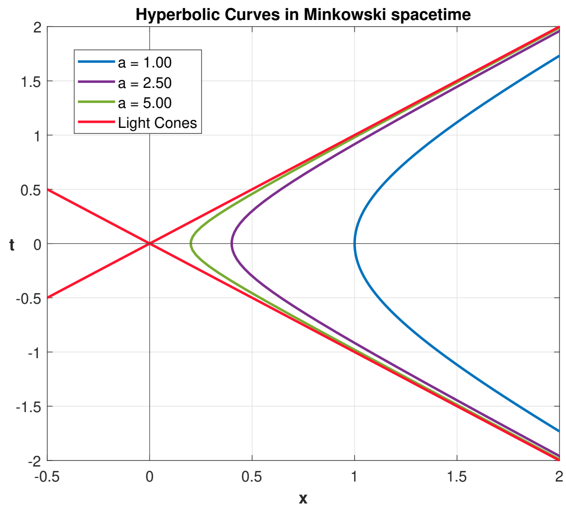


Figure 2: Hyperbolic curves formed in Minkowski spacetime from the perspective of an inertial observer at the origin by another observer accelerating at values $a = 1, 2.5, 5$.

Now we can consider an inertial observer that is instantaneously co-moving with our accelerating observer at every moment. The coordinate system of this observer is known as Rindler coordinates. Let ξ represent our position coordinate and η represent our time coordinate in the Rindler frame. We will start by defining light-cone coordinates in our Rindler spacetime,

$$\tilde{x}_+ = \frac{\xi + \eta}{\sqrt{2}} \quad \text{and} \quad \tilde{x}_- = \frac{\xi - \eta}{\sqrt{2}}. \tag{14}$$

Since the accelerated observer is stationary in their own frame,

$$x^\mu(\tau) = (\eta(\tau), \xi(\tau)) = (\tau, 0),$$

$$\tilde{x}_+(\tau) = \frac{1}{\sqrt{2}}\tau \quad \text{and} \quad \tilde{x}_-(\tau) = -\frac{1}{\sqrt{2}}\tau.$$

This allows us to relate the Minkowski light-cone coordinates and the Rindler light-cone coordinates,

$$x_+ = \frac{1}{a\sqrt{2}}e^{a\sqrt{2}\tilde{x}_+} \quad \text{and} \quad x_- = \frac{1}{a\sqrt{2}}e^{a\sqrt{2}\tilde{x}_-}. \tag{15}$$

From this relation, we can use the spacetime interval to derive the metric for this frame,

$$\frac{dx_+}{d\tilde{x}_+} = e^{a\sqrt{2}\tilde{x}_+} \quad \text{and} \quad \frac{dx_-}{d\tilde{x}_-} = e^{a\sqrt{2}\tilde{x}_-}$$

$$dx_+ = d\tilde{x}_+ e^{a\sqrt{2}\tilde{x}_+} \quad \text{and} \quad dx_- = d\tilde{x}_- = d\tilde{x_-}e^{a\sqrt{2}\tilde{x}_-}$$

$$ds^2 = 2dx_+dx_- = 2e^{a\sqrt{2}(\tilde{x}_+ + \tilde{x}_-)}d\tilde{x}_+d\tilde{x}_- = 2e^{2a\xi}d\tilde{x}_+d\tilde{x}_-.$$

This means that our Rindler spacetime metric is a conformal transformation of the Minkowski metric. Using **Equation (15)** we derive,

$$t = x - \sqrt{2}x_-, \; x = \sqrt{2}x_- + t$$
$$t = -x + \sqrt{2}x_+, \; x = \sqrt{2}x_+ - t.$$

Solving this system of equations yields,

$$t = \frac{x_+ - x_-}{\sqrt{2}}, \; x = \frac{x_- + x_+}{\sqrt{2}}.$$

$$t = \frac{e^{a\xi}\sinh a\eta}{a} \quad \text{and} \quad x = \frac{e^{a\xi}\cosh a\eta}{a}. \tag{16}$$

Quantum Scalar Fields in Flat Spacetime

Now that we have constructed the classical background, we can now construct our quantum field theory within this background. We start with the relativistically invariant Lagrangian density for a free real scalar field $\phi(x, t)$ of mass m (Klein-Gordon Lagrangian) in a (1+1)-D flat spacetime.

$$\mathcal{L}(\phi, \partial_\mu\phi) = -\frac{1}{2}\eta^{uv}\partial_u\phi\partial_v\phi - \frac{1}{2}m^2\phi^2 \tag{17}$$

The action is defined as the spacetime integral of the Lagrangian density,

$$S(\phi) = \int \mathcal{L}dxdt.$$

We can perform a variation of the Lagrangian density with respect to our field and obtain a differential equation for our field theory,

$$\delta_\phi\mathcal{L} = -\eta^{uv}(\partial_u(\delta\phi)\partial_v\phi) - m^2\phi$$

This corresponds to an action variation,

$$\delta S = -\int dxdt\left[\eta^{uv}(\partial_u(\delta\phi)\partial_v\phi) + m^2\phi\right]$$

We can integrate the first term by parts to pull out our $\delta\phi$ terms. Let $u = \partial_v\phi$, $v = \delta\phi$, $du = \partial_u\partial_v\phi$ and $dv = \partial_u(\delta\phi)$.

$$\delta S = -\int dxdt \left[\eta^{uv}\left(\delta\phi\partial_v\phi - \delta\phi\partial_u\partial_v\phi\right) + m^2\phi\right]$$

Our leading term $\delta\phi\partial_v\phi$ is zero by Stokes's Theorem. We can evaluate the variation at the boundary of the field, which we define to have $\delta\phi = 0$ at the boundary. This leaves,

$$\delta S = \int dxdt\delta\phi \left[\eta^{uv}\partial_u\partial_v\phi - m^2\phi\right] = 0.$$

Therefore, our differential equation is the Klein-Gordon equation,

$$\eta^{uv}\partial_u\partial_v\phi - m^2\phi = 0. \tag{18}$$

We write this differential equation with the 2-D d'Alembertian operator $\Box^{(2)} = \eta^{uv}\partial_u\partial_v$,

$$\Box^{(2)}\phi - m^2\phi = 0.$$

For the Unruh effect, we will be considering a massless scalar field, so we rewrite this as

$$\Box^{(2)}\phi = 0. \tag{19}$$

The Klein-Gordon equation is a wave equation and as a result, solutions are satisfied by plane waves and superpositions of plane waves.

$$\phi(x^\mu) = \phi_0 e^{ik_\mu x^\mu} = \phi_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

The wave vector is a function of components $k^\mu = (\omega, \mathbf{k})$. The frequency ω satisfies the dispersion relation as we are now working with fields not individual oscillating points.

$$\omega^2 = \mathbf{k}^2 + m^2$$

We are working in a massless field so this reduces to,

$$\omega^2 = \mathbf{k}^2.$$

Since we are now determining our frequency from the spatial wave vector $\mathbf{k}$ and the sign of ω, we must forego the notion of independent solutions for each oscillator. Therefore, our job now is to arrive at a general solution to the classical Klein-Gordon equation. To achieve this, we will construct a complete, orthonormal set of basis modes and express all solutions as a linear combination of these basis modes.This requires a defined inner product on our space of solutions, with which we can build our Fock space. We follow the methodology of Carroll [2], by defining the inner product over a constant-time hypersurface Σ_t, i.e. a space-like horizontal line connecting a string of events at a constant time t = C. The surface's unit normal vector is the time-like vector field ∂_t. Therefore, we can construct the inner product between two basis solutions ϕ_1 and ϕ_2 as,

$$(\phi_1, \phi_2) = -i\int_{\Sigma_t}\left(\phi_1\partial_t\phi_2^* - \phi_2^*\partial_t\phi_1\right)d^{n-1}x. \tag{20}$$

Let us evaluate the inner product with two of our basis modes,

$$\left(e^{ik_1^\mu x_\mu}, e^{ik_2^\nu x_\nu}\right) = -i\int_{\Sigma_t}\left(e^{-i\omega_1 t + i\mathbf{k}_1\cdot\mathbf{x}}\partial_t e^{i\omega_2 t - i\mathbf{k}_2\cdot\mathbf{x}} - e^{i\omega_2 t - i\mathbf{k}_2\cdot\mathbf{x}}\partial_t e^{-i\omega_1 t + i\mathbf{k}_1\cdot\mathbf{x}}\right)d^{n-1}x$$

$$= \int_{\Sigma_t}\left(\omega_2 e^{-i\omega_1 t + i\mathbf{k}_1\cdot\mathbf{x}} e^{i\omega_2 t - i\mathbf{k}_2\cdot\mathbf{x}} + \omega_1 e^{i\omega_2 t - i\mathbf{k}_2\cdot\mathbf{x}} e^{-i\omega_1 t + i\mathbf{k}_1\cdot\mathbf{x}}\right)d^{n-1}x$$

$$= (\omega_1 + \omega_2)e^{-i(\omega_1 - \omega_2)t}\int_{\Sigma_t} e^{i(\mathbf{k}_1 - k_2)\cdot\mathbf{x}} d^{n-1}x.$$

Using the relation,

$$\int e^{i\mathbf{k}\cdot\mathbf{x}} d^{n-1}x = (2\pi)^{n-1}\delta^{(n-1)}(\mathbf{k}),$$

and inputting the dimensionality of our (1+1)-D flat spacetime as $n = 2$, we obtain,

$$(\phi_1, \phi_2) = (2\pi)(\omega_1 + \omega_2)e^{-i(\omega_1 - \omega_2)t}\delta(\mathbf{k}_1 - \mathbf{k}_2).$$

Therefore, our orthonormal set of mode solutions is given by,

$$f_{\mathbf{k}}(x^\mu) = \frac{e^{ik^\mu x_\mu}}{\sqrt{4\pi\omega}}$$

Since the dispersion relation of $\mathbf{k}$ determines the frequency ω up to its sign, we want to find a way to deal with a negative frequency. We do this by holding $\omega > 0$ and making use of complex conjugation for our $f^*_{\mathbf{k}}(x^\mu)$, thus we can generate both positive and negative frequency modes by taking a complex conjugate.

$$\partial_t f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}, \text{ with } \omega > 0 \text{ (positive)}$$

$$\partial_t f^*_{\mathbf{k}} = i\omega f^*_{\mathbf{k}}, \text{ with } \omega > 0 \text{ (negative)}$$

It can be shown that the complex conjugates are orthogonal to the original frequency modes as well as orthonormal with a flipped sign. To take this classical field theory and make it a quantum field theory, we need to canonically quantize our system. We do this by promoting our field variables and conjugate momenta to quantum operators that act on a Hilbert space. We must impose the following commutation relations on equal time hypersurfaces (space-like hypersurfaces with constant time).

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')] = 0$$
$$[\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = 0$$
$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = i\delta^{(n-1)}(\mathbf{x} - \mathbf{x}')$$

The first two relations indicate that the spatial and momentum operators commute with themselves throughout space so long as they share a fixed time. The third relation states that the position and momentum operators of our quantum field may commute at the same given time t, so long as they are not operators at the same point in space. This is because space-like separated events along a constant time cannot influence one another, so if they share a constant time and are separate points, causality dictates that they cannot affect one another due to their space-like separation. We can expand our field operator in terms of the modes we derived earlier, and the coefficients are the creation and annihilation operators in our (1+1)-D Minkowski spacetime.

$$\hat{\phi}(t, x) = \frac{1}{\sqrt{4\pi}} \int dk \left[\hat{a}_{k_x} \frac{e^{ik_x \cdot x - i\omega t}}{\sqrt{\omega}} + \hat{a}^\dagger_{k_x} \frac{e^{-ik_x \cdot x + i\omega t}}{\sqrt{\omega}} \right]$$

We can make use of the simplification that our scalar field is massless and in (1+1)-D spacetime and rewrite k_x in terms of ω using the dispersion relation,

$$\hat{\phi}(t, x) = \frac{1}{\sqrt{4\pi}} \int d\omega \left[\hat{a}_\omega \frac{e^{i\omega \cdot x - i\omega t}}{\sqrt{\omega}} + \hat{a}^\dagger_\omega \frac{e^{-i\omega \cdot x + i\omega t}}{\sqrt{\omega}} \right].$$

Using our Minkowski light-cone coordinates, we can rewrite,

$$\hat{\phi}(t, x)_{MR} = \frac{1}{\sqrt{4\pi}} \int \frac{d\omega}{\sqrt{\omega}} \left[\hat{a}_\omega e^{i\omega\sqrt{2}x_-} + \hat{a}^\dagger_\omega e^{-i\omega\sqrt{2}x_-} \right], \tag{21}$$

$$\hat{\phi}(t, x)_{ML} = \frac{1}{\sqrt{4\pi}} \int \frac{d\omega}{\sqrt{\omega}} \left[\hat{a}_\omega e^{i\omega\sqrt{2}x_+} + \hat{a}^\dagger_\omega e^{-i\omega\sqrt{2}x_+} \right]. \tag{22}$$

These two operators describe the right and left moving positive and negative modes in our Minkowski spacetime. One can also demonstrate that these relations show that the creation and annihilation operators obey the commutation relations,

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0,$$
$$\left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = 0,$$

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k'}}\right] = \delta^{(n-1)}(\mathbf{k} - \mathbf{k}').$$

We can now define another quantum operator for each individual wave vector. This operator is called the number operator and is defined as,

$$\hat{n}_{\mathbf{k}} = \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}.$$

The eigenstates of the number operators form a basis for our Hilbert space which is known as the Fock space. Excitations in the Fock basis are interpreted as particles in space. Now let us repeat the procedure for our Rindler space. Following an analogous procedure, we find the right and left moving quantum field operators for our Rindler space to be,

$$\hat{\phi}(t,x)_{RR} = \frac{1}{\sqrt{4\pi}} \int \frac{d\Omega}{\sqrt{\Omega}} \left[\hat{b}_{\Omega} e^{i\Omega\sqrt{2}\tilde{x}_-} + \hat{b}^{\dagger}_{\Omega} e^{-i\Omega\sqrt{2}\tilde{x}_-}\right], \tag{23}$$

$$\hat{\phi}(t,x)_{RL} = \frac{1}{\sqrt{4\pi}} \int \frac{d\Omega}{\sqrt{\Omega}} \left[\hat{b}_{\Omega} e^{i\Omega\sqrt{2}\tilde{x}_+} + \hat{b}^{\dagger}_{\Omega} e^{-i\Omega\sqrt{2}\tilde{x}_+}\right]. \tag{24}$$

One can again demonstrate that these relations show that the creation and annihilation operators obey the commutation relations,

$$\left[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k'}}\right] = 0$$

$$\left[\hat{b}^{\dagger}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k'}}\right] = 0$$

$$\left[\hat{b}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k'}}\right] = \delta^{(n-1)}(\mathbf{k} - \mathbf{k}').$$

The number operator is then defined,

$$\hat{n}_{\mathbf{k}} = \hat{b}^{\dagger}_{\mathbf{k}}\hat{b}_{\mathbf{k}}.$$

Here we can define the vacuum state in Minkowski coordinates,

$$\hat{a}^{\dagger}_{\omega} \left|0_M\right\rangle = 0.$$

Likewise, the same can be established for Rindler coordinates,

$$\hat{b}^{\dagger}_{\Omega} \left|0_R\right\rangle = 0.$$

At this point we take our Minkowski coordinate system as the natural coordinate system for our space-time and $|0_M\rangle$ will be the natural ground state. This will result in particle production for the accelerated observer in the Rindler frame.

The Unruh Temperature

Quantum Field Theory states that a scalar field is equivalent to an infinite number of decoupled harmonic oscillators at each point in space. Within this framework, the existence of a particle is the result of frequency modes in the field. The ground or vacuum state corresponds to no particles. Positive modes or excited states correspond to real particles, and negative modes correspond to anti-particles. Therefore, the ground state of a quantum field is the result of destructive interference of these modes. In general, the vacuum state is coordinate dependent. By taking the Minkowski coordinate system to be the natural ground state and the Rindler frame to have particle production, we'd like to determine a representation of Rindler creation and annihilation operators in terms of the Minkowski creation and annihilation operators. Therefore, one observer's basis modes are used to form a linear combination representing the other observer's basis modes. We do this by using a Bogolyubov Transformation, which can be seen in the following equations,

$$\hat{b}_{\Omega} = \int_0^{\infty} d\omega [\alpha^*_{\Omega\omega}\hat{a}_{\omega} - \beta^*_{\Omega\omega}\hat{a}^{\dagger}_{\omega}], \tag{25}$$

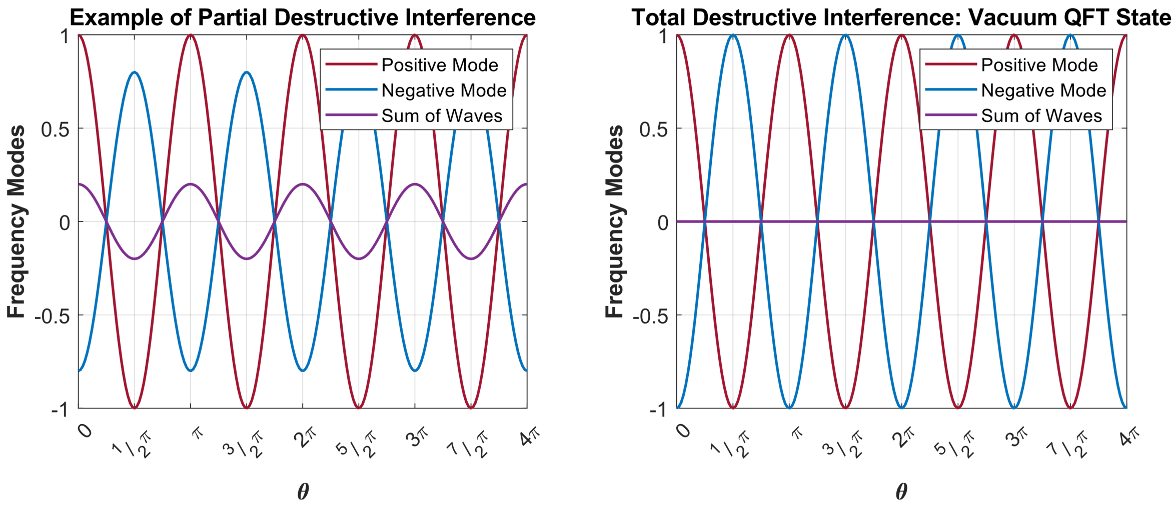


Figure 3: Simplified classical example of partial and total destructive interference. This is analogous to that of the quantum fields that lead to particle production.

$$\hat{b}^{\dagger}_{\Omega} = \int_0^{\infty} d\omega[\alpha_{\Omega\omega}\hat{a}^{\dagger}_{\omega} - \beta_{\Omega\omega}\hat{a}_{\omega}]. \tag{26}$$

We want to find the relation between the two Bogolyubov coefficients $\alpha_{\Omega\omega}$ and $\beta_{\Omega\omega}$. Let us insert these operator relations in our expression for the right moving wave operator in Rindler space, which should be equivalent to the right moving wave operator in Minkowski space.

$$\hat{\phi}(t,x)_{MR} = \frac{1}{\sqrt{4\pi}}\int \frac{d\omega}{\sqrt{\omega}}\left[\hat{a}_{\omega}e^{i\omega\sqrt{2}x_-} + \hat{a}^{\dagger}_{\omega}e^{-i\omega\sqrt{2}x_-}\right] = \hat{\phi}(t,x)_{RR} = \frac{1}{\sqrt{4\pi}}\int \frac{d\Omega}{\sqrt{\Omega}}\left[\hat{b}_{\Omega}e^{i\Omega\sqrt{2}\tilde{x}_-} + \hat{b}^{\dagger}_{\Omega}e^{-i\Omega\sqrt{2}\tilde{x}_-}\right],$$

$$= \frac{1}{\sqrt{4\pi}}\int \frac{d\Omega}{\sqrt{\Omega}}\left[\int_0^{\infty} d\omega[\alpha^*_{\Omega\omega}\hat{a}_{\omega} - \beta^*_{\Omega\omega}\hat{a}^{\dagger}_{\omega}]e^{i\Omega\sqrt{2}\tilde{x}_-} + \int_0^{\infty} d\omega[\alpha_{\Omega\omega}\hat{a}^{\dagger}_{\omega} - \beta_{\Omega\omega}\hat{a}_{\omega}]e^{-i\Omega\sqrt{2}\tilde{x}_-}\right]$$

Now we can group our $\hat{a}_{\omega}$ $\hat{a}^{\dagger}_{\omega}$ terms,

$$= \frac{1}{\sqrt{4\pi}}\int \frac{d\Omega}{\sqrt{\Omega}}\left[\int_0^{\infty} d\omega[\alpha^*_{\Omega\omega}\hat{a}_{\omega}e^{i\Omega\sqrt{2}\tilde{x}_-} - \beta^*_{\Omega\omega}\hat{a}^{\dagger}_{\omega}e^{i\Omega\sqrt{2}\tilde{x}_-} + \alpha_{\Omega\omega}\hat{a}^{\dagger}_{\omega}e^{-i\Omega\sqrt{2}\tilde{x}_-} - \beta_{\Omega\omega}\hat{a}_{\omega}e^{-i\Omega\sqrt{2}\tilde{x}_-}]\right]$$

$$= \frac{1}{\sqrt{4\pi}}\int \frac{d\Omega}{\sqrt{\Omega}}\left[\int_0^{\infty}[\alpha^*_{\Omega\omega}e^{i\Omega\sqrt{2}\tilde{x}_-} - \beta_{\Omega\omega}e^{-i\Omega\sqrt{2}\tilde{x}_-}]\hat{a}_{\omega}d\omega + \int_0^{\infty}[\alpha_{\Omega\omega}e^{-i\Omega\sqrt{2}\tilde{x}_-} - \beta^*_{\Omega\omega}e^{i\Omega\sqrt{2}\tilde{x}_-}]\hat{a}^{\dagger}_{\omega}d\omega\right]$$

Comparing these grouped terms to our expression for the Minkowski operator, one can notice,

$$\frac{e^{i\omega\sqrt{2}x_-}}{\sqrt{\omega}} = \int_0^{\infty}\frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha^*_{\Omega'\omega}e^{i\Omega'\sqrt{2}\tilde{x}_-} - \beta_{\Omega'\omega}e^{-i\Omega'\sqrt{2}\tilde{x}_-}\right),$$

$$\frac{e^{-i\omega\sqrt{2}x_-}}{\sqrt{\omega}} = \int_0^{\infty}\frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}e^{-i\Omega'\sqrt{2}\tilde{x}_-} - \beta^*_{\Omega'\omega}e^{i\Omega'\sqrt{2}\tilde{x}_-}\right).$$

Note I have relabeled Ω as Ω' for reasons that will become clear as we move forward. Let us solve the first equation for $\alpha_{\Omega'\omega}$. We start by multiplying by $e^{-i\Omega\sqrt{2}\tilde{x}_-}$, note that $\Omega \neq \Omega'$.

$$\frac{e^{i\omega\sqrt{2}x_- - i\Omega\sqrt{2}\tilde{x}_-}}{\sqrt{\omega}} = \int_0^{\infty}\frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}e^{-i(\Omega-\Omega')\sqrt{2}\tilde{x}_-} - \beta^*_{\Omega'\omega}e^{-i(\Omega+\Omega')\sqrt{2}\tilde{x}_-}\right)$$

Now let us integrate both sides over $d\tilde{x}_-$, making note of the identity,

$$\int_{-\infty}^{\infty} e^{i(\Omega-\Omega')\tilde{x}_-} d\tilde{x}_- = 2\pi\delta(\Omega-\Omega').$$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega\sqrt{2}x_- - i\Omega\sqrt{2}\tilde{x}_-}}{\sqrt{\omega}} d\tilde{x}_- = \sqrt{2}\int_{0}^{\infty} \frac{\alpha_{\Omega'\omega}(2\pi\delta(\Omega-\Omega') - \beta^*_{\Omega'\omega}(2\pi\delta(\Omega+\Omega')))}{\sqrt{\Omega'}} d\Omega'$$

Note that conveniently, our $\beta_{\Omega\omega}$ term must vanish as the delta function requires a $\Omega < 0$ to be non-zero, and we are not integrating in that range. This gives us,

$$\int_{-\infty}^{\infty} \frac{\exp\{i\omega\sqrt{2}x_- + i\Omega\sqrt{2}\tilde{x}_-\}}{\sqrt{\omega}} d\tilde{x}_- = \frac{2\sqrt{2}\pi\alpha_{\Omega\omega}}{\sqrt{\Omega}}.$$

Now we can solve for our coefficient,

$$\alpha^*_{\Omega\omega} = \frac{1}{2\sqrt{2}\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} \exp\left\{i\omega\sqrt{2}x_- - i\Omega\sqrt{2}\tilde{x}_-\right\} d\tilde{x}_-.$$

Here we change coordinate x_- to $\tilde{x}_-$,

$$x_- = \frac{1}{a\sqrt{2}} e^{a\sqrt{2}\tilde{x}_-}$$

$$\alpha^*_{\Omega\omega} = \frac{1}{2\sqrt{2}\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} \exp\left\{\frac{i\omega}{a} e^{a\sqrt{2}\tilde{x}_-} - i\Omega\sqrt{2}\tilde{x}_-\right\} d\tilde{x}_-.$$

Let us define,

$$t = e^{a\sqrt{2}\tilde{x}_-}$$

$$\frac{dt}{d\tilde{x}_-} = a\sqrt{2}e^{a\sqrt{2}\tilde{x}_-}$$

$$d\tilde{x}_- = \frac{dt}{a\sqrt{2}} e^{-a\sqrt{2}\tilde{x}} = \frac{1}{a\sqrt{2}}\frac{dt}{t}.$$

$$e^{-i\Omega\sqrt{2}\tilde{x}_-} = \left(\exp\left\{\sqrt{2}\tilde{x}_-\right\}\right)^{-i\Omega} = \left(\exp\left\{a\sqrt{2}\tilde{x}_-\right\}\right)^{-\frac{i\Omega}{a}} = t^{-\frac{i\Omega}{a}}$$

$$\alpha^*_{\Omega\omega} = \frac{1}{4\pi a}\sqrt{\frac{\Omega}{\omega}}\int_{0}^{\infty} t^{-\frac{i\Omega}{a}-1} e^{-\frac{i\omega}{a}t} dt$$

Here we are performing a contour integral and it is useful to rewrite $t = re^{\frac{i\pi}{2}}$, $r \in \mathbb{R}$. This makes our integral,

$$\alpha^*_{\Omega\omega} = \frac{1}{4\pi a}\sqrt{\frac{\Omega}{\omega}} e^{\frac{\Omega\pi}{2a}} \int_{0}^{\infty} r^{-\frac{i\Omega}{a}-1} e^{-\frac{\omega}{a}r} dr.$$

From here we can take $\theta = \dfrac{\omega}{a}r$ and $\Xi = -\dfrac{i\Omega}{a}$, to rewrite in terms of a Gamma function,

$$\alpha^*_{\Omega\omega} = \frac{1}{4\pi a}\sqrt{\frac{\Omega}{\omega}} e^{\frac{\Omega\pi}{2a}} \left(\frac{\omega}{a}\right)^{\frac{i\Omega}{a}} \int_{0}^{\infty} \theta^{\Xi-1} e^{-\theta} dr.$$

Now we can rewrite it as,

$$\alpha^*_{\Omega\omega} = \frac{1}{4\pi a}\sqrt{\frac{\Omega}{\omega}} \exp\left\{\frac{\Omega\pi}{2a}\right\} \exp\left\{\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)\right\} \Gamma\left(-\frac{i\Omega}{a}\right).$$

Likewise,

$$\beta_{\Omega\omega} = \frac{1}{4\pi a}\sqrt{\frac{\Omega}{\omega}} \exp\left\{-\frac{\Omega\pi}{2a}\right\} \exp\left\{\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)\right\} \Gamma\left(-\frac{i\Omega}{a}\right).$$

This gives us,

$$\alpha_{\Omega\omega} = -\frac{1}{4\pi a}\sqrt{\frac{\Omega}{\omega}}\exp\left\{\frac{\Omega\pi}{2a}\right\}\exp\left\{\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)\right\}\Gamma\left(-\frac{i\Omega}{a}\right), \tag{27}$$

$$\beta_{\Omega\omega} = \frac{1}{4\pi a}\sqrt{\frac{\Omega}{\omega}}\exp\left\{-\frac{\Omega\pi}{2a}\right\}\exp\left\{\frac{i\Omega}{a}\ln\left(\frac{\omega}{a}\right)\right\}\Gamma\left(-\frac{i\Omega}{a}\right). \tag{28}$$

From this, one can notice that,

$$|\alpha_{\Omega\omega}|^2 = e^{\frac{\pi\Omega}{a}}|\beta_{\Omega\omega}|^2. \tag{29}$$

Now, we can take the expectation value of the number operator to find the expected number of particles produced.

$$\langle N_b\rangle = \langle 0_M|\,\hat{b}^{\dagger}_{\Omega}\hat{b}_{\Omega}\,|0_M\rangle$$

We can rewrite,

$$\begin{aligned}\langle 0_M|\,\hat{b}^{\dagger}_{\Omega}\hat{b}_{\Omega}\,|0_M\rangle &= \int_0^{\infty} d\omega d\omega'\,\langle 0_M|\,[\alpha_{\Omega\omega}\hat{a}^{\dagger}_{\omega} - \beta_{\Omega\omega}\hat{a}_{\omega}][\alpha^{*}_{\Omega\omega'}\hat{a}_{\omega'} - \beta^{*}_{\Omega\omega'}\hat{a}^{\dagger}_{\omega'}]\,|0_M\rangle \\ &= \frac{1}{e^{\frac{2\pi\Omega}{a}}-1}\delta(0)\end{aligned}$$

We confine our system to a volume V to eliminate our UV divergent term and we obtain a number density,

$$\langle \hat{n}_{\omega}\rangle = \frac{\langle N_b\rangle}{V} = \frac{1}{e^{\frac{2\pi\Omega}{a}}-1} \tag{30}$$

This is the form of a Bose-Einstein distribution with temperature,

$$\boxed{T = \frac{a}{2\pi}.}$$

Hawking Radiation for a Schwarzschild Black Hole

Overview

In this section we derive the Hawking temperature for a Schwarzschild black hole. We do this by considering a (1+1)-D Schwarzschild black hole. This metric is conformally Minkowski, allowing us to apply our known quantum field theory from the prior chapter. This saves us the effort of having to repeat the computation of the Bogolyubov coefficients.

Schwarzschild and Kruskal-Szekeres Coordinates

In an analogous manner to our derivation of the Unruh effect, we will attempt to make a useful coordinate transformation between two frames. We will start by considering a (1+1)-D Schwarzschild line element,

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}}. \tag{31}$$

This coordinate system is defined only for regions $r > 2M$ and is singular at the event horizon $r = 2M$. It can be thought of as the perspective of an observer far outside the event horizon of the black hole where as $r \to \infty$, spacetime becomes asymptotically Minkowski. We want to find a conformal transformation to a coordinate system that can describe the full spacetime of the black hole to compare what an observer near the event horizon would observe. The appropriate choice for this purpose is the Kruskal-Szekeres coordinate system. The Schwarzschild spacetime as defined by the Kruskal-Szekeres coordinates gives a line element,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{2M}{r}\exp\Big[1 - \frac{r}{2M}\Big]\left[-dt^2 + dx^2\right]. \tag{32}$$

One can note that this coordinate system is a conformal transformation on Minkowski spacetime due to the presence of the $-dt^2 + dx^2$ term from the Minkowski line element and the presence of a multiplicative conformal factor. Also note that this coordinate system is only singular at the physical singularity point $r = 0$ and light travels at 45-degree angles.

One can note that in our Schwarzschild solution, we do not see the same properties of light traveling at 45-degree angles. This indicates that there is globally conformal mapping between the two coordinate systems. To resolve this, we introduce the tortoise coordinate.

$$r^* = r - 2M\left(1 - \ln\left(\frac{r}{2M} - 1\right)\right), \quad dr^* = \frac{dr}{1-\frac{2M}{r}}, \quad \text{for } r > 2M \tag{33}$$

This change of coordinate preserves light-cone angles at 45 degree angles and allows us to rewrite **Equation (31)** in terms of this new coordinate.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{\left(1-\frac{2M}{r}\right)^2}{1-\frac{2M}{r}}dr^{*2}$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)dr^{*2}$$

From **Equation (33)**,

$$r^* = r - 2M + 2M\ln\left(\frac{r}{2M} - 1\right) \to \frac{r^*}{2M} = \frac{r}{2M} - 1 + \ln\left(\frac{r}{2M} - 1\right).$$

$$\to e^{\frac{r^*}{2M}} = e^{\frac{r}{2M}-1}\left(\frac{r}{2M} - 1\right) \to e^{\frac{r^*}{2M}}e^{1-\frac{r}{2M}} = \frac{r}{2M} - 1 \to \left(1 - \frac{2M}{r}\right) = \frac{2M}{r}e^{\frac{r^*}{2M}}e^{1-\frac{r}{2M}}$$

$$ds^2 = \frac{2M}{r}\exp\left[\frac{r^*}{2M}\right]\exp\left[1-\frac{r}{2M}\right]\left[-dt^2+dr^{*^2}\right] \tag{34}$$

This change of coordinate has enabled us to rewrite our Schwarzschild spacetime in such a way that we can describe a conformal mapping between our two systems as angles are preserved globally.

To describe this transformation, let us return to light-cone coordinates. As before, our global coordinate system (Kruskal-Szekeres) will be denoted by x, t,

$$x_+ = \frac{x+t}{\sqrt{2}} \quad \text{and} \quad x_- = \frac{x-t}{\sqrt{2}}. \tag{35}$$

Our system in which an event horizon appears (Schwarzschild) will be denoted,

$$\tilde{x}_+ = \frac{r^*+t}{\sqrt{2}} \quad \text{and} \quad \tilde{x}_- = \frac{r^*-t}{\sqrt{2}}. \tag{36}$$

We can now rewrite **Equation (32)** and **Equation (34)** in terms of their respective light-cone coordinates.

$$ds^2 = \frac{2M}{r}\exp\left[1-\frac{r}{2M}\right]\left[-dt^2+dx^2\right] = \frac{4M}{r}\exp\left[1-\frac{r}{2M}\right]dx_+dx_-$$

$$ds^2 = \frac{4M}{r}\exp\left[1-\frac{r}{2M}\right]dx_+dx_- \tag{37}$$

$$ds^2 = \frac{2M}{r}e^{\frac{r^*}{2M}}\exp\left[1-\frac{r}{2M}\right]\left[-dt^2+dr^{*^2}\right] = \frac{4M}{r}\exp\left[\frac{\sqrt{2}\left(\tilde{x}_++\tilde{x}_-\right)}{4M}\right]\exp\left[1-\frac{r}{2M}\right]d\tilde{x}_+d\tilde{x}_-$$

$$ds^2 = \frac{4M}{r}\exp\left[\frac{\sqrt{2}\left(\tilde{x}_++\tilde{x}_-\right)}{4M}\right]\exp\left[1-\frac{r}{2M}\right]d\tilde{x}_+d\tilde{x}_- \tag{38}$$

Setting **Equation (37)** and **(38)** equal to one another allows us to solve for the conformal transformation for the non-global region $r > 2M$.

$$\frac{4M}{r}\exp\left[1-\frac{r}{2M}\right]dx_+dx_- = \exp\left[\frac{\sqrt{2}\left(\tilde{x}_++\tilde{x}_-\right)}{4M}\right]\exp\left[1-\frac{r}{2M}\right]d\tilde{x}_+d\tilde{x}_-$$

$$dx_+dx_- = \exp\left[\frac{\sqrt{2}\left(\tilde{x}_++\tilde{x}_-\right)}{4M}\right]d\tilde{x}_+d\tilde{x}_-$$

$$dx_+ = e^{\frac{\sqrt{2}\tilde{x}_+}{4M}}d\tilde{x}_+, \quad dx_- = e^{\frac{\sqrt{2}\tilde{x}_-}{4M}}d\tilde{x}_-$$

Now we integrate to obtain the coordinate transformations,

$$x_+ = \frac{4M}{\sqrt{2}}e^{\frac{\sqrt{2}\tilde{x}_+}{4M}} \quad \text{and} \quad x_- = \frac{4M}{\sqrt{2}}e^{\frac{\sqrt{2}\tilde{x}_-}{4M}}. \tag{39}$$

Now that we have arrived at our conformal transformation relations, we can begin to discuss the effects of quantum fields.

Quantum Scalar Fields in Schwarzschild Spacetime

With our classical background set up and a conformal transformation derived, we can now consider quantum field theory on the background of our curved Schwarzschild spacetime. We start with the Lagrangian density for a minimally coupled, free, massless, scalar field in our (1+1)-D curved spacetime.

$$\mathcal{L} = -\frac{\sqrt{-g}}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \tag{40}$$

This corresponds to the action,

$$S = -\int dtdx \frac{\sqrt{-g}}{2} \left[g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right]$$

We can perform a field variation under which our action remains invariant,

$$\delta S = -\int dtdx \frac{\sqrt{-g}}{2} \delta\phi \left[g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] = 0$$

$$= -\int dtdx \sqrt{-g} \left[g^{\mu\nu} \delta(\nabla_\mu \phi) \nabla_\nu \phi \right] = -\int dtdx \sqrt{-g} \left[g^{\mu\nu} \nabla_\mu \delta\phi \nabla_\nu \phi \right]$$

We can use u-substitution to factor out our variation. Let $u = \nabla_\nu \phi$, $v = \delta\phi$, $du = \nabla_\mu \nabla_\nu \phi$, and $dv = \nabla_\mu(\delta\phi)$.

$$\delta S = -\int dxdt \sqrt{-g} \left[\delta\phi \nabla_\mu \phi - \delta\phi \nabla_\mu \nabla_\nu \phi \right]$$

We can eliminate the first term in the square parenthesis using Stokes's theorem and knowing as we have field variation at the boundary $\delta\phi = 0$.

$$\delta S = -\int dtdx \sqrt{-g} g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0$$

From this we have the general form of Laplace's equation in curved space,

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = \nabla^\nu \nabla_\nu \phi = 0. \tag{41}$$

In our case with a spherically symmetric Schwarzschild black hole, this reduces to the d'Alembertian operator, and our fields are described by the same differential equation as for the Unruh effect **Equation (19)**.

$$\Box^{(2)} \phi = 0$$

This means we can follow the same procedure outlined in the Unruh effect to construct our plane wave solutions and quantize them. This will yield the exact same form of solution as before.

Hawking Radiation

Instead of repeating calculations, we will simply begin by comparing **Equations (15)** and **(39)**. Note that they are identical aside from,

$$a \to \frac{1}{4M}. \tag{42}$$

Additionally, instead of our Minkowski vacuum, we have a Kruskal vacuum, and instead of a Rindler vacuum, we have the Boulware vacuum (defined in Schwarzschild coordinates). This yields relations between the Bogolyubov transformation coefficients,

$$\alpha_{\Omega\omega} = -\frac{M}{\pi} \sqrt{\frac{\Omega}{\omega}} \exp\{2M\Omega\pi\} \exp\{4Mi\Omega \ln(4M\omega)\} \Gamma(-4Mi\Omega), \tag{43}$$

$$\beta_{\Omega\omega} = \frac{M}{\pi} \sqrt{\frac{\Omega}{\omega}} \exp\{-2M\Omega\pi\} \exp\{4Mi\Omega \ln(4M\omega)\} \Gamma(-4Mi\Omega). \tag{44}$$

As before, we can calculate the expected particle count to be,

$$\langle 0_K | \hat{b}_\Omega^\dagger \hat{b}_\Omega | 0_K \rangle = \frac{1}{\exp(8M\pi\Omega) - 1} \delta(0)$$

This gives a temperature,

$$\boxed{T = \frac{1}{8\pi M}.}$$

Holographic Black Holes

Overview

In this section we introduce the concept of holographic analogs for higher dimensional black holes. By reducing a more complicated (3+1)-D spherically symmetric metric to a (1+1)-D metric that is conformally Minkowski, we can continue to make use of our quantum field theory established in the first section. We start by demonstrating that this dimensional reduction works for the Schwarzschild black hole, for which we just derived the (1+1)-D case in the prior chapter. We then discuss how to reduce axially symmetric black holes. This will form the basis of our model for analyzing the Kerr-Newman case.

Representing (3+1)-D Black Holes as Holographic (1+1)-D Black Holes

In the prior section, we demonstrated that the methodology used in the derivation of the Unruh effect can be applied to the case of a conformally Minkowski (1+1)-D Schwarzschild metric. To continue using this methodology to explore more general black holes, we want to exploit a duality that enables us to represent all the relevant information about a (3+1)-D black hole holographically in our established (1+1)-D black hole system. This type of duality is known as a Kaluza-Klein reduction. More specifically, this duality states that the physics in the near-horizon region of a (3+1)-D black hole are identical to that of a (1+1)-D black hole coupled to a scalar field ϕ and an electromagnetic gauge potential $\mathcal{A} = A_\mu dx^\mu$ [12],

$$ds^2_{(3+1)-D} \xrightarrow{r\sim r_+} ds^2_{(1+1)-D}, \quad \phi, \quad \mathcal{A}. \tag{45}$$

The Kaluza-Klein form of the black hole is written as,

$$ds^2_{KK} = ds^2_{(1+1)-D} + \lambda^2 e^{-2\phi(r)} \left[d\theta^2 + \sin^2\theta(d\varphi + A_\mu dx^\mu)^2\right].$$

The value of λ is chosen such that $\lambda^2 e^{-2\phi(r)} = r^2$.

Let us now demonstrate this duality holds for our Schwarzschild black hole, yielding the same result as our previous section. This time we begin with the (3+1)-D metric for a Schwarzschild black hole.

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{46}$$

Next, we group the terms that do not involve dr or dt into a third term which we will call $r^2 d\Omega^2$. This is done primarily for notational purposes as it will allow us to avoid some unnecessary algebra.

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Next, we are going to relabel,

$$f(r) = \left(1 - \frac{2M}{r}\right).$$

This allows us to write our line element,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2.$$

Varying our action in the same way as in the prior sections, we yet again arrive at the generalized Laplacian in curved space **Equation (41)**,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0.$$

For our Schwarzschild metric in (3+1)-D **Equation (41)** becomes,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0$$

$$= \left[\partial_t\left(-\frac{r^2}{f(r)}\partial_t\right) + \partial_r\left(f(r)r^2\partial_r\right) + \partial_\Omega\left(\partial_\Omega\right)\right]\phi = 0$$

$$= \left[-\frac{\partial_t^2}{f(r)} + \frac{\partial_r}{r^2}\left(f(r)r^2\partial_r\right) + \partial_\Omega^2\right]\phi = 0$$

$$= \left[-\frac{\partial_t^2}{f(r)} + f(r)r^2\partial_r^2 + \frac{1}{r^2}\left(f'(r)r^2 + 2rf(r)\right)\partial_r + \partial_\Omega^2\right]\phi = 0$$

$$= \left[-\frac{\partial_t^2}{f(r)} + f(r)\partial_r^2 + \left(f'(r) + \frac{2f(r)}{r}\right)\partial_r + \partial_\Omega^2\right]\phi = 0$$

$$= \left[-\partial_t^2 + \partial_{r^*}^2 + \left(\frac{2r-2M}{r^2-2Mr}\right)\partial_{r^*} + f(r)\partial_\Omega^2\right]\phi = 0.$$

Here we have ∂_Ω as the angular component of our Laplacian,

$$\partial_\Omega^2 = \frac{1}{\sin^2\theta}\partial_\varphi^2 + \frac{\partial_\theta}{\sin\theta}(\sin\theta\partial_\theta).$$

From here, we can apply separation of variables to obtain our differential equation.

$$\phi(t, r^*, \varphi, \theta) = R(t, r^*)Y_{ml}(\varphi, \theta)$$

We arrive at our differential equation,

$$\left[\Box^{(2)} + V(r)\right]\phi = 0, \tag{47}$$

$$\Box^{(2)} = -\partial_t^2 + \partial_r^2, \quad V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right). \tag{48}$$

In the near horizon limit where $r \to r_+$, the potential term vanishes $V \to 0$. This means that our differential equation reduces to the traditionally expected form of $\Box^{(2)}\phi = 0$ and we make the same comparison made in the previous chapter, where $a \to (2r_+)^{-1} = (4M)^{-1}$ and we again obtain the same Hawking temperature as before,

$$T = \frac{1}{8\pi M}.$$

Extending to an Axially Symmetric Black Hole

In this section we will make use of the Kaluza-Klein reduction to consider a black hole that has axial symmetry, meaning either the Kerr or Kerr-Newman black hole. The axially symmetric line element is defined in a primed coordinate system as,

$$ds^2 = -\left(\frac{\Sigma\Delta}{\Pi}\right)dt'^2 + \Sigma\left(\frac{dr'^2}{\Delta} + d\theta'^2\right) + \frac{\sin^2\theta'}{\Sigma}\Pi\left(d\varphi' + \frac{2Mar'}{\Pi}dt'\right)^2 \tag{49}$$

in which,

$$\Sigma = r'^2 + a^2\cos^2\theta'$$
$$\Pi = \left(r'^2 + a^2\right)^2 - \Delta a^2\sin^2\theta'$$
$$\Delta = (r' - r_+)(r' - r_-)$$
$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}, \quad a = \frac{J}{M}$$

Notice that the only difference between the line elements of the Kerr black hole and the Kerr-Newman black hole is the presence of charge Q which appears in the expression for the horizons $r_\pm$. Setting charge

Q equal to zero in these expressions reduces the form from the Kerr-Newman to the Kerr case. In order to apply **Equation (41)** and perform our calculations on the action, we need to transform **Equation (49)** into a form more representative of our spacetime geometry near the horizon. We perform the transformation from the primed to un-primed coordinates by,

$$r' \to M + \lambda r, \quad t' \to \frac{t}{\lambda}, \quad \theta' \to \theta, \quad \varphi' \to \varphi + \frac{at}{l^2\lambda}, \quad l^2 = r_+^2 + a^2$$

Performing this transformation into the near-horizon regime is only mathematically stable for the extremal case, in which $M^2 = a^2 + Q^2$.The resulting near-horizon extremal metric is,

$$ds^2_{NHEKN} = \Psi\left[-\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + l^2 d\theta^2\right] + \frac{l^2\sin^2\theta}{\Psi}\left[d\varphi + \frac{2Mar}{l^4}dt\right]^2, \quad \Psi = 1 - \frac{a^2}{l^2}\sin^2\theta \tag{50}$$

However, this metric generates an issue in our temperature calculation, as the Hawking temperature for an extremal black hole is zero. Therefore, we must tune the metric with a perturbation to the near-extremal metric.

$$ds^2_{NHNEKN} = \Psi\left[-f(r)dt^2 + \frac{1}{f(r)}dr^2 + \Omega d\theta^2\right] + \frac{\Omega\sin^2\theta}{\Psi}\left[d\varphi + \frac{r-2M}{\Omega}dt\right]^2 \tag{51}$$

$$f(r) = \frac{\Delta}{\Omega}, \quad \Omega = r^2 + a^2 \tag{52}$$

It is worth noting that the value of $\Omega(r)$ in the extremal case at the horizon(meaning $r \to r_+$) will reduce to the scale parameter l^2. We now consider the matter field in our Kerr-Newman black hole background, which is very similar to a matter field for a Kerr black hole background.

$$S = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

By the principle of stationary action, we will vary the action with respect to our scalar field ϕ and integrate by parts in order to derive,

$$\begin{aligned}\delta S =& 0 = -\frac{1}{2}\int dx^4\delta\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)\\ 0 =& \frac{1}{2}\left[\int dx^3\delta\phi\sqrt{-g}g^{\mu\nu}\partial\phi - \int dx^4\delta\left(\phi\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\phi\right]\end{aligned}$$

In the above variation, the first integral will be zero under the common assumption that the variation $\delta\phi$ at the boundary will be zero. This leaves the second integral in the NHNEKN case,

$$\begin{aligned}S &= \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\\ &= -\frac{1}{2}\int dx^4\phi\left[\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\right]\phi\\ &= \int dx^4\phi\left[-\frac{\Omega(r_+)\sin\theta}{f(r)}\partial_t^2 + \sin\theta\partial_r\left(\Omega(r_+)f(r)\partial_r\right) + \left[\frac{\Psi^2}{\sin\theta} - \frac{(r-2M)^2\sin\theta}{f(r)\Omega(r_+)}\right]\partial_\varphi^2 + \frac{2\sin\theta(r-2M)}{f(r)}\partial_\varphi\partial_t\right]\phi\end{aligned}$$

From a visual inspection of the above action, is is clear that the inverse metric used in the action still provides a functional form which suggests a separation of the scalar field with its angular components. We can express the scalar field ϕ in terms of spherical harmonics $\phi(t,r,\theta,\varphi) = \sum_{lm} R_{lm}(t,r)Y_l^m(\theta,\varphi)$. By integrating with respect to the angular coordinates θ, φ, we obtain an equivalent action

$$S = \int dx^2\Omega(r_+)R^*(t,r)\left[-\frac{1}{f(r)}\left(\partial_t + \frac{ima}{\Omega(r_+)}\right)^2 + \partial_r\left(f(r)\partial_r\right)\right]R(t,r) \tag{53}$$

Let

$$\partial_{\bar{t}} = \partial_t - im\mathcal{A}_t, \quad \mathcal{A}_t = \frac{r - 2M}{\Omega(r_+)}$$

be a $U(1)$ gauge field, which transforms our action into,

$$S = \Omega(r_+) \int dx^2 R^*(t,r) \left[-\frac{1}{f(r)} \partial_{\bar{t}}^2 + \partial_r \left(f(r) \partial_r \right) \right] R(t,r)$$

This action mimics the Kerr case incredibly closely, with the only difference being the defined potential function $f(r)$. This is because the presence of charge in a black hole spacetime metric primarily influences the radial location of the event horizons. It does not strongly influence the deformation of the black hole's event horizons into an axially symmetric set of surfaces. That deformation is primarily caused by the black hole's rotation.

Hawking Radiation for a Kerr-Newman Black Hole

Overview

In this section, we apply our near-horizon near-extremal Kerr-Newman metric from the prior section to our quantum theory of a (1+1)-D black hole. We start by describing the Kerr-Newman black hole in general and then demonstrate that our methodology arrives at the appropriate result for the Hawking temperature.

The Kerr-Newman Black Hole

The Kerr-Newman black hole is the most general form of a black hole as described by Einstein's General Theory of Relativity. Its metric describes a stationary black hole possessing a mass M, angular momentum J, and electric charge Q. The metric describes an axially symmetric region of spacetime with several unique regions.

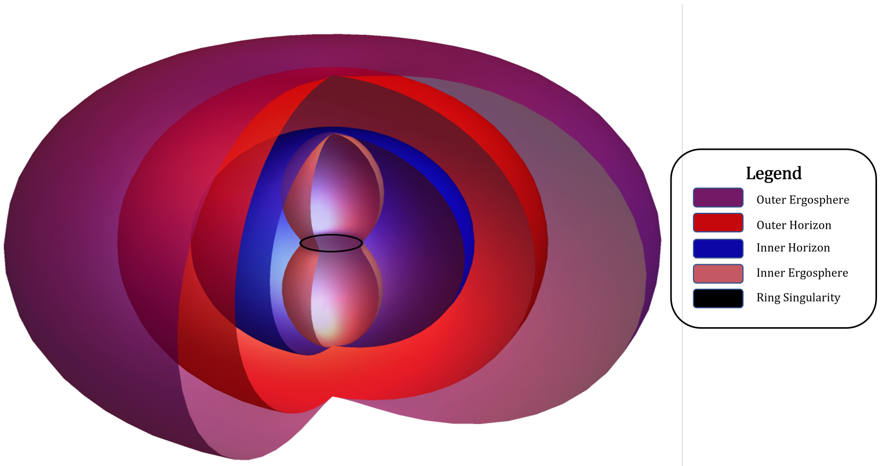


Figure 4: Geometry of the Kerr-Newman black hole.

Outside the black hole, there appears a surface known as the outer ergosphere (the purple region in **Figure 15**). In the region past the outer ergosphere, the rotation of the black hole causes spacetime itself to be dragged in a process known as "frame dragging". In this region, it is impossible for an object to remain stationary from the perspective of an far-distant outside observer without traveling faster than the speed of light.

Once past the outer ergosphere, we approach the outer horizon (the red sphere in **Figure 15**), the first of two event horizons that form due to the black hole mass rotating and possessing charge. This outer horizon is the same boundary for a causally disconnected region of the universe as seen in simpler black holes. Beyond this we approach the inner horizon (the blue sphere in **Figure 15**).

Once inside both horizons, we near the inner ergosphere (the pink region of **Figure 15**), a region which traces the singularity on the equatorial ($\theta = 0$) plane. Once inside we approach the singularity, which in the case of a rotating black hole, takes the form of a ring due to the incompatibility between a classical "point" possessing rotation and the corresponding angular momentum.

In the limit where the charge $Q \to 0$, the Kerr-Newman black hole reduces to the Kerr solution. When the angular momentum $J \to 0$, it reduces to the Reissner–Nordström solution. Finally, when both charge and angular momentum $Q \to 0$ and $J \to 0$, we obtain the Schwarzschild metric.

When the charge and angular momentum become sufficiently large relative to the mass such that,

$$M^2 = \frac{J^2}{M^2} + Q^2,$$

we have what is known as an "extremal Kerr-Newman black hole". In the extremal case, the inner and outer horizons converge at the location $r_+ = r_- = M$. For solutions such that,

$$M^2 < \frac{J^2}{M^2} + Q^2,$$

the event horizons vanish and what is known as a "naked singularity" occurs. This is a violation of the cosmological censorship hypothesis and such solutions are considered non-physical and impossible.

The line element describing a Kerr-Newman spacetime given in Boyer-Lindquist coordinates is given as,

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{J}{M}\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{1}{\rho^2}\left(\frac{J}{M}dt - \left(r^2 + \frac{J^2}{M^2}\right)\right)^2, \tag{54}$$

where,

$$\Delta = \left(r^2 + \frac{J^2}{M^2}\right) - 2Mr + Q^2 \quad \text{and} \quad \rho^2 = r^2 + \frac{J^2}{M^2}\cos^2\theta.$$

Applying the Unruh Quantum Field Theory to the NHNEKN Spacetime

In the previous section, we discussed using the Kaluza-Klein reduction as a method to reduce (3+1)-D axially symmetric black holes to (1+1)-D black holes with coupling to scalar and gauge fields. In this section, we return to **Equation (53)** describing the Near-Horizon-Near-Extremal Kerr-Newman black hole action dimensionally reduced to (1+1)-D.

$$S = \int dx^2 \Omega(r_+) R^*(t,r)\left[-\frac{1}{f(r)}\left(\partial_t + \frac{ima}{\Omega(r_+)}\right)^2 + \partial_r\left(f(r)\partial_r\right)\right] R(t,r)$$

It is known that every two dimensional pseudo-Riemannian manifold is conformally flat. Therefore the Kaluza-Klein reduction provides a powerful tool to enable us to make use of our existing quantum field theory derived for the Unruh Effect, thereby allowing us to not have to recalculate the Bogolyubov coefficients. This is the case because in the near-horizon limit, the quantum theory of an arbitrary axially symmetric metric becomes equivalent to that of a (1+1)-D black hole with an apparent scalar potential. What's more, the quantum field effects of any (1+1)-D black hole reduces a Minkowski flat spacetime in some certain coordinate system.

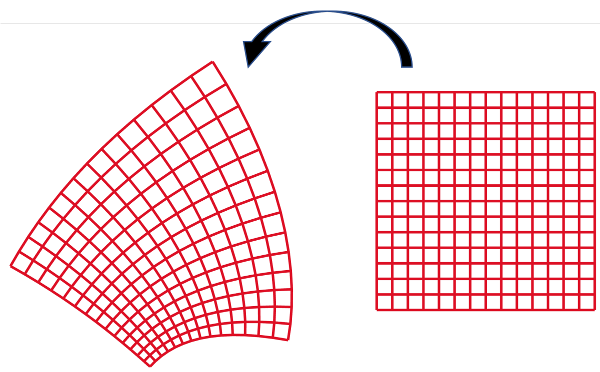


Figure 5: Diagram of conformal mapping in which angles are preserved.

Now all that is left to do is establish a coordinate transform to a coordinate system defined in the near-horizon Kerr-Newman metric regime, but specifically with its (1+1)-D Kaluza-Klein reduction. We begin with the (1+1)-D metric,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}$$

To define the transformation, we will define a coordinate system similar to tortoise coordinates in which,

$$\frac{dr}{dr^*} = f(r) \implies r^* = \int f(r)^{-1} dr, \quad dr^2 = f(r)^2 dr^{*2}$$
$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = f(r)\left[-dt^2 + dr^{*2}\right]$$

We define a set of light-cone coordinates as follows,

$$x^\alpha = (u, v), \quad u = r^* - t, \quad v = r^* + t, \quad \therefore\ ds^2 = f(r) du dv$$

The metric $g_{\alpha\beta}$ is conformally flat, therefore for any point $p \in M$ on our Minkowski spacetime manifold, there exists a neighborhood $\mathcal{U}$ surrounding p and a positive smooth function $a(r)$ defined on $\mathcal{U}$ such that,

$$g_{\alpha\beta} = e^{a(r)} \eta_{\alpha\beta}$$

We can choose our point p on the manifold to be near the horizon such that $\mathcal{U}$ contains the horizon. Recall that we can transform coordinates and so despite working within light cone coordinates (u, v), we can still define $a(r)$ in terms of r and use the transformation $r = f^{-1}\left(\frac{u+v}{2}\right)$ to define in terms of light-cone coordinates. We will also define another set of light-cone coordinates $X^\alpha = (U, V)$ defined on the locally Minkowski neighborhood $\mathcal{U}$. Since both light-cone coordinate systems are defined for the neighborhood $\mathcal{U}$, then there must be a transition function between the coordinate charts of x^α and X^α. This allows for an approximation of the two functions in the near-horizon region. Therefore we take a limit of our radial coordinate r to the outer horizon r_+,

$$\lim_{r \to r_+} e^{a(r)} \approx f(r)$$

This approximation is Taylor expanded about a point $p \in \mathcal{U}$ in order to find an expression equating the two functions $a(r)$ and $f(r)$,

$$e^{a(p)} + \frac{da(p)}{dr} e^{a(r_+)}(r - p) + \mathcal{O}(r^2) = f(p) + \frac{df(p)}{dr}(r - p) + \mathcal{O}(r^2)$$

In the expansion, the powers of $(r - p)$ must equate to each other. Therefore,

$$e^{a(p)} = f(p), \quad \frac{da(p)}{dr} e^{a(r_+)} = \frac{df(p)}{dr}, \implies \frac{da(p)}{dr} = \frac{df(r_+)}{dr} \frac{1}{f(p)}$$

By definition of our tortoise coordinate, $f(r) = \frac{dr}{dr^*}$. Thus $da = f'(r_+) dr^*$,

$$a(r) = f'(r_+) r^* + C, \quad \therefore\ e^a(r) = e^{f'(r_+) r^* + C} = A e^{f'(r_+) \frac{u+v}{2}} = f(r)$$

The above function equations are true for regions near the horizon. Therefore the map between the (1+1)-D black hole tortoise light-cone coordinates x^α and the locally Minkowski light-cone coordinates are transformed by,

$$U = A e^{\frac{f'(r_+)}{2} u}, \quad V = A e^{\frac{f'(r_+)}{2} v}$$

Recall in **Equation (15)**, the transformation between Minkowski light cone coordinates and Rindler frame coordinates contained similar exponential terms and were used to calculate a Hawking temperature in terms of the acceleration a of the frame. This same Bogolyubov calculation is equivalent in this case as well. All that must be done is replace $a \to \frac{f'(r_+)}{2}$.

The Bogolyubov coefficients therefore are,

$$\begin{cases} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{cases} = \int_{-\infty}^{\infty} \exp\left[\mp i\omega u + i\Omega U\right] du = \pm \frac{1}{\pi f'(r_+)} \sqrt{\frac{\Omega}{\omega}} \exp\left[\pm \frac{\pi\Omega}{f'(r_+)}\right] \exp\left[\frac{2i\Omega}{f'(r_+)} \ln \frac{2\omega}{f'(r_+)}\right] \Gamma\left(-\frac{2i\Omega}{f'(r_+)}\right).$$

The number density is therefore,

$$n_\Omega = \frac{\langle \hat{N}_\Omega \rangle}{V} = \left[\exp\left(\frac{4\pi\Omega}{f'(r_+)} - 1\right)\right]^{-1}, \quad \implies T_H = \frac{f'(r_+)}{4\pi}.$$

With our $f(r)$ for a Kerr-Newman black hole in the near-horizon region as,

$$f(r) = \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2}. \tag{55}$$

From this, we arrive at a Hawking temperature of,

$$\boxed{T = \frac{1}{2\pi}\left(\frac{\sqrt{M^2 - a^2 - Q^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}}\right).} \tag{56}$$

Black Hole Evaporation

Overview

In this section, we introduce the notion that Hawking radiation leads to black holes slowly evaporating over long periods of time. We demonstrate a simple analytical solution for the Schwarzschild black hole, then extend to solving more complicated systems via simulation.

How Black Holes Evaporate

As shown in the previous section, black holes radiate a thermal bath of particles with an associated temperature T. This radiation can be described as a radiating black-body by the Stefan-Boltzmann law.

$$L = \sigma T^4 A \tag{57}$$

As a consequence of this radiation, the black hole loses mass (energy) and evaporates. When $c = 1$, the change in mass with respect to time can be modeled by,

$$\frac{dM}{dt} = -L = -\sigma T_H^4 A_{BH}. \tag{58}$$

For a Schwarzschild black hole, solving this differential equation with the initial condition $M(t = 0) = M_0$ yields,

$$M(t) = M_0 \left(1 - \frac{t}{5120\pi M_0^3}\right)^{\frac{1}{3}} \tag{59}$$

where,

$$T_H = \frac{1}{8\pi M} \quad \text{and} \quad \sigma = \frac{\pi^2}{60}.$$

As the black hole continues to lose mass, the Hawking temperature increases since mass is inversely proportional to temperature.

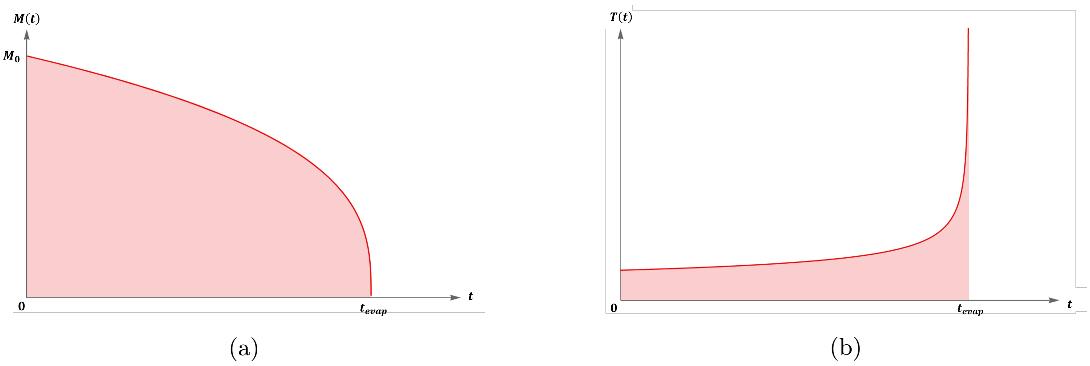


Figure 6: Mass (a) and Temperature (b) curves of a Schwarzschild black hole as it evaporates over time.

Having derived the general form for the Hawking temperature, we can explore the evaporation process of more complicated black holes that result in differential equations that are much more difficult to solve analytically.

Simulating an Evaporating Black Hole

In order to solve for these more complicated systems, we make use of simulations to numerically compute how these systems evolve over time. The simulation accepts a mass M in units of $M/M_\odot$, an electric charge q in terms of Coulombs, and a Kerr parameter $a = J/M$. After taking in these parameters, the simulation checks if this corresponds to a physical black hole solution, and whether the given black hole can evaporate further.

Simulation Methodology - Euler's Method

The simulation is written in Python and makes use of Euler's method as it is the most intuitive method. Euler's method is an iterative approach by which you define your variable x, how it changes with time $\frac{dx}{dt}$, and a step size Δt. With this, you can compute the n^{th} term from the $n-1^{th}$ term as follows.

$$x_n = x_{n-1} + \Delta t \frac{dx}{dt}$$

This algorithm is described below in pseudo-code.

$i \leftarrow N$
$arrayx \leftarrow [x_0]$
$\Delta t \leftarrow input$
$xprime \leftarrow input$
for $i = 0; i \leq N; i \leftarrow i+1$ **do**
  $x_n \leftarrow x_{n-1} + \Delta t \cdot xprime$
  $arrayx[i+1] \leftarrow (x_n)$
end for

Simulating the Evaporation of a Schwarzschild Black Hole

We simulated the evaporation process of a Schwarzschild black hole with mass equivalent to the mass of the Sun $M_{\odot}$.

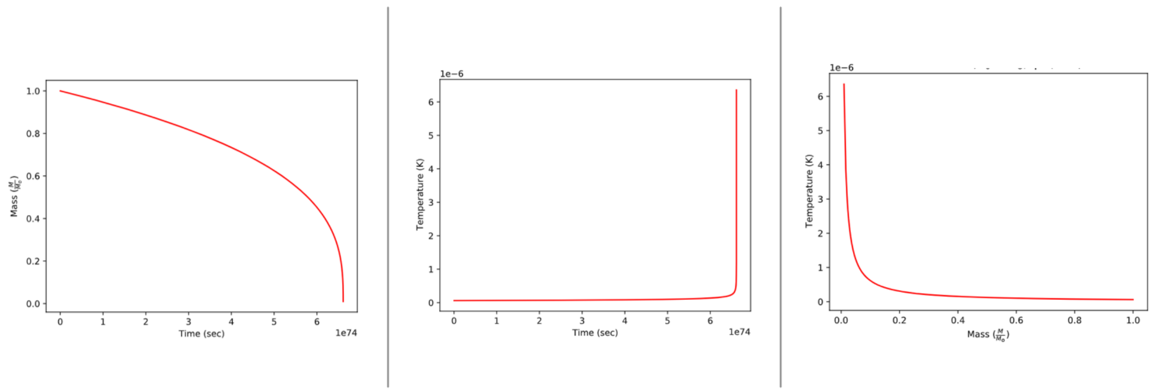


Figure 7: Graphs depicting black hole evaporation for a Schwarzschild black hole. Mass v.s. Time (left), Temperature v.s. Time (center), Temperature v.s. Mass (Right)

We found that the simulation outputs a lifetime on the order of $t \approx 10^{74}$ seconds, considerably longer than the age of the known universe.

Simulating the Evaporation of a Reissner–Nordström Black Hole

We simulated the evaporation of a Reissner–Nordström black hole with a mass $600M_\odot$ and electric charge of 4.20×10^4 C.

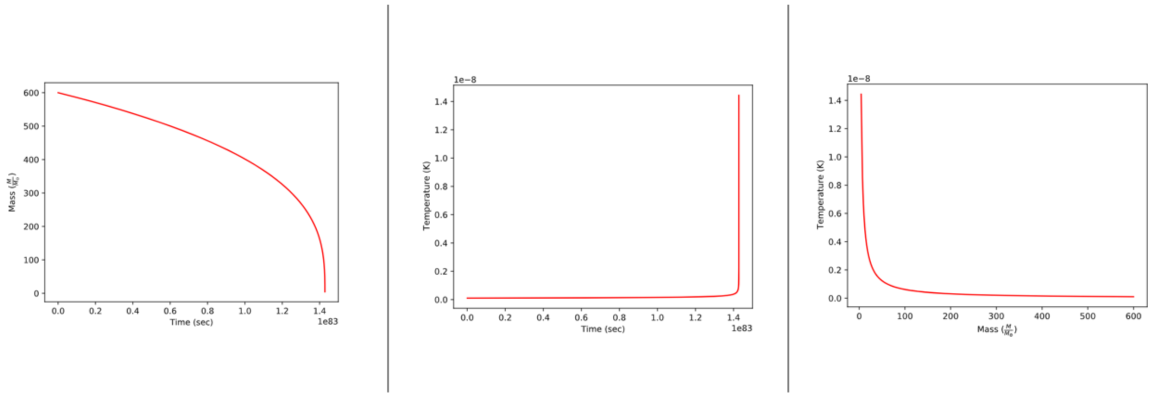


Figure 8: Graphs depicting black hole evaporation for a Reissner–Nordström black hole. Mass v.s. Time (left), Temperature v.s. Time (center), Temperature v.s. Mass (Right)

We found that this leads to a lifetime of the black hole on the order of $t \approx 10^{58}$ seconds.

Simulating the Evaporation of a Kerr Black Hole

We simulated the evaporation process of a Kerr black hole with parameters equivalent to those of the Earth $3.003 \times 10^{-6} M_\odot$ and Kerr-parameter $a = 3.93831$.

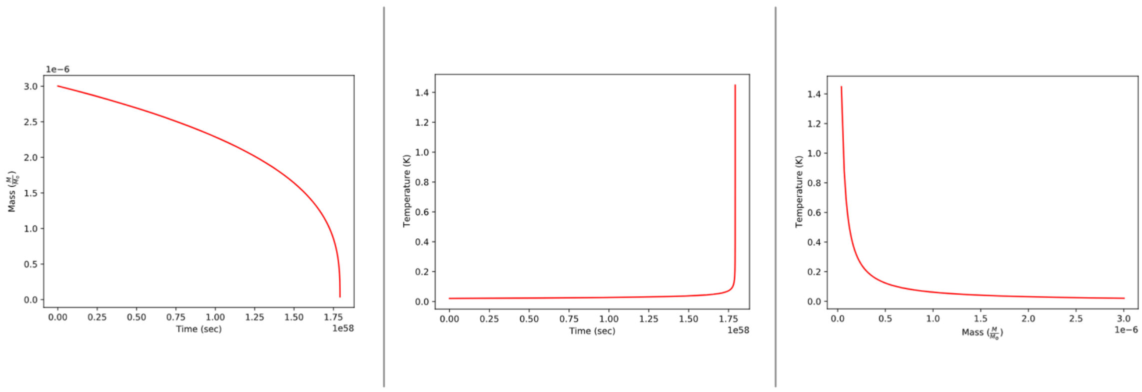


Figure 9: Graphs depicting black hole evaporation for a Kerr black hole. Mass v.s. Time (left), Temperature v.s. Time (center), Temperature v.s. Mass (Right)

For this black hole we found an evaporation period on the order of $t \approx 10^{58}$ seconds. This is consistent with predictions of the lifetime of a Schwarzschild black hole with the mass of the Earth. The reason for it not being drastically different is due to Earth's low angular momentum.

Simulating the Evaporation of a Kerr-Newman Black Hole

We simulated the evaporation process of a Kerr-Newman black hole with parameters equivalent to those of the Sun $M_\odot$, electric charge $q = 77$ C, and Kerr-parameter $a = 322.15$.

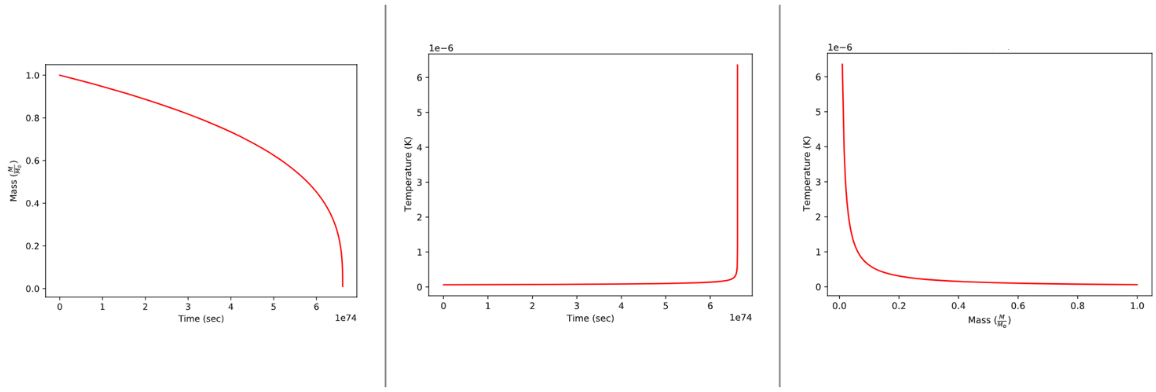


Figure 10: Graphs depicting black hole evaporation for a Kerr-Newman black hole. Mass v.s. Time (left), Temperature v.s. Time (center), Temperature v.s. Mass (Right)

We found that the evaporation period for this black hole was nearly identical to our simulation of the Schwarzschild. This is again due to the low angular momentum and charge of the system in comparison to the mass. In order to explore more unique examples of the Kerr-Newman black hole, we ran another simulation with new parameters. It had mass $7.28 \times 10^{24} M_\odot$, electric charge 3.14×10^{17}, and a Kerr parameter $a = 200$.

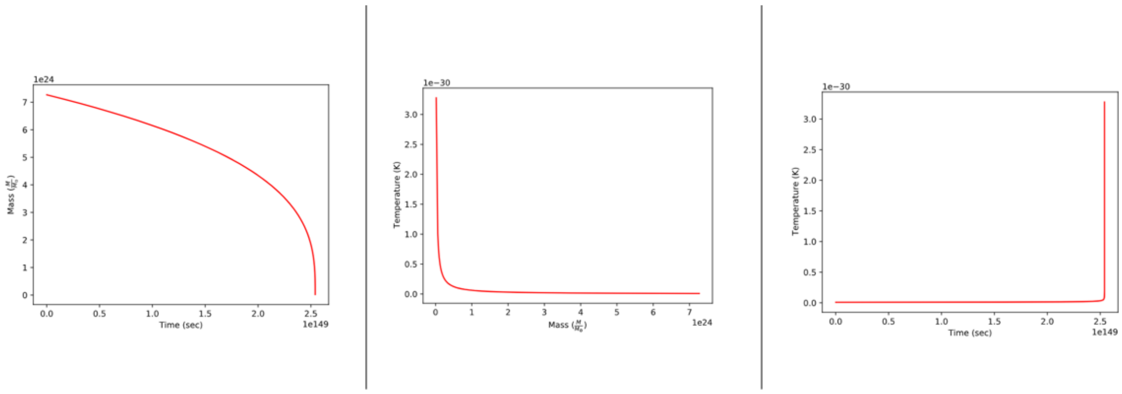


Figure 11: Graphs depicting black hole evaporation for a Kerr-Newman black hole. Mass v.s. Time (left), Temperature v.s. Time (center), Temperature v.s. Mass (Right)

This resulted in a drastically longer evaporation period before reaching extremality $t \approx 10^{149}$.

Conclusions and Future Directions

Discussion

In this work, we successfully reproduced the methodology and results of Chang [3] in using lower dimensional holographic physics to expedite the procedure for calculating the Hawking temperature for higher dimensional black holes. We further extended the method by deriving the relevant theory for the Kerr-Newman metric, enabling this method to be applied to a completely general black hole of arbitrary parameters. From our method, we derived the Hawking temperature to be that of **Equation (56)** repeated below.

$$T = \frac{1}{2\pi}\left(\frac{\sqrt{M^2 - a^2 - Q^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}}\right)$$

This equation agrees with established consensus for the Hawking temperature of a Kerr-Newman black hole and yields the same result as a variety of alternative methods of calculation [7]. In the limit where $Q \to 0$, we confirm the result found by Chang [3] for a Kerr black hole.

$$T = \frac{1}{2\pi}\left(\frac{\sqrt{M^2 - a^2}}{2M^2 + 2M\sqrt{M^2 - a^2}}\right)$$

In the limit where $a \to 0$, we obtain the Reissner–Nordström temperature [11].

$$T = \frac{1}{2\pi}\left(\frac{\sqrt{M^2 - a^2}}{2M^2 + 2M\sqrt{M^2 - Q^2}}\right)$$

Finally, in the limit where both $Q \to 0$ and $a \to 0$ we obtain the well known Hawking temperature for a Schwarzschild black hole [5].

$$T = \frac{1}{8\pi M}$$

After arriving at our analytical result for the Hawking temperature, we were able to probe the process by which black holes evaporate using simulations built in Python. These simulations yielded the correct evaporation curves for black holes with reasonable results.

Future Directions

There are several avenues for future work, including expanding on this existing project or exploring a tangential area to black hole thermodynamics.

Black Holes are Grey

As discussed in the third section, the approach by which we dimensionally reduce a (3+1)-D black hole to a conformally Minkowski one results in the appearance of a potential term $V(r)$.

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right)$$

We neglected this term by moving towards the near-horizon region where this potential drops to zero.

For a quantum wave to escape from the near-horizon region, it must overcome this potential. In addition, for incoming waves $r >> 2M$, the potential behaves repulsively, requiring waves entering the black hole to overcome this potential as well. This decreases the spectrum intensity by a greybody factor of $0 < \Gamma(\omega) < 1$ [14]. The presence of this potential also effects the rate at which a black hole evaporates as the black body radiation luminosity is now scaled by this greybody factor.

$$L = \Gamma(\omega)\sigma T_H^4 A$$

Therefore, in the evaporation of the black hole, our differential equation takes the form,

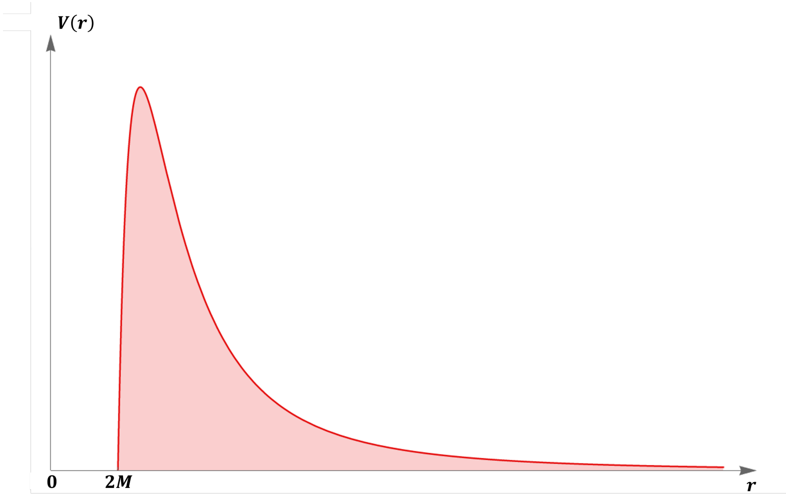


Figure 12: s-wave potential arising from dimensional reduction.

$$\frac{dM}{dt} = -\Gamma(\omega)\sigma T_H^4 A.$$

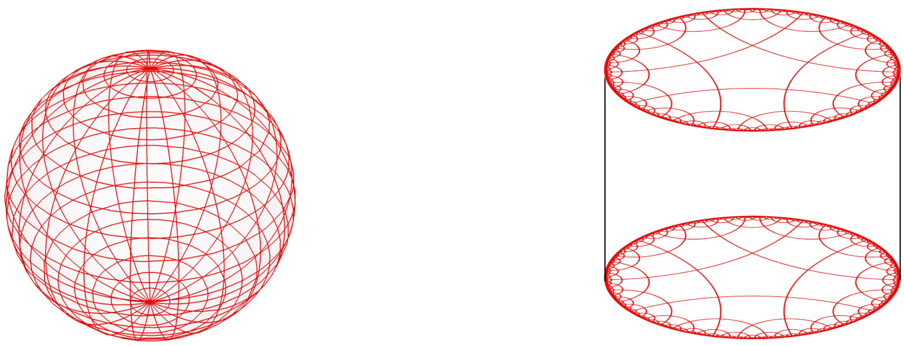
A future project may seek to calculate these greybody factors analytically for these black holes.

Black Holes in Different Cosmologies

Another investigation of interest is applying this method to spaces with different curvature. In particular, It would be of interest to solve the Einstein equations with a cosmological constant Λ.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

This would extend the method from an arbitrary black hole to an arbitrary black hole in an arbitrary cosmological space with nonzero curvature. Spacetimes of interest could include both de Sitter and Anti-de Sitter cosmologies as seen in **Figure (13)**.



(a) de Sitter Cosmology, analogous to an n-sphere.

(b) Anti-de Sitter Cosmology, represented as a hyperbolic space.

Figure 13: Cosmologies with non-zero cosmological constants.

Extending the Simulation of Evaporating Black Holes

There are a variety of more computer focused investigations that might serve as good follow-ups for this work. One such follow-up would be to include the greybody factors to more accurately simulate the evaporation process. This particular modification would pair well with the analytical derivations of these greybody factors. Additionally, investigations into the intensity of Hawking radiation as one moves radially out from the event horizon would be interesting as it would allow for calculations of observed

Hawking radiation at a distance. Another follow-up could explore the idea of matter entering the black hole as a function of time changing the properties $(M,\ J,\ Q)$ of the black hole, resulting in,

$$\frac{dM}{dt} = -\Gamma(\omega)\sigma T_H^4 A + f_M(t), \quad \frac{dQ}{dt} = f_Q(t), \quad \frac{da}{dt} = f_a(t).$$

Black Holes and Quantum Information

A tangential, but interesting avenue for further work would be to explore black hole evaporation in the context of quantum information theory. Exploration of topics such as the Page Curve, the Information Paradox, and entanglement entropy are all active areas of research and warrant investigation.

Appendix A: The Lagrangian and Hamiltonian Formalism and Field Theory

It is most natural to begin the Lagrangian Formalism by discussing the action principle.

Def. Action Principle: The action principle dictates that the state of a classical system is described by a set of generalized coordinates $q(t)$. For a system with N degrees of freedom, the state of that system is described by the degrees of freedom $q = \{q_1, q_2, \ldots, q_N\}$.

By working with fields, we are describing a system with infinite degrees of freedom since we are considering each point in the space. For a scalar field, we consider the degrees of freedom as a function of the field over time at a point indexed by the location $\mathbf{x}$.

$$q(t) = \{\phi_x(t)\}$$

The path taken that connects the system at two different points is time (t_1, t_2) is an extremum of what is called the **action functional**.

Def. Functional: A map from a vector space of functions into its field of scalar numbers. If a functional A maps a function $q(t)$ to a number a, we write $A[q] = a$ or $A[q(t)] = a$. Many functionals can be written as integrals,

$$A[q(t)] = \int_{t_1}^{t_2} F(q(t))dt.$$

Here, F is an ordinary function applied to the value of q [10].

The action functional in this context is written,

$$S[q(t)] = \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t), \ldots)dt.$$

Where our function $L(t, q(t), \dot{q}(t), \ldots)$ is known as the Lagrangian. For field theories, the Lagrangian is also a functional.

Def. Lagrangian: Put most simply, a Lagrangian is the difference between the kinetic and potential energy of a system.

$$L = T - V$$

Where T is our kinetic energy and V is our potential.

The equations of motion describing the time evolution of the system can be found by requiring that the function $q(t)$ extremizes the action in a way that leads to a differential equation describing the evolution of $q(t)$. We do this by varying the action by some small amount δS while demanding that the end points for $q(t_1) = q_1$ and $q(t_2) = q_2$. This differential equation is given by the Euler-Lagrange equation. Below is the derivation provided in *Introduction to Quantum Effects in Gravity* [10].

$$\delta S = S[q + \delta q] - S[q]$$

$$\delta S[q; \delta q] = S[q(t) + \delta q(t)] - S[q(t)] = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q(t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta \dot{q}(t) \right] dt$$

$$= \delta q(t) \frac{\partial L}{\partial \dot{q}} \bigg|_{t_1}^{t_2} + \int_{t_1}^{2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right] \delta q(t) dt.$$

The first term vanishes as variations δq at the boundaries are zero.

$$\delta S = \int_{t_1}^{2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right] \delta q(t) dt$$

δS must be second order in δq, therefore,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

This is the Euler-Lagrange equation. The variation of a functional can also be written in terms of a functional derivative $\frac{\delta S}{\delta q(t)}$,

$$\delta S = \int \frac{\delta S}{\delta q(t)} \delta q(t) dt.$$

Notice that this functional derivative is our Euler-Lagrange equation.

$$\frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

The Hamiltonian formulation of classical mechanics is similar to that of the Lagrangian, however instead of using the difference between the kinetic and potential, we are considering the total energy of the system. The Hamiltonian formalism is based on the Legendre transform of the Lagrangian with respect to the velocity of a given generalized coordinate.

Def. Legendre Transform: For some function $f(x)$, one can introduce a new variable p instead of x,

$$p = \frac{df}{dx},$$

and create a mapping from $f(x)$ to a new function $g(x)$ $H : Im(f(x)) \mapsto Im(g(x))$.

$$g(p) = px(p) - f(x(p))$$

Here x is now a function of p.

Let us perform a Legendre transform on our Lagrangian. We replace the velocity $\dot{q}$ with a variable p known as the canonical momentum.

$$p = \frac{\partial L}{\partial \dot{q}}$$

One can then solve this for velocity,

$$\dot{q} = v(p; q, t)$$

We can now define our Hamiltonian as,

$$H(p, q, t) = [p\dot{q} - L(t, q, \dot{q})]_{\dot{q}=v(p;q,t)}$$

Here we have changed our equations of motion. Instead of obtaining second-order differential equations for $q(t)$ from the Lagrangian, we now obtain first-order differential equations for the two variables $q(t)$ and $p(t)$. We can define,

$$\dot{p} = \left. \frac{\partial L(t, q, \dot{q})}{\partial q} \right|_{\dot{q}=v(p;q,t)}, \qquad \dot{q} = v(p; q, t)$$

Using these and rearranging our equation for our Hamiltonian we find,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$H(p, q, t) = [p\dot{q} - L(t, q, \dot{q})]_{\dot{q}=v(p;q,t)} = T + V.$$

Like the Lagrangian, the Hamiltonian follows the action principle and is extremized by independently varying S_H by $q(t)$ and $p(t)$.

For Lagrangian field theory, our functional is the action integral, which measures a physical systems change over time. Where S is our action quantity, and L is our Lagrangian. We can also describe the action as a functional of the Lagrangian density $\mathcal{L}$

$$S = \int dtL$$

$$S = \int dtdV\mathcal{L}$$

Our choice of Lagrangian density $\mathcal{L}$ will depend on our physical system and could involve any number of different fields. Our invariant condition means that even under linear transformations of those fields, our resulting field equations of the system will always take the same form. We will use the classical gravielectric field equations to provide an example. Let us start by defining our fields.

$$\vec{g} = -\nabla\Phi - \frac{\partial\vec{V}}{\partial t}$$

$$\vec{K} = \nabla \times \vec{V}$$

We can derive our action using the Lorentz Invariant:

$$S = \int dVdt\left[\frac{1}{2}\left(-\frac{1}{4\pi G}\vec{g}^2 - \frac{1}{\mu_g}\vec{K}^2\right) - \Phi\rho + \vec{V}\cdot\vec{\mathcal{P}}\right]$$

We want to vary our action with respect to our two fields to generate the equations of motion for that field. Let us start by varying our scalar field.

$$\delta S = \int dVdt\delta\left[\frac{1}{2}\left(-\frac{1}{4\pi G}\vec{g}^2 - \frac{1}{\mu_g}\vec{K}^2\right) - \Phi\rho + \vec{V}\cdot\vec{\mathcal{P}}\right]$$

$$\delta S = \int dVdt\left[\frac{1}{2}\left(-\frac{1}{4\pi G}\delta(\vec{g}^2) - \frac{1}{\mu_g}\delta(\vec{K}^2)\right) - \delta(\Phi\rho) + \delta(\vec{V}\cdot\vec{\mathcal{P}})\right]$$

$$\delta S = \int dVdt\left[\frac{1}{2}\left(-\frac{1}{4\pi G}\delta(\vec{g}^2)\right) - \delta(\Phi\rho)\right]$$

$$\delta S = \int dVdt\left[\frac{1}{2}\left(-\frac{1}{4\pi G}2\vec{g}\delta(\vec{g})\right) - \delta(\Phi\rho)\right]$$

$$\delta S = \int dVdt\left[-\frac{1}{4\pi G}\vec{g}\delta(\vec{g}) - \delta(\Phi\rho)\right]$$

$$\delta S = \int dVdt\left[\frac{1}{4\pi G}\vec{g}\cdot\nabla(\delta\Phi) - \delta\Phi\rho\right]$$

We can integrate by parts,

$$\delta S = \int dt\{\frac{\vec{g}}{4\pi G}\delta\Phi - \int dV\frac{1}{4\pi G}(\nabla\cdot\vec{g})\delta\Phi - \delta\Phi\rho\} = 0$$

Total derivative terms drop off,

$$\delta S = -\int dtdV\{\frac{1}{4\pi G}\left(\nabla\cdot\vec{g}\right)\delta\Phi + \delta\Phi\rho\} = 0.$$

$$\frac{1}{4\pi G}\left(\nabla\cdot\vec{g}\right)\delta\Phi = -\delta\Phi\rho$$

We have derived the first field equation for the gravielectric field,

$$\boxed{\nabla\cdot\vec{g} = -4\pi G\rho.}$$

Now we will vary the vector field to derive the next field equation. Let us return to our action.

$$S = \int dVdt\left[\frac{1}{2}\left(-\frac{1}{4\pi G}\vec{g}^2 - \frac{1}{\mu_g}\vec{K}^2\right) - \Phi\rho + \vec{V}\cdot\vec{\mathcal{P}}\right]$$

$$\delta S = \int dV dt \delta \left[\frac{1}{2} \left(-\frac{1}{4\pi G} \vec{g}^2 - \frac{1}{\mu_g} \vec{K}^2 \right) - \Phi\rho + \vec{V} \cdot \vec{\mathcal{P}} \right]$$

$$\delta S = \int dV dt \left[\frac{1}{2} \left(-\frac{1}{4\pi G} \delta(\vec{g}^2) - \frac{1}{\mu_g} \delta(\vec{K}^2) \right) - \delta(\Phi\rho) + \delta(\vec{V} \cdot \vec{\mathcal{P}}) \right]$$

$$\delta S = \int dV dt \left[\frac{1}{2} \left(-\frac{1}{4\pi G} (2\vec{g}) \delta\vec{g} - \frac{1}{\mu_g} \delta\vec{K}(2\vec{K}) \right) + \delta(\vec{V} \cdot \vec{\mathcal{P}}) \right]$$

$$\delta S = \int dV dt \left[\frac{1}{4\pi G} (\vec{g}) \cdot (\frac{\partial}{\partial t}(\delta\vec{V})) - \frac{1}{\mu_g} \cdot (\vec{K})(\nabla \times \delta\vec{V}) + \delta\vec{V} \cdot \vec{\mathcal{P}} \right]$$

Here we integrate by parts on one term,

$$= \int dV \{ -\frac{1}{4\pi G} \vec{g} \cdot \delta\vec{V} - \int \{ dt \frac{1}{4\pi G} \{ \delta\vec{V} \cdot \frac{\partial}{\partial t}(\vec{g}) \} \} \} - \int dV dt \frac{1}{\mu_g} \vec{K} \cdot (\nabla \times \delta\vec{V}) - \delta\vec{V} \cdot \vec{\mathcal{P}}$$

Now we can drop the total derivative term and use the vector identity given in the book to find,

$$= \int dV dt \left[-\frac{1}{4\pi G} \frac{\partial\vec{g}}{\partial t} \cdot \partial\vec{V} - \frac{1}{\mu_g} (\nabla \times \vec{K}) \cdot \delta\vec{V} + \delta\vec{V} \cdot \vec{\mathcal{P}} \right] = 0$$

Now we can pull out the field equation,

$$\boxed{\nabla \times \vec{K} = \mu_g \vec{\mathcal{P}} - \frac{\mu_g}{4\pi G} \frac{\partial\vec{g}}{\partial t}.}$$

Next we want to discuss conserved quantities in field theories. For this we should be familiar with Noether's Theorem.

Theorem. *Noether's Theorem. Continuous symmetries of the action in a physical theory generate conserved quantities.*

To derive our conservation equation, we need to perform a gauge transform. In this case, a small coordinate transformation (diffeomorphism). Our gauge transforms leave $\vec{g}$ and $\vec{K}$ invariant. We also want it to leave the action invariant, therefore, let us perform our transformations, enter them into the action and solve.

$$\Phi \to \tilde{\Phi} = \Phi - \frac{\partial\Lambda}{\partial t}$$

$$\vec{V} \to \tilde{\vec{V}} = \vec{V} + \nabla\Lambda$$

$$S = \int dV dt \left[\frac{1}{2} \left(-\frac{1}{4\pi G} \vec{g}^2 - \frac{1}{\mu_g} \vec{K}^2 \right) - \left(\Phi - \frac{\partial\Lambda}{\partial t} \right) \rho + \left(\vec{V} + \nabla\Lambda \right) \cdot \vec{\mathcal{P}} \right]$$

$$S = \int dV dt \left[\frac{1}{2} \left(-\frac{1}{4\pi G} \vec{g}^2 - \frac{1}{\mu_g} \vec{K}^2 \right) - \Phi\rho + \vec{V} \cdot \vec{\mathcal{P}} + \frac{\partial\Lambda}{\partial t} + \nabla\Lambda \cdot \vec{\mathcal{P}} \right]$$

This can be reduced,

$$= S(\Phi, \vec{V}) + \int dV dt \left[\frac{\partial\Lambda}{\partial t} + \nabla\Lambda \cdot \vec{\mathcal{P}} \right].$$

For our action to remain invariant,

$$\left[\frac{\partial\Lambda}{\partial t} + \nabla\Lambda \cdot \vec{\mathcal{P}} \right] = 0.$$

We can integrate by parts and drop total derivative terms,

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \vec{\mathcal{P}}.$$

$$\boxed{\nabla \cdot \vec{\mathcal{P}} = -\frac{\partial \rho}{\partial t}}$$

We have shown conservation of jumps out as a requirement of the action being invariant for a gauge transformation on the fields.

Appendix B: Hilbert Spaces and Quantum Mechanics

Def. (Complex) Vector Space: [8]: Let $\mathbb{C}$ denote the field of complex numbers. A **vector space over** $\mathbb{C}$ is a set V which is endowed with two operations. **Vector addition** $V \times V \to V$, denoted by $(v, w) \mapsto v + w$, and **scalar multiplication** $\mathbb{C} \times V \to V$, denoted by $(a, v) \mapsto av$. This set V also satisfies the following properties.

1. V is an abelian group under vector addition
2. Scalar multiplication satisfied the identities,
   - $a(bv) = (ab)v \quad \forall v \in V, a, b \in \mathbb{C}$
   - $1v = v \quad \forall v \in V$
3. Scalar multiplication and vector addition are distributive,
   - $(a + b)v = av + bv \quad \forall v \in V, a, b \in \mathbb{C}$
   - $a(v + w) = av + aw \quad \forall v, w \in V, a \in \mathbb{C}$

Def. Normed Vector Space: [4] A **normed vector space** is a pairing $(V, \|\cdot\|)$ where V is a vector space over $\mathbb{C}$ and a function $\|\cdot\|$ called a **norm** on V with the following properties.

1. (positive definite) $\|\cdot\| = 0 \iff x = 0$.
2. (homogeneous) $\|\alpha x\| = |\alpha|\|x\|\ \forall x \in V$ and $\forall \alpha \in \mathbb{C}$.
3. (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|\ \forall x, y \in V$.

Def. Cauchy Sequence (Vectors): [4] A sequence $\{v_n\}_{n=1}^{\infty}$ is called a **Cauchy sequence**if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\|v_n - v_m\| < \epsilon\ \forall n, m \geq N$.

Def. Banach Space: [4] A pairing $(V, \|\cdot\|)$ is called **complete** if every Cauchy sequence (of vectors) in V converges to some vector in V. A **banach Space** is a complete normed vector space.

Def. Inner Product: [4] A function $\langle x, y\rangle$ on pairs (x, y) of vectors in $V \times V$ taking values in $\mathbb{C}$ satisfying:

1. (positive definiteness) $\langle x, x\rangle \geq 0\ \forall x \in V$ and $\langle x, x\rangle = 0$ only if $x = 0$.
2. (symmetry) $\langle x, y\rangle = \langle y, x\rangle\ \forall x, y \in V$.
3. (bilinearity) $\forall x, y, z \in V$ and scalars $\alpha, \beta \in \mathbb{C}$, $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$.

Def. Inner Product Space: [4] A vector space with an inner product.

Def. Orthogonal: [4] Two vectors x and y are orthogonal if $\langle x, y\rangle = 0$.

Def. Orthonormal Vectors: [4] A collection of vectors $\{e_n : n \in S\}$ is called **orthonormal** if $\|e_n\| = 1$ $\forall n \in S$ and $\langle e_n, e_m\rangle = 0$ for $n \neq m \in S$.

Def. Dirac Notation: Short-hand for calculations in both finite- or infinite-dimensional vector spaces. Vectors in this notation are denoted with $|+\rangle$. This is called a *ket*. We can create linear super-positions of these states,

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle\,.$$

The dual space of this are given the notation $\langle +|$. This is called a *bra*. Linear super-positions of these dual vectors can also be written in the same way as for vectors.

Def. Dual Space: A vector space V has a **dual space** containing covectors. Covectors act on a vector to output a complex number ($f : V \mapsto \mathbb{C}$). As discussed above, they are notated using the bra symbol.

$$\langle f| (a |v\rangle + b |w\rangle) = a \langle f|v\rangle + b \langle f|w\rangle \in \mathbb{C}.$$

The space of these covectors forms the dual space.

Def. Scalar Product: A mapping of two vectors into a (complex) number. A scalar product establishes a bijection between vectors and covectors. A vector corresponds to its covector if,

$$\langle v|w\rangle = (|v\rangle , |w\rangle)$$

Def. Hermitian Scalar Product: Scalar product satisfying,

$$(|v\rangle , |w\rangle) = (|v\rangle , |w\rangle)^*,$$

where the asterisk denotes a complex conjugate.

Def. Hilbert Space: [4] A complete inner product space. A complete, separable vector space with a Hermitian scalar product.

Rmk. Scalar products of two bra vectors in a dual space is equal to that of the corresponding scalar product of the ket vectors. Therefore, the dual space is also a Hilbert space. A Hilbert space is isomorphic to its dual space. Thus $\langle v|$ and $|v\rangle$ are different symbols referring to the same quantum state.

Rmk. Vectors in finite-dimensional space are given by a collection of components.

$$\vec{x} = \langle x_1, x_2, x_3 \rangle , x_k \in \mathbb{C}, \forall k \in \{1, 2, 3\}$$

Vectors in infinite-dimensional spaces have infinitely many complex components.

$$\vec{x} = \langle x_1, x_2, x_3, \dots \rangle , x_k \in \mathbb{C}, \forall k \in \mathbb{N}$$

Now that we have this information, we can elaborate on quantizing our Hamiltonian. In a vector space, our non-commuting operators are represented as linear transformations on the space of quantum states. Additionally, the vector space of quantum states must be infinite-dimensional and be defined over the field of complex numbers $\mathbb{C}$. This can be shown by using our commutation relation between $\hat{q}$, $\hat{p}$.

$$Tr([\hat{q}(t), \hat{p}(t)]) = Tr(\hat{q}(t)\hat{p}(t)) - Tr(\hat{p}(t)\hat{q}(t) + = i\hbar Tr(\hat{I})) \neq 0.$$

However, should these operators be finite dimensional, the trace is well defined and

$$Tr([\hat{q}(t), \hat{p}(t)]) = Tr(\hat{q}(t)\hat{p}(t)) - Tr(\hat{p}(t)\hat{q}(t) + = i\hbar Tr(\hat{I})) = 0.$$

Appendix C: Quantization of Classical Fields

In a classical scalar field, the Lagrangian density of an infinite number of coupled harmonic oscillators is,

$$\mathcal{L} = \frac{1}{2}\partial_t^2\phi - \frac{1}{2}\partial_x^2\phi - \frac{1}{2}m\phi^2 - V(\phi)$$

The conjugate momentum of the field is the derivative of the Lagrangian density with respect to the time derivative of the field.

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t\phi)}$$

Under canonical quantization, the field variables ϕ, π are now treated as operators $\hat{\phi}, \hat{\pi}$ following the canonical commutator relations.

$$[\phi(x,t), \phi(y,t)] = [\pi(x,t), \pi(y,t)] = 0, \quad [\phi(x,t), \pi(x,t)] = i\hbar\delta(x-y)$$

Under a classical scalar field, a free scalar field representing harmonic oscillators can be expressed as in terms of a superposition of frequency modes. A free scalar field means that the potential $V(\phi) = 0$. To calculate this, perform a Fourier transform on the fields.

$$\phi_k = \int \phi(x)e^{-ikx}dx, \quad \pi_k = \int \pi(x)e^{-ikx}dx, \quad \phi_{-k} = \phi_k^\dagger, \quad \pi_{-k} = \phi_k^\dagger$$

The commutator relations also apply to the transformed field variables $[\hat{\phi_{k_1}}, \hat{\pi_{k_2}}^\dagger] = i\hbar\delta(k_1 - k_2)$. And through this transformation, the new quantum operators can be more conveniently expresses as creation and annihilation operators of the normal modes.

$$a_k = \frac{1}{\sqrt{2\pi\omega_k}}(\omega_k\hat{\phi_k} + i\hat{\pi_k}), \quad a_k^\dagger = \frac{1}{\sqrt{2\pi\omega_k}}(\omega_k\hat{\phi_k}^\dagger - i\hat{\pi_k}^\dagger)$$
$$[a_{k_1}, a_{k_2}] = [a_{k_1}^\dagger, a_{k_2}^\dagger] = 0 \quad [a_{k_1}, a_{k_2}^\dagger] = \delta(k_1 - k_2)$$

Appendix D: Basics of Lorentzian Manifolds

The mathematical formalism for Einstein's general theory of relativity are tensor fields defined on a Lorentzian manifold to represent spacetime. In this paper, it is assumed that the reader has previous exposure to differential geometry and tensors. However, for those without much exposure to these topics, this appendix will briefly review some of the mathematical formalism used in general relativity.

Typically, early exposures to gravity as a result of curved spacetime is exemplified via diagrams of gridlines which bend around massive objects. The simplest example would be a diagram of gridlines surrounding a star with a planet following the gridlines in a gravitational orbit around the star (following a concept known as spacetime geodesics). The core issue of concern is that earlier introductory notions of physics such as vectors require mathematical definitions which are significantly more involved. Introductory physics typically takes the background space to be an immutable given backdrop. This is convenient because with an unchanging background, we can define a coordinate system and recognize locations of objects within the space in reference to this coordinate system. In general relativity, we have lost this immutable background as now the presence of matter curves spacetime and the curves of spacetime influence the dynamics of matter; And our mathematical descriptions of this must all be self-contained and make no reference to any outer immutable background beyond spacetime.

We first require a mathematical description of a manifold that allows for calculus to be performed. What's more, it's desirable for our choice of a manifold to be as relatable as possible to a more humanistic description of reality, such as an Euclidean space (i.e., our three-dimensional intuition of length, width, and height). We will describe the basic definitions required to describe a manifold and outline the specific properties we desire for a manifold to be used in general relativity. We begin with the assumption that the reader is aware of basic set theory and the basic definitions of a topological space. More detail can be found in Appendix A of [8].

Def. Topological Manifolds: [8] Let M be a topological space. We specify that M is a topological manifold of dimension n if it has the following properties:

- M is a **Hausdorff space**: For every pair of distinct points $p, q \in M$, there are disjoint open subsets, $U, V \subseteq M$ such that $p \in M$ and $q \in M$.
- M is **Second Countable**: There exists a countable basis for the topology of M.
- M is **locally Euclidean of dimension n**: Each point of M has a neighborhood U that is homeomorphic to an open subset of $\mathbb{R}^n$. Which means that for each point $p \in M$, we can find...
  1. an open subset $U \subseteq M$ such that $p \in M$.
  2. an open subset $\hat{U} \subseteq \mathbb{R}^n$
  3. a homeomorphism $\phi : U- > \hat{U}$

This description of a manifold doesn't clearly indicate how it can be used in general relativity, nor does it appear like a physical space in which one can exist and experience gravity. In order to better demonstrate this, we will make use of **coordinate charts**. A coordinate chart is a way of expressing the points of a small neighborhood on a manifold as coordinates in Euclidean space $\mathbb{R}^n$. In general relativity, typically the only manifolds considered to represent spacetime are locally approximately Euclidean space. This is because physical laws on a humanistic scale fit incredibly well into a Euclidean space, and it is only on large cosmological scales in which the curvature of spacetime must be considered to correctly describe physical laws. However, we will not consider this assumption to be true at first as we are defining everything more generally.

Def. Coordinate Chart: [8] Let M be a topological n-manifold. A **coordinate chart** on M is a pair (U, φ), where $U \subseteq M$ is open, and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$. U is known as the coordinate domain. φ is known as the coordinate map.

Based on the definition of a topological manifold, each point on the manifold $p \in M$ is contained in the domain of some chart (U, φ). If $\varphi(p) = 0$, the chart is referred to as "centered at p." This is useful for defining the tangent bundle $T(M)$ seen in [2].

Def. Homeomorphism: [8] A continuous bijective map $f : X \to Y$ with a continuous inverse is called a homeomorphism.

While this defines a topological manifold and a methodology to relate that manifold to Euclidean space $\mathbb{R}^n$, we don't yet have a structure to define calculus concepts such as differentiation or integration. In order to do so, we will define some more structures that we wish to imbue onto a manifold and use that to define a smooth manifold.

Def. Smooth Functions and Diffeomorphism: [8] Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open. A function $F : U \to V$ is **smooth** (or $\boldsymbol{C^\infty}$, or **infinitely differentiable**) if each of the functions' component functions has continuous partial derivatives of all orders. If F is also bijective and has a smooth inverse map, then it is a homeomorphism known as a **diffeomorphism**.

This definition of smooth is limited to functions with a domain in Euclidean space. To get around this constraint, we want to define the notion of a smooth chart to map from the manifold to Euclidean space. Let M be an n-manifold. As outlined above, $\forall p \in M, \exists \varphi : U \to \hat{U} \subseteq \mathbb{R}^n$ coordinate map such that $p \in U$. We can define a smooth function on the manifold $f : M \to \mathbb{R}$, if and only if the composite function $f \circ \varphi^{-1} : \hat{U} \to \mathbb{R}$. is smooth with the above definition. However, a choice of point in M could be contained in multiple coordinate charts, and we'd like to construct this definition such that it does not depend on the choice of coordinate chart. We can do so with a collection of "smooth charts" imprinted on M.

Def. Transition Maps and Smooth Charts: [8] Let M be an n-manifold. Let $(U, \varphi), (V, \psi)$ be two charts such that $U \cap V = \varnothing$. The **transition map from $\boldsymbol{\varphi}$ to $\boldsymbol{\psi}$** is the composition map $\psi \circ \varphi^{-1}$: $\varphi(U \cap V) \subseteq \mathbb{R}^n \to \psi(U \cap V) \subseteq \mathbb{R}^n$. Two charts are **smoothly compatible** if $U \cap V = \varnothing$ or more interestingly, the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. Since we are located in $\mathbb{R}^n$, smoothness is interpreted still as the continuous partial derivatives of all orders.

Def. Smooth Atlas: [8] An **atlas** is a collection of charts whose collective domains cover all of M. A **smooth atlas** $\mathcal{A}$ has the extra property that any two charts in the atlas are smoothly compatible with each other.

By constructing a smooth atlas, we can create a smooth structure to overlay onto our manifold M, and we can use the charts in this atlas to satisfy our earlier stated condition that $f : M \to \mathbb{R}$ is smooth if and only if the composite function $f \circ \varphi^{-1} : \hat{U} \to \mathbb{R}$ is smooth. However, while we have narrowed down our choice of charts to a convenient collection via the smooth atlas, there can very easily be multiple smooth atlases. We'd like to not have any ambiguity when defining this smooth structure for M, so we restrict our choice of atlas slightly by also requiring it to be **maximal**.

Def. Maximal Atlas: [8] A smooth atlas $\mathcal{A}$ on M is **maximal** if it is not contained inside any other larger smooth atlas. This is equivalent to the statement that any chart which is smoothly compatible with all the charts inside $\mathcal{A}$ is already contained in $\mathcal{A}$.

It can be proven that every smooth atlas for M is contained inside the unique maximal smooth atlas, thus making it the idea choice. Now we are finally ready to define a smooth manifold, not as a unique manifold object, but rather as a manifold with an additional structure(in this case, the atlas) that is used to define smooth functions.

Def. Smooth Manifold: [8] A **smooth manifold** is a pair $(M, \mathcal{A})$ where M is a manifold and $\mathcal{A}$ is the maximal smooth atlas on M.

Def. Smooth Functions on a Manifold: [8] Suppose M is a smooth n-manifold and $f : M \to \mathbb{R}^k$. We say f is a **smooth function** if $\forall p \in M$, $\exists (U, \varphi)$ smooth chart for M whose domain contains p such that $f \circ \varphi^{-1}$ is smooth on the open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The set of smooth function on a manifold is labeled as $\boldsymbol{C^\infty(M)}$

With this definition of a smooth manifold, we now have a space for physical reality to exist, which is quite relieving. However, by allowing this manifold to change in response to the presence of matter and energy, we do not have a traditional static background and the ability to conveniently use vectors to denote magnitude and direction in a dynamic system. We can't define a vector space by non-linearly adding curved paths of particles. The best solution to this problem is to carefully reconstruct these same intuitive concepts by defining a vector space known as the tangent space to a manifold at a point.

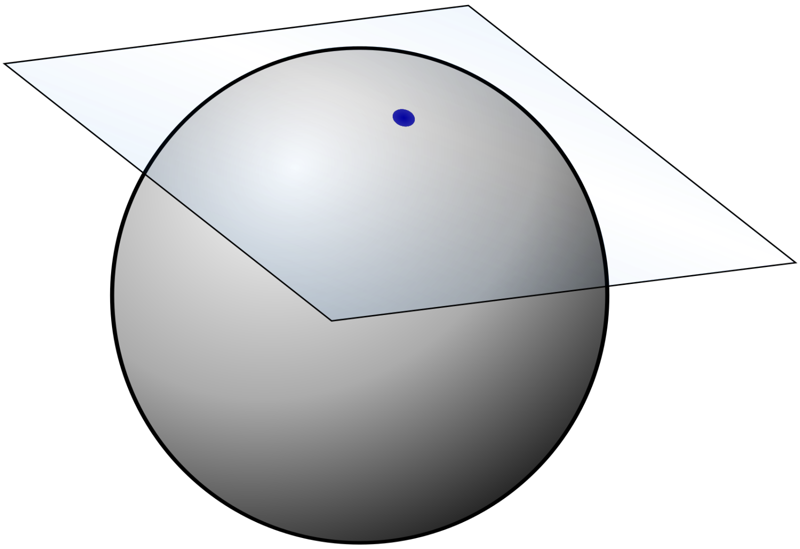


Figure 14: Tangent Plane located at a point on a Sphere[15].

To begin, we will describe a tangent space in physics terms as a space of possible velocities for a particle moving on the manifold. This is akin to an example of having a particle restricted to a three dimensional curved surface embedded in $\mathbb{R}^3$ and the possible velocities existing on a tangent place $\mathbb{R}^2$.

If a particle located at this point on the sphere were to be travelling downward towards the reader, we could represent its velocity as a vector on the plane pointing towards the reader. This might seem like an unnecessary complication, however that is because we are used to the convenience of Euclidean space $\mathbb{R}^n$ in which the tangent spaces of each point all overlap. However, if you were to place the point in the diagram above at a different location of the sphere, the tangent plane is a completely different space. This problem occurs in most general manifolds and tangent spaces, and as such we need to define the tangent space at a point such that we can have a convenient reference between tangent spaces in different points across the manifold.

Def. Tangent Space-Derivation: [8] Suppose M is a $\boldsymbol{C^\infty}$ manifold. A **derivation at** $\boldsymbol{x \in M}$ is defined as a linear map $D : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz identity:

$$\forall f, g \in C^\infty(M) : \quad D_x(fg) = D_x(f) * g(x) + f(x) * D_x(g)$$

Notice that derivations are linear maps defined for a specific point on the manifold. By the Leibniz identity, we also have the following properties of the set of all derivations at x denoted as T_xM.

- $f = const, \quad D(f) = 0$
- $(D_{1,x} + D_{2,x})(f) := D_{1,x}(f) + D_{2,x}(f)$
- $(\lambda \cdot D_x)(f) := \lambda \cdot D_x(f)$

These properties satisfy the criteria of defining T_xM as a vector space.

While this defines a vector space known as the tangent space, it doesn't provide any intuitive notions for how we can relate this to our conventional definitions of vectors in Euclidean space. However, this definition which is independent of Euclidean space is mathematically necessary. In order to connect this definition to a Euclidean space, we will create an equivalent definition of the tangent space using coordinate charts, and these two definitions will create spaces which are isomorphic with each other.

Def. Tangent Space-Coordinate Chart: [8] Suppose M is a $\boldsymbol{C^\infty}$ manifold and $x \in M$. Let $\varphi : U \to \mathbb{R}^n$ be a coordinate chart where $x \in U \subseteq M$. We will parameterize two curves traveling through the manifold as $\gamma_1, \gamma_2 : (-1, 1) \to M$ with $\gamma_1(0) = x = \gamma_2(0)$. The curves are chosen such that $\varphi \circ \gamma_1, \varphi \circ \gamma_2$ are differentiable. We define an equivalence relation where curves γ_1, γ_2 are equivalent at parameter value 0 if the derivatives of $\varphi \circ \gamma_1, \varphi \circ \gamma_2$ coincide. The equivalence class of these curves $\gamma'(0)$ define tangent vectors of M on x. The space of all these vectors is the tangent space T_xM.

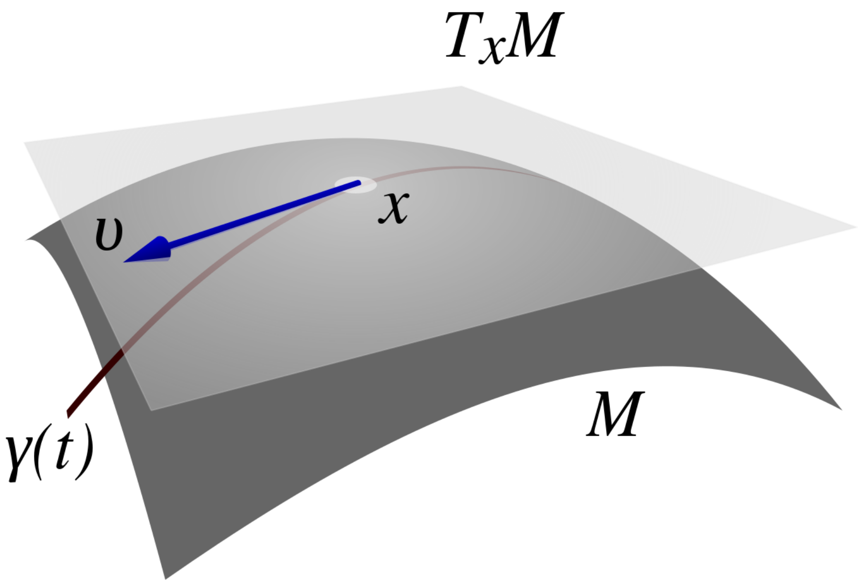


Figure 15: Tangent Space T_xM and tangent vector $v \in T_xM$, along a curve $\gamma(t)$ traveling through $x \in M$ [9].

Consider an unspecified coordinate system of an n-dimensional manifold as x^μ, in which $\mu = 1, ..., n$. We can define a parameterized curve $x^\mu(\lambda)$ through this manifold as coordinate functions of a parameter λ. The tangent vector to this curve will have the following components and the entire vector is expressed as an Einstein summation over the tangent space basis vectors,

$$V^\mu(\lambda) = \frac{dx^\mu}{d\lambda}, \quad V = V^\mu \hat{e}_{(\mu)}$$

In general relativity, these vectors are usually just denoted by their components and are known as **contravariant vectors**. As an example, consider the manifold $\mathbb{R}^2$. Express in Cartesian coordinates the straight line "curve" from the origin $(0,0)$ to $(3,4)$ as parameterized by λ from 0 to 1.

$$x^1(\lambda) = 3\lambda, \quad x^2(\lambda) = 4\lambda, \quad V^1 = 3, \quad V^2 = 4, \quad V = (3,4)$$

With this definition of a vector space, we will detour briefly to describe how the principle of general covariance influences our choices for constructing the mathematics of general relativity. In short, this principle states that physical laws will be invariant under arbitrary differentiable coordinate systems. The idea hearkens back our previous statement that there is no static backdrop or universal coordinate system imbued into reality. Coordinate systems are simply a way to assign numbers to describe and predict a physical system, and human choice in labeling should have no bearing on how a physical system will progress. This requirement, combined with the geometric nature of spacetime indicates that general relativity should be formulated with tensors. In short, tensors are algebraic objects that describe a multilinear mapping between sets of algebraic objects related to vector spaces. We will define them more exactly soon, but the important thing to note is that tensors are defined by the way they are transformed, not in regards to any coordinate system. As such, we can describe the mathematics of our system with tensors, and then express these tensors in any basis of any coordinate system that we want, and they will all be equivalent expressions. To do this, we must define a dual space to the tangent space.

Def. Covector and Dual Space: [8] Let V be a finite-dimensional vector space. **A covector on V** is a real-valued linear functional on V, meaning a linear map $\omega : V \to \mathbb{R}$. The space of all covectors on V is itself a real vector space under the operations of pointwise addition and scalar multiplication. This space of maps, denoted as $V*$ is also known as the **dual space of V**.

Def. Cotangent Space: [8] Let M be a smooth manifold and $x \in M$. **The Cotangent Space at x**, denoted as $T*_x M$, is the dual space of the tangent space T_xM.

By the definition of a dual to a vector space being a vector space, we can continue to use Einstein summation notation to denote the linear map ω as a summation over basis vectors,

$$\omega = \omega_\nu \hat{\theta}^{(\nu)}, \quad \hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^\nu_\mu \quad \therefore \omega(V) = \omega_\nu \hat{\theta}^{(\nu)}(V^\mu \hat{e}_{(\mu)}) = \omega_\mu V^\mu \in \mathbb{R}$$

Similarly to the vectors in the tangent space, we will denote covectors by their components ω_μ. However, we switch the height of the index in our notation in order to denote which space these vectors exist in. Covectors are also known as dual vectors, one-forms, and covariant vectors. You might notice in the above equation that notationally the expression of applying the covariant vector linear map ω to the contravariant vector V is ordered, however it doesn't have to be. These expressions are actually commutative as the Einstein summation notation is simply addition of the element-wise multiplication of vectors. It is equally true that the contravariant vector V^μ of the tangent space is a linear map of the covariant vectors in the cotangent space.

$$V(\omega) = \omega(V) = V^\mu \omega_\mu = \omega_\mu V^\mu \in \mathbb{R}$$

Now that these two vector spaces associated with specific points are defined, we can readily define the tensor; The most useful mathematical concept for general relativity.

Def. Tensor: [2] A tensor T of type (k, l) is a multilinear map, meaning linear in all it's arguments, from a collection of contravariant vectors and covariant vectors to $\mathbb{R}$. Let $\times$ denote the Cartesian product.

$$T : T^*_x M \times ... \; k \text{ times} ... \times T^*_x M \times T_x M \times \; l \text{ times} ... \times T_x M \to \mathbb{R}$$

To demonstrate multi-linearity, consider a tensor T of type $(1, 1)$ and contravariant vectors $V, W \in T_x M$ and covariant vectors $\omega, \eta \in T *_x M$.

$$T(a\omega + b\eta, cV + dW) = acT(\omega, V) + adT(\omega, W) + bcT(\eta, V) + bdT(\eta, W)$$

It is worth noting that **tensors**, by being defined as a map from the domain of the tangent and cotangent spaces of a point on the manifold, are defined only on specific points for each manifold. In general relativity, it is far more useful to consider **tensor fields**, in which tensor components are considered functions of a point in the manifold and we have one tensor object which will change according to each point in the manifold. General relativity almost entirely works with tensor fields, and are refereed to in shorthand as tensors.

Tensors are used throughout all of general relativity to express quantities and derive the Einstein field equations, which govern how matter and energy will curve spacetime. However for this paper, we only need to use the metric tensor to define the Riemannian manifold and express the action of a massless free scalar field.

Def. Riemannian Metric and Manifold: [8] Let M be a smooth manifold. A **Riemannian metric on M** is a smooth symmetric covariant $(0, 2)$ tensor field on M that is positive definite at each point. A **Riemannian manifold** is a pairing of structures (M, g) where M is the smooth manifold and g is the Riemannian metric on M.

The simplest Riemannian metric is the Euclidean metric $\bar{\eta}$ on $\mathbb{R}^n$, which in general coordinates is expresses by,

$$\bar{\eta} = \delta_{\mu\nu} dx^\mu dx^\nu = (dx^1)^2 + ... + (dx^n)^2$$

where $\bar{\eta}$ is the Kronecker delta tensor.

In general relativity, a pseudo-Riemannian manifold is used instead. This manifold is very similar to a Riemannian manifold, however the requirement that the metric be positive-definite is relaxed and instead replaces with the requirement that the metric be non-degenerate. This is what allows for the metric in general relativity to be used to calculate the spacetime interval, which can be positive, negative, or zero, instead of only positive. The pseudo-Riemannian manifold which is used is known as the Lorentzian manifold. This manifold uses a specific metric tensor, and is refereed to simply as the metric tensor as other metrics are not commonly used.

Def. Lorentzian Manifold: [2] In general relativity, spacetime is a four-dimensional smooth manifold M. **The metric tensor**, denoted as g, is a $(0, 2)$ symmetric tensor on M. The metric is required to be nondegenerate with signature (-,+,+,+). Nondegenerate means that the metric $g(V, W) = 0, \forall V \in T_x M \setminus \{0\}$implies that $W = 0$. A manifold with this metric is considered to be a **Lorentzian Manifold**.

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