Explorations in the Modification of Newton's Law of Gravitation and Einstein's Field Equations: Accounting for Dark Energy and Dark Matter

A Major Qualifying Project
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Equation 3. $\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$

Equation 4. $g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}$

Equation 5. $$\begin{aligned} dV^1 &= dX^1 dX^2 \left(V^1\{\ldots\} + V^2\{\ldots\} + V^3\{\ldots\}\right) \\ dV^2 &= dX^1 dX^2 \left(V^1\{\ldots\} + V^2\{\ldots\} + V^3\{\ldots\}\right) \\ dV^3 &= dX^1 dX^2 \left(V^1\{\ldots\} + V^2\{\ldots\} + V^3\{\ldots\}\right) \end{aligned}$$

Equation 6. $$\begin{aligned} dV^1 &= dX^1 dX^2 \left(V^1 R^1{}_{112} + V^2 R^1{}_{212} + V^3 R^1{}_{312}\right) \\ dV^2 &= dX^1 dX^2 \left(V^1 R^2{}_{112} + V^2 R^2{}_{212} + V^3 R^2{}_{312}\right) \\ dV^3 &= dX^1 dX^2 \left(V^1 R^3{}_{112} + V^2 R^3{}_{212} + V^3 R^3{}_{312}\right) \end{aligned}$$

Equation 7. $R_{\mu\nu} = \begin{bmatrix} R_{00} & R_{01} & R_{02} & R_{03} \\ R_{10} & R_{11} & R_{12} & R_{13} \\ R_{20} & R_{21} & R_{22} & R_{23} \\ R_{30} & R_{31} & R_{32} & R_{33} \end{bmatrix}$

Equation 8. $p = \gamma m v$

Equation 9. $$\begin{aligned} p^0 &= \gamma m V^0 \quad = \frac{\gamma m c^2}{c} = \frac{E}{c} \\ p^1 &= \gamma m V^1 \\ p^2 &= \gamma m V^2 \\ p^3 &= \gamma m V^3 \end{aligned}$$

Equation 10. $T^{\mu\nu} = \begin{bmatrix} V^0p^0 & V^0p^1 & V^0p^2 & V^0p^3 \\ V^1p^0 & V^1p^1 & V^1p^2 & V^1p^3 \\ V^2p^0 & V^2p^1 & V^2p^2 & V^2p^3 \\ V^3p^0 & V^3p^1 & V^3p^2 & V^3p^3 \end{bmatrix}$

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Equation 15. $G = \frac{Mm}{\left(\frac{a}{a_0}\right)r^2}$

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List of Variables

Quantity	Unit	Symbol
Constants		C_1, C_2, C_3, C_4
Cosmological Constant	m^{-2}	Λ
Distance	m	r
Einstein Gravitational Constant	N^{-1}	κ
Energy	J	E
Energy-Momentum Density		$T^{\mu\nu}$
Energy-Stress-Momentum Tensor		$T_{\mu\nu}$
Force	N	F
Gravitational Acceleration	m/s^2	a
Gravitational Constant	$N \cdot m^2/kg^2$	G
Lorentz Factor		γ
Mass	kg	M, m
Metric Tensor		$g_{\mu\nu}$
Momentum	$kg \cdot m/s$	p
MONDian Constant	m/s^2	a_0
Ricci Scalar		R
Ricci Tensor	m^{-2}	$R_{\mu\nu}$
Speed of Light	m/s	c
Velocity	m/s	v, v_f

List of Constants

Quantity	Value	Symbol
Cosmological Constant	$1.1056 \times 10^{-52}\ m^{-2}$	Λ
Gravitational Constant	$6.67 \times 10^{-11}\ N \cdot m^2/kg^2$	G
MONDian Constant	$1.2 \times 10^{-10}\ m/s^2$	a_0
Speed of Light	$2.998 \times 10^8\ m/s$	c

EXECUTIVE SUMMARY

This paper conducts an exploration of gravitational theory, from Newton's Law of Universal Gravitation to Einstein's Theories of Special and General Relativity, and examines the potential unification of relativity and quantum mechanics through Modified Newtonian Dynamics (MOND). Einstein's theories introduce dark elements, called dark matter and dark energy, which are attributed to the acceleration of the universe. The problem with these elements lies in their lack of interaction with any part of the electromagnetic spectrum, so their existence is merely observed across large distances. Mordehai Milgrom, an Israeli physicist, proposed Modified Newtonian Dynamics (MOND) as a means to reconcile relativity and quantum mechanics, by replacing dark constituents with an additional acceleration constant.

Without proper instrumentation to conduct observational experiments and collect real astronomical data, further understanding of these theories was achieved through extensive research and simulations. The research was conducted through the use of Google Scholar and other search engines. The simulations were plotted using Microsoft Excel for simplicity and basic modeling. The simulations displayed comparisons between Newton's Law of Universal Gravitation and MOND. Four graphs were modeled: acceleration due to gravity versus distance, acceleration due to gravity versus mass, force-to-max force ratio versus distance, and force-to-max force ratio versus mass. Said graphs can be found below in Section 4 Results and Discussion. The behavior displayed in each graph was consistent with the literature. Notably, MOND acceleration was greater than Newtonian acceleration across distance and mass, resulting in a greater MOND force ratio than the Newtonian counterpart. The acceleration plots indicated a point of deviation or convergence, where the behavior between the theories changed. The change occurred at very small distances and extremely large masses. The behavior of these graphs aligns with the research; however, this paper does not aim to conclusively prove any theory, but rather to discuss where they differ. From the simulations, the points of deviation and convergence could be areas of future research. Additionally, the current ΛCDM model seems to be favored, albeit incomplete.

Abstract

This study explores the initial gravitational theory proposed by Isaac Newton and the subsequent advancements that led to Einstein's Theories of Special and General Relativity, as well as the modern alternative, Modified Newtonian Dynamics. Although General and Special Relativity more accurately explain the behavior of gravity, Einstein's "cosmological constant" continues to perplex physicists, inhibiting the progress of a unified theory of physics. A new, emerging theory, MOND, claims to effectively integrate dark energy and the Theories of Relativity and may contain answers on unifying these theories with quantum mechanics. Through plotting gravitational acceleration graphs against various parameters, this paper analyzes gravity under Newtonian, Einsteinian, and MONDian dynamics. The study utilized acceleration due to gravity versus distance and mass graphs for both Newton's Law of Universal Gravitation and MOND to explore deviations between these methods. Without conducting observational experiments, this paper does not conclude which theory is correct; however, based on extensive research, the consensus seems to show a majority in favor of the current ΛCDM model.

Acknowledgements

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I would also like to thank WPI's IT Services for their patience and helpful advice.

1 INTRODUCTION

The infinite universe is at the forefront of scientific study. The foundational force connecting the fabric of space and astrophysical phenomena is known as gravity. The Lambda Cold Dark Matter (ΛCDM) theory is the currently accepted model of gravity. In the ΛCDM model, Einstein's Theories of Special and General Relativity contribute to a large aspect of astrophysics (regarding massive bodies), while Quantum Mechanics refers to the microscopic, particle interactions. Currently, relativity cannot be extended to apply to quantum mechanics, accurately, and quantum mechanics cannot be extrapolated to portray the physics of relativity correctly. The disconnect between these branches of physics gives way to the "Theory of Everything," the idea of a unified theory that describes all of the fundamental forces in one singular framework. In addition to the discrepancies between relativity and quantum mechanical theories, the concepts of "dark energy" and "dark matter" have perplexed scientists for decades. The need for the cosmological constant in the equations of relativity furthers the idea that these theories prohibit progress toward the unified field theory.

Considering Einstein's "cosmological constant" and Newton's Universal Law of Gravitation, physicists are researching a new theory that may correctly predict gravitation according to Newton, while extrapolating these results to account for Einstein's missing mass problem, which is dark energy and matter. Within the last few years, many studies have been published both in support of and in opposition to this new theory of Modified Newtonian Dynamics (MOND). The first author of a study supporting MOND, physicist Indranil Banik published a paper addressing answers that MOND provides to questions long unanswered. Through data analysis and simulations, Banik presented conclusions suggesting the current universe model is not accurate. However, a few years later, Banik authored a study that considered the same test but improved the process through refined uncertainties. The result of this study was not in favor of his previous work; Banik abandoned MOND and suggested that his new work does not necessarily support the current ΛCDM model, but does favor Einstein and Newtonian principles of gravitation rather than the latter.

Alongside research, graphical representations of each theory explored in this study were used to gain a deeper understanding of the strengths and limitations of each. Through comparison of plots, the nature of the behavior of gravity under Newtonian and MONDian dynamics was explored and areas of deviation were noted. The results of these graphs represent very basically the behavior of each method and were not modeled using real astronomical data, and therefore were not heavily considered in concluding MOND. Without proper instrumentation, this paper aims to expose potential areas for future research that could aid in uniting the theories of gravity, as well as the ΛCDM model and Quantum Mechanics.

2 BACKGROUND

I NEWTONIAN AND EINSTEINIAN DYNAMICS

In the 16^{th} century, physicist Isaac Newton demonstrated the fundamentals of gravity on Earth and extrapolated his findings to apply to the universe. Newton's discovery became the model for gravitational interactions for centuries until Albert Einstein found the instantaneity of these interactions to be improper. In 1905, Einstein published his Theory of Special Relativity, with the groundbreaking discovery that light travels at a finite speed. Einstein's new theory postulated that space and time are interconnected, creating the space-time continuum. In this model, Einstein states that as a result of light traveling at a finite speed, observers could experience events at different times. The Theory of Special Relativity was revolutionary for the advancement of science, however, Einstein did not apply the finite speed of light to gravitation [12]. In 1915, Einstein published the Theory of General Relativity, where massive bodies warp the fabric of space-time.

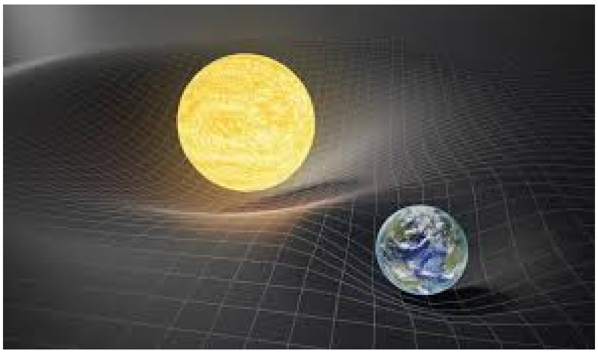

Figure 1. A visualization of the warping of spacetime, where the Sun and Earth are massive bodies that disrupt the structure of the fabric of spacetime [13].

With the curving of the fabric of space-time as seen in Figure 1 above, Einstein stated that impacts on the massive body would impact the gravitational field in a ripple effect. The idea that gravitation follows the limitation set on the universe by the finite speed of light completely contradicted the fundamentals of Newton's theory; instantaneity does not abide by the limit set on space travel. Einstein's Theory of General Relativity became widely accepted as the new model for gravity, governed by the Einstein Field Equations. Although Einstein had described the role of gravity in the universe, his field equations demonstrated that the universe was expanding. Einstein was troubled by this result and included a "cosmological constant" in his equations explaining this expansion [2].

II THE COSMOLOGICAL CONSTANT AND EINSTEIN'S FIELD EQUATIONS

The idea of this unknown constant was not accepted by the scientific community until decades later after Einstein had rescinded his statement. In 1998, the cosmological constant (now known as "dark energy") was discovered by two international teams. The astronomers compared observational data to mathematical predictions to calculate the universe's deceleration. However, supernova 1997ff (the astronomical feature studied in this case) appeared dimmer than the predicted calculated luminosity [3]. In a decelerating universe, astronomical phenomena would appear brighter as the distance between features decreased. Therefore, the only solution as to why luminosity would be decreasing over time, is that the distance between objects was increasing, thus the universe was not decelerating, but was accelerating. Based on the curvature of spacetime as a result of the existence of massive bodies, gravitational is strictly an attractive force, so a question arose from the observation that the universe is expanding: what force is opposing gravity, and how strong is this force to have the ability to overcome gravity? Coined "dark energy" by Michael Turner, this new force is "dark" as it does not interact with any waves in the electromagnetic spectrum [2]. Scientists are still unable to fully define this phenomenon and only prove its existence by measuring its opposition to gravity. The cosmological constant appears in Einstein's Field Equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \qquad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, Λ is the cosmological constant, κ represents the constant $\frac{8\pi G}{c^4}$, and $T_{\mu\nu}$ is the energy-stress-momentum tensor [6]. Both the Ricci tensor R and the Ricci scalar R relate to the curvature of spacetime.

CURVATURE AND GEODESICS IN DIMENSIONAL SPACES

The left-hand side of the equation relates to the geometry of the curvature of spacetime. Before discussing the curvature of a four-dimensional space, understanding the curvature of a three-dimensional space is important. The curvature of a three-dimensional space is easily understood using a cylinder as an example [21].

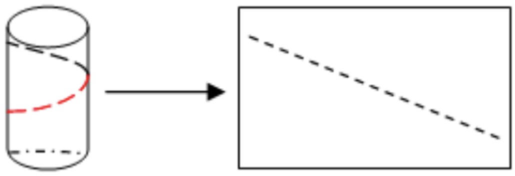

Figure 2. The curvature on a three-dimensional figure (cylinder) and the path represented on a two-dimensional figure (plane) to demonstrate the linearity of a geodesic [11].

Considering this example, when the two-dimensional plane is curved around the cylinder, the straight line curves. The curving of the fabric of spacetime follows this same concept. The mass of celestial bodies causes the warping (or curving) of spacetime in the same manner as depicted in Figure 2, although not in cylindrical terms, but in more spherical terms. Now it is important to understand movement on curved surfaces. In two-dimensional space, to move from Point A to Point B, an object would travel along a straight line, as seen below.

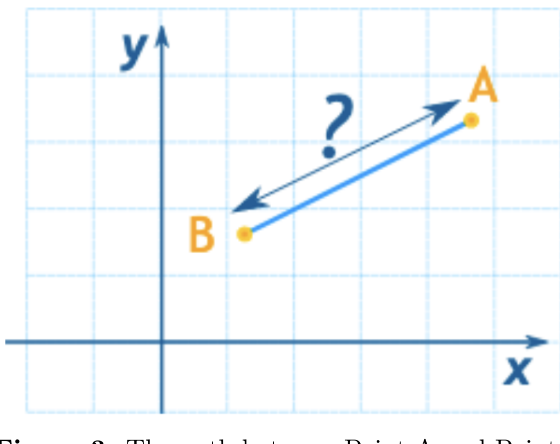

Figure 3. The path between Point A and Point B in a two-dimensional coordinate plane to demonstrate the linearity of the shortest path in two dimensions [5].

However, to travel from Point A to Point B along a curved surface, the path would be curved as well [7]. Although that seems intuitive, it is difficult to visualize at a point on the surface. Consider traveling from the United States to somewhere in Europe. As we are accustomed to the flat map of the Earth, it would seem like the shortest path to travel would be a straight line, however, the flight path appears curved on a two-dimensional plane in order to show the actual path is not straight (Figure 4).

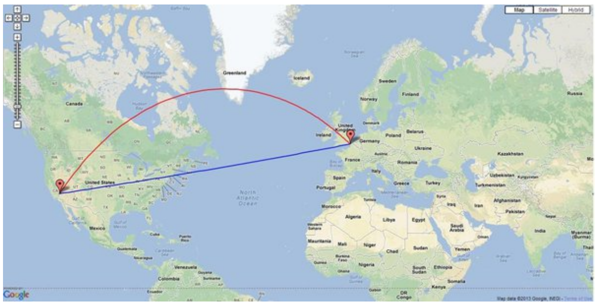

Figure 4. The straight path (in blue) from the United States to Europe and the actual path (in red), where the red line is the geodesic and displays the true three-dimensional path along a curved surface [24].

The actual path appears as a curved line, as traveling in a "straight" line on Earth is, in reality, traveling along the curve of the sphere of Earth. Traveling directly from the US to Europe along the red line depicted in Figure 4 is the shortest possible path for the flight to move along in a curved space, called a geodesic [7]. Below is a three-dimensional visualization of this movement from Point A to Point B along a curved object.

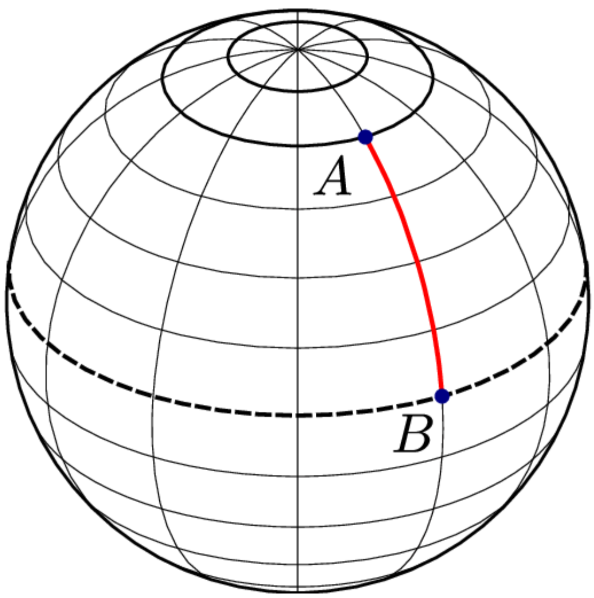

Figure 5. The geodesic from Point A to Point B on a sphere, where geodesic refers to the shortest path between points on a curved surface [8].

CALCULATING DISTANCE IN DIMENSIONAL SPACES

In two-dimensional space, calculating the distance between two points is easily accomplished using the Pythagorean Theorem, where $a^2+b^2=c^2$. However, in higher-dimensional space, the distance between two points on a curved surface, or the length of the geodesic, cannot be calculated using the Pythagorean Theorem, as there is no right angle. Instead, the length of the geodesic, known as the spacetime interval, will be dependent on the metric tensor, $g_{\mu\nu}$, following the Pythagorean Theorem using Riemann metrics:

$$C_1(\Delta X)^2 + C_2(\Delta X)(\Delta Y) + C_3(\Delta Y)(\Delta X) + C_4(\Delta Y)^2 \quad (2)$$

where C_1, C_2, C_3, and C_4 are constants, and ΔX and ΔY are the changes in the distance along the x and y axes, respectively. The above equation is the Pythagorean Theorem using Riemann metrics in two-dimensional space, where C_2 and C_3 are equal to zero. However, considering this equation in its current form, a matrix can be made using the constants as components [7].

$$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \quad (3)$$

This matrix is the metric tensor g for two-dimensional space. The metric tensor in Einstein's Field Equation seen in Equation 4 represents four-dimensional space, where time is the fourth dimension.

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}, \quad (4)$$

where the indices represent the dimensions, and time is considered to be the zeroth dimension.

The Ricci tensor $R_{\mu\nu}$ represents the measurement of the deviation in volume of a curved space from Euclidean geometry. Consider moving in two-dimensional space. An object may move in any direction and return to its original position in the same orientation, resulting in a deviation of zero [7]. However, in a curved space, direction is an important factor, for as an object moves along a geodesic, the angle is constantly changing, so when returned to its original position, the object may have rotated, and therefore there is a deviation.

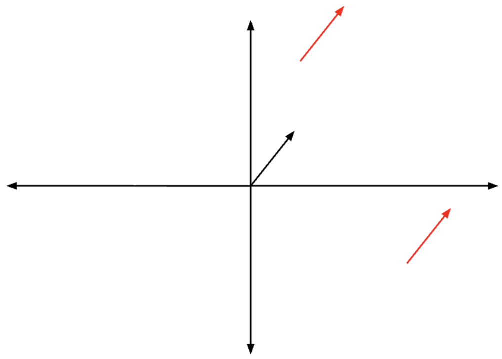

Figure 6. Two-dimensional movement, where the black arrow is the original position and the red arrows are translations.

As seen in Figure 6 above, both translations result in the arrow pointing in the same direction, so when returned to the position of the black arrow, the direction will be the same as the original.

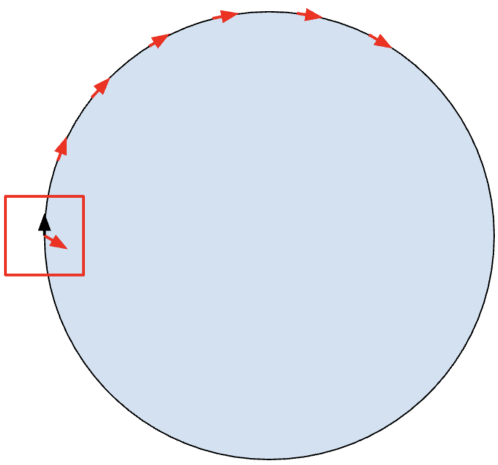

Figure 7. Three-dimensional movement along a curved surface, where the black arrow is the original position and the red arrows are translations.

As seen in Figure 7 above, each translation points in a new direction. The final translation is seen in the box with the original arrow but points in a completely different direction. The deviation between the original arrow and the final translation arrow is what the Ricci tensor represents [7]. A representation of the Ricci tensor is shown below.

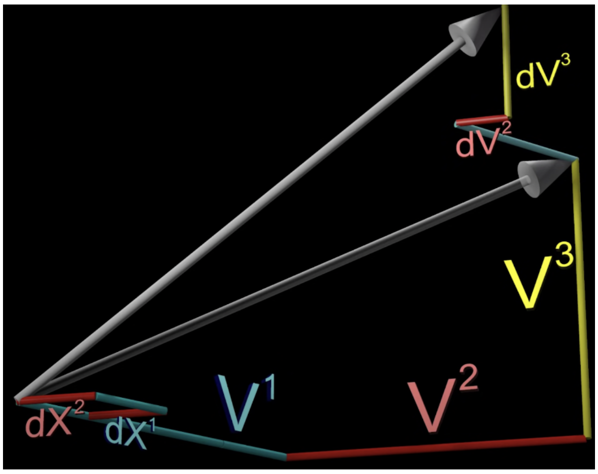

Figure 8. A visualization of the components used for calculating the deviation of a curved space from Euclidean space [7].

In Figure 8, V^1, V^2, and V^3 represent vectors in each of the three dimensions, where indices represent the dimension and are not exponents, dX_1 and dX_2 are the changes in their respective dimensions, and dV_2 and dV_3 are the changes of the vector's direction in their respective dimensions. The mathematical representation of the calculation of these changes in vector direction is

$$\begin{aligned} dV^1 &= dX^1 dX^2 \left(V^1\{\ldots\} + V^2\{\ldots\} + V^3\{\ldots\}\right) \\ dV^2 &= dX^1 dX^2 \left(V^1\{\ldots\} + V^2\{\ldots\} + V^3\{\ldots\}\right) \\ dV^3 &= dX^1 dX^2 \left(V^1\{\ldots\} + V^2\{\ldots\} + V^3\{\ldots\}\right), \end{aligned} \tag{5}$$

where colors are included to demonstrate the difference between indices of the same number. The complete equation below includes the Ricci scalar, which is the measurement of the curvature of a Riemannian space, or the curvature of a space that follows Riemannian geometry rather than Pythagorean geometry [7].

$$\begin{aligned} dV^1 &= dX^1 dX^2 \left(V^1 R^1{}_{112} + V^2 R^1{}_{212} + V^3 R^1{}_{312}\right) \\ dV^2 &= dX^1 dX^2 \left(V^1 R^2{}_{112} + V^2 R^2{}_{212} + V^3 R^2{}_{312}\right) \\ dV^3 &= dX^1 dX^2 \left(V^1 R^3{}_{112} + V^2 R^3{}_{212} + V^3 R^3{}_{312}\right) \end{aligned} \tag{6}$$

These scalar values of curvature are a part of the components of the Ricci tensor once again in four-dimensional space to represent our three-dimensional universe with time as the fourth dimension.

$$R_{\mu\nu} = \begin{bmatrix} R_{00} & R_{01} & R_{02} & R_{03} \\ R_{10} & R_{11} & R_{12} & R_{13} \\ R_{20} & R_{21} & R_{22} & R_{23} \\ R_{30} & R_{31} & R_{32} & R_{33} \end{bmatrix}, \tag{7}$$

where $R_{00} = R^0_{000} + R^1_{101} + R^2_{202} + R^3_{303}$, and so on. As stated above, Λ denotes the cosmological constant, which has a value of $\approx 1.056 \times 10^{-52}\mathrm{m}^{-2}$.

ENERGY-RELATED DENSITIES

Now consider the right-hand side of the equation, $\kappa = \frac{8\pi G}{c^4}$, denoted as Einstein's gravitational constant which includes Newton's gravitational constant of G = 6.674×10^{-11} Nm^2kg^2. $T_{\mu\nu}$ is known as the energy-stress-momentum tensor. From Newtonian physics, momentum, p, is equivalent to the product of mass, m, and velocity, v. However, p = mv assumes the momentum is in the system's reference frame. For relativity, m refers to the rest mass and v refers to the velocity relative to an observer, therefore gamma, γ, is multiplied as the relativistic factor, or the Lorentz factor:

$$p = \gamma m v \tag{8}$$

where $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ to account for the mass changing as velocity changes according to an observer [7].

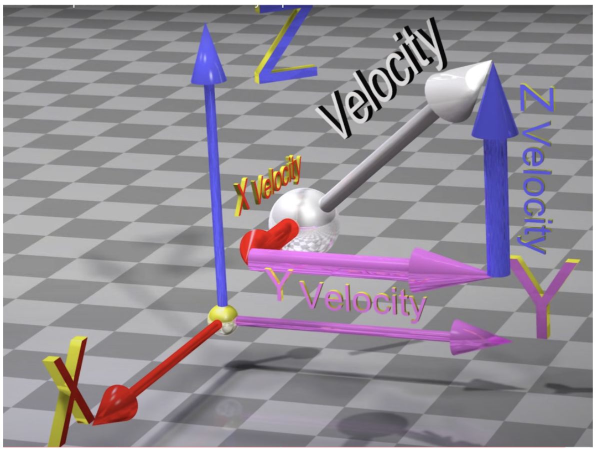

Figure 9. A depiction of velocity and its components in the x-y-z directions in three-dimensional space [7].

From Figure 9, velocity has three components, one in each of the three spatial dimensions, where V1, V2, and V3 represent the velocity in the x-, y-, and z- directions, respectively. To extend the velocity vector to four dimensions, V0 is used to denote the velocity through time, or the speed of light, c. Apply these velocity components to the relativistic momentum equation to calculate the momentum in each of the four dimensions.

$$
\begin{aligned}
p^0 &= \gamma m V^0 \quad = \frac{\gamma m c^2}{c} = \frac{E}{c} \\
p^1 &= \gamma m V^1 \\
p^2 &= \gamma m V^2 \\
p^3 &= \gamma m V^3,
\end{aligned} \tag{9}
$$

where p^0 is the momentum in time and p1, p2, and p3 are the momentums in the x-, y-, and z- directions, respectively [7]. From Equation 9, the momentum in time is equivalent to Einstein's famous equation of energy, $E = mc^2$ divided by the velocity through time, c. By multiplying each momentum by its respective velocity, the energy density in that dimension is yielded.

$$
T^{\mu\nu} = \begin{bmatrix} V^0p^0 & V^0p^1 & V^0p^2 & V^0p^3 \\ V^1p^0 & V^1p^1 & V^1p^2 & V^1p^3 \\ V^2p^0 & V^2p^1 & V^2p^2 & V^2p^3 \\ V^3p^0 & V^3p^1 & V^3p^2 & V^3p^3 \end{bmatrix} \tag{10}
$$

$$
T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix} \tag{11}
$$

where each component is multiplied by another γ to account for length contraction when calculated per unit volume. $T_{\mu\nu}$ represents energy density and is calculated as

$$
T^{\mu\nu} = \gamma m (V^{\mu\nu})^2 \tag{12}
$$

per unit volume. Einstein's Field Equations do not include $T^{\mu\nu}$ but include $T_{\mu\nu}$, which does not denote the same value.

$$
\begin{aligned}
T_{\mu\nu} = T^{00} g_{0\mu} g_{0\nu} + T^{01} g_{0\mu} g_{1\nu} + T^{02} g_{0\mu} g_{2\nu} + T^{03} g_{0\mu} g_{3\nu} \\
+ T^{10} g_{1\mu} g_{0\nu} + T^{11} g_{1\mu} g_{1\nu} + T^{12} g_{1\mu} g_{2\nu} + T^{13} g_{1\mu} g_{3\nu} \\
+ T^{20} g_{2\mu} g_{0\nu} + T^{21} g_{2\mu} g_{1\nu} + T^{22} g_{2\mu} g_{2\nu} + T^{23} g_{2\mu} g_{3\nu} \\
+ T^{30} g_{3\mu} g_{0\nu} + T^{31} g_{3\mu} g_{1\nu} + T^{32} g_{3\mu} g_{2\nu} + T^{33} g_{3\mu} g_{3\nu}
\end{aligned} \tag{13}
$$

The value of $T^{\mu\nu}$ represents the density of energy and momentum at any point in spacetime. Finally, the stress-energy-momentum tensor and each of the previously discussed variables in Einstein's Field Equations can be understood together to demonstrate that gravity is the result of spacetime being curved by mass and energy [7]. Einstein's theory of gravity is currently accepted in the standard model of cosmology, called the Lambda Cold Dark Matter Theory (ΛCDM), where Λ represents the cosmological constant.

III THE LAMBDA COLD DARK MATTER MODEL OF COSMOLOGY

The accepted model of cosmology is known as the Lambda Cold Dark Matter Theory, or briefly as the ΛCDM Theory, where lambda represents the cosmological constant, now known as dark energy. The ΛCDM Theory models the universe's inflation beginning at the Big Bang and includes dark energy, dark matter, and ordinary matter.

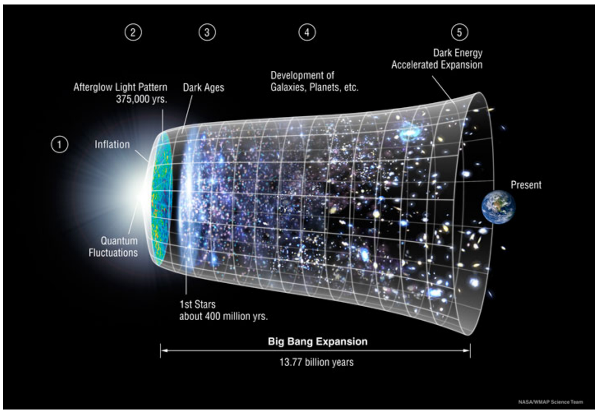

Figure 10. A visualization of the Lambda Cold Dark Matter (ΛCDM) Model of Cosmology where Einstein's Theory of General Relativity is the Accepted Model of Gravity [16].

In this model, Einstein's general relativity is considered the correct theory of gravity. Beginning at the Big Bang, the universe underwent a period of inflation, where the universe expanded at a rate faster than the speed of light with extreme heat, measuring about 10^{28} K. Before the Big Bang, the universe was a singularity, a point of infinite density where the laws of physics do not apply. Throughout the lifetime of the universe, nuclei, elements, atoms, stars, galaxies, black holes, and finally, human life were formed [18]. During each stage of formation, the universe continued to cool until its current temperature of about 2.7 K and expanded due to the force of the cosmological constant, known as dark energy. The ΛCDM model understands dark energy as a mysterious force that does not interact with any part of the electromagnetic spectrum but can be measured as the opposing force to gravity. The dependence on dark energy to explain the standard model of cosmology without a proper understanding of what it is has prompted many scientists to consider an alternate theory of gravity, Modified Newtonian Dynamics (MOND).

IV MODIFIED NEWTONIAN DYNAMICS (MOND)

The uncertainty behind the origin and composition of dark energy and dark matter led to the development of a new theory, Modified Newtonian Dynamics (MOND). First introduced in 1982 by Mordehai Milgrom, MOND considers the idea that previously unaccounted-for gravitational interactions can replace the effects of the cosmological constant. Milgrom proposed that tweaking Newton's equations to account for this external gravitation could explain the extra force in Einstein's equations [17]. Below are both Newton's equation of gravitation and the modified equation of gravitation.

Law of Gravitation $$G = \frac{Mm}{r^2} \tag{14}$$

Modified Newtonian Equation $$G = \frac{Mm}{\left(\frac{a}{a_0}\right) r^2} \tag{15}$$

The extra variable in the modified equation is the ratio of the actual acceleration due to gravity of the object to the expected acceleration due to gravity as predicted by Newton's Law of Universal Gravitation. Newtonian gravity is representative of high-acceleration environments but breaks down when applied to those of low-acceleration. Milgrom proposed this ratio of acceleration to agree with Newton's Second Law, while also supporting his new theory regarding low-acceleration environments [17]. Below are mathematical representations of the modified Newtonian equation when a0 is and is not equal to a (the gravitational acceleration predicted by Newton).

MOND Equation, where $a_0 = a$ $\quad G = \frac{Mm}{\left(\frac{a}{a}\right) r^2}$

MOND Equation, where $a_0 \neq a$ $\quad G = \frac{Mm}{\left(\frac{a}{a_0}\right) r^2}$

The premise of this new variable of acceleration is to propose the weak gravitation experienced by objects at the perimeter of galaxies and solar systems is responsible for the extra factors of energy and mass in accordance with the external field effect.

EXTERNAL FIELD EFFECT

The external field effect (EFE) is the effect of a large system on a much smaller system.

$$M = \frac{RV^2}{G} \tag{16}$$

$$M = \frac{g_{ex}}{a_0}\frac{RV^2}{G} \tag{17}$$

Consider a binary star system. According to Newton, the gravitational force between the two bodies is calculated by his Law of Gravitation seen in Equation 13 above [20].

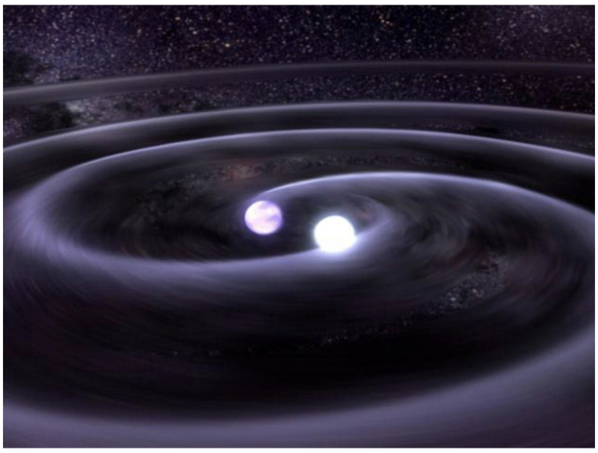

Figure 11. A binary star system with gravitational wave emission to display the system extensively studied by MOND [1].

Newton's Law of Gravitation does not consider the impact of external objects' gravitation on the system in consideration. However, MOND proposes that the sum of the very small accelerations due to gravity induced by bodies not involved in the system could be the true force of dark energy [20].

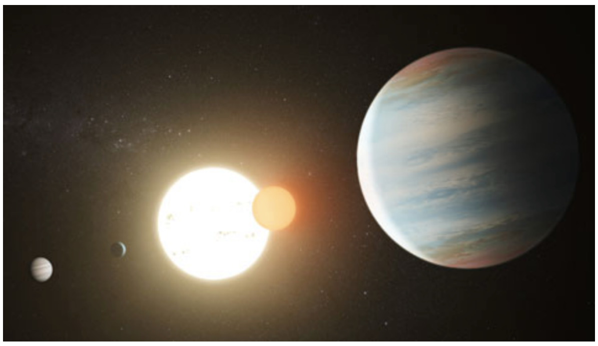

Figure 12. An inaccurate portrayal of a binary star system to show the presence of other massive bodies whose gravitational force is considered in MOND [4].

Visualizing this concept, Figure 12 displays a binary star system between three other massive bodies. In Figure 11, only the forces between the two stars were considered, however, Figure 12 demonstrates the proximity of external bodies, in which, according to MOND, their effects of gravity on the said system must be factored in.

V BANIK ET AL. "THE GLOBAL STABILITY OF M33 IN MOND"

In 2020, first author Banik published his paper "The Global Stability of M33 in MOND" alongside other scholars explaining and proving the theory of MOND. Under the ΛCDM model of gravity, the stability of disk galaxies is sustained by dark matter halos (DM). However, Banik et al. explored the possibility that the MOND theory of gravity more accurately explains the stability of disk galaxies and can be extrapolated [14].

PROPERTIES OF DISK GALAXIES: ROTATION CURVES AND STABILITY

Constituting roughly 80% of all galaxies, disk galaxies exhibit a physical similarity to disks, hence the name, where most celestial objects orbit the galactic core on the same plane, as seen below in Figure 13.

Figure 13. A visual representation of a disk galaxy portraying the "flatness" of the shape [22].

In 1973, Ostriker and Peebles demonstrated through N-body simulations (a simulation of particles under force, such as gravity) that disk galaxies can be stabilized by dark matter halos. The model suggests that such disk galaxies are not self-gravitating, but are constrained by the DM halos. The gravitational effect from the DM halos increases the disk's rotation curve (RC), therefore simulating a "flattening" effect on the outskirts of the galaxy [14]. Although this model is considered to be an essential part of the ΛCDM model, scientists have encountered problems in matching the properties of RCs and the long-term stability of such galaxies. Banik et al. believe that compatibility between observation and theory does not prove the theory true, and other explanations should be considered. An alternative possibility explored by Banik et al. considers the idea that Newtonian dynamics may not be accurate on galactic scales. As Newtonian dynamics are based solely on data collected from the Solar System, another explanation is plausible that utilizes data that can be more accurately extrapolated [14]. In this theory, the gravitational field strength can be demonstrated by the equation below.

$$g = \frac{\sqrt{GMa_0}}{R}, R \gg \sqrt{\frac{GM}{a_0}} \tag{18}$$

where a_0 is about $1.2 \times 10^{-10}\ ms^{-2}$. The above equation and MOND are consistent with properties of polar ring and shell galaxies, further supporting the plausibility of the theory. Discrepancies between quantum mechanics and gravity are thought to have the potential to be reconciled by the MOND theory [14]. As mentioned above, an increase in RC results in the ends of the galaxy becoming asymptotically flat; MOND describes this relation in terms of velocity, v_f.

$$v_f = \sqrt[4]{GMa_0} \tag{19}$$

According to angular momentum, the velocity is greatest near the center and gradually decreases towards the ends. Equation 18 describes this relationship mathematically in terms of MOND.

EXTERNAL FIELD EFFECT

Once again, the EFE is specific to the MOND theory of gravity, where internal gravitational accelerations of a sub-system do not determine the effect that an external field will have on said sub-system. Banik et al. explore the idea that MOND cannot be superposed, creating a weakening effect on the galaxy's gravitational field that is independent of and, therefore, does not affect the inertial mass of the galaxy. MOND with the EFE is compatible with the curve of Galactic escape velocity, providing further proof the EFE exists. In addition, the data from the Gaia mission demonstrates that MOND must exist alongside an EFE. Using the velocities and observations of wide binary stars, Banik et al. conclude that MOND must include the EFE [14].

VI “STRONG CONSTRAINTS ON THE GRAVITATIONAL LAW FROM GAIA DR3 WIDE BINARIES”

Although the article explored above suggested strong evidence in support of MOND, first author Banik has since refuted this theory. In 2023, Banik et al. published “Strong Constraints on the Gravitational Law from GAIA DR3 Wide Binaries” contradicting their previous work. Analysis of data from GAIA DR3 (Global Astrometric Interferometer for Astrophysics Data Release 3), specifically regarding wide binary systems, along with more quality cuts and accurate uncertainties led the group to the conclusion that Newtonian dynamics and the ΛCDM model better portray the physics of the universe [15]. GAIA is a European mission conducted by the European Space Agency (ESA) that maps and models the motion of the stars in the Milky Way Galaxy. Figure 14 below depicts some of these models from GAIA DR3.

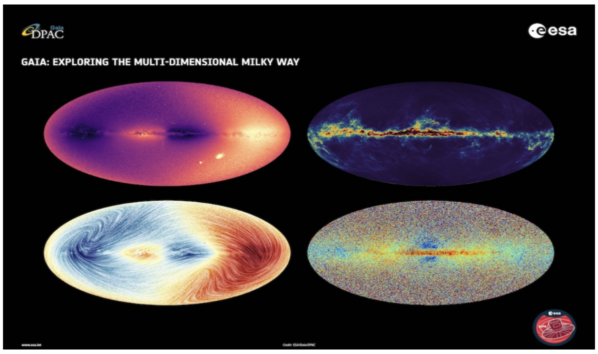

Figure 14. GAIA DR3 imaging of the Milky Way Galaxy, whose data was analyzed as a method of disproving the MOND theory [9].

BAYESIAN STATISTICS

Physicists conducting this experiment ensured reliable results by focusing on accurate uncertainties. Benefits of Bayesian Statistics include the probability distribution of hypotheses, and combining observed data with previous information. Rather than calculating the likelihood of the data given the hypothesis, Bayesian Statistics suggests collecting the data and calculating the likelihood of the hypothesis. Bayes’ Theorem has been used in astronomy to correct measurement errors, allowing for the hypothesis and the data to correlate with more accurate uncertainties [19].

WIDE BINARY TEST

From the wide binary test using a sample size of 8611, Banik et al. conclude that the Newtonian model drastically outperformed the model for MOND. Their work demonstrates the inadequacy of MOND, however, Banik et al. stress the fact that disproving MOND does not prove the ΛCDM model to be fully accurate. Regardless of the plausibility of MOND, the authors of this article conclude that our current understanding of the dynamics of the universe is incomplete [15].

3 METHODS

I RESEARCH

This project required understanding the basic principles of the geometry and physics that govern Newton’s Universal Law of Gravitation and Einstein’s Field Equations. Using Google Scholar, information regarding geodesics and curvature in metric spaces was studied. In addition to retrieving information regarding the two governing theories of gravity in our universe, an introduction to a new theory of gravity, Modified Newtonian Dynamics (MOND), has also been studied. Without proper instrumentation, a conclusion on the plausibility of the MOND theory in place of the current ΛCDM model cannot be drawn, so a consideration of the areas in which they differ was explored through graphical representations.

II GRAPHICAL REPRESENTATIONS

Graphing each theory of gravity (Einstein’s Field Equations, Newton’s Law of Universal Gravitation, and MOND) in the proper spatial dimensions requires the use of software capable of decoding and representing differential equations. Due to a lack of time and coding experience, computation and the use of Microsoft Excel were chosen instead. For graphs modeling the acceleration due to gravity, one variable was held constant. Data from our solar system was used as the model for constant mass and constant distance. Calculating the force then required a smaller, constant mass, and the mass of the moon was selected to understand the impacts on a system on the scale of our Solar System. When needed, the use of a logarithmic scale was implemented to better visualize the behaviors of the graphs on a more equal scale. The goal of said plots was to explore the differences between the theories and suggest that deviation could be an important area for further research.

PLOTTING NEWTON'S LAW OF UNIVERSAL GRAVITATION

To graph the acceleration due to gravity against both mass and distance, Newton's Law must be in terms of acceleration. The relationship is demonstrated by the sequences of equations that reformulate Newton's Law to be in terms of acceleration due to gravity instead of in terms of the force of gravitational attraction.

$$F = \frac{GMm}{r^2} \tag{20}$$

$$ma = \frac{GMm}{r^2} \tag{21}$$

$$a = \frac{GM}{r^2} \tag{22}$$

For the graph representing the acceleration due to gravity against distance, the mass (M) was held constant. The same was done for the graph representing the acceleration due to gravity against mass; the distance (r) was held constant. Modeling the ratio of the force of gravitational attraction to the maximum force required another constant mass (m). Using Newton's Second Law, the acceleration calculated from Eq. 21 was multiplied by mass, m, to yield the force, F.

To graph the force to maximum force ratio against distance and mass, the maximum force yielded by Newton's Second Law and the forces over distance and mass were used. The ratio was calculated with the constant mass and distance used in plotting the acceleration graphs.

PLOTTING MODIFIED NEWTONIAN DYNAMICS

To graph the acceleration due to gravity against mass and distance, again, the formula must be in terms of acceleration. The following MOND equation in terms of acceleration was used.

$$a = \frac{\sqrt{GMa_0}}{r} \tag{23}$$

Both the accelerations due to gravity and the force-to-maximum force ratio were calculated using the same data as above in the Plotting Newton's Law of Universal Gravitation section.

4 RESULTS AND DISCUSSION

Graphical representations of the theories of gravity were utilized to exploit the point where deviation occurs. Deviation between theories may signify an area of research that could be beneficial in uniting the theories of gravity and quantum mechanics.

I ACCELERATION DUE TO GRAVITY VS DISTANCE

After calculating the acceleration due to gravity for both Newton's Law of Universal Gravitation and MOND using a constant mass (the mass of the Earth), the ranges of the accelerations differ greatly. As seen in Figure 15 below, the acceleration according to the MOND theory ranges from powers of the negative 6^{th} order of magnitude to the negative 19^{th} order of magnitude.

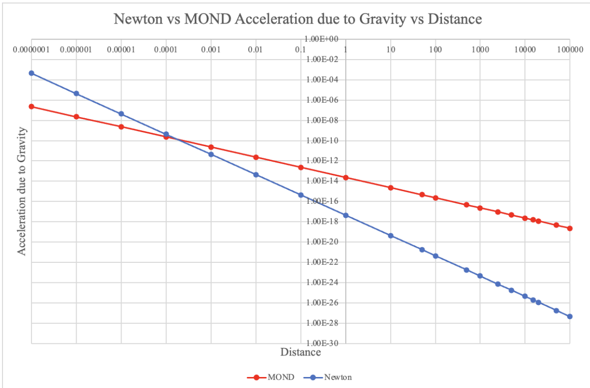

Figure 15. Acceleration due to Gravity vs Distance Graphs Plotting Newton's Universal Law of Gravitation (Blue) Against MOND (Red) to Demonstrate Where Acceleration Differs Between the Theories.

The range mentioned is many orders of magnitude greater than that of Newton's Law, where Newton's Law has acceleration ranging from powers of the negative 4^{th} order of magnitude to the negative 28^{th} order of magnitude. Newton's Law of Universal Gravitation spans a much larger range consisting of much smaller accelerations. Considering the fact that MOND accounts for the effects of dark energy and matter within its acceleration constant, a0, the larger accelerations calculated are logical. The distances spanned from 0.0000001 light years to 1,000,000 light years, where one light year equals $9.46 \times 10^{15} m$. Recall that on a smaller scale, MOND breaks down to Newton's Law. Distances less than 0.0001 light years, according to Figure 15, display Newton's Law to be greater than MOND, which is logical as dark energy and dark matter are observed on a large scale, meaning 0.0001 light years may be too short to observe such effects. The acceleration due to gravity according to MOND decreases at a significantly lower rate as opposed to Newton's Law. Without conducting observational experiments, a full analysis cannot be explained in this paper, however, the small range of orders of magnitude for acceleration according to MOND as well as the distance at which the theories deviate from each other could be an area of consideration for the plausibility of the theory.

II ACCELERATION DUE TO GRAVITY VS MASS

After calculating the acceleration due to gravity for Newton's Law of Universal Gravitation and MOND, similar to the distance graphs in the section above, the range of accelerations differed greatly. As demonstrated in Figure 16 below, the acceleration due to gravity according to Newton's Law has a much steeper increase and spans a greater range (about 10^{-24} to 10^{-11} than MOND.

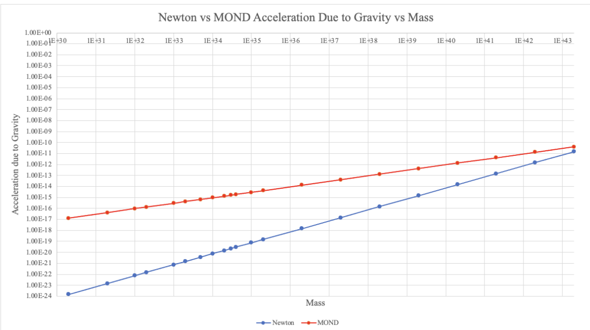

Figure 16. Acceleration due to Gravity vs Mass Graphs Plotting Newton's Universal Law of Gravitation (Blue) Against MOND (Red) to Demonstrate Where Acceleration Differs Between the Theories.

According to MOND, the acceleration increases over about seven orders of magnitude (about 10^{-17} to 10^{-10}). Considering the fact that the range of mass is from 1 solar mass to 10,000,000,000,000 solar masses, the increase in accelerations in both Newton's Law and MOND is understandable. Relating this relationship to the one explored in the previous section, acceleration due to gravity under MOND over mass is less than that over distance. The acceleration according to MOND is vastly greater than that according to Newton over distance, which is understandable as MOND also accounts for the effect of dark energy and dark matter, which is better seen on a larger scale. Similarly, over mass, the acceleration under MONDian dynamics is vastly greater than that under Newtonian dynamics. Dark energy and dark matter effects are observed over a larger scale (as above with distance), however, such effects are expected to be observed over larger masses as well, as seen above.

When first simulating the graph shown in Figure 16, the distance of 0.0001 light years resulted in Newton's Law being greater than MOND over mass. Adjusting the distance to 1,000,000 light years yielded more accurate results aligned with the expectation that MOND accelerations should be greater than Newton accelerations. Due to the MONDian acceleration being well in excess of Newtonian acceleration over distance, dark energy may be responsible for this increase in MOND. Similarly, dark constituents may be responsible for the increases in MOND over mass, as well. From Figures 15 and 16, it appears that distance has a greater impact on MONDian and Newtonian dynamics, perhaps due to the effects of dark energy and dark matter. Figure 16 displays a converging behavior between MOND and Newton's Law. As distance affected the results of deviation most, the distance of 0.0001 light years could be an area for future research, as well as the mass at which convergence occurs.

III FORCE-TO-MAX FORCE RATIO

From the accelerations calculated in the sections above, the force was calculated under Newtonian and MONDian dynamics. The new mass in the calculation is equal to the mass of the moon, again for an understanding of the behavior on a well-known scale. After calculating the force across various distances, the maximum force was identified and used in the subsequent calculation. A ratio between the force at a certain distance and this maximum force was calculated and the relationship can be seen in Figure 17 below.

All simulations were modeled using Microsoft Excel and were manually coded, which may introduce slight errors in the values. However, based on extensive research the behavior depicted in the simulations should remain accurate.

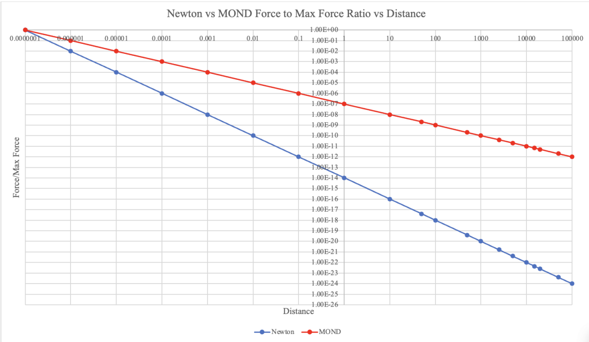

Figure 17. Force to Max Force Ratio vs Distance Graph Plotting Newton's Universal Law of Gravitation (Blue) Against MOND (Red) to Highlight the Differences in How Force Decreases Over Distance.

According to the graph above, the ratio under MOND is much greater than that under Newton's Law. As force over distance decreases, the starting value of 1 aligns with behavioral expectations. With the mass of the moon multiplied by acceleration, the force under MOND is greater than that under Newton's Law as MOND acceleration is greater. Each acceleration, and therefore force, under MOND, is greater, explaining the behavior seen in Figure 17 above. From Figure 17, the conclusion seems to be consistent, MOND is greater than Newton's Law of Gravitation due to the addition of dark elements.

The multiplication of acceleration and the mass of the moon was calculated for the force-to-max force ratio vs mass as well. As the behavior of the acceleration vs mass graph was increasing, the force and the force-to-max ratio plots exhibited this behavior as well.

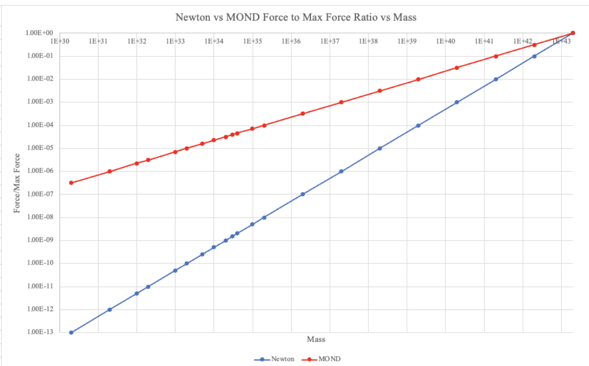

Figure 18. Force to Max Force Ratio vs Mass Graph Plotting Newton's Universal Law of Gravitation (Blue) Against MOND (Red) to Highlight the Differences in How Force Decreases Against Mass.

Figure 18 above displays the MOND force ratio to be greater than Newton's force ratio, which, again, is consistent with the literature. Demonstrated in this plot is also the steep incline in the force ratio under Newtonian dynamics, where the force ratio under MOND increases more gradually. In each of the four graphs, MOND acceleration and force is overall greater than Newton's Law by quite a few orders of magnitude, most probably explained by the inclusion of dark energy and dark matter in the MONDian acceleration constant.

5 CONCLUSION AND RECOMMENDATIONS

I CONCLUSION

Through simulations of both Newton's Law of Universal Gravitation and MOND, the accelerations due to gravity are greater under MONDian dynamics, which is in line with expectations of MOND as MOND accounts for dark constituents. The two methods have points of deviation and convergence when considered over varying distances and masses. Perhaps these points could be further studied to better understand which method more accurately explains such behavior. This paper did not utilize real astronomical data or conduct observational experiments and therefore does not aim to prove any theories, but merely explore any differences. However, based on extensive research, including the papers authored by Banik et al. previously discussed, many seem to be in favor of our current model. MOND has since been dismissed by many, but physicists continue to think outside the box and stretch the bounds of astrophysics. Although this theory may not hold the answers to unite general relativity and quantum mechanics, understanding its shortcomings could help steer physics in the right direction.

II RECOMMENDATIONS

For future research regarding the differing theories of gravity, the ability to use observational data or instrumentation to retrieve such data could be very useful in examining specific scenarios. Studying specific situations provides a deeper understanding of the limits each theory possesses. For future graphical representations, a two- or three-dimensional visual is recommended to better model the behavior of gravity under each method. To simulate such models, proficiency in multiple coding languages is strongly advised. For proficient coders, the use of the Einstein Toolkit may provide better visuals and easier coding than Excel, as modeling Einstein's Field Equations is the purpose of such software. However, if using the Einstein Toolkit, I strongly recommend using Windows or Linux, as MacOS is a bit harder to use for downloading multiple coding languages and packages. Perhaps in the future WPI may be able to provide this resource for students pursuing research in astronomy and astrophysics.

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