Chapter 1
$L^\infty-$extremal mappings in AMLE and Teichmüller theory

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Abstract These lecture focus on two vector-valued extremal problems which have a common feature in that the corresponding energy functionals involve L^∞ norm of an energy density rather than the more familiar L^p norms. Specifically, we will address (a) the problem of extremal quasiconformal mappings and (b) the problem of absolutely minimizing Lipschitz extensions.

1.1 Introduction

These notes originate from a C. I. M. E. mini course held by the author in July 2012 in Cetraro, Italy. They are meant to provide a quick introduction to two model L^∞ variational problems involving mappings, i.e. where the set of competitors is not scalar but vector-valued.

The first concerns a classical problem in geometric function theory that first arose in 1928 in the work of Grotzsch [32]

Problem 1. Among all orientation preserving quasiconformal homeomorphisms $w:\Omega\to\Omega'$ whose traces agree with a given mapping $u_0:\partial\Omega\to\partial\Omega'$, find one which minimizes the functional

$$u\to\left\|\frac{|du|}{(\det du)^{\frac{1}{n}}}\right\|_\infty .$$

Variants of this problem occur when, instead of using boundary data, the class of competitors is defined in terms of a fixed homotopy class or by requesting that the traces map quasi-symmetrically boundary into boundary.

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The second problem has also a classical flavor. It goes back to the work of Whitney [68] and the work of MacShane [48] in 1934, and leads to the recent theory of *absolutely minimal Lipschitz extensions* (AMLE) [5].

Problem 2. Let $\Omega \subset \mathbb{R}^n$, $F \subset \bar{\Omega}$ be a compact set and let $g \in Lip(F, \mathbb{R}^m)$. Among all Lipschitz extensions of F to Ω is there a canonical unique extension that in some sense has the smallest possible Lipschitz norm?

Some natural questions arise in connection to these problems

- Do minimizers exist?
- Are minimizers unique?
- What is the structure of the minimizers? In which norm there is continuity with respect to the data?

The last few decades have seen intense activity from different communities of mathematicians in the study of both problems. However at this time there does not seem to be much synergy and communication between these communities, both in terms of shared techniques used in the study of these problems and in terms of common point of views. One of the goals of these notes is to foster such synergies by outlining some of the common features in these problems. The notes (as well as the lectures) are mainly addressed to graduate students and because of this we have included some very basic material and maintained throughout an informal style of exposition. Since there are no original results in this survey, all proofs are merely sketched, and references to the detailed arguments are provided.

There are several other sources that discuss more extensively either extremal quasiconformal mappings or vector valued AMLE, but we are not aware of a reference striving for a unified point of view. The (possibly too optimistic) goal of this set of notes is to provide such perspective. Regarding other pertinent references: For classical extremal quasiconformal mappings we recommend the surveys of Strebel [64] and [63]. Two very clear and extremely well-written accounts of the classical Teichmüller theory can be found in [1] and [14]. The paper of Grotzsch [32] is at the origin of the subject and Hamilton's dissertation [33] provided an interesting development. The reader will also benefit from reading the classic monograph [3] as well as the more recent [7]. For the higher dimensional theory of quasiconformal mappings and the corresponding extremal problems I recommend the following fundamental contributions by Gehring and Vaisala [29], [30], [66], as well as the more recent comprehensive book by Iwaniec and Martin [42]. Various aspects of the extremal problem can be found in [9], [8], [6], [61], [60], [59] and [23]. There is considerably less literature on the vector valued extremal Lipschitz extension problem: A good introduction is in the papers of Barron, Jensen and Wang [12] and [11]. More recent developments can be found in the work of Naor and Sheffield [51], Sheffield and Smart [62], Katzourakis [47], Ou, Troutman, and Wilhelm [52]. We also want to point out two relevant references that, in our opinion, have great potential for applications to the problems discussed here: Dacorogna and Gangbo [20] and Evans, Gangbo and Savin [21].

Although these notes do not involve specific applications, the topic of L^∞ variational problems arises naturally in mathematical models of several real-world phenomena. To this regard, we conclude this introduction with a quote from Robert Jensen's seminal paper [44]

> The importance of variational problems in L^∞ is due to their frequent appearance in applications. The following examples give just a small sample of these. In the engineering of a load-bearing column it is preferable to minimize the maximal stress (i.e., the L^∞ norm of the stress) in the column rather than some average of the stress. When constructing a rocket, the maximal acceleration applied to the payload is an important factor in the design. Optimal operation of a heating-cooling system for an office building requires control of the maximal and minimal temperature within the building rather than the average temperature. Windows on airplanes are made without corners to prevent high pointwise stress concentrations. These considerations motivate the study of the issues of existence, uniqueness, and regularity etc. etc.

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1.2 Notation and preliminaries

In this section we set the notation for the rest of the notes and include some basic, elementary definitions and results that will be needed later on.

1.2.1 Notation: Topology

- An *homeomorphism* between two topological spaces is a continuous bijection whose inverse is also continuous.
- A *topological manifold* of dimension $n \in \mathbb{N}$ is a topological space for which every point has a neighborhood homeomorphic to $\mathbb{R}^n$.
- A *smooth manifold* of dimension n is an $n-$dimensional topological space along with a collection of charts $(U_\alpha, f_\alpha)_{\alpha\in A}$ with $U_\alpha \subset M$ open and such that they cover M, $f_\alpha : U_\alpha \to f_\alpha(U_\alpha) \subset \mathbb{R}^n$ homeomorphism and such that $f_\alpha \circ f_\beta^{-1}$ is smooth on its domain.
- An *homotopy* between two continuous functions f, g between two topological spaces X and Y is a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
- The *fundamental group* $\pi_1(M, p)$ of a topological manifold M with $p \in M$ is the quotient of the space of loops at p through the equivalence relation $\gamma \approx \eta$ if and only if $\gamma \circ \eta^{-1}$ is homotopic to the identity. If $\pi_1(M, p) = 0$ then M is simply connected.

- A *triangle* T in a surface S is a closed set obtained as the homeomorphic image of a planar triangle. The image of vertices and edges of the planar triangle are also called vertices and edges. A *triangulation* of a compact surface S is a finite set of triangles $T_1,...,T_m$ such that $\cup_{i=1}^m T_i = S$ and every pair T_i, T_j is either disjoint or intersects at a single point (vertex) or a shared edge.
- The *Euler characteristic* of a triangulated compact surface S is given by $\chi = v - e + f$ where v is the number of vertices of the triangulation, e is the number of edges and f the number of triangles. This number does not depend on the specific triangulation of S. The *genus* of S is the number g obtained from the identity $\chi = 2 - 2g$.

  Example 1. The sphere has genus zero, as does the unit disc. The torus has genus 1. Roughly, for general orientable surfaces, the genus is the number of *handles* in the surface.

1.2.2 Notation: Differentials and dilation of mappings

The background for fine properties of mappings, their dilation and much more can be found in the monograph [42]. Let $\Omega \subset \mathbb{R}^n$ and denote by

$$u = (u^1,...,u^n) : \Omega \to \mathbb{R}^n \tag{1.1}$$

a $W^{1,n}_{loc}(\Omega)$ orientation-preserving homeomorphism. At points $x \in \Omega$ of differentiability of u we denote by $du(x) : \mathbb{R}^n \to \mathbb{R}^n$ the *differential of* u. In coordinates one has that for $v \in \mathbb{R}^n$ the action of the differential is[1] $[du(x)(v)]^i = du_{ij}v_j$, $i = 1,...,n$ where we have let $(du)_{ij} = \partial_{x_j} u^i$. Set $|du|^2 = trace(du^T du) = du_{ij}du_{ij}$. At points of differentiability, the pull-back $du^*(x)g_E$ of the Euclidean metric g_E is given by $d(u^*(x)g_E)_{ij} = [du^T du]_{ij} = \partial_{x_i} u^k \partial_{x_j} u^k$, for $i, j = 1,...,n$.

If $n = 2$ it is convenient to use complex notation: Set $u = u^1 + \mathbf{i}u^2$, and

$$\partial_z u = \frac{1}{2}(\partial_x u - \mathbf{i}\partial_y u) \text{ and } \partial_{\bar z} u = \frac{1}{2}(\partial_x u + \mathbf{i}\partial_y u).$$

Note $\partial_{\bar z} u = \overline{\partial_z \bar u}$. We also let $dz = dx + \mathbf{i}dy$ and note that $dz(\partial_z) = 1$ while $dz(\partial_{\bar z}) = 0$. Similarly $d\bar z = dx - \mathbf{i}dy$ and $d\bar z(\partial_z) = 0$ and $d\bar z(\partial_{\bar z}) = 1$.

Next we introduce different ways in which one can quantify how a differentiable homeomorphism $u : \Omega \to \mathbb{R}^n$ can distort the ambient geometry. We start by considering linear bijections $A : \mathbb{R}^n \to \mathbb{R}^n$ expressed in coordinates as $y^i = A_{ij}x^j$ for $i = 1,...,n$. Denote by $|A|_O := \max_{|V|=1} |AV|$ the *operator norm* of A and consider the following quantities

- the *linear dilation* of A is

[1] Implicit summation on repeated indices is used throughout the paper

$$H(A) = \frac{\max_{|h|=1} |Ah|}{\min_{|h|=1} |Ah|}. \tag{1.2}$$

- the *outer dilation* of A is

$$H_o(A) = \frac{|A|_O^n}{|det\ A|}. \tag{1.3}$$

- the *trace dilation* of A is

$$\mathbb{K}(A)^n = \frac{|\sum_{ij} A_{ij}^2|^{n/2}}{|det\ A|}. \tag{1.4}$$

If u is as in (1.1) then we set $\mathbb{K}_u(x) = \mathbb{K}(du(x))$.

1.2.3 Notation: Complex analysis

Basic references for the complex analysis background are the classical book of Ahlfors [2] and Jost's monograph [45].

- A C^1 function $w = u + iv : \mathbb{C} \to \mathbb{C}$ is *holomorphic* if

$$\partial_{\bar{z}} w = u_x + iu_y + iv_x - v_y = 0$$

Equivalently w must satisfy the *Cauchy-Riemann equations* $u_x = v_y$ and $u_y = -v_x$.
- *Conformal invariance of harmonic functions.* If $w = h(z)$ is a holomorphic function and $f : \mathbb{C} \to \mathbb{C}$ is smooth then

$$\frac{\partial^2}{\partial z \partial \bar{z}} f \circ h(z) = \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial w}\frac{\partial h}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{h}}{\partial \bar{z}}\right]$$

$$= \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial \bar{w}}|_{h(z)}\frac{\partial \bar{h}}{\partial \bar{z}}\right] = \left[\frac{\partial^2}{\partial w \partial \bar{w}} f\right]|_{h(z)} \partial_z h \partial_{\bar{z}} \bar{h}.$$

- A holomorphic map $u : U \subset \mathbb{C} \to \mathbb{C}$ is called *conformal* if $\partial_z u \neq 0$ at every point in U.

Example 2. Set

$$D = \{z \in \mathbb{C} |\ |z| < 1\} \text{ and } H = \{z = x + \mathbf{i}y |\ y > 0\}$$

These are conformally equivalent under the map $H \to D$ given by $z \to \frac{z - z_0}{z - \bar{z}_0}$.

Theorem 1. *Every $f : D \to D$ (or $f : H \to H$) which is biholomorphic (i.e., conformal and bijective) is a Möbius transformation, i.e. there are $a, b, c, d \in \mathbb{C}$ such that*

$$f(z) = \frac{az + b}{cz + d}.$$

For any ring R define the group

$$SL(2,R) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | ad - cb = 1 \}$$

while $PSL(2,R)$ denotes its quotient by the sub-group generated by $\pm Id$. Every element in $PSL(2,R)$ defines a Möbius tranformation $H \to H$.

Definition 1. A group G acts as a transformation group on a manifold M if there is a map $G \times M \to M$ denoted as $(g,x) \to gx$ with $(g_1 g_2)(x) = g_1(g_2 x)$ and $ex = x$. The isotropy group of $x \in M$ is a subgroup of G which fixes x.

Example 3. The group $PSL(2,\mathbb{R})$ acts as a transformation group of H. The isotropy group of each element is isomorphic to $SO(2)$.

Both D and H can be given a (non-euclidean) metric structure through the *hyperbolic metric*

$$\frac{1}{y^2} dz d\bar{z} \text{ on } H \quad \text{and} \quad \frac{1}{(1-|z|^2)^2} dz d\bar{z} \text{ on } D.$$

An isometry between two Riemannian manifolds

$$u : (M,g) \to (M',g')$$

is a map such that

$$g'_{u(x)}(d_x uV, d_x uW) = g_x(V,W)$$

for any $x \in M$ and $V,W \in T_x M$.

Theorem 2. *All isometries between the hyperbolic H and D are Möbius transformations. The isometry group of H is $PSL(2,\mathbb{R})$.*

Definition 2. A group action G on M is *properly discontinuous* if every $x \in M$ has a neighborhood U such that $\{g \in G | gU \cap U \neq 0\}$ is finite and if x,y are not in the same orbit then they have neighborhoods U_x, U_y such that $gU_x \cap U_y = \emptyset$ for all $g \in G$.

Definition 3. Let $\Gamma \subset PSL(2,\mathbb{R})$ be properly discontinuous subgroup and $z_1, z_2 \in H$. We say that z_1 and z_2 are equivalent if there exists $g \in \Gamma$ such that $gz_1 = z_2$. Consider H/Γ the space of quotient classes equipped with the quotient topology.

Proposition 1. *Let $\Gamma \subset PSL(2,\mathbb{R})$. If the action of Γ on H properly discontinuous and does not fix points ($gx \neq x$ for all $x \in H$ and all $g \neq id$) then the quotient H/Γ can be given a Riemann surface structure.*

1.3 Conformal deformations

An a.e. differentiable homeomorphism $u : \mathbb{R}^n \to \mathbb{R}^n$ is *conformal* if there exists a scalar function λ such that at every point of differentiability

$$du^T du = \lambda Id \tag{1.5}$$

The pull-back of the Euclidean metric dx^2 is a scalar multiple of dx^2, i.e. *angles are preserved*. Equivalently, a.e. in Ω, one must have

$$g := \frac{du^T du}{(\det du)^{\frac{2}{n}}} = Id. \tag{1.6}$$

The function $\sqrt{trace(g)} = |du|/(det\ du)^{1/n}$ is called *dilation* of u.

Remark 1. At every point of differentiability for u one has $\mathbb{K}_u = trace(g) \geq n$, with the equality being achieved if and only if $g = Id$.

Definition 4. Following Ahlfors (see also [42]) we define the *distortion tensor of u at a point of differentiability* $x \in \mathbb{R}^n$

$$S(g) := \frac{g+g^T}{2} - \frac{trace(g)}{n} Id = g - \frac{trace(g)}{n} Id, \tag{1.7}$$

and denote by

$$\mathbb{K}(u,\Omega) = ||\mathbb{K}_u||_{L^\infty(\Omega)} = ||\sqrt{trace(g)}||_{L^\infty(\Omega)}, \tag{1.8}$$

the *maximal dilation* of u in Ω.

Proposition 2. *With the notation above, one has that a diffeomorphism u is conformal if and only if* $S(g) = 0$ *and if and only if* $\mathbb{K}(u,\Omega)^2 = \mathbb{K}_u^2 = trace(g) = n$ *identically in* Ω.

Remark 2. It is not difficult to show that if $\mathbb{K}_u = K_0 > \sqrt{n}$, then there exists $\varepsilon = \varepsilon(K_0) > 0$ so that

$$\varepsilon \leq |S(g)|^2 \leq \mathbb{K}_u^4 \Big(1 - \frac{1}{n}\Big).$$

When $n = 2$, if we denote by $0 \leq \lambda_1 \leq \lambda_2$ be the eigenvalues of g, then one can find an explicit lower bound. In this case, $\lambda_1\lambda_2 = 1$ and

$$|S(g)|^2 = \lambda_1^2 + \lambda_2^2 - \frac{1}{2}(\lambda_1+\lambda_2)^2 = \frac{1}{2}(\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2 = \frac{1}{2}(\mathbb{K}_u^4 - 4).$$

Remark 3. Denote by $CO_+(n)$ the space of differentials of orientation preserving conformal mappings, then its tangent space $TCO_+(n)$ at the identity is

$$\Big\{A \in \mathbb{R}^{n\times n} s.t.\ S(A) = \frac{A+A^T}{2} - \frac{trace(A)}{n} Id = 0\Big\}.$$

Accordingly we have that the distance of a matrix A from $CO_+(n)$ satisfies

$$d^2(A, CO_+(n)) = c|S(A-I)|^2 + O(|A-I|^4).$$

This shows that the operator S arises naturally when considering the linearization of the distance of a deformation from being conformal. For more results from this point of view, including a remarkable geometric rigidity result in the spirit of Frieseke, James and Müller [24], see the work of Faraco and Zhong [22].

Three remarkable properties of conformal deformations

- *Conformal implies smooth.* If an homeomorphism $u \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies $\mathbb{K}(u,\Omega) = \sqrt{n}$ then a result of F. Gehring [28] implies that $u \in C^\infty(\Omega)$. The proof is based on regularity of weak solutions to the $n-$Laplacian, via the De Giorgi-Nash-Moser theorem. See the discussion below on Liouville theorem for more details. Moreover, if f is a weak solution to the $n-$Laplacian and u is conformal then $f \circ u$ is also $n-$harmonic. This is the so-called *morphism* property.
- For $n = 2$; *Conformal transformation are holomorphic diffeomorphism and viceversa*. In particular the space of conformal planar deformations is infinite dimensional.
- *Riemann Mapping Theorem* Any non-empty, simply connected open planar set can be mapped conformally to the disc (uniquely if one prescribes target for *three* points).

Rigidity of conformal deformations Despite the flexibility of the Riemann mapping theorem and the usefulness in changes of variables arguments, conformal mappings exhibit aspects of rigidity that make it too restrictive for many applications.

- *Liouville Theorem* For $n \geq 3$ conformal deformations are compositions of translations, rotations, dilations and inversions. The theorem was proved originally by J. Liouville (J. Math. Pures Appl. 15 (1850), 103) with the hypothesis that the fourth order derivatives of the maps be continuous. Gehring [28] and Reshtnyak [56] established remarkable generalizations respectively to quasiconformal and to quasiregular mappings in $W^{1,n}_{lot}$. For $n = 2l$ a sharp form of the Liouville theorem was established by Iwaniec and Martin in [41]. In this paper, among other things, it is proved that for $l > 1$, every $u \in W^{1,l}_{loc}(\Omega, \mathbb{R}^{2l})$ with $\det du \geq 0$ (or $\det du \leq 0$) a.e. and such that $H(du) = 1$, i.e. $||du||_O \leq \min_{|v|=1} |duv|$ a.e. is either constant or the restriction of a Möbius transformation to Ω. The Sobolev exponent l is optimal in the sense that there are weak $W^{1,p}_{loc}$ solutions of the Cauchy-Riemann equations with $p < l$, which are not Möbius.
- *Rigidity with respect to boundary data.* Even in the plane, despite the Riemann mapping theorem one *cannot prescribe boundary data* (more than three points) when mapping conformally one domain into the other. For instance, in mapping one rectangular box into another, sending sides to sides, one can achieve this through a conformal deformation only if the boxes are similar (see the next section).

The intrinsic rigidity of conformal mappings provided a motivation for the extension to a larger family, that of *quasiconformal mappings*. Quoting F. Gehring [29],

> ... quasiconformal mappings constitute a closed class of mappings interpolating between homeomorphisms and diffeomorphisms for which many results of geometric topology hold regardless of dimension.

In the next section we will see how at the genesis of the theory of quasiconformal mappings lies a L^∞ variational problem.

1.4 Grötzsch problem and quasiconformal deformations

Let R and R' be two rectangles with sides a,b and a',b', that are not similar (i.w. $a/b \neq a'/b'$). It is then easy to see that there is no conformal deformation mapping R to R' sending edges to edges. In connection to this observation, in 1928 H. Grötzsch [32] posed the following question

Problem 3. Is there a *most nearly conformal mapping* between R and R'?

Quoting L. Ahlfors [1] in relation to this problem

> This calls for a measure of approximate conformality, and in supplying such a measure Grötzsch took the first step toward the creation of a theory of q.c. mappings.

To address Grötzsch's question one would need to identify a quantitative way of determining how non-conformal a mapping can be and then find an extremal point for this quantity in a suitable class of competitors. For such a general scheme to work it is of paramount importance to have good compactness properties for the class of competitors. Such considerations hint at the *need of introducing a more general class of deformations that are less rigid, yet retain some of the useful features of conformal mappings*. One also would like to have an instrument to quantify how far a given deformation is from being conformal.

Definition 5. Let $\Omega \subset \mathbb{R}^n$ be an open set. If $u \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ is an homeomorphism then we say it is *quasiconformal* if

$$\mathbb{K}(u,\Omega) = ||\sqrt{trace(g)}||_{L^\infty} = \left\| \frac{|du|}{(\det du)^{1/n}} \right\|_{L^\infty} < \infty.$$

We say u is $K-$quasiconformal if $K = \|H_O(du)\|_\infty = \||du|_O/\det du^{1/n}\|_\infty$.

Example 4. In the following we list some simple examples of quasiconformal mappings.

- Linear bijections $x \to Ax$ with $A \in \mathbb{R}^{n\times n}$.
- Diffeomorphisms with non-vanishing Jacobians are locally quasiconformal.
- For $a \neq 0$ consider the family of quasiconformal mappings $u(x) = |x|^{a-1}x$. For $a = -1$ this is the conformal inversion.

- In cylindrical coordinates (r,ϕ,z) set $D_\alpha = \{\phi \in (0,\alpha)\}$ and define $f : D_\alpha \to D_\beta$ as $f(r,\phi,z) = (r,\beta\phi/\alpha,z)$ (*folding map*).
- In spherical coordinates (R,ϕ,θ) define C_α a cone of angle α by $0 \le \theta < \alpha$. Set $f : C_\alpha \to C_\beta$ as $f(R,\phi,\theta) = (R,\phi,\beta\theta/\alpha)$. The map is quasiconformal for $\beta < \pi$ and but fails to be quasiconformal for $\beta = \pi$.

Definition 5 seems to require a-priori information on a.e. differentiability of the mapping which are counterintuitive in relation to the need for compactness of the class of competitors we referred to. There are in fact previous equivalent definitions for quasiconformality which do not require *any a priori differentiability*.

Definition 6. *Geometric definition* Let $r > 0$ and $x \in \Omega$. Set

$$L(x,r) = \sup_{y\in\Omega|\ |x-y|\le r} |u(x) - u(y)|$$

and

$$l(x,r) = \inf_{y\in\Omega|\ |x-y|\ge r} |u(x) - u(y)|$$

The homeomorphism u is quasiconformal if there exists $H \ge 1$ such that for *every* $x \in \Omega$ the *linear dilation* satisfies

$$H(x,u) := \limsup_{r\to 0} \frac{L(x,r)}{l(x,r)} \le H < \infty. \tag{1.9}$$

Remark 4. If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection then $H(x,A) = H(A)$ with $H(A)$ defined as in (1.2). If $u : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable at the point x with non-vanishing Jacobian determinant then $H(x,u) = H(du(x))$.

Remark 5. The differential $du(x)$ transforms circles centered at the origin into similar ellipses. The quantity $H(du)$ is the ratio of the axis of such ellipse. Thus quasiconformal deformations map infinitesimal circles into ellipses with a bounded ratio of the axis.

Homemorphism for which (1.9) holds with $\limsup$ substituted by $\sup$ are called *quasisymmetric*. It was F. Gehring [27] who first proved that quasiconformal implies quasisymmetric if $n \ge 2$. See also [34] for quantitative estimates and extensions to more general metric spaces.

Theorem 3. *Consider an homeomorphism $u : \Omega \to \Omega'$, then the quantity $||H(x,u)||_{L^\infty}$ is finite if and only if $u \in W^{1,n}_{lot}(\Omega,\mathbb{R}^n)$ and $\mathbb{K}(u,\Omega)$ is finite.*

Theorem 4. *(Gehring, 1962) For every $K \ge 1$ and $n \in \mathbb{N}, n \ge 2$ there exits $\theta_{n,K} : (0,1) \to \mathbb{R}$ increasing, with $\lim_{r\to 0} \theta_{n,K}(r) = 0$ and $\lim_{r\to 1} \theta_{n,K}(r) = \infty$ such that for every $f : \Omega \to \Omega'$ $K-$quasiconformal one has*

$$\frac{d(f(x),f(y))}{d(f(x),\partial\Omega')} \le \theta_{n,K}\left(\frac{d(x,y)}{d(x,\partial\Omega)}\right),$$

for all distinct $x, y \in \Omega$ sufficiently close. Moreover for r sufficiently small, one can choose $\theta_{n,K}(r) = c_n r^\alpha$ with $\alpha = K^{1/(1-n)}$.

In complex notation one denotes the map as

$$z = x + \mathbf{i}y \in \Omega \subset \mathbb{C} \to \zeta(z) = \xi + i\eta,$$

and set $p = \partial_z \zeta$ and $q = \partial_{\bar z}\zeta$, so that $d\zeta = pdz + qd\bar z$. The mapping $d\zeta$ is affine and satisfies

$$\Big||p| - |q|\Big||dz| \le |d\zeta| \le \Big||p| - |q|\Big||dz|$$

From the latter we see that the ratio of the axes of the ellipse obtained as image of a circle under $d\zeta$ is given by the maximal dilation

$$K = \sup_\Omega \frac{|p| + |q|}{|p| - |q|}.$$

We also define the maximal excentricity $\kappa = \frac{K-1}{K+1} = \sup \frac{|q|}{|p|}$. Note that ζ is conformal iff $K = 1$, $\kappa = 0$. The Jacobian determinant of the map ζ is $J = |p|^2 - |q|^2$.

Remark 6. Since the derivatives of the inverse map $\zeta \to z(\zeta)$ are given by

$$p' = J^{-1}\bar p \text{ and } q' = -J^{-1}q$$

then the mapping $\zeta = \zeta(z)$ and $z = z(\zeta)$ have the same dilation at corresponding points, hence the same maximal dilation. Moreover, the dilation is invariant by conformal deformation in both the z and the ζ planes.

1.4.1 Grötzsch problem

Let us return to Grötzsch original question. Consider two rectangles R, R' with sides parallel to the axis and with one vertex at the origin, as illustrated in figure 1.1.

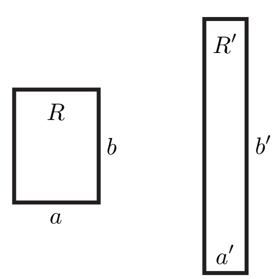

Fig. 1.1 Grötzsch problem

Remark 7. The affine transformation mapping $R \to R'$ that maps edges to edges is the *anisotropic dilation* $\xi = \frac{a'}{a}x$ *and* $\eta = \frac{b'}{b}y$ i.e.,

$$\zeta = \frac{1}{2}\left([\frac{a'}{a} + \frac{b'}{b}]z + [\frac{a'}{a} - \frac{b'}{b}]\bar{z} \right). \tag{1.10}$$

The dilation of the affine map is a constant

$$K = \begin{cases} \frac{a'b}{b'q} & \text{if} \quad \frac{a'}{b'} > \frac{a}{b} \\ \frac{ab'}{ba'} & \text{if} \quad \frac{a'}{b'} < \frac{a}{b} \end{cases} \tag{1.11}$$

Proposition 3. *Every diffeomorphism from R to R′, mapping edges to edges, has dilation larger or equal than the dilation of the affine transformation* (1.10)*. Moreover, if its dilation is the same as that in* (1.11) *then the diffeomorphism must coincide with* (1.10).

Proof. Let $\zeta : R \to R'$ be diffeomorphism from R to R', mapping edges to edges. Recall that for $\xi(0,y) = 0$ while $\xi(a,y) = a'$. A simple computation shows that the integral of the differential form $\xi(x,y)dy$ along the boundary of R yields $a'b = \int_{\partial R} \xi(x,y)dy$. On the other hand Stokes theorem yields

$$a'b = \int_{\partial R} \xi(x,y)dy = \int_R \partial_x \xi(x,y)dxdy = \int_R Re(p+q)dxdy$$

(since $(|p|+|q|)^2 \leq K(|p|^2 - |q|^2)$)

$$\leq \int_R |p|+|q|dxdy \leq \sqrt{|R|}\left(\int_R (|p|+|q|)^2 dxdy \right)^{1/2} \leq \sqrt{K|R||R|'}.$$

In conclusion $\frac{a'b}{ab'} \leq K$. Reverting the role of z and ζ and recalling that the two maps have the same dilation one has $\frac{ab'}{a'b} \leq K$. *In both cases the affine map has minimal dilation*.

To prove the second part of the proposition we notice that the only way one may have equality in the previous computation is if

$$Re(p+q) = |p|+|q| \text{ and } (|p|+|q|)/(|p|-|q|)$$

is constant. The former yields $Im(p) = Im(q) = 0$ and consequently $\partial_y \xi = \partial_x \eta = 0$. The latter yields that $\partial_x \xi(x) = K\partial_y \eta(y)$ which has as immediate consequence that $\partial_x \xi(x) = const$ and $\partial_y \eta(y) = const$. Hence any extremal map must have the form $\zeta(z) = \alpha x + \mathbf{i}\beta y$ and α, β must match the values for the affine map for ζ to map R into R', vertex by vertex.

1.4.2 Grötzsch problem revisited

Consider two Jordan regions $Q, Q' \subset \mathbb{C}$ with distinguished boundary points $p_1, ..., p_4 \in \partial Q$ and $p'_1, ..., p'_4 \in \partial Q'$.

Problem 4. Among all diffeomorphism $\zeta : Q \to Q'$ mapping $\zeta(p_i) = p'_i$, $i = 1, ..., 4$, find the one with minimal maximal dilation.

As illustrated in figure 1.2, the Riemann mapping theorem yields two rectangles R, R' and conformal transformations $\phi : Q \to R$ and $\psi : Q' \to R'$ of the domains Q, Q' to R, R' with the points p_i, p'_i mapped to the vertices of the rectangles. Since the dilation is conformally invariant any map $\zeta : R \to R'$ has the same dilation of its lift $\phi \circ \zeta \circ \psi^{-1} : Q \to Q'$. The previous argument yields the following conclusion: *The extremal map for the revisited Grötzsch problem is a composition of an affine anisotropic dilation with conformal transformations $\phi \circ affine \circ \psi^{-1}$. Such map is unique modulo conjugation with conformal transformations.*

Such transformations are examples of *Teichmüller mappings.*

1.5 Teichmüller theorem and extremal quasiconformal mappings

Let $u : \Omega \to \mathbb{R}^n$ be a quasiconformal mapping. Since $\mathbb{K}(u, \Omega) = \sqrt{n}$ if and only if the mapping u is conformal, we will interpret Grötzsch's *closest to conformal* requirement as *minimizing* $\mathbb{K}(u, \Omega) = ||\sqrt{trace(g)}||_{L^\infty}$ *among all competitors*. The $n = 2$ setting has been studied in depth by many mathematicians. Here we recall the work of Reich [53], Reich and Strebel [54], [55], Strebel [64] and Gardiner [26].

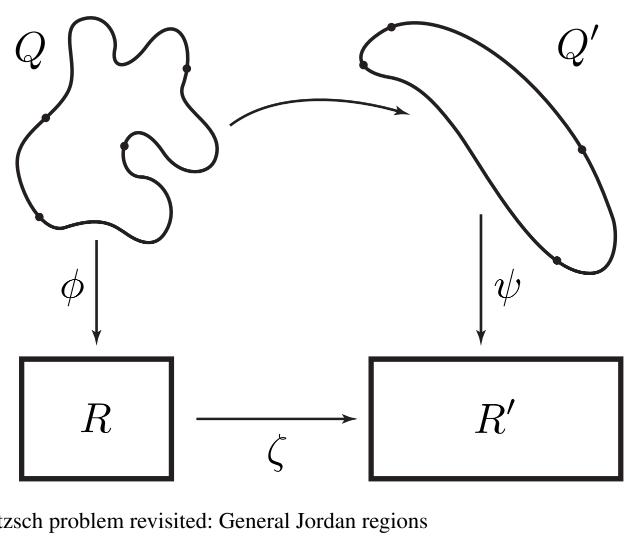

Fig. 1.2 Grötzsch problem revisited: General Jordan regions

In the following we will briefly highlight the set of ideas, techniques and results loosely known as *Teichmüller theory*. This important theory has as a starting point a similar L^∞ variational problem in the context of holomorphisms between Riemann surfaces of same genus ($g > 3$) and where the constraint defining the class of competitors is given by membership in the same homotopy class. In this setting one has existence, uniqueness and some amount of regularity for the minimizers.

For $n > 2$ less is known. The fundamental reference by Gehring and Vaisala [30] establishes the problem in a more general setting and provides some existence results. The higher dimensional analogue of Grörtzsch problem was solved by Fehlmann [23]. A great amount of recent literature focuses on the L^p variational problems, which we will briefly describe through the work of Astala, Iwaniec, Onninen, Martin [9]. We also recall related work of Balogh-Fässler-Platis [10] and Astala, Iwaniec and Martin [8].

In a (rough) comparison with similar problems in elasticity, *conformal deformations correspond to isometries.* Accordingly, the variational problems stated above corresponds to finding deformations *closest to isometries* in given classes of competitors.

Some of the main obstacles in studying this L^∞ problem are

- Lack of convexity.
- L^∞ functionals are not sensitive to deformations of functions away from their maximum. Unlike L^p averages they are not "local". *This makes uniqueness unlikely*.
- The problem is vector-valued, and as such not approachable through the established techniques from game theory or viscosity solutions.
- There is a topological constraint.

1.5.1 Riemann surfaces

A *Conformal Atlas* on a two dimensional smooth manifold is an atlas $(U_\alpha, z_\alpha)_{\alpha\in A}$ with $z_\alpha : U_\alpha \to \mathbb{C}$ local charts such that the transition maps $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ are holomorphic.

An atlas (U_α, z_α) is compatible to another atlas (V_β, w_β) if their union is still a conformal atlas. A *conformal structure* is the union of an atlas with all other compatible charts.

Definition 7. A *Riemann surface* is a two dimensional smooth manifold with a conformal structure.

Example 5. The *Riemann sphere* $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$. To show that S^2 is a Riemann surface we consider an atlas with open sets U_1, U_2 obtained from the whole sphere minus respectively the north pole $(0,0,1)$ and the south pole $(0,0,-1)$. Define the *stereographic projections* charts

$$z_1(x) = \frac{x_1 + ix_2}{1 - x_3} \text{ on } U_1 \text{ and } z_2(x) = \frac{x_1 - ix_2}{1 + x_3} \text{ on } U_2$$

Note that $z_1(U_1 \cap U_2) = z_2(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$ and that

$$z_2 \circ z_1^{-1}(a+ib) = \frac{a+ib}{a^2+b^2} = \frac{1}{a+ib}$$

Example 6. The *Riemann Torus* is defined as follows: Set $w_1, w_2 \in \mathbb{C}$ two non-zero vectors and define an equivalence relation in $\mathbb{C}$ by saying that $a+ib \approx x+iy$ if there exists two rational numbers m,n such that $a+ib = x+iy+mw_1+nw_2$. The discrete abelian subgroup $M = \{mw_1+nw_2\}$ is called a *lattice*. If we let π be the projection to the quotient space then $\mathbb{T} = \pi(\mathbb{C})$ is a Riemann surface. To define an atlas we consider open sets $O \subset \mathbb{C}$ containing to equivalent pairs (for instance a subset of a fundamental domain) and define the chart $U = \pi(O)$ and $z = \pi|_O^{-1}$. Since $z_\alpha \circ z_\beta^{-1}$ is a translation then this is a conformal atlas.

We have already seen that if $\Gamma \subset PSL(2,\mathbb{R})$ and the action of Γ on H is properly discontinuous and does not fix points then the quotient H/Γ can be given a Riemann surface structure. Viceversa one has the following

Theorem 5. *(Uniformization theorem) Let Σ be a compact Riemann surface of genus p. There exists a conformal diffeomorphism $f : \Sigma \to \Sigma'$ where Σ' is either*
(i) of the form H/Γ if $p \geq 2$;
(ii) A torus $\mathbb{C}/M$ if $p = 1$;
(iii) the Riemann sphere if $p = 0$.

As corollary, the universal cover of a compact Riemann surface is conformally equivalent to S^2, $\mathbb{C}$ or D.

Definition 8. A continuous map $u : S \to S'$ is holomorphic if it is so when expressed (locally) through conformal charts. If these local expression have non vanishing $\partial_z u$ derivative then u is conformal.

Let us recall the topological classification of compact Riemann surfaces

Theorem 6. *Two differentiable, orientable, compact triangulated surfaces*[2] *are homeomorphic if and only if they have the same genus.*

Every Riemannian metric $g_{ij} dx_i dx_j$ on a oriented surface can be written locally in complex coordinates as

$$\sigma(z)|dz + \mu(z) d\bar{z}|^2 = \sigma(z)(dz + \mu d\bar{z})(\bar{z} + \bar{\mu} dz)$$

where $\sigma > 0$ (real) and $|\mu| < 1$.

Theorem 7. *Every oriented Riemannian surface admits a conformal structure and a conformal Riemann metric $\lambda dz d\bar{z}$. A local system of holomorphic coordinates is given by the solutions of the equation $\partial_{\bar{z}} u = \mu \partial_z u$ and $\lambda = \frac{\partial \sigma}{\partial_z u \partial_{\bar{z}} \bar{u}}$.*

[2] Recall that every Riemann surface is orientable and any conformal atlas yields a triangulation.

1.5.2 Teichmüller Theorem

The focus of Teichmüller theorem is on a classification of all possible conformal structures of a given Riemann surface S, and on establishing a structure theorem for such a moduli space. The natural candidate for *space of all conformal structures* is

Definition 9. Given a compact Riemann surface S with genus p, we define the *moduli space* M_p of conformal structures on S where (S,g_1) and (S,g_2) are identified if there exists a conformal diffeomorphism between them.

However this moduli space does not have a manifold structure and its topology is very complicated. To somewhat simplify the structure Teichmüller proposed a new notion of moduli space of conformal structures, known today as Teichmüller space.

Definition 10. Given a compact Riemann surface S with genus p, we define the *moduli space* T_p of conformal structures on S where (S,g_1) and (S,g_2) are identified if there exists a conformal diffeomorphism between them wich is homotopic to the identity.

The first step in studying the structure of this space is given by the following existence theorem

Theorem 8. *(Existence Theorem) Given S,S' closed Riemann surfaces of same genus and $\alpha : S \to S'$ an homeomorphism, there exists a quasiconformal mapping $\zeta : S \to S'$ homotopic to α and minimizing the maximal dilation in the homotopy class of α.*

What is needed next is a uniqueness result for such minimizers as well as an algebraic characterization that would allow to define a manifold structure on the moduli space. In the next sections we will will state such uniqueness results and then proceed to sketch Ahlfors' proof of this remarkable characterization.

1.5.3 Coverings and group action

If S,S' are Riemann surfaces of same genus $g > 1$ realized as $D/G, D'/G'$. The quotient map $p : D \to D/G = S$ is a covering map and the group G, which acts on D is called a Fuchsian group. The disc D is the universal cover of S.

Theorem 9. *Any homeomorphism map $\zeta : S \to S'$ can be lifted to a family of mappings $\zeta : D \to D'$ with the property that for every $g \in G$ there exists a unique $g' := \alpha(g) \in G'$ such that*

$$\zeta(g(z)) = g'(\zeta(z)).$$

Viceversa, any homeomorphism $\zeta : D \to D'$ which satisfies the identity above induces an homeomorphism $\zeta : S \to S'$. Lifts of quasiconformal mappings are quasiconformal with the same dilation. The maps $g \to \alpha(g)$ are group isomorphisms. Any two lifts are related by a inner automorphism of G or G'.

Theorem 10. *Any two homeomorphism maps $S \to S'$ are homotopic if and only if the determine isomorphisms $G \to G'$ which differ only by an inner automorphism. So essentially, modulo renormalization, there exists a one-to-one correspondence between homotopy classes of homeomorphisms and isomorphisms $G \to G'$.*

This result allows to reframe Teichmüller theorem and the variational problem only in terms of mappings $\zeta : D \to D'$ which satisfy the functional equation $\zeta(g(z)) = g'(\zeta(z))$.

Theorem 11. *(Existence Theorem reformulated) Given $\zeta_0 : D \to D'$ a fixed homeomorphism, let $\alpha : G \to G'$ denote the induced isomorphism. There exists a quasiconformal mapping $\zeta : D \to D'$ minimizing the maximal dilation in the class of all homeomorphisms satisfying the function equation*

$$\zeta(g(z)) = \alpha(g)(\zeta(z)).$$

Consider the set of all quasiconformal mappings satisfying the identity (this is not empty since the surfaces are diffeomorphic) and with dilation less than a fixed number K. Gehring's theorem implies that this is a normal family, and hence any minimizing sequence will converge to either a quasiconformal mapping satisfying the same functional identity or a constant. Constant are ruled out by the functional identity and the fact that no element of D' is fixed by every $g' \in G'$. Uniqueness of the representative is provided by a deep connection between extremal quasiconformal mappings and *quadratic differentials*.

1.5.4 Quadratic differentials

Consider a $1-$form $dz = dx + \mathbf{i}dy$ in $\mathbb{C}$. If $F : \mathbb{C} \to \mathbb{C}$ is a conformal map and we denote $z(w) = F(w)$ then for any $(a,b) \in \mathbb{R}^2$ we can compute the action of $dF : \mathbb{R}^2 \to \mathbb{R}^2$ in complex notation as $dF \cdot (a,b) = \frac{dF}{dw}(a+ib)$. In view of this then the pull-back F^*dz is given by $F'(w)dw = \frac{dz}{dw}dw$, in fact the action of F^*dz on any complex tangent vector $a+ib$ can be computed through

$$F^*dz(a+ib) = dz(dF(a+ib)) = dz(F'(w)(a+ib)) = F'(w)dw(a+ib)$$

Through a similar computation one sees that the symmetric 2-tensor $\phi(z)dz^2$ pulls back to $\phi(z(w))(\frac{dz}{dw})^2dw^2$.

These computations motivate the following

Definition 11. Let S be a Riemann surface and $\{(U_\alpha, h_\alpha)\}$ denote its conformal structure. A meromorphic (resp. holomorphic) *quadratic differential* h in S is a set of meromorphic (res. holomorphic) functions f_α in the local coordinates given by $z_\alpha = h_\alpha(p)$ with $p \in S$ satisfying the transformation law

$$f_\alpha(z_\alpha) = f_\beta(z_\beta)\left(\frac{\partial z_\beta}{\partial z_\alpha}\right)^2,$$

for all charts (U_α, z_α) and (U_β, z_β) around a point $p \in S$.

Recalling the formula for the pull back of a complex $2-$tensor described earlier we can write the definition above as

$$h_\alpha(z_\alpha)dz_\alpha^2 = h_\beta(z_\beta)dz_\beta^2$$

Observe that quadratic differentials are holomorphic sections of the bundle of holomorphic symmetric tensors.

Let $Q(S)$ denote the space of all quadratic differentials on a given compact Riemann surface S. Since the sum of two quadratic differentials as well as the multiplication by a scalar of a quadratic differential are still elements of $Q(S)$ then the latter is a complex vector space. The following is a consequence of the Riemann-Roch Theorem

Theorem 12. *The space $Q(S)$ has finite dimension (over $\mathbb{R}$) $6p-6$.*

Theorem 13. *(Structure of minimizers) Given S, S' closed Riemann surfaces of same genus $p > 1$ and $\alpha : S \to S'$ an homeomorphism. Denote by $\zeta : S \to S'$ a quasiconformal mapping homotopic to α and minimizing the maximal dilation in the homotopy class of α. Either ζ is analytic or there exists a quadratic differential $f dz^2$ on S and a constant $\kappa \in (0,1)$, such that ζ is differentiable away from the zero set of f (with non-vanishing complex derivatives q, p) and satisfies*

$$\frac{q}{p} = \kappa \frac{\bar{f}}{|f|}.$$

The quadratic differential is uniquely represented up to a positive constant factor and κ represents the (constant) eccentricity of the extremal mapping.

Theorem 14. *(Uniqueness) Every map ζ whose complex derivatives satisfy*

$$\frac{q}{p} = \kappa \frac{\bar{f}}{|f|}$$

has a maximal dilation which is strictly smaller than the dilation of any other mapping (not conformally equivalent to ζ).

These result yield that for every homotopy class one has existence of a unique minimizer for the maximal dilation and associated to this minimizer there is a unique pair of quadratic differentials. This correspondence gives a manifold structure to the Teichmüller space, with the same dimension $6p-6$ as the space of quadratic differentials.

1.5.5 Ahlfors' proof of existence and uniqueness

In [1], Ahlfors considers the following $m-$mean distortion functional: For every a. e. differentiable map $\zeta = \xi + \mathbf{i}\eta : D \to D'$ and $m \geq 1$,

$$I_m(\zeta) = \frac{1}{\pi}\int\int_{D'} \left(\frac{|p|^2+|q|^2}{|p|^2-|q|^2}\right)^m \Bigg|_{\zeta^{-1}(\xi+\mathbf{i}\eta)} d\xi d\eta.$$

The customary $1-$parameter deformations used in the calculus of variations to derive the Euler-Lagrange equations are of the form $\zeta_s = \zeta + s\psi$ where $s \in (-\varepsilon, \varepsilon)$ and $\phi \in C_0^\infty(D, D')$ serves as a test function. However if ζ is merely quasiconformal, in particular not a C^1 diffeomorphism then the deformation $\zeta_s = \zeta + s\psi$ may fail to be an homeomorphism and hence be outside of the set of competitors, making it useless for the purpose of deriving a PDE which describes the behavior of minimizers.

To circumvent this problem one may choose to do a different set of perturbations, acting on the domain of the map, rather than one the image, thus setting:

$$z = H(z', \varepsilon) := s' + \varepsilon h(z') + o(\varepsilon),$$

yielding

$$\partial_z H = 1 + \varepsilon\partial_z h + o(\varepsilon); \text{ and } \partial_{\bar z} H = \varepsilon\partial_{\bar z} h + o(\varepsilon).$$

If G is a Fuchsian group acting on a Riemann surface S then for H to determine a deformation of the surface one needs $H(gz, \varepsilon) = gH(z, \varepsilon)$ for every $g \in G$. This eventually yields $h(gz) = \partial_z g\ h(z)$, which characterizes all *infinitesimal deformations of S*.

Remark 8. A brief digression: If one considers the Dirichlet energy

$$\int\int_D |\partial_z w|^2 + |\partial_{\bar z} w|^2 dxdy$$

then the usual *exterior* deformations lead to the Laplacian $\partial_z\partial_{\bar z} w = 0$.

If instead one proceeds as in Ahlfors (and Hopf, Morrey, and many others) and carries out *inner variations* as described earlier then one obtains the PDE

$$\partial_{\bar z}\left(\partial_z\ w\ \overline{\partial_{\bar z}\ w}\right) = 0,$$

which is of a very different nature from Laplace's equation. To the best of our knowledge, currently the sharpest regularity result known for such PDE is Lipschitz continuity, see Iwaniec, Kovalev and Onninen [40]. See also earlier work of Bauman, Owen and Phillips [13].

In Ahlfors' argument the inner variation produces the equation in weak form

$$\mathrm{Re}\int\int_{D'}\left(\frac{|p|^2+|q|^2}{|p|^2-|q|^2}\right)^m\frac{p\bar{q}}{|p|^2+|q|^2}\partial_{\bar{z}}hd\xi d\eta.$$

Changing variables $z'=\zeta(z)$ we obtain

$$\mathrm{Re}\int\int_{D\cap\{|p|>|q|\}}\left(\frac{|p|^2+|q|^2}{|p|^2-|q|^2}\right)^{m-1}p\bar{q}\partial_{\bar{z}}hdz\wedge d\bar{z}.$$

Set

$$U_m=\begin{cases}\left(\frac{|p|^2+|q|^2}{|p|^2-|q|^2}\right)^{m-1} & \text{if } |p|>|q|\\ 0 & \text{otherwise.}\end{cases}$$

and $\rho=\sum_{g'\in G'}|\partial_z g'|$ one obtains a reformulation of this PDE in terms of integration over the original surface

$$\int\int_S U_m\rho\, p\bar{q}\partial_{\bar{z}}h=0.$$

$$\int\int_S U_m\rho\, p\bar{q}\partial_{\bar{z}}h=0.$$

in particular this yields

Lemma 1. *The function $f_m=U_m\rho\, p\bar{q}$ is holomorphic and so describes a holomorphic quadratic differential*

$$f_m(z)dz^2$$

in D.

We let $C_m>0$ be constants defined so that

$$U_m\rho\, p\bar{q}=c_m f_m(z)$$

with $\int_S|f_m|dxdy=1$

1.5.6 Normal family of mappings with integrable distortion

Theorem 15. *If $f\in W^{1,n}(\Omega,\mathbb{R}^N)$ is a mapping whose distortion is $m-$integrable with $m>n-1$, then f is continuous and the modulus of continuity depends only on the L^m norm of the distortion.*

In this form and in this setting, this result is due to Ahlfors [1]. It is also a consequence of work of Iwaniec and Sverak [43] and of Manfredi and Villamor [67]. See also the work of Koskela, Iwaniec, and Onninen [38], [39].

Given any diffeomorphism $\alpha:D\to D'$, then in view of Ascoli-Arzela and Theorem 15 one has that for every m there exists (a possibly not unique) $\zeta_m:D\to D'$

in the same homotopy class as α and which minimizes I_m. We denote by p_m, q_m its complex differentials and set

$$\min_{\zeta} I_m(\zeta) = I_m(\zeta_m) = \int\int_{D'} \left(\frac{|p_m|^2 + |q_m|^2}{|p_m|^2 - |q_m|^2} \right)^m .$$

Denote the quantity above by πK_m^m.

In view of Hölder inequality K_m is monotone increasing and bounded (by the dilation of α) hence $K_m \to K < \infty$. Let $0 \le \kappa < 1$ be defined by

$$K = \frac{1+\kappa^2}{1-\kappa^2}.$$

Normality and a diagonalization argument yields:

Lemma 2. *For a subsequence one has $\zeta_m \to \zeta$, as $m \to \infty$, uniformly on compact sets, with ζ quasiconformal.*

Lemma 3. *For a subsequence one has*

$$(C_m)^{\frac{1}{m}} \to K$$

as $m \to \infty$.

Lemma 4. *For a subsequence one has*

$$\int\int_D \left| |q_m| - \kappa |p_m| \right| dz \wedge d\bar{z} \to 0,$$

as $m \to \infty$.

Note that the relation

$$C_m f_m = \left(\frac{|p|^2 + |q|^2}{|p|^2 - |q|^2} \right)^{m-1} \rho(\zeta_m) p_m \bar{q}_m$$

yields

$$\frac{f_m}{|f_m|} = \frac{p_m \bar{q}_m}{|p_m||q_m|}$$

and consequently

$$\left| \frac{f_m}{|f_m|} q_m - \kappa p_m \right| = \left| \frac{p_m}{|p_m|} |q_m| - \kappa p_m \right| = ||q_m| - \kappa |p_m||$$

Passing to a subsequence then we can assume that f_m tend to a limit f and that $\zeta_m \to \zeta$ uniformly on compact sets. The limit mapping ζ has complex derivatives p, q which are limit of p_m, q_m and thus satisfy

$$\frac{f}{|f|}q = \kappa p$$

1.5.7 Teichmüller mappings in local parameters

A homeomorphism

$$z \to \zeta(z) : D \to D$$

whose complex derivatives satisfy

$$\kappa \frac{p}{q} = \frac{f}{|f|}$$

is called a *Teichmüller mapping*.

We show that there exists local parameters (i.e. a local set of conformal coordinates) ζ^*, z^* such that in this coordinates the map reads as the composition of two conformal transformations conjugating an affine mapping (just as in Grötzsch's problem). Denote by $\zeta \to z$ the inverse mapping and by p', q' its complex derivatives.

Differentiating the formula $z = z(\zeta(z))$ along z and $\bar{z}$ one can see that p, q, p', q' are related by the formula

$$p' = \frac{\bar{p}}{|p|^2 - |q|^2} \text{ and } q' = -\frac{q}{|p|^2 - |q|^2}.$$

If $\zeta(z)$ is quasiconformal extremal then so is $z(\zeta)$ and its associated quadratic differential $\phi(\zeta)$ satisfies:

$$\frac{\phi}{|\phi|} = \kappa \frac{p'}{q'}.$$

Consequently it follows that if $z \to \zeta$ is Teichmüller then

$$\frac{\bar{q}}{p} = \kappa \frac{\phi}{|\phi|}$$

Next we introduce two new local charts

$$z^* := \int \sqrt{f} dz \text{ and } \zeta^* := \int \sqrt{\zeta} d\zeta$$

This can be done in a sufficiently small neighborhood of a point where f, ϕ do not vanish and with fixed branches of the square roots and arbitrary integration constants.

Keeping in mind the expression $\zeta^*(\zeta(z(z^*)))$ then in terms of these new variables one has

$$p^* = \frac{d\zeta^*}{dz^*} = \frac{d\zeta^*}{d\zeta}\frac{d\zeta}{dz}\frac{dz}{dz^*} = \frac{\sqrt{\phi}}{\sqrt{f}}p$$

text and similarly

$$q^* = \frac{\sqrt{\phi}}{\sqrt{\bar{f}}}q$$

Next, observe that

$$q^* = \frac{\sqrt{\phi}}{\sqrt{\bar{f}}}q = \kappa p\frac{\bar{f}}{|f|}\frac{\sqrt{\phi}}{\sqrt{\bar{f}}} = \kappa p^*\frac{\bar{f}}{|f|}\frac{\sqrt{f}}{\sqrt{\bar{f}}} = \kappa p^*.$$

Similarly, if we use

$$\frac{\bar{q}}{p} = \kappa\frac{\phi}{|\phi|}$$

then we obtain

$$\bar{q}^* = \kappa p^*.$$

Since $q^* = \bar{q}^*$ then q^* *is real and so is* p^*.

Next, observe that

$$\frac{d}{d\bar{z}^*}(\zeta^* - \kappa\bar{\zeta}^*) = q^* - \kappa p^* = 0$$

Hence $\zeta^* - \kappa\bar{\zeta}^*$ is holomorphic and its complex derivative is

$$\frac{d}{dz^*}(\zeta^* - \kappa\bar{\zeta}^*) = p^* - \kappa q^* = p^* - \kappa\kappa p^* = p^*(1-\kappa^2).$$

Since derivatives of a holomorphic functions are also holomorphic then p^* is both real and holomorphic, hence it must be constant.

Consequently one has that

$$\zeta^*(z^*) = p^*z^* + q^*\bar{z}^* = A(z^* + \kappa\bar{z}^*) + B$$

for some constants $A, B \in \mathbb{C}$, proving our statement on the local structure of Teichmüller mappings.

1.5.8 Uniqueness (rough idea)

The uniqueness part follows from a Grötzsch-like argument (more complicated in view of possible singularities). The analogues of the *rectangular regions* arise in the following way: We consider a Riemannian metric

$$ds^2 = \phi^2 d\zeta d\bar{\zeta}$$

where ϕ is the quadratic differential in the target region associated to a Teichmüller mapping. This metric is complete and non-positively curved, thus yielding unique geodesics between any pair of points. By the geodesic equation the quantity $\sqrt{\phi}d\zeta$ is constant along geodesics.

We call *horizontal arcs* those geodesics along which the argument of $\sqrt{\phi}d\zeta$ is zero. Likewise we call *vertical* those for which the argument is π.

The local charts z^*, ζ^* we have introduced earlier transforms rectangular boxes in D defined by horizontal and vertical arcs into actual rectangles in the complex plane, while at the same time the Teichmüller mapping is affine, transforming one rectangle into the other, when read in those coordinates.

An argument similar to the one we have used for Grötzsch problem yields the uniqueness and the extremality of the Teichmüller mapping.

1.6 A variation on the theme: Extremal mappings of finite distortion

The integral version of the extremal mapping problem, as well as the notion of map with integrable power of the distortion have appeared in the work of Ahlfors in 1954 [1] and later in several papers from the russian school, in particular Semenov [59], [60], [61] and references therein. As we have seen, in Ahlfors's approach to the extremal mapping problems he used a relaxation of the L^∞ variational problem, where the interest is shifted to minimizers of the L^p norm of the dilation, rather than to the L^∞ norm.

Problem 5. L^p variational problem Let $u_0 : \Omega \to \Omega'$ be a homeomorphism of finite distortion. Among all homeomorphisms $u : \Omega \to \Omega'$ whose extension to $\partial\Omega$ coincide with u_0 find one minimizing

$$\int_\Omega \psi\left(\frac{|du|}{(det(du))^{1/n}}\right)dx,$$

where $\psi : [1,\infty) \to [1,\infty)$ strictly increasing convex function with $\psi(1) = 1$.

This problem, along with generalizations to more general boundary data, has recently been studied in a sequence of papers by Astala, Iwaniecz, Martin, Onninen and several collaborators, see [9] and [8]. In the following we give a quick survey of their work.

Remark 9. From a Calculus of Variations point of view, one can see that following [42, Section 8.8.2] the functional

$$\mathscr{F}(du,\Omega) = \int_\Omega \psi\left(\frac{|du|}{(det(du))^{1/n}}\right)dx,$$

although not convex, is indeed *quasiconvex*, i.e. for every constant differential $A \in \mathbb{R}^{n\times n}$ and for any $\phi \in C_0^\infty(\Omega)$ one has

$$F(A,\Omega) \le F(A+d\phi,\Omega).$$

This notion was introduced by Morrey in 1952, see [49], [50]. Quasiconvex energy densities are those for which affine deformations are minimizers with respect to their own boundary conditions. We recall that quasiconvexity plus some growth estimates are roughly equivalent to lower semicontinuity, see Giaquinta's book [31] for a more detailed statement. Hence quasi-convexity is used often to prove existence of minimizers (as well as regularity of the extremals).

In the case at hand there are two problems:

- The growth conditions are not satisfied.
- There is a topological constraint: The space of competitors is not a vector space.

In conclusion, the results currently available from Calculus of Variations are not sufficient to attack the problem and new techniques are needed.

1.6.1 The finite distortion version of Grötzsch problem

Let $R = [0,1]\times[0,1]$ and $R' = [0,2]\times[0,1]$. The same argument holds for any pair of rectangles. Consider the set $\mathscr{F}$={ all homeomorphisms $u : R \to R'$ such that $u \in W^{1,1}_{loc}(R,\mathbb{R}^2)$ taking vertices into vertices}.

Theorem 16. *[9] There is a unique minimizer for the L^1 variational problem:*

$$\min_{u\in\mathscr{F}} \int_R \left| \frac{|du|^2}{det(du)} \right|$$

Remark 10. The affine map $(x,y) \to (2x,y)$ sends R to R' by mapping vertices to vertices and has distortion

$$\frac{|du|^2}{\det du} = \frac{5}{2}.$$

If we were to measure distortion using the operator norm we would have obtained

$$|du|_O = 2$$

and hence

$$\frac{|du|_O^2}{\det du} = 2.$$

Proof. The proof is very similar to the proof of Grötzsch problem we presented earlier: Setting $u = \alpha + \mathbf{i}\beta$ and arguing as we did then yields

$$\int_{\partial R} \alpha dy = 2 \text{ and } \int_{\partial R} \beta dx = 1$$

Using Stokes Theorem yields

$$\int_R \alpha_x dxdy = 2 \text{ and } \int_R \beta_y dxdy = 1$$

and

$$\int_R (2\alpha_x + \beta_y) dxdy = 5.$$

$$\begin{aligned} 5 = \int_R (2\alpha_x + \beta_y) dxdy &\leq \int_R \sqrt{2^2+1} \int_R \sqrt{\alpha_x^2 + \beta_y^2} dxdy \\ &\leq \sqrt{5} \int_R ||du|| dxdy = \sqrt{5} \int_R K(u,z) \sqrt{det(du)} dxdy \\ &\leq \sqrt{5} \sqrt{\int_R K^2(u,z) dxdy} \sqrt{\int_R det(du) dxdy} = \sqrt{10} \sqrt{\int_R K^2(u,z) dxdy}. \end{aligned}$$

This shows that the minimum of the functional is $5/2$, which is achieved by the linear map $(x,y) \to (2x,y)$. An examination of the case when the inequalities above are equalities yields that the minimum can only be achieved by this linear map.

1.6.2 Trace norm vs. operator norm

In [9], Astala, Iwaniecz, Martin and Onninen show that if the above problem one substitutes the operator norm $|A|_O = \max_{|v|=1} |Av|$ to the Hilbert-Schmidt norm, i.e. one studies minimizers of

$$\int_R \left| \frac{|du|_O^2}{det(du)} \right|$$

then the situation changes completely and one can find infinitely many minimizers.

To see this first one uses the argument above to show that

$$2 \leq \int_R \left| \frac{|du|_O^2}{det(du)} \right|.$$

Next we observe that there is a 1-parameter family of minimizers for $a \in [0,1)$,

$$U(x,y) = \begin{cases} x + \mathbf{i}y & \text{for } x + \mathbf{i}y \in [0,a] \times [0,1] \\ \frac{2-a}{1-a}x - \frac{a}{1-a} + \mathbf{i}y & \text{for } x + \mathbf{i}y \in [a,1] \times [0,1]. \end{cases}$$

1.6.3 Affine boundary data

Following Astala, Iwaniecz, Martin and Onninen [9] we consider more general affine boundary data in higher dimension. Let us start with the case of affine orientation preserving data $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ prescribed on a domain Ω with $(n-1)$ rectifiable boundary.

Theorem 17. *Given any homeomorphism of finite distortion $u : \bar{\Omega} \to \bar{\Omega}'$ such that $u = u_0$ on $\partial\Omega$ then*

$$\int_\Omega \psi\left(\frac{|du_0|^n}{det\ du_0}\right)dx \le \int_\Omega \psi\left(\frac{|du|^n}{det\ du}\right)dx$$

with equality if and only if $u = u_0$ in Ω.

Sketch of the proof. We first recall two basic estimates

1. the sub-gradient inequality

$$\psi(t) - \psi(t_0) \ge \psi'(t_0)(t-t_0)$$

   valid for a.e. $t, t_0 \in [1,\infty)$.
2. The function

$$(x,y) \to x^\alpha / y^\beta$$

   defined for $x, y \in \mathbb{R}$ and $\alpha \ge \beta + 1 \ge 1$ is convex. In particular

$$\frac{x^\alpha}{y^\beta} - \frac{a^\alpha}{b^\beta} \ge \alpha \frac{a^{\alpha-1}}{b^\beta}(x-a) - \beta \frac{a^\alpha}{b^{\beta+1}}(y-\beta)$$

Using these estimates one can easily prove that

$$\psi\left(\frac{|du|^n}{det\ du}\right) - \psi\left(\frac{|du_0|^n}{det\ du_0}\right) \ge \psi'\left(\frac{|du_0|^n}{det\ du_0}\right)\frac{|du_0|^{n-2}}{det\ du_0}\langle du_0, du - du_0\rangle$$

$$+\psi'\left(\frac{|du_0|^n}{det\ du_0}\right)\frac{|du_0|^n}{(det\ du_0)^2}(det\ du_0 - det\ du)$$

Integrating the latter over Ω yields

$$\int_\Omega \left[\psi\left(\frac{|du|^n}{det\ du}\right) - \psi\left(\frac{|du_0|^n}{det\ du_0}\right)\right]dx$$

$$\ge \int_\Omega \left[\psi'\left(\frac{|du_0|^n}{det\ du_0}\right)\frac{|du_0|^{n-2}}{det\ du_0}\langle du_0, du - du_0\rangle\right.$$

$$+\psi'\left(\frac{|du_0|^n}{det\ du_0}\right)\frac{|du_0|^n}{(det\ du_0)^2}(det\ du_0 - det\ du)\Bigg]dx$$

Observe that since $du_0 = const$ and $u = u_0$ on $\partial\Omega$ then the first term on the LHS vanishes. As for the second term, note that $\int_\Omega det\ du = \int_\Omega \det\ du_0 = |\Omega'|$. Thus the LHS has non-negative integral proving the first assertion. Uniqueness follow from a careful analysis of the consequences of having an identity in the above argument.

1.6.4 More general boundary data

The case of more general boundary data is still open. In [9], Astala, Iwaniecz, Martin and Onninen prove the following remarkable theorem:

Theorem 18. *Let $\Omega \subset \mathbb{R}^2$ be a convex domain and set*

$$\mathscr{C} = \{u \in W^{1,2}(\Omega,\mathbb{R}^2) \text{ homeomorphism of finite distortion for which } \int_\Omega \frac{|du|^2}{\det du} \text{ is finite}\}. \quad (1.12)$$

Let $u_0 \in \mathscr{C}$.

There exists a unique smooth diffeomorphism solution to the minimization problem

$$\min_{u\in\mathscr{C},u=u_0 \text{ in } \partial\Omega} \int_\Omega \frac{|du|^2}{\det du}dx$$

The key idea in the proof is to put in relation the extremal problem above with the classical Dirichlet problem

Problem 6. ($n-$*harmonic mappings*) Given $h_0 \in W^{1,n}(\Omega',\mathbb{R}^n)$, minimize the $n-$energy

$$\int_{\Omega'} |dh|_O^n dy$$

over the class $h \in h_0 + W_0^{1,n}(\Omega',\mathbb{R}^n)$.

The link between the two problems rests on the following theorem in [9].

Theorem 19. *Let $u \in W^{1,n}_{lot}(\Omega,\Omega')$ be a homeomorphism of finite distortion with*

$$\int_\Omega \frac{|du^{-1}|_O^n}{\det\ du^{-1}}(x)dx < \infty$$

The inverse map $h : \Omega' \to \Omega$ belongs to $W^{1,n}(\Omega',\Omega)$ and moreover satisfies

$$\int_{\Omega'} |dh(y)|_O^n dy = \int_\Omega \frac{|du^{-1}|_O^n}{\det\ du^{-1}}(x)dx.$$

See also recent developments by Hencl, Koskela, and Onninen [35], [37], [36] and by Fusco, Moscariello, and Sbordone [25] and by Csörnyei, Maly [19].

The proof of Theorem 19 is based on the a.e. differentiability result of Vaisala [65] and on a change of variable formula for homeomorphisms in Sobolev spaces due to Reshetnyak [57].

The two previous theorems state that the minimization problem for the $n-$energy

$$\int_{\Omega'} |dh|_O^2 dy$$

of $h : \Omega' \to \Omega$ in $h_0 + W_0^{1,2}(\Omega', \Omega)$ is equivalent to a minimization problem (with corresponding boundary data) for the inner distortion

$$\int_\Omega \frac{|du^{-1}|_O^2}{\det\ du^{-1}}(x)dx.$$

However, when $n = 2$ one has that inner and outer distortion agree, so that

$$\frac{|du|_O^2}{det\ du} = \frac{|du^{-1}|_O^2}{\det\ du^{-1}},$$

hence minimizing the Dirichlet energy

$$\int_{\Omega'} |dh|_O^2 dy$$

is equivalent to minimizing the mean dilation

$$\int_\Omega \frac{|du|_O^2}{det\ du} dx.$$

To return to the Hilbert-Schmidt norm from the operator norm we observe that in $n = 2$

$$\frac{|A|^2}{\det\ A} = \left(\frac{|A|_O^2}{\det\ A} + \frac{\det A}{|A|_O^2}\right)$$

and that the mapping

$$K \to K + \frac{1}{K}$$

is monotone. Consequently

$$\left\|\frac{|du|^2}{\det\ du}\right\|_\infty = \left(\left\|\frac{|du|_O^2}{\det\ du}\right\|_\infty + \frac{1}{\left\|\frac{|du|_O^2}{\det\ du}\right\|_\infty}\right)$$

then the minimization problem for the operator norm has the same solution as the one for the Hilbert-Schmidt norm. Other generalization of Grötzsch problem in higher dimension have appeared in the work of Fehlmann [23].

1.7 Minimal Lipschitz extensions

We start by looking at a (relatively) simpler functional, which has been extensively studied in the last few decades.

Problem 7. Consider two sufficientlys smooth bounded open sets $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^N$. Among all Lipschitz mappings $u \in Lip(\Omega, \Omega')$ with prescribed trace, find one which minimizes the functional

$$u \to \left\| du \right\|_\infty .$$

The problem is related to that of finding a canonical (unique) Lipschitz extension of the boundary map. We are interested in questions of existence, uniqueness and continuous dependence from the data.

1.7.1 Aronsson's approach in the scalar case $N = 1$

In the following we describe the $N = 1$ scalar case for this L^∞ variational problem. Since $||\nabla u||_\infty$ is equivalent to the Lipschitz norm of the scalar function $u : \Omega \to \mathbb{R}$ this leads to the following

Definition 12. *A minimizing Lipschitz extension* is an extension of a Lipschitz scalar function $f : \partial\Omega \to \mathbb{R}$ to $u : \Omega \to \mathbb{R}$ with $u = f$ on $\partial\Omega$ and

$$Lip(u, \Omega) = \sup_{x \neq y, x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|} = \mathrm{Lip}(f, \partial\Omega)$$

In 1934, independently E. J. MacShane [48] and H. Whitney [68] noted the following:

Theorem 20. *Such extensions always exist but are not unique.*

The proof of existence is based on the following observation: Assume that an extension u exists and let $\lambda = Lip(f, \partial\Omega)$. Since $Lip(u, \Omega) = Lip(f, \partial\Omega)$ then for all $x \in \partial\Omega$ and all $y \in \Omega$ one must have

$$-\lambda \leq \frac{|u(y) - f(x)|}{d(x,y)} \leq \lambda,$$

and hence

$$f(x) - \lambda d(x,y) \leq u(y) \leq f(x) + \lambda d(x,y).$$

Since for all $x \in \partial\Omega$ and $y \in \Omega$

$$f(x) - \lambda d(x,y) \leq u(y) \leq f(x) + \lambda d(x,y).$$

if we define the upper and lower functions

$$L(y) = \sup_{x\in\partial\Omega} (f(x) - \lambda d(x,y)) \text{ and } U(y) = \inf_{x\in\partial\Omega} (f(x) + \lambda d(x,y))$$

then these are minimizing Lipschitz extensions of f and so is any u such that $L \le u \le U$ in Ω.

Remark 11. We recall an example due to Jensen [44], showing that the problem of minimal Lipschitz extension does not have a unique solution. Let $\Omega = B(0,1)$ and $f(x,y) = 2xy$. For every $0 \le \alpha \le 1/2$ set

$$u^\alpha(x,y) = \begin{cases} 0 & \text{for } x^2+y^2 \le \alpha^2 \\ \frac{2xy(\sqrt{x^2+y^2}-\alpha)}{(1-\alpha)(x^2+y^2)} & \text{for } \alpha^2 \le x^2+y^2 \le 1. \end{cases}$$

Note that for $x^2+y^2 = 1$ we have $u^\alpha(x,y) = 2xy$ and more over

$$Lip(u^\alpha,\Omega) = Lip(f,\partial\Omega)$$

So there are infinitely many distinct minimal Lipschitz extensions.

Problem 8. Is there a special class of *canonical* extensions for which uniqueness holds?

In 1967 G. Aronsson (see [4]) proposed a way to *localize* the functional by introducing the formal approximation scheme:

- Consider minimzers u_p of
$$\int |\nabla u|^p$$
They are $p-$*harmonic*, i.e. weak solutions of the equation
$$\Delta_p u_p = div(|\nabla u_p|^{p-2}\nabla u_p) = 0$$
- In case $u \in C^2$ then we can rewrite this PDE in non-divergence form
$$(p-2)|\nabla u|^{p-4}\left(u_i u_j u_{ij} + \frac{|\nabla u|^2}{p-2}\Delta u\right) = 0$$
- Let $p \to \infty$ and formally obtain the $\infty-$*Laplacian*
$$\Delta_\infty u = u_{ij}u_i u_j = \frac{1}{2}\langle\nabla|\nabla u|^2, \nabla u\rangle = 0.$$

Remark 12. Note that a priori there is no link between solutions of the non-linear, degenerate elliptic PDE

$$u_i u_j u_{ij} = 0$$

and the problem of minimal Lipschitz extensions. *The previous computation is purely formal.*

In [4], Aronsson established a link between sufficiently smooth solutions of the infinity Laplacian and correspondingly smooth minimizers of the L^∞ variational problem.

Theorem 21 ($\infty-$harmonic implies AMLE [4]). *C^2 solutions of $\Delta_\infty u = 0$ are Absolute Minimizing Lipschitz Extensions (AMLE), i.e. they minimize $Lip(u,D)$ on every subdomain $D \subset \Omega$*

$$Lip(u,D) = Lip(u,\partial D) \text{ for every } D \subset \Omega$$

In some sense, the localization built-in in the notion of AMLE is inherited from the L^p problem. The key observation in the proof is that for C^2 solutions one has $|\nabla u|$ is constant along integral curves of ∇u in $D \subset \Omega$. Aronsson proved that such curves cannot vanish in the interior of the domain and cannot wind up infinitely many times within the domain, hence they have to reach the boundary.

As a converse to the previous theorem, Aronsson also proved

Theorem 22. *Every C^2 AMLE is $\infty-$harmonic.*

Regarding existence of AMLE, Aronsson established the following

Theorem 23. *(Existence of AMLE) Given any $\Omega \subset \mathbb{R}^n$ and $f \in Lip(\partial\Omega)$ there exists always a AMLE.*

We say that a minimal Lipschitz extension $u \in Lip(\Omega)$, with boundary values $f \in Lip(\partial\Omega)$, has the property $\mathscr{A}$, if for every $D' \subset D$ one has $u \leq U'$ in D' where U' is the upper function in D' with respect to the boundary value u.

The AMLE corresponding to $f \in Lip(\partial\Omega)$ is then defined as

$$u(x) := \inf_g g(x)$$

where the inf is taken over all functions with the $\mathscr{A}$ property with respect to f.

In 1968 Aronsson proved that there can be at most one $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solution of the $\infty-$Laplacian. Thus showing that there can be at most one C^2 AMLE.

Remark 13. Aronsson shows that C^2 solutions have nowhere vanishing gradient, however any C^2 solution with boundary data $2xy$ must have a critical point. So there may not be C^2 solutions of the $\infty-$Laplacian for this data.

The C^2 hypothesis in Aronsson's work was a severe limitation until in 1993 Jensen (see [44]) removed it using the theory of viscosity solutions, and eventually proving *uniqueness of AMLE.*

Definition 13 (Viscosity solution). A continuous function u is $\infty-$subharmonic in the viscosity sense if for any point $x \in \Omega$ and $\phi \in C^2(\Omega)$ such that $\phi - u$ has a minimum at x one has $\phi_{ij}\phi_i\phi_j \geq 0$. Supersolutions and solutions are defined in a similar fashion.

See [18], [5], and [17] for a broad exposition and a detailed list of references.

Theorem 24 (Bhattacharya, Di Benedetto, Manfredi [15]). *Given fixed boundary data, $p-$harmonic functions converge to viscosity $\infty-$harmonic functions as $p \to \infty$.*

Theorem 25 (Jensen [44]). *AMLE are viscosity $\infty-$harmonic functions.*

Theorem 26 (Jensen [44]). *The Dirichlet problem for viscosity $\infty-$harmonic functions has a unique solution.*

At present, thanks to the work of Armstrong, Barron, Champion, Crandall, De Pascale, Evans, Gariepy, Jensen, Juutinen, Manfredi, Oberman, Parviainen, Rossi, Smart, Wang, Yu (to quote just a few) as well as the more recent approach of Naor, Peres, Sheffield, Schramm and Wilson one can prove the uniqueness of AMLE in a variety of ways. In particular, this can be achieved without directly using the $\infty-$Laplacian operator and viscosity solutions for PDE. See [5] and [17] for a detailed account of these developments.

However, at present, out of this multitude of approaches there is no method that can be immediately extended to approach *uniqueness* in the vector valued case.

1.7.2 Aronsson's approach in the vector-valued case $N > 1$

Existence of minimizing Lipschitz extensions for mappings follows from the classical

Theorem 27. *(Kirszbraun's Theorem) Let X, Y be two Hilbert spaces and $U \subset X$ and open set. If $f : U \to Y$ is a Lipschitz mapping then there exits an extension $F : X \to Y$ with the same Lipschtiz constant.*

However, as we have seen, such extensions may not be unique. Generalizing Aronsson's approach beyond the real-valued mappings setting, and in particular to the vector-valued case $u : \Omega \to \mathbb{R}^N$, is very challenging but, aside from being an important problem in its own right, may have several potential applications in image processing (specifically image inpainting and surface reconstruction).

A first step in this direction was taken by Naor and Sheffield in [51] where the focus is on absolutely minimizing Lipschitz extensions in the context of tree-valued mappings. Their main result in [51] consists in existence of a unique AMLE of any prescribed Lipschitz mappings from a subset of a locally compact length metric space to a metric theory. Among other things, the authors also introduce a general definition of discrete infinity harmonic function and prove existence of infinity harmonic extensions.

Shortly afterwards, in [62], Sheffield and Smart considered minimizing extensions of the Lipschitz norm

$$Lip(u, \Omega) = \sup_{x,y \in \Omega} \frac{d(u(x), u(y))}{d(x, y)} = \sup_{\Omega} |du|_O.$$

as well as its *discrete* analogue for mappings $u : G \to R^N$ where $G = (E,X,Y)$ is a finite graph

$$Su(x) := \sup_{y \approx x} d(u(x),u(y))$$

and two vertices $x, y \in E$ are in relation if they are separated by an edge. The subset of vertices $Y \subset X$ here plays the role of the domain for the mapping to be extended.

Definition 14. A mapping is said to be discrete $\infty-$harmonic at $x \in X \setminus Y$ if there is no way to decrease $Su(x)$ by changing the value of u at x.

Peres, Schramm. Sheffield and Wilson have shown that for any Lipschitz $f : Y \to \mathbb{R}$ there exists a unique Lipschitz extension $u : X \to \mathbb{R}$ which is $\infty-$harmonic. In [62], Sheffield and Smart prove that the uniqueness fails for the vector valued case.

To recover uniqueness in [62] Sheffield and Smart introduce a new notion, that of *tight extension* that is stronger than discrete $\infty-$harmonic:

Definition 15. (Tightness) Consider mappings $u, v : X \to \mathbb{R}^N$ that agree on Y. The mapping v is tighter that u on G if

$$\sup\{Su|\ Su > Sv\} > \sup\{Sv|\ Sv > Su\}.$$

The mapping u is *tight* on G if there is no tighter v.

Theorem 28 (Sheffield and Smart [62]). *Let $G = (E,X,Y)$ be a finite connected graph.*

- *Every Lipschitz $f : Y \to \mathbb{R}^N$ has a unique tight extension $u : X \to \mathbb{R}^N$. Moreover u is tighter than every other extension of f.*
- *For every $p > 0$ consider a minimizer $u_p : X \to Y$ of $I_p(w) := \sum_x (Sw(x))^p$, with $u_p = f$ on Y. As $p \to \infty$ the mappings $u_p \to u$ pointwise, where u is the tight extension of f.*

Motivated by this result, Sheffield and Smart introduced a notion of tight extension in the continuous setting: First one sets $Lu(x) = \inf_{r>0} Lip(u, \Omega \cap B(x,r))$.

Definition 16. Let $u, v \in C(\bar{\Omega}, \mathbb{R}^N)$ be two Lipschtiz functionn which agree on $\partial\Omega$. We say that v is tighter that u if

$$\sup\{Lu|\ Lv < Lu\} > \{Lv|\ Lv > Lu\}.$$

A mapping u is called *tight* if there is no tighter v.

Definition 17. A principal direction for for a mapping $u \in C^1(\Omega, \mathbb{R}^N)$ is a continuous, unit vector field in Ω such that at each point it spans the the principal eigenspace of $du^T du$. If $N = 1$ then the field is $-\nabla u/|\nabla u|$. Note that the existence of a principal direction field implies that the largest eigenvalue for $du^T du$ is simple.

Recall that the linear transformation $y \to du(x)y$ sends spheres into ellipsoids. The principal direction corresponds to the largest axis of such ellipsoid.

Theorem 29 (Sheffield and Smart [62]). *Let $u \in C^3(\Omega, \mathbb{R}^N)$ have a principal direction field $a \in C^2(\Omega, \mathbb{R}^n)$. The mapping u is tight if and only if*

$$(u^i_j a_j)_k a_k = 0.$$

Theorem 30 (Sheffield and Smart [62]). *Let $\Omega \subset \mathbb{C}$ be a bounded open set and $u : \Omega \to \mathbb{C}$ be analytic in a neighborhood of $\bar{\Omega}$. The mapping u is tight if and only if either*

(i) $\partial_z \partial_z u = 0$ in Ω (i.e., u is affine); or

(ii) The meromorphic function

$$Re\left(\frac{u_z u_{zz}}{(u_{zzz})^2}\right) \leq 2,$$

in the set where $u_{zz} \neq 0$.

If u is a diffeomorphism and u_{zz} never vanishes then part (ii) can be rewritten as $(\Delta - \Delta_\infty)\log|u_z^{-1}| \leq 0$. In other words, the level sets of $|u_z^{-1}|$ are convex.

1.7.3 A refinement of the Aronsson equation

If we use Aronsson's scheme in the scalar case then we have seen as (with sufficient regularity) the approximating $p-$harmonic functions satisfy

$$(p-2)|\nabla u|^{p-4}\left(u_i u_j u_{ij} + \frac{|\nabla u|^2}{p-2}\Delta u\right) = 0$$

In the vector case, using the Euclidean norm this time, it is easy to see that one obtains instead

$$du_{lk}\partial_{jk}u^l \partial_j u + \frac{|du|^2 \Delta u}{p-1} = 0. \tag{1.13}$$

If one lets $p \to \infty$ then formally we obtain the $\infty-$Laplacian system

$$u^l_{jk} u^l_k u^i_j = 0 \text{ for } i = 1, ..., N.$$

$$u^l_{jk} u^l_k u^i_j = 0 \text{ for } i = 1, ..., N.$$

Theorem 31 (Katzourakis [47]). *There exists distinct solutions of the $\infty-$Laplacian above with the same boundary data.*

The explicit counterexamples are all $1-$dimensional, with $\Omega \subset \mathbb{R}$. In view of such examples it appears that the $\infty-$Laplacian analogue may not an appropriate PDE to characterize unique extremals. In [47], Katzourakis observed that one can recover more information, leading to an augmented (formal) Aronsson system: Recall

$$du_{lk}\partial_{jk}u^l\partial_j u+\frac{|du|^2\Delta u}{p-1}=0.$$

Notice that the term $du_{lk}\partial_{jk}u^l\partial_j u$ lies in the image of du. Consequently for (1.13) to hold we must also have

$$\pi_{N(du)}\Delta u=0$$

where $N(du)=\{v\in\mathbb{R}^N|\ duv=0\}$ is the null-space of the linear application $v\to duv$ and π_N denotes the orthogonal projection in $\mathbb{R}^N$ onto such space. Thus a more complete choice for the Aronsson system would be the coupled system

$$u^l_{jk}u^l_k u^i_j=0 \text{ and } \pi_{N(du)}\Delta u=0, \tag{1.14}$$

As noted in [47], this system may have discontinuous coefficients even for smooth du, since the rank of du may change from point to point. Although the previous derivation is purely formal one has the following variational interpretation

Theorem 32 (Katzourakis [47]). *Let $\Omega\subset\mathbb{R}^n$ and $u\in C^2(\Omega,\mathbb{R}^n)$ be diffeomorphism with non-vanishing Jacobian. The mapping u solves* (1.14) *if and only for every subdomain $D\subset\subset\Omega$ and for every $g\in Lip_0(D,\mathbb{R})$ and $\xi\in\mathbb{R}^n$ one has*

$$||\nabla u||_{L^\infty(D)}\le||\nabla(u+g\xi)||_{L^\infty(D)}$$

The actual result is more general and involves C^2 mappings $u:\Omega\to\mathbb{R}^N$ and an additional variational characterization.

1.8 Aronsson's approach for the extremal dilation problem

In this final section we return to the extremal problem for quasiconformal mappings and recall recent results by Raich and the author [16] in which the Aronsson's approximation scheme is used to introduce a notion of absolute minimizers in the quasiconformal setting. The goal here is to find a candidate PDE that would play the role similar to that of the infinity-Laplacian in the AMLE theory. Following the approach of Sheffield and Smart in [62] we focus on the C^2 case. Although this is an unnatural smoothness hypothesis for quasiconformal mappings, it does provide some insights into the general problem.

The first step in this approach consists in studying extremal mappings for the corresponding L^p problem. If $p>1$, $\Omega\subset\mathbb{R}^n$ and the diffeomorphism $u\in C^2(\Omega,\mathbb{R}^n)$ is a critical point of the functional

$$\mathscr{F}_p(u,\Omega):=\int\frac{|du|^{np}}{(det du)^p}dx \tag{1.15}$$

then the mapping u satisfies the system of Euler-Lagrange equations

$$(L_p u)^i = np\partial_j\left(\left[trace(g)\right]^{\frac{np-2}{2}} du^{-1,T}S(g)\right)_{ij}, \text{ for } i = 1,...,n$$

If we let $p \to \infty$ then formally one obtains the Aronsson PDE,

$$(L_\infty u)^i = S(g)_{ij}\partial_j\sqrt{trace(g)} = 0 \text{ for } i = 1,...,n. \tag{1.16}$$

This PDE tells us that the *dilation of the mapping u (i.e.* $\sqrt{trace(g)}$ *) is constant along curves tangent to the sub-bundle generated by the rows of* $S(g)$.

Problem 9. What is the lowest regularity for the mapping u for which the PDE

$$(L_\infty u)^i = S(g)_{ij}\partial_j\sqrt{trace(g)} = 0 \text{ for } i = 1,...,n$$

is meaningful?

Remark 14. It is tempting to define solutions of (1.16) as quasiconformal mappings such that their dilation $trace(g)$ is constant along all curves tangent to the sub-bundle generated by the rows of $S(g)$. Observe that for this definition to be meaningful at the very least one would need regularity for u such that constant linear combinations of the rows of $S(g)$ generate integral flows (for instance $S(g) \in BV$) and the quantity $trace(g)$ must be continuous (so it can be evaluated along such integral curves) .

It is important to note that classical solutions of the extremal quasiconformal problems, e.g. Teichmüller mappings, solve (1.16) in the regions where they are C^2 smooth.

Proposition 4. *(1) Any Teichmüller map of the form* $u := \psi \circ v \circ \phi^{-1}$ *with* ψ, ϕ *conformal and* v *affine is a solution of* $L_\infty u = 0$. *(2) the quasiconformal mappings* $u(x) = |x|^{\alpha-1}x$ *for* $\alpha > 0$ *solve* $L_\infty u = 0$ *away from the origin. (3) Let* $0 < \alpha < 2\pi$ *and* (r,θ,z) *be cylindrical coordinates for* $x = (x_1,...,x_n)$ *where* $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ *and* $x_j = z_j$, $3 \le j \le n$. *The quasiconformal mapping*

$$u(r,\theta,z) = \begin{cases} (r, \pi\theta/\alpha, z) & 0 \le \theta \le \alpha \\ (r, \pi + \pi\frac{\theta-\alpha}{2\pi-\alpha}, z) & \alpha < \theta < 2\pi \end{cases} \tag{1.17}$$

solves $L_\infty u = 0$ *away from the set* $r = 0$.

The following theorem establishes a link between the formal PDE (1.16) and the L^∞-variational problem.

Theorem 33 ([16]). *If* $u : \Omega \to \mathbb{R}^n$ C^2 *is a quasiconformal solution of* $L_\infty u = 0$ *in* Ω*, then*

(i) For every $D \subset \Omega$ *one has* $\sup_{\bar D}\sqrt{trace(g)} = \sup_{\partial D}\sqrt{trace(g)}$.

(ii) For every $D \subset \Omega$ *one has* $\inf_{\bar D}\sqrt{trace(g)} = \inf_{\partial D}\sqrt{trace(g)}$.

(iii) There exists $C = C(n) > 0$ *such that for every* C^2 *domain* $D \subset \Omega$ *and* $w : \bar D \to \mathbb{R}^n$ C^2 *quasiconformal such that* $u = w$ *on* ∂D *one has* $\sup_D\sqrt{trace(g(u))} \le C\sup_D\sqrt{trace(g(w))}|$.

(iv) If $n = 2$ the dilation $|g|$ is constant in Ω and if u is affine in a neighborhood of $\partial\Omega$ then u must be an affine transformation throughout Ω.

Sketch of the proof Show that any interior maximum points for $|g|$ propagate along curves tangent to the span of the rows of $S(g)$ until they reach the boundary. This is achieved by using the fact that $S(g)$ is either vanishing or has at least rank higher than two. This is used to construct a non self-intersecting curve of this kind and showing that (i) its total length must be finite; (ii) the curve cannot vanish in Ω. Points (i) and (ii) imply then that the curve must reach the boundary. ☐

Theorem 34 ([16]). *If $u : \Omega \to \mathbb{R}^n$ C^2 is a quasiconformal absolute extremal, i.e. for every $D \subset \Omega$ and $w : \bar{D} \to \mathbb{R}^n$ C^2 quasiconformal such that $u = w$ on ∂D one has $\sup_D \sqrt{trace(g(w))} \leq \sup_D \sqrt{trace(g(w))}$, then $L_\infty u = 0$ in Ω.*

Sketch of the proof Arguing by contradiction we assume there is a ball $B \subset\subset \Omega$ s.t. $L_\infty u \neq 0$ in B. We construct a *better competitor* for the variational problem: i.e. a C^2 quasiconformal diffeomorphism $V : \bar{B} \to \mathbb{R}^n$ with same boundary values as u on ∂B and $\sup_B trace\ g(V) < \sup_B trace\ g(u)$. This is done by perturbing u with a finite number of "bumps" that reduce the dilation near the boundary. ☐

Remark 15. A similar result was proved much earlier by Barron, Jensen and Wang in their important work [11] with a different, less constructive proof. The advantage of the approach in [16] is that it provides a competitor which is also quasiconformal.

Remark 16. Recently in [46], Katzourakis applied the refined derivation technique we described earlier to the quasiconformal setting and obtained the formal extended system:

$$du_{ak} J_{ki} du_{bl} J_{lj} \partial_{kl} u^b + |du|^2 [\pi_{N(duJ)}]_{ab} J_{ij} \partial_{ij} u^b = 0$$

where $J = g^{-1} S(g)$ and $g = du^T du$. The equation is composed of two linearly independent parts. The first, in the case of diffeomorphisms between domains of $\mathbb{R}^n$ coincides with the system we have described earlier. The second component is new but it is not yet clear how it relates to the variational problem. The paper [46] also provides a necessary and sufficient condition for C^2 mappings to satisfy this system.

1.8.1 A gradient flow approach

Let $\Omega \subset \mathbb{R}^n$ is a bounded, $C^{2,\alpha}$ smooth, open set. Currently we do not know how to prove existence of solutions of (1.16) or how to attack the extremal problems for a fixed homotopy class of quasiconformal mappings. A possible strategy for a proof consists in finding solutions of a gradient flow $u_p(x,t)$ for the functional $\mathscr{F}_p(u,\Omega)$ defined in (1.15). The long term existence and suitable estimates (independent of p as $p \to \infty$) for such flow then would yield the existence of the asymptotic mapping $w_p(x) = \lim_{t\to\infty} u_p(x,t)$ which would be a candidate for the L^p minimization problem within the homotopy class of the initial data. The solution to the L^∞ problem then could be achieved by establishing estimates on w_p independent of p and letting $p \to \infty$.

For a fixed diffeomorphism $u_0 : \Omega \to \mathbb{R}^n$, we want to study diffeomorphism solutions $u(x,t)$ of the initial value problem

$$\begin{cases} \partial_t u = -L_p u & \text{in } \Omega \times (0,T). \\ u = u_0 & \text{at } \Omega \times \{t = 0\} \end{cases} \tag{1.18}$$

where we recall that

$$(L_p u)^i = np\partial_j \left(|g|^{\frac{np-2}{2}} du^{-1,T} S(g) \right)_{ij}, \text{ for } i,j = 1,...,n$$

If there is a $T > 0$ such that a solution $u \in C^2(\Omega \times (0,T))$ exists with $\det du > 0$ in $\Omega \times (0,T)$, then

$$\frac{d}{dt}\mathscr{F}_p(u,\Omega) = -\left(\frac{1}{|\Omega|} \int_\Omega |L_p u|^2 dx \right) \leq 0,$$

i.e., the p-distortion is nonincreasing along the flow. Hence we obtain

Proposition 5. *If* $u \in C^2(\Omega \times [0,T), \mathbb{R}^n) \cap C^1(\bar{\Omega} \times [0,T), \mathbb{R}^n)$ *is a solution of* (1.18) *with* $\det du > 0$ *in* $\bar{\Omega} \times [0,T)$*, then for all* $0 \leq t < T$*,* $\|\mathbb{K}_{u_p}\|^p_{L^p(\Omega)} = \|\mathbb{K}_u\|^p_{L^p(\Omega)} - \int_0^T \|L_p u(\cdot,t)\|_{L^2(\Omega)} dt$ *and consequently*

$$\|\mathbb{K}_u\|_{L^p(\Omega \times \{t\})} \leq \|\mathbb{K}_{u_0}\|_{L^p(\Omega)}. \tag{1.19}$$

It is immediate to show that the functional $\mathscr{F}_p(u,\Omega)$ is invariant by conformal deformation. Therefore, if we let $s \mapsto F_s : \mathbb{R}^n \to \mathbb{R}^n$ be a one-parameter semi-group of conformal transformations, then solutions to the PDE system

$$\partial_t u = L_p u + \frac{d}{ds} F_s(u) \bigg|_{s=0}$$

would also satisfy (1.19). Recall that the flow F_s is conformal if

$$S(d\mathscr{D}) = \frac{d\mathscr{D} + d\mathscr{D}^T}{2} - \frac{1}{n} \text{ trace } (d\mathscr{D}) I_n = 0$$

where $\mathscr{D} = (\frac{d}{ds}F_s)\Big|_{s=0} \circ F_0^{-1} = (\frac{d}{ds}F_s)\Big|_{s=0}$ and S denotes the Ahlfors operator. If $n = 2$ then this amounts to $\partial_{\bar{z}}\mathscr{D} = 0$. If $n \geq 3$ there is more rigidity and conformality requires that

$$\mathscr{D}(x) = a + Bx + 2(c \cdot x)x - |x|^2 c$$

for any vectors a, c and matrix B with $S(B) = 0$ (see [58]).

We observe that in light of conformal invariance, if $u(x,t)$ is a solution of (1.18) in $\Omega\times(0,T)$ and $v(x,t)=\delta u(\lambda x,\delta^{-2}t)$ for some $\lambda,\delta>0$, then $v(x,t)$ is still a solution with initial data $v_0(x)=\delta u_0(\lambda x)$ in an appropriately scaled domain. Applying inversions in a suitable way will also yield new solutions from $u(x,t)$.

Usual elliptic/parabolic PDE techniques do not apply. The main difficulty consists in the fact that the functional is *not convex* but only quasi-convex (in the sense of Morrey). In order to study the gradient flow it helps to rewrite the system in non-divergence form[3].

$$(L_p u)^i = A^{ik}_{j\ell}(du)u^k_{j\ell}.$$

with

$$A^{ik}_{j\ell}(q) = -p\frac{|q|^{np-2}}{(\det q)^p}\left[np(q_{k\ell}q^{ji}+q_{ij}q^{\ell k}) - n(np-2)\frac{q_{ij}q_{k\ell}}{|q|^2} - |q|^2(q^{\ell i}q^{jk}+pq^{\ell k}q^{ji}) - n\delta_{ki}\delta_{j\ell}\right].$$

This form of the PDE has a remnant of ellipticity in the form of the so-called *Legendre-Hadamard* property: There exists constants $C_1, C_2>0$ depending respectively only on n and on p and on n such that for a.e. $q\in\mathbb{R}^{n\times n}$ and for all $\xi,\eta\in\mathbb{R}^n$

$$C_1(n,p)p|\eta|^2|\xi|^2\frac{|q|^{np-2}}{(\det q)^p} \le A^{ik}_{j\ell}(q)\eta_i\xi^j\eta_k\xi^\ell$$
$$\le C_2(n)p^2|\eta|^2|\xi|^2\left(\frac{|q|^{np-2}}{(\det q)^p} + \frac{|q|^{n(p+2)-2}}{(\det q)^{p+2}}\right)$$

Using the latter, Raich and the author established in [16] certain Schauder type estimates for the gradient flow (i.e. a gain of two derivatives with respect to the regularity of the right hand side and the coefficients of the PDE). The Schauder estimates in turn allow to rephrase the system (1.18) as a fixed point problem for a contraction map, leading to the short time existence and uniqueness result

Definition 18. Let $\Omega\subset\mathbb{R}^n$ be a smooth bounded domain and for $T>0$ let $Q=\Omega\times(0,T)$. The *parabolic boundary* is defined by $\partial_{par}Q=(\Omega\times\{t=0\})\cup(\partial\Omega\times(0,T))$. The *parabolic distance* is $d((x,t),(y,s)) := \max(|x-y|,\sqrt{|t-s|})$. For $\alpha\in(0,1)$ we define the *parabolic Hölder class* $C^{0,\alpha}(Q) := \{u\in C(Q,\mathbb{R})|\ \|u\|_{C^\alpha(Q)} := [u]_\alpha + \|u\|_0 < \infty\}$, where

$$[u]_\alpha := \sup_{(x,t),(y,s)\in Q \text{ and } (x,t)\neq(y,s)} \frac{|u(x,t)-u(y,s)|}{d((x,t),(y,s))^\alpha}$$

and $|u|_0 = \sup_Q |u|$. For $K\in\mathbb{N}$ we let $C^{K,\alpha}(Q) = \{u : Q\to\mathbb{R}|\ \partial_{x_{i_1}}\cdots\partial_{x_{i_K}}u \in C^{0,\alpha}(Q)\}$.

Proposition 6. *Let $u_0:\Omega\to\mathbb{R}^n$ be a $C^{2,\alpha}$ diffeomorphism for some $0<\alpha<1$ with $\det du_0\ge\varepsilon>0$ in $\bar\Omega$. Assume that $L_pu_0=0$ for all $x\in\partial\Omega$. There exist constants*

[3] To do this however one has to assume existence of two derivatives for the solution

$C,T>0$ depending on $p,n,\Omega,\varepsilon,\|u_0\|_{C^{1,\alpha}(\bar{\Omega})}$, and a sequence of diffeomorphisms $\{u^h\}$ in $C^{2,\alpha}(Q)$ with $Q=\Omega\times(0,T)$ so that

(a) $\det u^h \geq \frac{\varepsilon}{2}$ *for all* $(x,t)\in Q$*,*
(b) $\|u^h\|_{C^{2,\alpha}(Q)}+\|\partial_t u^h\|_{C^{0,\alpha}(Q)} \leq C\|u_0\|_{C^{2,\alpha}(\Omega)}$*,*
(c) $\begin{cases} \partial_t u^{h,i} - A^{ik}_{jl}(du^{h-1})\partial_j\partial_l u^{h,k} = 0 & \text{in } Q \\ u^h = u_0 & \text{on } \partial_{par}Q. \end{cases}$

Theorem 35. *If $u(x,0)\in C^{2,\alpha}$+boundary conditions then there exists a unique $C^{2,\alpha}_1(\Omega\times(0,T),\mathbb{R}^n)$ solutions for small $T=T(p,n,u_0,\Omega)>0$.*

Although the previous result establishes short time existence, the dependence of the interval of existence from p remains an obstacle to the study of the asymptotic limit $p\to\infty$. In order to carry out the program we outlined earlier, one would need a global existence result, as well as estimates independent of p as $p\to\infty$. Currently there is very little literature about gradient flows of quasi-convex functionals but a an important paper of Evans-Gangbo-Savin [21] lays out a strategy to obtain global estimates: Following [21], Raich and the author in [16] let $\beta = det\ du^{-1}$ then show that β solves the scalar PDE

$$\partial_t\beta = [a_{ij}(du)\beta]_{ij}$$

with

$$a_{ij} = p\left(\delta_{ij} - n\frac{du_{jk}du_{ik}}{|du|^2}\right)\sqrt{|g|}^{np}.$$

Although the lack of a sign in the symbol prevents us from using the maximum principle and establishing immediate global bounds, this PDE is a starting point for the study of global estimates.

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