Crack Propagation and the Wave Equation on time dependent domains

A Major Qualifying Project Report submitted to the Faculty of WPI in partial fulfillment of the requirements for the Degree of Bachelor of Science in Mathematics.

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1 Abstract

This project investigates some problems associated with solutions of the wave equation on time dependent domains, a problem whose solution is relevant to modelling crack propagation in materials. Mainly, the use of bijective transformations from varying domains to a fixed domain complicate the PDE, and the relationship between the behavior of the solution to the original equation and the transformed solution becomes difficult to pin down. Despite these issues, uniqueness of a solution to the problem can still sometimes be proved.

2 Introduction

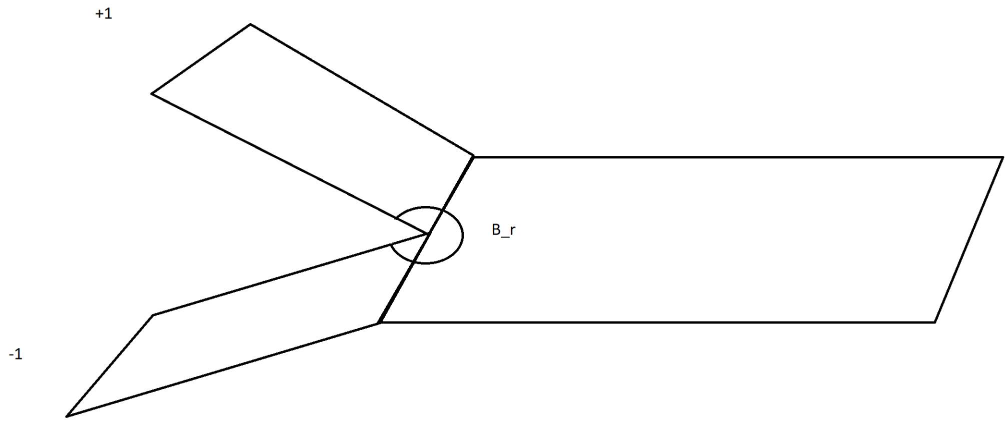
Modelling and understanding growing cracks remains a mathematical challenge. In order to possibly explain the branching effect commonly exhibited in physical models, we would like to figure out the direction of the largest singularities in tangential derivatives local to the crack formation. Similarly, to estimate the speed at which the crack propagates at initiation, we estimate the energy local to the crack formation in time interal for a ball of radius r. In order to perform these calculations, the boundary value problem with a time dependent domain can be reformulated as a more complicated one with a fixed domain, but only if the speed of the crack, c, is less than the wave speed [1]. Despite this, we seek to prove uniqueness of a solution that does not require this reformulation or the restriction of speed c.

3 Calculating energy flowing into B_r in $[0, \tau]$

In order to estimate speed at initiation, we estimate the energy in the ball of radius r local to the crack formation. The integral we wish to calculate is

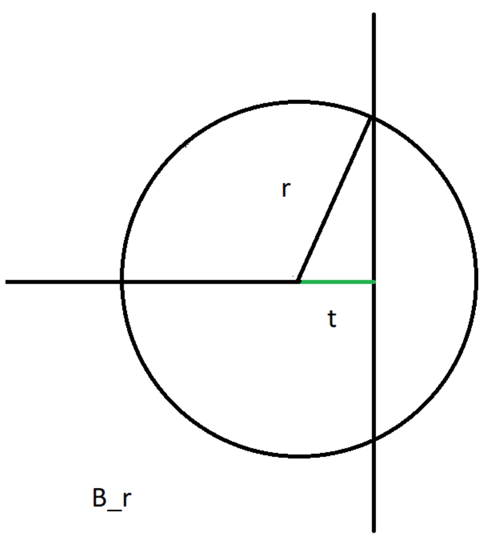
$$E(B_r, \tau) = \int_0^\tau \int_{\partial B_r} u_t(\nabla u \cdot \partial_{B_r} n), \tag{1}$$

where B_r is the ball centered at the tip (considered to be the origin $(0, 0)$) with radius r, and $0 \leq \tau \leq r$.



At time t between 0 and τ , we have the following:

On the top half of ∂B_r where u_t and ∇u are nonzero (say, for θ between $\theta_0(t)$ and π where $\theta_0(t)$ is between 0 and $\frac{\pi}{2}$) we have that $u_t = 1, \nabla u = (-1, 0)$.



We parametrize as follows:

$$\theta_0(t) \leq \theta \leq \pi,\ x(\theta) = r\cos(\theta),\ y(\theta) = r\sin(\theta) \tag{2}$$

In which $\theta_0(t) = \arccos(\frac{t}{r}) = \arcsin(\frac{\sqrt{r^2-t^2}}{r})$. Then note that we get for a fixed t between 0 and τ:

$$\int_{\partial B_r} u_t(\nabla u \cdot \partial_{B_r} n) = \int_{\theta_0(t)}^{\pi} u_t(\theta)(\nabla u(\theta) \cdot \partial_{B_r} n(\theta)) \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \quad (3)$$

which simplifies to

$$\int_{\theta_0}^{\pi} 1((-1,0) \cdot (\cos\theta, \sin\theta) \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2} d\theta = \int_{\theta_0}^{\pi} -r\cos\theta d\theta \quad (4)$$

But

$$\int_{\theta_0}^{\pi} -r\cos\theta d\theta = -r\left[\sin\theta\right]|_{\theta_0(t)}^{\pi} = -r(0 - \frac{\sqrt{r^2-t^2}}{r}) = \sqrt{r^2-t^2} \quad (5)$$

By symmetry, our integral then becomes:

$$E(B_r, \tau) = 2\int_0^{\tau} \sqrt{r^2-t^2} dt \quad (6)$$

Making the substitution $t = r\sin(u)$ we obtain

$$\begin{aligned} E(B_r, \tau) &= 2\int_0^{\arcsin(\tau/r)} \left(\sqrt{r^2 - r^2\sin^2(u)}\right)(r\cos(u)du) \\ &= 2r^2 \int_0^{\arcsin(\tau/r)} \cos^2(u)du \\ &= 2r^2 \int_0^{\arcsin(\tau/r)} \frac{1+\cos(2u)}{2} du \\ &= 2r^2\left(\frac{1}{2}\arcsin\left(\frac{\tau}{r}\right) + \frac{1}{4}\sin\left(2\arcsin\left(\frac{\tau}{r}\right)\right)\right) \\ &= r^2\arcsin\left(\frac{\tau}{r}\right) + \frac{1}{2}r^2\sin\left(2\arcsin\left(\frac{\tau}{r}\right)\right)) \end{aligned} \quad (7)$$

Note though that

$$\begin{aligned} \sin\left(2\arcsin\left(\frac{\tau}{r}\right)\right) &= 2\sin\left(\arcsin\left(\frac{\tau}{r}\right)\right)\cos\left(\arcsin\left(\frac{\tau}{r}\right)\right) \\ &= 2\frac{\tau}{r}\sqrt{1-\left(\frac{\tau}{r}\right)^2} \end{aligned} \quad (8)$$

In all,

$$E(B_r,\tau) = r^2\left(\arcsin\left(\frac{\tau}{r}\right) + \frac{\tau}{r}\sqrt{1-\left(\frac{\tau}{r}\right)^2}\right) \tag{9}$$

And in particular,

$$E(B(0,r),r) = r^2\left(\frac{\pi}{2}\right) \tag{10}$$

Since this energy is at most (in fact equal) to $r^2\frac{\pi}{2}$, the length of the crack at time t is at most t^2, since the energy of the crack is its length, and total energy of the system is conserved.

4 Modified PDE for a solution to the wave equation

One method of dealing with partial differential equations on time dependent domains is translating and scaling the domain in such a way so that it is fixed. However, such transformations often complicate the partial differential equation, making it difficult to relate the two functions. We demonstrate by example with the wave equation in $\mathbb{R}^2$.

Suppose that $\Omega \subset \mathbb{R}^2$ is open, $T > 0$ and $u \in C^2(\Omega \times [0,T])$ satisfies the wave equation, i.e.,

$$\ddot{u} = \Delta u \tag{11}$$

For $c > 0$ given and $\alpha > 0$ yet to be determined, we define the function

$$w(x_1,x_2,t) = u(\alpha x_1 + ct, x_2, t) \tag{12}$$

We seek to find a value for α such that $\ddot{w} - \Delta w$ is conveniently expressable in terms of the partial derivatives of u and w. We proceed directly by calculating, and define for convenience $\vec{v} := (x_1, x_2, t)$ and $\vec{z} := (\alpha x_1 + ct, x_2, t)$:

$$\begin{aligned} w_{x_1}(\vec{v}) = \alpha u_{x_1}(\vec{z}),\ \ w_{x_1,x_1}(\vec{x}) = \alpha^2 u_{x_1,x_1}(\vec{z}) \\ w_{x_2}(\vec{v}) = u_{x_2}(\vec{z}),\ \ w_{x_2,x_2}(\vec{v}) = u_{x_2,x_2}(\vec{z}) \end{aligned}$$

and for $\ddot{w}$:

$$\begin{aligned} w_t(\vec{v}) &= cu_{x_1}(\vec{z}) + u_t(\vec{z}) \\ w_{t,t}(\vec{v}) &= c\left[cu_{x_1,x_1}(\vec{z}) + u_{x_1,t}(\vec{z})\right] + cu_{t,x_1}(\vec{z}) + u_{t,t}(\vec{z}) \\ &= c^2 u_{x_1,x_1}(\vec{z}) + cu_{x_1,t}(\vec{z}) + cu_{t,x_1}(\vec{z}) + u_{t,t}(\vec{z}) \end{aligned}$$

Combining the above, we obtain that

$$(\ddot{w} - \Delta w)(\vec{v}) = u_{t,t}(z)+c^2 u_{x_1,x_1}(\vec{z})+c\left[u_{x_1,t}(\vec{z}) + u_{t,x_1}(\vec{z})\right]-\alpha^2 u_{x_1,x_1}(\vec{z})-u_{x_2,x_2}(\vec{z})$$

If we take $\alpha = \sqrt{c^2+1}$, then $c^2 - \alpha^2 = -1$ and thus

$$\begin{aligned}(\ddot{w} - \Delta w)(\vec{v}) &= \ddot{u} - \Delta u c\left[u_{x_1,t}(\vec{z}) + u_{t,x_1}(\vec{z})\right] \\ &= c\left[u_{x_1,t}(\vec{z}) + u_{t,x_1}(\vec{z})\right]\end{aligned}$$

Since $u \in C^2(\Omega \times [0,T])$ we have that

$$(\ddot{w} - \Delta w)(\vec{v}) = 2cu_{x_1,t}(\vec{z}) \tag{13}$$

Although the PDE in (13) has few terms, it also complicates the process of understanding the behavior of u when we only know the behavior of w (with for example, finding tangential derivatives).

4.1 Finding the tangential derivative of a transformed solution of the wave equation

On domains changing in time, like $\Omega_t := \Omega \setminus \Gamma(t)$, formal calculation suggests that if u solves the wave equation on Ω and $\partial_n u = 0$ (where n is the unit normal to $\Gamma(t)$) on $\Gamma(t)$, a crack growing at speed c, then u admits a decomposition as follows [2]:

If $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ then there is a $w \in H^2(\Omega)$ and a $c \in \mathbb{R}$ such that

$$u = w + c\psi \tag{14}$$

where ψ given by

$$\psi : \mathbb{R}^2 \to \mathbb{R}, \quad \psi(r,\theta) := \sqrt{r}\sin(\theta/2) \tag{15}$$

Using this function, we demonstrate the complexity in understanding the behavior of a simple transformation (that is, translation and scaling) in the x_1 direction. Define by ψ^c the function ψ as given in cartesian coordinates, i.e.,

$$\psi^c(x_1,x_2) = \sqrt{x_1^2 + x_2^2}\sin\left(\frac{\arctan(x_1/x_2)}{2}\right)$$

and define the function

$$f^c(x_1,x_2) := \psi^c(\frac{x_1 - ct}{\beta}, x_2)$$

For $0 < c < 1$ and $\beta = \sqrt{1-c^2}$. Our goal in this section is to calcuate f_θ for time $t = 0$ where f is the function f^c written in polar coordinates.

To write f as a composite of functions, we define the map

$$\begin{aligned}
\phi &: \mathbb{R}^2 \to \mathbb{R}^2 \\
\phi(r,\theta) &= (\sqrt{\left(\frac{r\cos\theta}{\beta}\right)^2 + (r\sin\theta)^2}, \arctan\frac{r\sin\theta}{r\cos\theta}\beta) \\
&= (r\sqrt{\frac{\cos^2\theta}{\beta^2} + \sin^2\theta}, \arctan(\beta\tan\theta))
\end{aligned}$$

So that in particular,

$$f = \psi \circ \phi(r,\theta) \tag{16}$$

and that

$$f_\theta(r,\theta) = \psi_r(\phi(r,\theta))\phi^r_\theta(r,\theta) + \psi_\theta(\phi(r,\theta))\phi^\theta_\theta(r,\theta) \tag{17}$$

where ϕ^r and ϕ^θ are the first (r) and second (θ) components of the function ϕ respectively.

Calculating directly, we obtain

$$\begin{aligned}
\phi^r_\theta &= \frac{r}{2\sqrt{\frac{\cos^2\theta}{\beta^2} + \sin^2\theta}}\left[\frac{-2\cos\theta\sin\theta}{\beta^2} + 2\cos\theta\sin\theta\right] \\
&= \frac{r\cos\theta\sin\theta}{\sqrt{\frac{\cos^2\theta}{\beta^2} + \sin^2\theta}}\left[1 - \frac{1}{\beta^2}\right]
\end{aligned}$$

and

$$\begin{aligned}
\phi^\theta_\theta(r,\theta) &= \frac{1}{(\beta\tan\theta)^2 + 1}\beta\sec^2\theta \\
&= \frac{\beta}{\beta^2\sin^2\theta + \cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
\psi_r(r,\theta) &= \frac{1}{2\sqrt{r}}\sin(\theta/2) \\
\psi_\theta(r,\theta) &= \frac{\sqrt{r}}{2}\cos(\theta/2)
\end{aligned}$$

and thus

$$\psi_r(\phi(r,\theta)) = \frac{1}{2\sqrt{r}\left(\frac{\cos^2\theta}{\beta^2} + \sin^2\theta\right)^{1/4}} \sin\left(\frac{\arctan(\beta\tan\theta)}{2}\right)$$

$$\psi_\theta(\phi(r,\theta)) = \frac{\sqrt{r}\left(\frac{\cos^2\theta}{\beta^2} + \sin^2\theta\right)^{1/4}}{2} \cos\left(\frac{\arctan(\beta\tan\theta)}{2}\right)$$

and thus we may calculate $f_\theta(r,\theta)$ directly for $\theta = 0, \pi/4, \pi/2$.

4.2 $f_\theta(r,0)$

$$\phi^r_\theta(r,0) = 0$$
$$\phi^\theta_\theta(r,0) = \frac{\beta}{0+1} = \beta$$

and

$$\begin{aligned}\psi_r(\phi(r,0)) &= \frac{1}{2\sqrt{r}\left(\frac{1}{\beta^2}\right)^{1/4}} \sin\left(\frac{\arctan 0}{2}\right) \\ &= 0 \\ \psi_\theta(\phi(r,0)) &= \frac{\sqrt{r}}{2\sqrt{\beta}} \cos\left(\frac{\arctan 0}{2}\right) \\ &= \frac{\sqrt{r}}{2\sqrt{\beta}}\end{aligned}$$

Altogether then,

$$\begin{aligned}f_\theta(r,0) &= \psi_r(\phi(r,0))\phi^r_\theta(r,0) + \psi_\theta(\phi(r,0))\phi^\theta_\theta(r,0) \\ &= 0 + \frac{\sqrt{r}}{2\sqrt{\beta}}\beta \\ &= \frac{\sqrt{r\beta}}{2}\end{aligned}$$

4.3 $f_\theta(r,\pi/2)$

Note for $\theta = \pi/2$ we may extend by (continuity) the definition of $\cos\left(\frac{\arctan(\beta\tan\theta)}{2}\right)$ to include $\pi/2$ so that $\frac{\arctan(\beta\tan(\pi/2))}{2} = \cos(\pi/4) = \cos(-\pi/4) = \sqrt{2}/2$. Although the same cannot be applied to $\sin\left(\frac{\arctan(\beta\tan\theta)}{2}\right)$, this will not matter since $\phi^r_\theta(r,\pi/2) = 0$ and since both $\lim_{\theta\to\pi/2^+} \sin\left(\frac{\arctan(\beta\tan\theta)}{2}\right)$ and $\lim_{\theta\to\pi/2^-} \sin\left(\frac{\arctan(\beta\tan\theta)}{2}\right)$ exist in $\mathbb{R}$.

$$\begin{aligned}\phi^r_\theta(r,\pi/2) &= 0\\ \phi^\theta_\theta(r,\pi/2) &= \frac{\beta}{\beta^2} = \frac{1}{\beta}\end{aligned}$$

and

$$\begin{aligned}\psi_r(\phi(r,\pi/2)) &= \frac{1}{2\sqrt{r}\,(1)^{1/4}}\sin(\pm\pi/4)\\ &= \pm\frac{1}{2\sqrt{2r}}\\ &\text{(cannot be extended by continuity, but is bounded with respect to left and right limits)}\\ \psi_\theta(\phi(r,\pi/2)) &= \frac{\sqrt{r}\,(1)^{1/4}}{2}\cos\left(\frac{\arctan(\pi/2)}{2}\right)\\ &= \frac{\sqrt{r}}{2\sqrt{2}}\end{aligned}$$

Altogether then,

$$\begin{aligned}f_\theta(r,\pi/2) &= \psi_r(\phi(r,\pi/2))\phi^r_\theta(r,\pi/2) + \psi_\theta(\phi(r,\pi/2))\phi^\theta_\theta(r,\pi/2)\\ &= 0 + \frac{\sqrt{r}}{2\sqrt{2}}\frac{1}{\beta}\\ &= \frac{\sqrt{r}}{2\sqrt{2}\beta}\end{aligned}$$

4.4 Comments

Note that as $c \to 1$, we have that $\beta \to 0$ and thus

$$\begin{aligned}f_\theta(r,0) &= \frac{\sqrt{r\beta}}{2} \to 0\\ f_\theta(r,\pi/2) &= \frac{\sqrt{r}}{2\sqrt{2}\beta} \to +\infty\end{aligned}$$

In the physical model of crack propagation, this larger derivative for $\theta \neq 0$ for large enough c corresponds to angles in which cracks are more likely to continue progressing. This may explain the phenomena of branching in crack propagation.

5 Uniqueness of Solution to Wave Equation with Time-Dependent Domain

While the calculation above depends on the assumption that c be less than the speed of the wave, we seek in this section to prove uniqueness of a solution independent of the speed c for the simple case that the crack is a line growing in the positive x_1 direction in time.

Let $\Omega \subset \mathbb{R}^2$ be an open set, and let $\Gamma : [0,T] \to \mathcal{P}(\Omega)$ be an increasing set function ($s \leq t \Rightarrow \Gamma(s) \subset \Gamma(t)$) with $\mathcal{H}^1(\Gamma(t))$ bounded. Define the set $\Omega_t := \Omega \setminus \Gamma(t)$, and the time-dependent bilinear form for $t \in [0,T]$ and $f, g \in H_0^1(\Omega_t)$

$$B[f,g;t] := \int_{\Omega_t} \nabla f \cdot \nabla g$$

Moreover, we work in the function spaces which are useful for solving weak formulations of the wave equation [1]

$$V_t := \{v \in H^1(\Omega_t) \mid \; v = 0 \text{ on } \partial\Omega\}$$

Consider the following initial/boundary value problem:

$$\ddot{u}(t) - \Delta u(t) = 0 \text{ on } \Omega_t \tag{18}$$

$$u(t) = 0 \text{ on } \partial\Omega \tag{19}$$

$$u(0) = g, \; \dot{u}(0) = f, \;\; g \in H_0^1(\Omega_0), \;\; f \in L^2(\Omega_0) \tag{20}$$

A **weak solution** of the wave equation with a time-dependent domain [1] is a function $u(x_1, x_2, t)$ such that

$$\begin{aligned}
&u \in H^1(0,T;V_T) \cap W^{1,\infty}(0,T;L^2(\Omega)) \\
&\forall t \in [0,T] \;\; u(t) \in V_t \\
&\forall s \in [0,T) \;\; u \in W^{2,\infty}(s,T;V_s^*) \\
&\sup_{s\in[0,T)} ||\ddot{u}||_{L^\infty(s,T;V_s^*)} < +\infty \\
&\forall s \in (0,T) \text{ the functions} \\
&t \to \frac{1}{h}||\dot{u}(t) - \dot{u}(t-h)||^2_{L^2(\Omega)}, \; h \in (0,s) \\
&\;\text{are equiintegrable on } (s,T)
\end{aligned}$$

and u satisfies

$$\langle \ddot{u}(t), v\rangle_{H^{-1}(\Omega_t)} + B[u(t), v; t] = 0 \;\; \forall v \in H_0^1(\Omega_t)$$

$$u(0) = g, \;\; \dot{u}(0) = f$$

Proposition 1. *For any $t \in [0,T]$, $B[f,g;t]$ is a continuous bilinear form on $H_0^1(\Omega_t)$.*

Proof. For $f, g \in H_0^1(\Omega_t)$ we have that

$$\begin{aligned}
|B[f,g;t]| &= |\int_{\Omega_t} f_{x_1} g_{x_1} + \int_{\Omega_t} f_{x_2} g_{x_2}| \\
&\leq |\int_{\Omega_t} f_{x_1} g_{x_1}| + |\int_{\Omega_t} f_{x_2} g_{x_2}| \\
&\leq ||f_{x_1}||_{L^2(\Omega_t)} ||g_{x_1}||_{L^2(\Omega_t)} + ||f_{x_2}||_{L^2(\Omega_t)} ||g_{x_2}||_{L^2(\Omega_t)} \\
&\leq 2\sqrt{||f_{x_1}||^2_{L^2(\Omega_t)} ||g_{x_1}||^2_{L^2(\Omega_t)} + ||f_{x_2}||^2_{L^2(\Omega_t)} ||g_{x_2}||^2_{L^2(\Omega_t)}} \\
&\leq 2\sqrt{\left(||f||^2_{L^2(\Omega_t)} + ||f_{x_1}||^2_{L^2(\Omega_t)} + ||f_{x_2}||^2_{L^2(\Omega_t)}\right)\left(||g||^2_{L^2(\Omega_t)} + ||g_{x_1}||^2_{L^2(\Omega_t)} + ||g_{x_2}||^2_{L^2(\Omega_t)}\right)} \\
&= 2||f||_{H_0^1(\Omega_t)} ||g||_{H_0^1(\Omega_t)}
\end{aligned}$$

Hence, B is continuous. □

Proposition 2. *If $v \in H^1(0,T; H_0^1(\Omega_0))$ where spt $v \subset\subset \Omega$ then $\frac{d}{dt} B[v(t), v(t); 0]$ exists and equals $2B[v(t), \dot{v}(t), 0]$ for t a.e. in $[0,T]$.*

Proof. For $h \neq 0$, we have that

$$\begin{aligned}
\frac{1}{h}\left[B[v(t+h), v(t+h); 0] - B[v(t), v(t); 0]\right] &= \frac{1}{h}\left[B[v(t+h) - v(t), v(t+h); 0] - B[v(t), v(t+h) - v(t)]\right] \\
&= \left[B[\frac{v(t+h) - v(t)}{h}, v(t+h); 0] - B[v(t), \frac{v(t+h) - v(t)}{h}\right]
\end{aligned}$$

Then since $v \in H^1(0,T; H_0^1(\Omega_0))$, we have that v is absolutely continuous (and hence continuous), and differentiable almost everywhere [3]. Hence, as $h \to 0$, we obtain for t a.e. in $[0,T]$,

$$\begin{aligned}
&\frac{v(t+h) - v(t)}{h} \to \dot{v}(t) \text{ in } H_0^1(\Omega_0) \\
&v(t+h) \to v(t) \text{ in } H_0^1(\Omega_0)
\end{aligned}$$

We then see from Proposition 1 that

$$\begin{aligned}
&B[\frac{v(t+h) - v(t)}{h}, v(t+h); 0] \to \dot{B}[\dot{v}(t), v(t); 0] \\
&B[v(t), \frac{v(t+h) - v(t)}{h}; 0] \to \dot{B}[v(t), \dot{v}(t); 0]
\end{aligned}$$

For t a.e. in $[0,T]$. Since B is symmetric, the claim is proven.

□

Proposition 3. *If $u \in H^1(0,T;H_0^1(\Omega_T))$, $u(t) \in H_0^1(\Omega_t)\ \ \forall t \in [0,T]$, and $\Gamma(s)$ has measure zero for all $s \in [0,T]$, then $\frac{d}{dt}||u(t)||^2_{L^2(\Omega_t)}$ exists a.e. for $t \in [0,T]$ and*

$$\frac{d}{dt}||u(t)||^2_{L^2(\Omega_t)} = 2\langle \dot{u}(t), u(t)\rangle_{L^2(\Omega_t)}$$

Proof. As before, for $h \neq 0$ we calculate

$$\begin{aligned}\frac{1}{h}\left(||u(t+h)||^2_{L^2(\Omega_{t+h})} - ||u(t)||^2_{L^2(\Omega_t)}\right) &= \frac{1}{h}\left(\langle u(t+h), u(t+h)\rangle_{L^2(\Omega_{t+h})} - \langle u(t), u(t)\rangle_{L^2(\Omega_t)}\right) \\ &= \frac{1}{h}\left(\langle u(t+h), u(t+h)\rangle_{L^2(\Omega_t)} - \langle u(t), u(t)\rangle_{L^2(\Omega_t)}\right) \\ &\text{(Since } \Omega_{t+h} \setminus \Omega_t \text{ and } \Omega_t \setminus \Omega_{t+h} \text{ have measure zero)} \\ &= \langle \frac{u(t+h)-u(t)}{h}, u(t+h)\rangle_{L^2(\Omega_t)} + \langle u(t), \frac{u(t+h)-u(t)}{h}\rangle_{L^2(\Omega_t)}\end{aligned}$$

Again, since $u \in H^1(0,T;H_0^1(\Omega_T))$, u is continuous in t and differentiable almost everywhere for $t \in [0,T]$ [3]. Hence as $h \to 0$ we get

$$\begin{aligned}&\frac{u(t+h)-u(t)}{h} \to \dot{u}(t) \text{ in } H_0^1(\Omega_t) \\ &u(t+h) \to v(t) \text{ in } H_0^1(\Omega_t)\end{aligned}$$

Recall that $H_0^1(\Omega_t)$ convergence implies $L^2(\Omega_t)$ convergence, so by Cauchy Schwarz, we see that

$$\begin{aligned}\langle \frac{u(t+h)-u(t)}{h}, u(t+h)\rangle_{L^2(\Omega_t)} &\to \langle \dot{u}(t), u(t)\rangle_{L^2(\Omega_t)} \\ \langle u(t), \frac{u(t+h)-u(t)}{h}\rangle_{L^2(\Omega_t)} &\to \langle u(t), \dot{u}(t)\rangle_{L^2(\Omega_t)}\end{aligned}$$

for t a.e. in $[0,T]$, proving the claim. □

Proposition 4. *Assume that $u(x_1,x_2,t)$ is a weak solution to the above with $g = f = 0$, where $\Gamma(t)$ is a line extending in the positive x_1 direction whose position at time t is given by $\phi(t)$ where ϕ is an increasing continuous function from $\mathbb{R}$ to $\mathbb{R}$. In particular, $\Gamma(l)$ has measure zero for each $l \in [0,T]$. Assume also that $u(t)$ is compactly supported in Ω_t for each $t \in [0,T]$ and that $spt(u(t)) - \phi(t) \subset \Omega_0$ for each $t \in [0,T]$. Then $u = 0$.*

Proof. We modify a proof from Evans [4]. Define first the function

$$w(x_1,x_2,t) := u(x_1+\phi(t),x_2,t) \tag{21}$$

and remark that since $u(t) \in H_0^1(\Omega_t)$ and by the restrictions of $\mathrm{spt}(u(t))$ we have that $w(t) \in H_0^1(\Omega_0)$ for each $t \in [0,T]$. Fix some $s \in (0,T]$ and define the following function:

$$v(t) := \chi_{[0,s]}(t) \int_t^s w(\tau) d\tau$$

For the interval $[0,s]$, note that

$$\begin{aligned} v(t) &= \int_t^s w(\tau) d\tau && (22) \\ &= v(s) + \int_s^t -w(\tau) d\tau && (23) \\ & && (24) \end{aligned}$$

Then since $-w(\tau) \in H_0^1(\Omega_0)$ we get that $v \in H^1(0,s;H_0^1(\Omega_0))$ (see proposition 2.4 - (iii) in [3]). Since $v(s) = 0$ and v is extended to 0 for $t \geq s$, then $v \in H^1(0,T;H_0^1(\Omega_0))$. From Propositions 2 and 3 we have that

$$\begin{aligned} &\frac{d}{dt}\left(B[v(t);v(t);0]\right) = 2B[v(t),\dot{v}(t);0] \\ &\frac{d}{dt}||u(t)||^2_{L^2(\Omega_t)} = 2\langle \dot{u}(t), u(t)\rangle_{L^2(\Omega_t)} \end{aligned}$$

Moreover, note that for $t \in [0,s]$, $\dot{v}(t) = -w(t) = -u(x_1+\phi(t),x_2,t)$. Hence

$$\begin{aligned} B[v(t),\dot{v}(t);0] &= -\int_{\Omega_0} \nabla v(t) \cdot \nabla u(x_1+\phi(t),x_2,t) \\ &= -\int_{\Omega_t} \nabla v(x_1-\phi(t),x_2,t) \cdot \nabla u(t) \\ &= -B[u(t), v(x_1-\phi(t),x_2,t);t] \end{aligned}$$

Similarly,

$$\begin{aligned} \langle \dot{u}(t), u(t)\rangle_{L^2(\Omega_t)} &= -\langle \dot{u}(t), \dot{v}(x_1-\phi(t),x_2,t)\rangle_{L^2(\Omega_t)} \\ &= \langle \ddot{u}(t), v(x_1-\phi(t),x_2,t)\rangle_{H^{-1}(\Omega_t)} \end{aligned}$$

Combining these two, we get that

$$\begin{aligned} \int_0^s \frac{d}{dt}\left(\frac{1}{2}||u(t)||^2_{L^2(\Omega_t)} - \frac{1}{2}B[v(t);v(t);0]\right) &= \int_0^s \langle \dot{u}(t), u(t)\rangle_{L^2(\Omega_t)} - B[v(t),\dot{v}(t);0]dt \\ &= \int_0^s \langle \ddot{u}(t), v(x_1-\phi(t),x_2,t)\rangle_{H^{-1}(\Omega_t)} + B[u(t), v(x_1-\phi(t),x_2,t);t]dt \end{aligned}$$

Now $v(x_1-\phi(t),x_2,t)\in H_0^1(\Omega_t)$ so that we may apply the definition of weak solution to obtain

$$\int_0^s \frac{d}{dt}\left(\frac{1}{2}||u(t)||^2_{L^2(\Omega_t)}-\frac{1}{2}B[v(t);v(t);0]\right)=0 \tag{25}$$

However, we also note that

$$\begin{aligned}0=\int_0^s \frac{d}{dt}\left(\frac{1}{2}||u(t)||^2_{L^2(\Omega_t)}-\frac{1}{2}B[v(t);v(t);0]\right) &= \left(\frac{1}{2}||u(t)||^2_{L^2(\Omega_t)}-\frac{1}{2}B[v(t);v(t);0]\right)|_{t=0}^{t=s} \\ &= \frac{1}{2}||u(s)||^2_{L^2(\Omega_s)}+\frac{1}{2}B[v(0),v(0);0]\end{aligned}$$

since $||u(0)||_{L^2(\Omega_0)}=B[v(s),v(s);0]=0$. Finally, this implies $||u(s)||_{L^2(\Omega_s)}=0$, showing that $u=0$.

□

6 Conclusion

Studying PDEs on time-dependent domains is difficult since many methods of transforming domains do not work. However, such PDEs are important to physical modeling problems, such as crack propagation. While transforming the domain of the PDE can alleviate this issue in some sense, it poses its own problems: these transformed solutions solve new and sometimes more complicated PDEs, and their relationship between the behavior of the original solutions and the transformed solutions is not always readily seen. In particular, these transformation methods require the crack speed to be below the wave speed, yet existence can be shown independently of the crack speed [1]. Here we have shown that for a straight crack, uniqueness also holds independent of crack speed.

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