Continuous Loss Development Factors

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Background

When setting reserves, there are numerous ways and reasonings available to set reserves. One method is called the Development Method. Also known as the Chain Ladder Method, the Development Method uses past data to forecast future claim costs.

Below is an example that will be used to demonstrate how the Development Method is used to set reserves. The following table is the cumulative data from past years used to forecast future claims starting in 2010.

Cumulative Reported Claims

	A	B	C	D	E	F	G	H	I	J	K
1	Maturity	12	24	36	48	60	72	84	96	108	120
2	Average Age	6	18	30	42	54	66	78	90	102	114
3	AY 2000	$42,000	$60,000	$71,000	$80,000	$88,000	$93,000	$96,000	$98,000	$99,000	$ 100,000
4	AY 2001	$40,000	$59,000	$68,000	$78,000	$87,000	$92,000	$95,000	$98,000	$99,000	
5	AY 2002	$36,000	$55,000	$65,000	$76,000	$85,000	$91,000	$94,000	$97,000		
6	AY 2003	$38,000	$57,000	$66,000	$77,000	$86,000	$92,000	$95,000			
7	AY 2004	$39,000	$58,000	$67,000	$77,000	$87,000	$92,000				
8	AY 2005	$41,000	$60,000	$70,000	$79,000	$88,000					
9	AY 2006	$42,000	$61,000	$70,000	$79,000						
10	AY 2007	$43,000	$62,000	$72,000							
11	AY 2008	$43,000	$63,000								
12	AY 2009	$42,000									

Figure 1: Actual Data

In Figure 1, Column A represents the year that the accident happened, the first accident year is the year 2000 and the most recent accident year is 2009 (AY 2009).

Each column represents how many months have passed since the start of the accident year. Usually, the columns would be time intervals such as 0-12, 12-24, …108-120, but we will use 6, 18, …114 since those are the midpoints of each time period (every 6 months). This reason is because the claims are assumed to come in uniformly for the whole year. For example, in 2010, we assume the average accident date is July 1, 2010. Therefore, at the end of the first 12-month development period, we assume the known claims, are, on average, 6 months old.

Cells B2 to K11 represent claims for every 12 months per accident year. These data points are the dollars of claims that have already come in. The first row 9 (Row 2) is already filled in since 10 years have passed since the accident year started in 2000 and there is only one year of data for 2009 since only one year has passed since 2009. In this example, the current year is 2010. For example, Cell F6, $87,000, means that there were $87,000 worth of claims at the end of 60 months (5 years) for accidents that happened in 2004. Note that in this example, all claims are assumed closed at the end of 120 months (10 years).

$$Link\ Ratio = \frac{c_{AY:\,y}}{c_{AY:\,x}} = \frac{c_{2000:18}}{c_{2000:6}} = \frac{\$60000}{\$42000} = 1.43$$

Figure 2: Link Ratio Formula and Example

We will use this formula for each of the known data points, $c_{AY:\,x,y}$ where time y is one time period after time x. We will do this for each known data point in the table.

Using these fractions, shown below, we are able to determine the rate of which claims are being submitted.

	A	B	C	D	E	F	G	H	I	J	K
1	Year/Maturity	6	18	30	42	54	66	78	90	102	114
2	2000	1.43	1.18	1.13	1.10	1.06	1.03	1.02	1.01	1.01	
3	2001	1.48	1.15	1.15	1.12	1.06	1.03	1.03	1.01		
4	2002	1.53	1.18	1.17	1.12	1.07	1.03	1.03			
5	2003	1.50	1.16	1.17	1.12	1.07	1.03				
6	2004	1.49	1.16	1.15	1.13	1.06					
7	2005	1.46	1.17	1.13	1.11						
8	2006	1.45	1.15	1.13							
9	2007	1.44	1.16								
10	2008	1.47									
11	2009										
12		6-18	18-30	30-42	42-54	54-66	66-78	78-90	90-102	102-114	
13	Average Link Ratio	1.471	1.163	1.145	1.116	1.062	1.033	1.028	1.010	1.010	

Figure 3: Link Ratio Table

From Figure 3, the columns per each month are very similar for each year. We can average these values together to get a representative link ratio for each development period. There are other ways to choose the link ratios. For example, if any of these numbers seemed like outliers, they would be omitted. Also, if there were a trend, the link ratios could be chosen such that it followed that trend. For example, the more recent years were trending such that the link ratios were getting larger or smaller than the rest of the policy years for that 12-month period.

$$\mu_{x+1} = LR_{x:x+1} \cdot \mu_x$$

Figure 4: Development Method Formula for Unknown Data Points

The average link ratios are used to forecast claim development. In this formula, LR is the average link ratio calculated in the Link Ratio Table. We will do this for each unknown data point until we reach ultimate for every year.

Year/Maturity	6	18	30	42	54	66	78	90	102	114 - ult
2000										
2001										$ 100,000
2002									$ 97,990	$ 98,980
2003								$ 97,670	$ 98,667	$ 99,664
2004							$ 95,000	$ 97,671	$ 98,667	$ 99,664
2005						$ 93,493	$ 96,542	$ 99,256	$ 100,269	$ 101,281
2006					$ 88,144	$ 93,646	$ 96,700	$ 99,418	$ 100,433	$ 101,447
2007				$ 82,451	$ 91,995	$ 97,738	$ 100,925	$ 103,762	$ 104,821	$ 105,879
2008			$ 73,287	$ 83,925	$ 93,639	$ 99,484	$ 102,729	$ 105,616	$ 106,694	$ 107,772
2009		$ 61,793	$ 71,882	$ 82,317	$ 91,845	$ 97,578	$ 100,760	$ 103,592	$ 104,649	$ 105,706

Figure 5: Estimated Claims Development

Using the selected link ratios, Figure 5 shows the estimated claims in the future. Each cell is multiplied by the link ratio in the same column to get the next cell in the same row or accident year.

	A	B	C	D	E	F	G	H	I	J	K
1	Year/Maturity	6	18	30	42	54	66	78	90	102	114
2	2000	$ 42,000	$ 60,000	$ 71,000	$ 80,000	$ 88,000	$ 93,000	$ 96,000	$ 98,000	$ 99,000	$ 100,000
3	2001	$ 40,000	$ 59,000	$ 68,000	$ 78,000	$ 87,000	$ 92,000	$ 95,000	$ 98,000	$ 99,000	$ 100,000
4	2002	$ 36,000	$ 55,000	$ 65,000	$ 76,000	$ 85,000	$ 91,000	$ 94,000	$ 97,000	$ 97,990	$ 98,980
5	2003	$ 38,000	$ 57,000	$ 66,000	$ 77,000	$ 86,000	$ 92,000	$ 95,000	$ 97,670	$ 98,667	$ 99,664
6	2004	$ 39,000	$ 58,000	$ 67,000	$ 77,000	$ 87,000	$ 92,000	$ 95,000	$ 97,671	$ 98,667	$ 99,664
7	2005	$ 41,000	$ 60,000	$ 70,000	$ 79,000	$ 88,000	$ 93,493	$ 96,542	$ 99,256	$ 100,269	$ 101,281
8	2006	$ 42,000	$ 61,000	$ 70,000	$ 79,000	$ 88,144	$ 93,646	$ 96,700	$ 99,418	$ 100,433	$ 101,447
9	2007	$ 43,000	$ 62,000	$ 72,000	$ 82,451	$ 91,995	$ 97,738	$ 100,925	$ 103,762	$ 104,821	$ 105,879
10	2008	$ 43,000	$ 63,000	$ 73,287	$ 83,925	$ 93,639	$ 99,484	$ 102,729	$ 105,616	$ 106,694	$ 107,772
11	2009	$ 42,000	$ 61,793	$ 71,882	$ 82,317	$ 91,845	$ 97,578	$ 100,760	$ 103,592	$ 104,649	$ 105,706
12	Selected Link Ratio	1.47	1.16	1.15	1.12	1.06	1.03	1.03	1.01	1.01	
13	LDF	2.52	1.71	1.47	1.28	1.15	1.08	1.05	1.02	1.01	

Figure 6: Completed Table

The completed table is shown in Figure 6. It shows the past data, the forecasted claims (highlighted in orange), the selected link ratios (red), and the LDF (Loss Development Factor) (blue). The LDF shows how much the current 12-month period number of claims would need to be multiplied by to reach ultimate at age 120.

	A	B	C	D	E	F	G	H	I	J	K	L
1	Year/Maturity	6	18	30	42	54	66	78	90	102	114	Reserve
2	AY 2000	$42,000	$60,000	$71,000	$80,000	$88,000	$93,000	$ 96,000	$ 98,000	$ 99,000	$ 100,000	$ -
3	AY 2001	$40,000	$59,000	$68,000	$78,000	$87,000	$92,000	$ 95,000	$ 98,000	$ 99,000	$ 100,000	$ 1,000
4	AY 2002	$36,000	$55,000	$65,000	$76,000	$85,000	$91,000	$ 94,000	$ 97,000		$ 98,980	$ 1,980
5	AY 2003	$38,000	$57,000	$66,000	$77,000	$86,000	$92,000	$ 95,000			$ 99,664	$ 4,664
6	AY 2004	$39,000	$58,000	$67,000	$77,000	$87,000	$92,000				$ 99,664	$ 7,664
7	AY 2005	$41,000	$60,000	$70,000	$79,000	$88,000					$ 101,281	$ 13,281
8	AY 2006	$42,000	$61,000	$70,000	$79,000						$ 101,447	$ 22,447
9	AY 2007	$43,000	$62,000	$72,000							$ 105,879	$ 33,879
10	AY 2008	$43,000	$63,000								$ 107,772	$ 44,772
11	AY 2009	$42,000									$ 105,706	$ 63,706
12											Total Reserve	$ 193,393

Figure 7: Calculating Reserve

Now that we have our ultimates, we can calculate how much our reserve needs to be. To do this, all we need to do is add up the ultimates(yellow) and subtract them by the diagonal, or last known data point for each year(red). Thus, our reserve is $193,393.

We want a function that would tell us what percentage of the claims are reported by time x.

$$G(x) = \frac{1}{LDF_x} = \frac{1}{LDF_{60}} = \frac{1}{1.15} = 86.89\%$$

Figure 8: G(x) Formula and Example

For the Development Method the variable $G(x)$ changes the Link Ratio to a percent of the ultimate claims reported to date. From the example, the table for $G(x)$ at certain points in the development is shown below.

Time	LDF_x	Percent Reported to Date: G(x)		
12-ult	2.52	39.73%	=	1/2.52
24-ult	1.71	58.46%	=	1/1.71
36-ult	1.47	68.00%	=	1/1.47
48-ult	1.28	77.87%	=	1/1.28
60-ult	1.15	86.89%	=	1/1.15
72-ult	1.08	92.31%	=	1/1.08
84-ult	1.05	95.32%	=	1/1.05
96-ult	1.02	98.00%	=	1/1.02
108-ult	1.01	99.00%	=	1/1.01

Figure 9: LDF and G(x) Table

Another way of setting reserves is by using the Cape Cod Method. Also known as the Stanard-Buhlmann Method, the Cape Cod Method uses past premiums to set reserves. We will use the same data as the example of the Development Method to keep things consistent.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	0	1	2	3	4	5	6	7	8	9	10	on level earned premium	% Reported
2	2000	$ 42,000	$ 18,000	$ 11,000	$ 9,000	$ 8,000	$ 5,000	$ 3,000	$ 2,000	$ 1,000	$ 1,000	$ 125,001	100.00%
3	2001	$ 40,000	$ 19,000	$ 9,000	$ 10,000	$ 9,000	$ 5,000	$ 3,000	$ 3,000	$ 1,000		$ 126,745	97.64%
4	2002	$ 36,000	$ 19,000	$ 10,000	$ 11,000	$ 9,000	$ 6,000	$ 3,000	$ 3,000			$ 127,793	94.88%
5	2003	$ 38,000	$ 19,000	$ 9,000	$ 11,000	$ 9,000	$ 6,000	$ 3,000				$ 129,622	91.61%
6	2004	$ 39,000	$ 19,000	$ 9,000	$ 10,000	$ 10,000	$ 5,000					$ 131,175	87.67%
7	2005	$ 41,000	$ 19,000	$ 10,000	$ 9,000	$ 9,000						$ 132,852	82.80%
8	2006	$ 42,000	$ 19,000	$ 9,000	$ 9,000							$ 128,923	76.60%
9	2007	$ 43,000	$ 19,000	$ 10,000								$ 131,625	68.38%
10	2008	$ 43,000	$ 20,000									$ 138,678	56.79%
11	2009	$ 42,000										$ 135,767	38.67%

Figure 10: Cape Cod Data

This example has on level earned premium which is the premium for each year changed such that it represents the premium if it were the same as the current year due to inflation, changes in claim costs, or other things. This data also has percent reported which is not needed in the Development Method. Now that we have the data, we will need the Cape Cod Loss Ratio to determine the reserve.

$$Earned\ Loss\ Ratio = \frac{\sum Reported\ Claims}{\sum On\ Level\ Earned\ Premium \cdot \%\ Reported}$$

Figure 11: Cape Cod Loss Ratio Formula

We already have the reported claims, on level earned premium, and the percent reported so now we just need the sums of each.

$\sum Reported\ Claims$	\$827,000
$\sum On\ Level\ Earned\ Premium \cdot \%\ Reported$	\$1,033,750

Figure 12: Cape Cod ELR Table

Using these calculations, we can obtain the Earned Loss Ratio.

$$Earned\ Loss\ Ratio = \frac{\$827{,}000}{\$1{,}033{,}750} = 0.8$$

Figure 13: Earned Loss Ratio

Using the Earned Loss Ratio, we can now go back to the data and use the ratio to estimate our ultimates. For the sake of our example, each on level earned premium will have the same loss ratio. The diagonal is the last observed data point for each accident year.

$$Ultimate_x = Diagonal + \left(1 - \frac{1}{LDF}\right) \cdot On\ Level\ Earned\ Premium_x \cdot Earned\ Loss\ Ratio_x$$

Figure 14: Calculating Ultimates Using Cape Cod Method

Accident Year	Earned Premium	Ultimate Claims
AY 2000	$ 125,001	$ 100,000
AY 2001	$ 126,745	$ 100,086
AY 2002	$ 127,793	$ 99,219
AY 2003	$ 129,622	$ 99,927
AY 2004	$ 131,175	$ 99,931
AY 2005	$ 132,852	$ 101,937
AY 2006	$ 128,923	$ 102,220
AY 2007	$ 131,625	$ 105,181
AY 2008	$ 138,678	$ 105,471
AY 2009	$ 135,767	$ 103,109
	Total Ultimate Claims	$ 1,017,082

Figure 15: Cape Cod Ultimates

We can now calculate the reserve by subtracting the ultimate claims by the reported claims.

$$Reserve = \$190{,}082$$

Figure 16: Cape Cod Reserve

The Development Method and Cape Cod Method are classic ways to establish the reserves. However, there are other problems that arise when using these methods.

1. Interpolation

One such problem is that it is unknown as to what happens in between each 12-month period. It is known how many claims are expected to be submitted at the end of each 12-month period, but the Development Method nor the Cape Cod Method do not show at what rate or when the claims will come in during those 12 months.

2. Variance

Another shortcoming with the Development Method and Cape Cod Method is that they both lack variance estimates and point estimates associated with its estimations. If one wants to have a degree of certainty, then they need variance.

Process

David R. Clark's paper, LDF Curve-Fitting and Stochastic Reserving: A Maximum Likelihood Approach helps model claim distributions arises. Primarily the incremental claim table is much more useful to model curves than the cumulative claim counts. As such, below is the incremental table using the data from the background. This table is comprised of data representing claims reported for each 12-month period.

Incremental Table

Year/Maturity	6	18	30	42	54	66	78	90	102	114
2000	$ 42,000	$ 18,000	$ 11,000	$ 9,000	$ 8,000	$ 5,000	$ 3,000	$ 2,000	$ 1,000	$ 1,000
2001	$ 40,000	$ 19,000	$ 9,000	$ 10,000	$ 9,000	$ 5,000	$ 3,000	$ 3,000	$ 1,000	
2002	$ 36,000	$ 19,000	$ 10,000	$ 11,000	$ 9,000	$ 6,000	$ 3,000	$ 3,000		
2003	$ 38,000	$ 19,000	$ 9,000	$ 11,000	$ 9,000	$ 6,000	$ 3,000			
2004	$ 39,000	$ 19,000	$ 9,000	$ 10,000	$ 10,000	$ 5,000				
2005	$ 41,000	$ 19,000	$ 10,000	$ 9,000	$ 9,000					
2006	$ 42,000	$ 19,000	$ 9,000	$ 9,000						
2007	$ 43,000	$ 19,000	$ 10,000							
2008	$ 43,000	$ 20,000								
2009	$ 42,000									

Figure 17: Incremental Table of Data from Background

We will use this data to create a curve to fit some distributions that best represent the rate at which claims are developed. The two curves that we will be fitting are the Weibull and Loglogistic Distributions. We will use the Weibull and Loglogistic curves as examples for this exercise because they are similar to how we expect our claims to develop. To fit the data to the two distributions, we will need to use the function $G(x)$, which is the percent of the ultimate claims at time x. As mentioned before, $G(x) = \frac{1}{LDF_x}$. However, this does not accurately represent the missing data, this would make the data linear, which is unlikely. To better represent the data, we will model $G(x)$ using the parameters for the Weibull and Loglogistic curves, (ω, θ) to calculate its value. The difference between the two curves is that the Weibull curve has a smaller tail than the Loglogistic curve. The equations for the Weibull curve and Loglogistic curve are shown below.

$$Weibull: G(x|\omega, \theta) = 1 - e^{-(x/\theta)^{\omega}}$$

Figure 18: Weibull Distribution Formula

$$Loglogistic: G(x|\omega, \theta) = \frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}$$

Figure 19: Loglogistic Distribution Formula

Starting with the Weibull Distribution, reasonably approximate ω and θ to get a cumulative distribution function that will approximate the curve. This data, in Figure 20, was changed to the best fitting curve when we found the best fitting ω and θ after completing the Maximum Loglikelihood Estimation (MLE) in Figure 28.

	A	B	C	D	E	F	G	H	I	J	K	L
1	Maturity	0	6	18	30	42	54	66	78	90	102	114
2	Weibull Distribution	0.00	0.36	0.52	0.63	0.71	0.77	0.82	0.85	0.88	0.90	0.92
3	Weibull Distribution/Last term	0/0.92	0.36/0.92	0.52/0.92	0.63/0.92	0.71/0.92	0.77/0.92	0.82/0.92	0.85/0.92	0.88/0.92	0.90/0.92	0.92/0.92
4	Weibull Distribution Truncated	0.00	0.39	0.57	0.69	0.78	0.84	0.89	0.93	0.96	0.98	1.00
5	G(x) with theta and omega assumed											

Figure 20: Weibull Distribution truncated

Row 2 is $G(x)$ for the Weibull distribution. Notice that our ultimate, 114, is not equal to 1 since the Weibull distribution has infinite domain and is asymptotic to 1. As a result, we needed to truncate the distribution so that it reaches 1 during the last time interval at 114. To do this we will divide the distribution by the last term(red). We need to do this since our ultimate is the total number of claims and there can't be any claims after that, so by truncating the Weibull distribution, it will match our data distribution. The new distribution that will be used is in yellow.

$$\mu_{AY:x,y} = ULT_{AY} \cdot [G(y|\omega,\theta) - G(x|\omega,\theta)]$$

Figure 21: Expected Incremental Loss Formula

This formula is the expected incremental loss for each accident year between time x and time y based on the ultimate for each accident year and the type of distribution. This equation will have n+2 parameters where n is the number of ultimates, (one ultimate per accident year) and the 2 are ω and θ.

$$Loglikelihood\ Function\!:\ \ \sum c_{AY:x,y} \cdot \ln(\mu_{AY:x,y}) - \mu_{AY:x,y}$$

Figure 22: Loglikelihood Function

See Appendix 1 for where this came from

This is the Loglikelihood Function that we will use to best estimate $\mathrm{G}(x)$. $c_{AY:x,y}$ is the actual incremental data that we have from Figure 17: Incremental Table and $\mu_{AY:x,y}$ is the expected incremental data. We will use this term to best estimate ω and θ that best fits the data. Each incremental claim is assumed to follow an overdispersed poisson distribution which is why we use Appendix 1 for the Loglikelihood Function.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1		Year/Maturity	6	18	30	42	54	66	78	90	102	114		Ultimate
2	(G(x)-G(x-1))*Ultimate	2000	38.99	18.21	12.02	8.57	6.36	4.85	3.77	2.97	2.36	1.90		100.00
3		2001	39.35	18.38	12.13	8.65	6.42	4.89	3.80	2.99	2.38	1.92		100.92
4		2002	39.50	18.45	12.18	8.69	6.45	4.91	3.81	3.00	2.39	1.92		101.32
5		2003	39.93	18.65	12.31	8.78	6.52	4.97	3.86	3.04	2.42	1.94		102.40
6		2004	40.30	18.82	12.43	8.86	6.58	5.01	3.89	3.06	2.44	1.96		103.36
7		2005	40.77	19.04	12.57	8.96	6.65	5.07	3.94	3.10	2.47	1.99		104.56
8		2006	39.59	18.49	12.21	8.71	6.46	4.92	3.82	3.01	2.40	1.93		101.55
9		2007	40.55	18.94	12.51	8.92	6.62	5.04	3.92	3.08	2.46	1.97		104.01
10		2008	42.94	20.06	13.24	9.44	7.01	5.34	4.15	3.27	2.60	2.09		110.14
11		2009	42.00	19.61	12.95	9.23	6.86	5.22	4.06	3.19	2.54	2.05		107.72
12														
13														
14		G(x)-G(x-1)	0.39	0.18	0.12	0.09	0.06	0.05	0.04	0.03	0.02	0.02		
15			this is the expected incremental loss table that we can match up to each actual loss using Weibull											

Figure 23: Obtaining Expected Incremental Loss using Weibull

When calculating the formula in Figure 22: Loglikelihood , we need to first obtain the data in the correct form, starting with $\mu_{AY:x,y}$, from Figure 21. In this table, we calculate

$[G(y|\omega, \theta) - G(x|\omega, \theta)]$, where y is any time period after x. Thus, why it is $G(x) - G(x - 1)$ resulting in incremental data. Using the Weibull distribution for the function $G(x)$, we multiplied the ultimate of each accident year by the Weibull distribution to get $\mu_{AY:x,y}$.

	A	B	C	D	E	F	G	H	I	J	K	L
1	ln(u)	Year/Maturity	6	18	30	42	54	66	78	90	102	114
2		2000	3.66	2.90	2.49	2.15	1.85	1.58	1.33	1.09	0.86	0.64
3		2001	3.67	2.91	2.50	2.16	1.86	1.59	1.33	1.10	0.87	0.65
4		2002	3.68	2.91	2.50	2.16	1.86	1.59	1.34	1.10	0.87	0.65
5		2003	3.69	2.93	2.51	2.17	1.87	1.60	1.35	1.11	0.88	0.66
6		2004	3.70	2.93	2.52	2.18	1.88	1.61	1.36	1.12	0.89	0.67
7		2005	3.71	2.95	2.53	2.19	1.90	1.62	1.37	1.13	0.90	0.69
8		2006	3.68	2.92	2.50	2.16	1.87	1.59	1.34	1.10	0.87	0.66
9		2007	3.70	2.94	2.53	2.19	1.89	1.62	1.37	1.13	0.90	0.68
10		2008	3.76	3.00	2.58	2.25	1.95	1.68	1.42	1.18	0.96	0.74
11		2009	3.74	2.98	2.56	2.22	1.93	1.65	1.40	1.16	0.93	0.72

Figure 24: Natural Log of Expected Incremental Claims

After obtaining $\mu_{AY:x,y}$ from the Figure 23, we can proceed to the next step in getting the Figure 22: Loglikelihood Function. In this step, we took the natural log of each $\mu_{AY:x,y}$ which is the second term of the Loglikelihood.

	A	B	C	D	E	F	G	H	I	J	K	L
1	c*ln(u)	Year/Maturity	6	18	30	42	54	66	78	90	102	114
2		2000	153.86	52.23	27.36	19.34	14.81	7.89	3.98	2.17	0.86	0.64
3		2001	146.90	55.31	22.46	21.58	16.74	7.94	4.00	3.29	0.87	
4		2002	132.35	55.38	25.00	23.78	16.77	9.55	4.02	3.30		
5		2003	140.11	55.59	22.60	23.89	16.87	9.62	4.05			
6		2004	144.16	55.76	22.68	21.82	18.84	8.06				
7		2005	152.03	55.98	25.32	19.74	17.06					
8		2006	154.50	55.43	22.52	19.48						
9		2007	159.21	55.88	25.26							
10		2008	161.68	59.97								
11		2009	156.98									

Figure 25: Actual Data multiplied by Natural Log of Expected Incremental Claims

This next step is where we multiplied c, our actual data for known incremental claims, by Figure 24: Natural Log of Expected Incremental Claims. This will complete the first two parts of the Loglikelihood Function, which was $c_{AY:x,y} \cdot \ln(\mu_{AY:x,y})$. Now, we just need to subtract $\mu_{AY:x,y}$.

	A	B	C	D	E	F	G	H	I	J	K	L
1	c*ln(u)-u	Year/Maturity	6	18	30	42	54	66	78	90	102	114
2		2000	114.87	34.03	15.33	10.76	8.44	3.04	0.21	-0.79	-1.50	-1.26
3		2001	107.55	36.93	10.33	12.93	10.32	3.05	0.21	0.30	-1.51	
4		2002	92.85	36.94	12.82	15.09	10.33	4.64	0.20	0.30		
5		2003	100.18	36.94	10.28	15.12	10.35	4.65	0.19			
6		2004	103.86	36.94	10.25	12.96	12.26	3.05				
7		2005	111.26	36.94	12.74	10.77	10.40					
8		2006	114.91	36.94	10.31	10.77						
9		2007	118.66	36.94	12.76							
10		2008	118.73	39.91								
11		2009	114.98									

Figure 26: Loglikelihood Estimate Table

In this step, we subtract $\mu_{AY:x,y}$ from Figure 25. Putting all the previous steps together, we get the Loglikelihood Estimate Table, which is the Loglikelihood Function in tabular format. The final step is to take the sum of this table.

$$MLE\ term = \sum c_{AY:x,y} \cdot \ln(\mu_{AY:x,y}) - \mu_{AY:x,y}$$

$$MLE = 1690.45$$

Figure 27: The Maximum Loglikelihood Estimate Development Weibull

This completes the formula for Figure 22: Loglikelihood Function and we are now able to use this term to find a θ and ω for the Weibull Distribution that will match our incremental data (Figure 17).

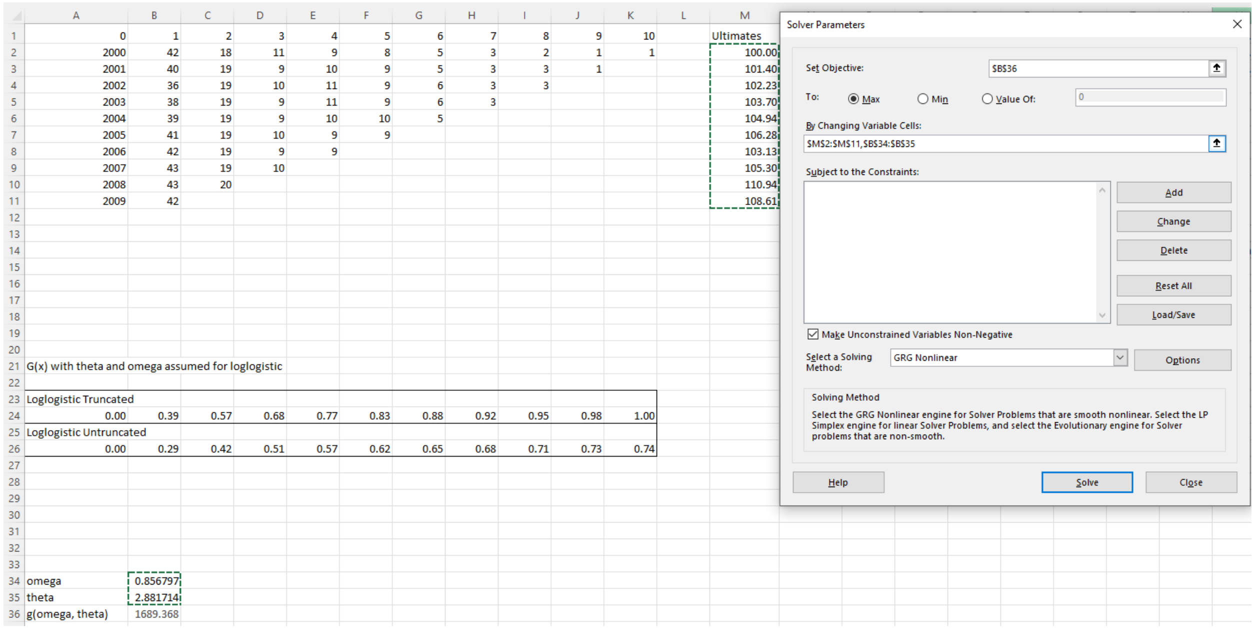

Figure 28: Using Solver to find the best ω and θ

When using Solver in Microsoft Excel, we first set an equation for G(θ, ω), which is the Loglikelihood Function in Figure 22: Loglikelihood . This will be our objective in Solver. Next, we need to choose the variables in order to maximize the objective. These parameters (n+2) for the Weibull Distribution were the number of ultimate's (n), ω and θ. As expected from Figure 27: The Maximum Loglikelihood Estimate .

ω	0.75
θ	2.98

Figure 29: ω and θ using the Weibull Distribution

For the Loglogistic Distribution, we made a video going over the same steps as the Weibull Distribution in Excel. The Loglogistic Distribution has the same parameters as the Weibull Distribution, number of ultimate's (n), ω and θ.

Development Method Loglogistic Distribution Video

Figure 30: Loglogistic Method Video

To recap, using the equation from Figure 19: Loglogistic , we assumed that we knew the ω and θ. We were then able to get the Loglogistic distribution from $x = [0: 120]$ as shown in Figure 31, with the top distribution being the truncated Loglogistic distribution such that the curve reaches 1 at $x = 10$.

G(x) with theta and omega assumed for loglogistic										
Loglogistic Truncated										
0.00	0.39	0.57	0.68	0.77	0.83	0.88	0.92	0.95	0.98	1.00
Loglogistic Untruncated										
0.00	0.29	0.42	0.51	0.57	0.62	0.65	0.68	0.71	0.73	0.74

Figure 31: Loglogistic Distribution Development from X= [0,10]

We used this data to get the incremental table for $\mu_{AY:x,y}$. This table is shown below in Figure 32: Expected Incremental Loss using Development Loglogistic.

M	N	O	P	Q	R	S	T	U	V	W	X
Ultimates	G(x)-G(x-1)	0.39	0.18	0.12	0.08	0.06	0.05	0.04	0.03	0.03	0.02
100.00		38.67	18.12	11.59	8.22	6.20	4.87	3.94	3.27	2.76	
101.40		39.21	18.37	11.75	8.34	6.29	4.94	4.00	3.31		
102.23		39.54	18.52	11.85	8.41	6.34	4.98	4.03			
103.70		40.10	18.79	12.02	8.53	6.43	5.05				
104.94		40.58	19.01	12.16	8.63	6.51					
106.28		41.10	19.25	12.32	8.74						
103.13		39.88	18.68	11.95							
105.30		40.72	19.07								
110.94		42.90	20.10								
108.61		42.00									
			this is the expected incremental loss table that we can match up to each actual loss								

Figure 32: Expected Incremental Loss using Development Loglogistic

From this table we are able to do the same steps to get the MLE for the Loglogistic Distribution as we did to get the MLE using the Weibull Distribution. As before this value is the same as when using the solver function in Excel. This value was 1689.368. The solver function did not show up in the recording it is shown below in Figure 34: Solver Function for Loglogistic. As before, the solver function maximizes G(ω, θ) by using the ultimate claims in column M and changing (ω, θ) in B34 and B35 which are ω and θ respectively. These cells were highlighted in the video. The final values of ω and θ are shown below.

ω	0.8568
θ	2.881

Figure 33: ω and θ using the Loglogistic Distribution

Below in Figure 34: Solver Function for Loglogistic shows the solver function used to obtain the Maximum Loglikelihood Estimate by changing the values of ω and θ.

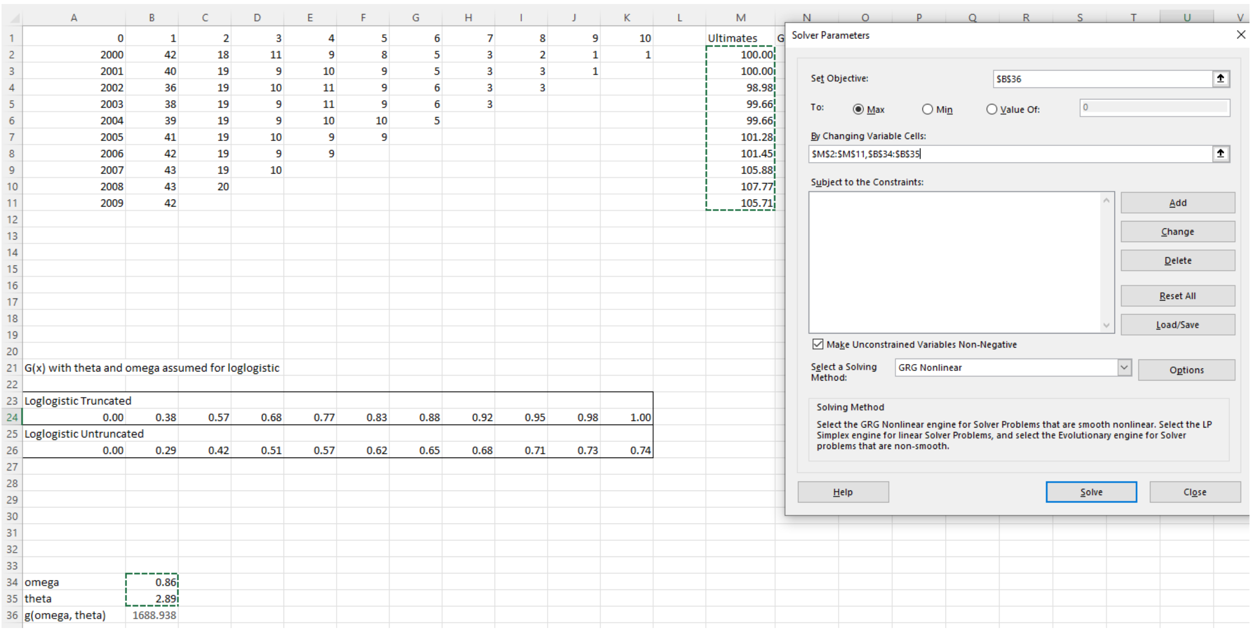

Figure 34: Solver Function for Loglogistic

We will now do these same steps for the Cape Cod Method using the Loglogistic Distribution.

Loglogistic Untruncated											
	0.00	0.20	0.29	0.35	0.39	0.43	0.46	0.49	0.51	0.54	0.55
Loglogistic Truncated											
	0.00	0.35	0.51	0.63	0.71	0.78	0.84	0.89	0.93	0.97	1.00

Figure 35: Loglogistic Distribution Truncated Cape Cod from x= [0,10]

Using this distribution, we can obtain $\mu_{AY:x,y}$ our expected incremental data.

O	P	Q	R	S	T	U	V	W	X	Y	Z
Ultimates	G(x)-G(x-1)	0.35	0.16	0.11	0.09	0.07	0.06	0.05	0.04	0.04	0.03
100.00		35.46	16.00	11.14	8.54	6.89	5.74	4.90	4.25	3.74	3.33
101.40		35.95	16.23	11.30	8.66	6.99	5.82	4.97	4.31	3.79	3.38
102.23		36.25	16.36	11.39	8.73	7.04	5.87	5.01	4.35	3.83	3.40
103.70		36.77	16.59	11.55	8.86	7.15	5.96	5.08	4.41	3.88	3.45
104.94		37.21	16.79	11.69	8.96	7.23	6.03	5.14	4.46	3.93	3.49
106.28		37.68	17.01	11.84	9.08	7.32	6.10	5.21	4.52	3.98	3.54
103.14		36.57	16.51	11.49	8.81	7.11	5.92	5.05	4.39	3.86	3.43
105.30		37.34	16.85	11.73	8.99	7.26	6.05	5.16	4.48	3.94	3.51
110.94		39.34	17.75	12.36	9.48	7.64	6.37	5.44	4.72	4.15	3.69
108.61		38.51	17.38	12.10	9.28	7.48	6.24	5.32	4.62	4.06	3.62
		this is the expected incremental loss table that we can match up to each actual loss									

Figure 36: Expected Incremental Loss using Cape Cod Loglogistic

Following the same steps as the previous two distributions using the Development Method, we get the Loglikelihood Estimate.

$$Maximum\ Loglikelihood\ Estimate = 1684.63$$

Figure 37: Maximum Loglikelihood Estimate Cape Cod Loglogistic

Using the Solver Function we can change the parameters, Earned Loss Ratio, ω, and θ to maximize the loglikelihood estimate.

ELR	0.80
ω	0.71
θ	7.35

Figure 38: ELR, ω and θ using Cape Cod Loglogistic

Now that we have our parameters to fit our models, we need to calculate the variance.

Process Variance

Our next step is to find the variance occurring around the incremental data. When doing so, we will be calculating process variance and parameter variance. The first type, process variance, is the variance around the expectation of $G(x)$, the expected loss emergence pattern. This is the random amount for each of the expected claims that come in for each time period and the ultimates. This is denoted by the following formula.

$$\frac{Variance}{Mean} = \sigma^2 = \frac{1}{n-p} \cdot \Sigma_{AY}^{n} \left(\frac{\left(c_{AY} - \mu_{AY}\right)^2}{\mu_{AY}} \right)$$

Figure 39: Variance scale factor

In this formula, σ^2 is a scale factor (not actual variance). The actual incremental loss emergence is c_{AY}, and the expected incremental loss emergence is μ_{AY}. This formula is the variance around each of the expected values around the distribution calculated in the Process section.

n	55
Number of Ultimates	10
θ	1
ω	1
p= (Ultimates+ω+θ)	12
n – p	43

Figure 40: n is number data points, p is the number of parameters

This table is the number of variables we have for the Weibull and Loglogistic Distributions using the Development Method when calculating the process variance. From the previous formula, n is the number of unknown data points and p is the number of parameters. The process variance for the reserve is calculated differently than the data points are.

$$Process\,Variance = \sigma^2 \cdot Reserve$$

Figure 41: Process Variance of the Reserve

In this formula, the process variance of the reserve is calculated using σ^2 from Figure 39: where $\mu_{AY:x,y}$ is the expected incremental loss for that specific year. As done for the reserves, σ^2 could be multiplied by the entire table so that the process variance for each estimated point is that point of data multiplied by σ^2.

$$\sigma^2 = 0.45$$

Figure 42: Scale Factor for Variance Development Loglogistic

We found σ^2 and now we will need to find what the reserve is.

Reserve	0	0	0	0	0	0	0	0	0	0	Total Reserve in Thousands
	0	0	0	0	0	0	0	0	0	2.40	
	0	0	0	0	0	0	0	0	2.82	2.42	219.54
	0	0	0	0	0	0	0	3.39	2.86	2.45	
	0	0	0	0	0	0	4.14	3.43	2.89	2.48	
	0	0	0	0	0	5.18	4.19	3.47	2.93	2.51	
	0	0	0	0	6.39	5.02	4.07	3.37	2.84	2.44	
	0	0	0	8.66	6.53	5.13	4.15	3.44	2.90	2.49	
	0	0	12.86	9.12	6.88	5.40	4.37	3.62	3.06	2.62	
	0	19.67	12.59	8.93	6.73	5.29	4.28	3.55	3.00	2.57	

Figure 43: Development Loglogistic Total Reserve Calculation in Thousands

$$\text{Total Reserve} = \$219{,}536$$

Figure 44: Total Reserve Development Loglogistic

From the figure above, we found the expected incremental loss to be 219.54. So now we just need to multiply σ^2 and the sum of the future reserves together to get the process variance.

$$\text{Process Variance} = \sigma^2 \cdot \text{Reserve} = 0.45 \cdot \$219{,}536 = \$99{,}367$$

Figure 45: Process Variance of the Reserve Loglogistic Development

The process variance for the Cape Cod Method is calculated the same way as the Development Method except for the parameters.

n	55
ELR	1
θ	1
ω	1
p = (ELR, ω, θ)	3
n – p	52

Calculating the process variance for the Cape Cod Method using the Loglogistic Distribution will follow the same steps.

$$\sigma^2 = 0.50$$

Figure 46: Process Variance Scale Factor Cape Cod Loglogistic

The next step is to find the reserve.

Reserve	0	0	0	0	0	0	0	0	0	0	Total Reserve in Thousands
	0	0	0	0	0	0	0	0	0	3.38	
	0	0	0	0	0	0	0	0	3.83	3.40	255.82
	0	0	0	0	0	0	0	4.41	3.88	3.45	
	0	0	0	0	0	0	5.14	4.46	3.93	3.49	
	0	0	0	0	0	6.10	5.21	4.52	3.98	3.54	
	0	0	0	0	7.11	5.92	5.05	4.39	3.86	3.43	
	0	0	0	8.99	7.26	6.05	5.16	4.48	3.94	3.51	
	0	0	12.36	9.48	7.64	6.37	5.44	4.72	4.15	3.69	
	0	17.38	12.10	9.28	7.48	6.24	5.32	4.62	4.06	3.62	

Figure 47: Cape Cod Loglogistic Total Reserve Calculation in Thousands

$$Total\ Reserve = \$255{,}818$$

Figure 48: Total Reserve Cape Cod Loglogistic

$$\text{Process Variance} = \sigma^2 \cdot \text{Reserve} = 0.50 \cdot \$255{,}818 = \$129{,}035$$

Figure 49: Process Variance of the Reserve Loglogistic Cape Cod

Parameter Variance

The next step is to account for the variance of the parameters themselves. We call the variance of the parameters, parameter variance. For the Development Method, if we were to the process variance for the Development Method, we would need to find the covariance matrix for θ, ω, and the 10 ultimates which would result in a 12x12 covariance matrix. After we find the parameter variance, we will add the parameter variance to the process variance to get the total variance. This process is more complicated as it will be an information matrix and σ.

$$I = \begin{bmatrix} \sum \frac{\partial^2 l_{1,t}}{\partial ULT_1^2} & 0 & \cdots & 0 & \sum \frac{\partial^2 l_{1,t}}{\partial ULT_1 \partial\omega} & \sum \frac{\partial^2 l_{1,t}}{\partial ULT_1 \partial\theta} \\ 0 & \sum \frac{\partial^2 l_{2,t}}{\partial ULT_2^2} & \dots & 0 & \sum \frac{\partial^2 l_{2,t}}{\partial ULT_2 \partial\omega} & \sum \frac{\partial^2 l_{2,t}}{\partial ULT_2 \partial\theta} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \sum \frac{\partial^2 l_{n,t}}{\partial ULT_n^2} & \sum \frac{\partial^2 l_{n,t}}{\partial ULT_n \partial\omega} & \sum \frac{\partial^2 l_{n,t}}{\partial ULT_n \partial\theta} \\ \sum \frac{\partial^2 l_{1,t}}{\partial\omega \partial ULT_1} & \sum \frac{\partial^2 l_{2,t}}{\partial\omega \partial ULT_2} & \dots & \sum \frac{\partial^2 l_{n,t}}{\partial\omega \partial ULT_n} & \sum \frac{\partial^2 l_{y,t}}{\partial\omega^2} & \sum \frac{\partial^2 l_{y,t}}{\partial\omega \partial\theta} \\ \sum \frac{\partial^2 l_{1,t}}{\partial\theta \partial ULT_1} & \sum \frac{\partial^2 l_{2,t}}{\partial\theta \partial ULT_2} & \dots & \sum \frac{\partial^2 l_{n,t}}{\partial\theta \partial ULT_n} & \sum \frac{\partial^2 l_{y,t}}{\partial\theta \omega} & \sum \frac{\partial^2 l_{y,t}}{\partial\theta^2} \end{bmatrix}$$

Figure 50: LDF Method Information Matrix

The information matrix uses second order partial derivatives of the twelve parameters based on the distribution. We will do the parameter variance based on the

Loglogistic Distribution. We will then use the inverse of the Information Matrix and the scale factor, σ^2 to get the covariance matrix.

$$\Sigma = \begin{bmatrix} VAR(ULT_i) & COV(ULT_i, \omega) & COV(ULT_i, \theta) \\ COV(\omega,\ ULT_i) & Var(\omega) & COV(\omega, \theta) \\ COV(\theta, ULT_i) & COV(\theta, \omega) & VAR(\theta) \end{bmatrix} \geq\ -\sigma^2 \cdot I^{-1}$$

Figure 51: LDF Covariance Matrix

After finding the parameter variance for each ultimate, we will then find the parameter variance of the total reserve. The parameter variance of the reserve, R, uses the covariance matrix, $\sum$.

$$Var(E[R]) = (\partial R)' \cdot \sum \cdot (\partial R)$$

Figure 52: Parameter Variance of the Reserve

In order to find the Parameter Variance of the reserve, we need to find the partial derivative of the reserve.

$$\partial R =\ \left\langle \left\{ \frac{\partial R}{\partial ULT_i} \right\}_i^n, \frac{\partial R}{\partial \theta}, \frac{\partial R}{\partial \omega} \right\rangle$$

Figure 53: Partial Derivative of the Reserve with Respect to the Parameters

Now we need to find the partial derivatives of the reserve with respect to each of the parameters.

$$\frac{\partial R}{\partial ULT_i} = \sum \left(G(y_i) - (G(x_i) \right)$$

Figure 54: Partial of the Reserve with Respect to each Ultimate

$$\frac{\partial R}{\partial \theta} = \sum ULT_i \cdot \left(\frac{\partial G(y_i)}{\partial \theta} - \frac{\partial G(x_i)}{\partial \theta} \right)$$

Figure 55: Partial of the Reserve with Respect to θ

$$\frac{\partial R}{\partial \omega} = \sum ULT_i \cdot \left(\frac{\partial G(y_i)}{\partial \omega} - \frac{\partial G(x_i)}{\partial \omega} \right)$$

Figure 56: Partial of the Reserve with Respect to ω

The Cape Cod Method has three parameters calculated in the parameter variance. These three parameters are the Earned Loss Ratio, θ and ω. The Cape Cod parameter variance doesn't impact the reserve, it only impacts the loss emergence. The Cape Cod parameter variance uses a similar Covariance Matrix.

$$\Sigma = \begin{bmatrix} VAR(ELR) & COV(ELR,\omega) & COV(ELR,\theta) \\ COV(\omega,ELR) & Var(\omega) & COV(\omega,\theta) \\ COV(\theta,ELR) & COV(\theta,\omega) & VAR(\theta) \end{bmatrix} \geq -\sigma^2 \cdot I^{-1}$$

Figure 57: Cape Cod Covariance Matrix

The Cape Cod Information Matrix is a lot simpler than the Development Method Information Matrix. As opposed to a 12x12 matrix, the Cape Cod Information Matrix is a 3x3.

$$I = \begin{bmatrix} \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial ELR^2} & \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial ELR \partial \omega} & \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial ELR \partial \theta} \\ \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial \omega \partial ELR} & \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial \omega^2} & \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial \omega \partial \theta} \\ \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial \theta \partial ELR} & \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial \theta \partial \omega} & \sum_{\mathrm{y,t}} \frac{\partial^2 l_{y,t}}{\partial \theta^2} \end{bmatrix}$$

Figure 58: Cape Cod Information Matrix

The information matrix uses partial derivatives of the three parameters based on the distribution. We will do the parameter variance of Cape Cod based on the Loglogistic Distribution. We will then use this to get the covariance matrix.

Since there are second degree partial derivatives, there are specific formulas needed for each part of the Information Matrix. Using Figure 58 and Appendix 2, we get the following Information Matrix.

$$I = \begin{bmatrix} -1292.19 & -99.53 & -44.00 \\ -99.53 & -1734.68 & 10.46 \\ -44.00 & 10.46 & -23.59 \end{bmatrix}$$

Figure 59: Cape Cod Information Matrix Completed

After calculating the Information Matrix, we need to take the inverse of it for the Covariance Matrix.

$$I^{-1} = \begin{bmatrix} -7.31 \cdot 10^{-4} & 5.03 \cdot 10^{-5} & 1.39 \cdot 10^{-3} \\ 3.38 \cdot 10^{-5} & -5.80 \cdot 10^{-4} & -3.20 \cdot 10^{-4} \\ -1.35 \cdot 10^{-3} & -1.63 \cdot 10^{-4} & -4.00 \cdot 10^{-2} \end{bmatrix}$$

Figure 60: Inverse Information Matrix Cape Cod

Now that we have the Inverse Information Matrix, we multiply it by the $-\sigma^2$ calculated from Figure 46: Process Variance Scale Factor Cape Cod Loglogistic

$$\Sigma \geq \begin{bmatrix} 3.70 \cdot 10^{-4} & -2.55 \cdot 10^{-5} & -7.01 \cdot 10^{-4} \\ -1.71 \cdot 10^{-5} & 2.94 \cdot 10^{-4} & 1.62 \cdot 10^{-4} \\ 6.83 \cdot 10^{-4} & 8.28 \cdot 10^{-5} & 2.02 \cdot 10^{-2} \end{bmatrix} = -\sigma^2 \cdot I^{-1}$$

Figure 61: Cape Cod Covariance Matrix Calculation

In our example, the parameter variance is very low, close to 0 for all parameters. In order to get the parameter variance, we only need to take the ELR variance as that is the only one that affects the reserve.

$$Parameter\ Variance = VAR(ELR) \cdot Premium^2 = (3.70 \cdot 10^{-4}) \cdot \$1{,}308{,}182^2$$

$$Parameter\ Variance = \$633{,}226{,}958$$

Figure 62: Parameter Variance of the Reserve

This uses the same reserve calculation from Figure 48: Total Reserve Cape Cod Loglogistic and the first term from Figure 61: Cape Cod Covariance Matrix Calculation.

Total Variance

The total variance is the combined variance of the process variance and the parameter variance. The Figure 49: Process Variance of the Reserve Loglogistic Cape Codand Figure 62: Parameter Variance of the Reserve are needed to obtain the final variance

$$Total\ Variance = Process\ Variance + Parameter\ Variance$$

Figure 63: Total Variance Formula

$$Total\ Variance = \$129{,}035 + \$633{,}226{,}958 = \$633{,}355{,}993$$

Figure 64: Total Variance Cape Cod Loglogistic

The total variance of the Cape Cod Method using the Loglogistic Distribution to forecast future data is $129,129. Almost all the variance comes from the process variance which is half of the total reserve.

$$Standard\ Deviation = \$25,167$$

Figure 65: Standard Deviation Cape Cod Loglogistic

The total standard deviation is 9.84% of the total estimated reserve.

Conclusion

We were able to model both the reserve using both the Development Method and the Cape Cod Method using the Weibull and Loglogistic Distributions. Using these distributions, we were able to obtain an ω and θ that would more accurately represent the rate at which claims will emerge. We then found the process variance for both the Development Method and the Cape Cod Method modeled from the Loglogistic Distribution in which the Development Method had a lower process variance than the Cape Cod Method. Finally, we calculated the parameter variance of the model that was formed from the Cape Cod Method and Loglogistic Distribution. We found that the total standard deviation of the model used by the Cape Cod Method and Loglogistic Distribution was less than 10% of the estimated reserve.

Appendix 1

It is unlikely that we will find parameters for our expected distribution that perfectly matches our given data at each point in the development. To account for this we need to allow dispersion between what we expect at each point in the development for incremental loss and the actual incremental loss. However how we weigh the error between our expectation and the actual data will change where our expected distribution is most accurate. For example we could decide a distribution for u based on minimizing the total error defined $\sum_{i=1}^{n} |c_i - \mu_i|$ or the relative error defined $\sum_{i=1}^{n} \frac{|c_i - \mu_i|}{c_i}$, but what we are actually trying to capture is a distribution where of our actual incremental loss data points occur in other words an estimation that maximizes the joint likelihood or MLE. The joint likelihood is the likelihood that n multiple events all occur and assuming they are independent then $p(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} p(A_i)$ So we want to find ω, θ and the ELR/Ultimates. We don't want to bias our distribution to the beginning of the distribution or to the end so want to assume that the randomness of incremental claim count loss is proportional to the claim count. Thus, we would want to model incremental claims in over-dispersed Poisson of $\frac{e^{-\lambda}\lambda^x}{x!}$ as variance is equal to and thus proportional to the mean. Taking the maximum of the logarithm of this joint density gives us the most likely parameters.

Appendix 2

Derivatives of the Loglikelihood Function

$$\frac{\partial^2 l}{\partial ELR^2} = \sum\left(\frac{-\mathrm{c_{i,t}}}{ELR^2}\right)$$

$$\frac{\partial^2 l}{\partial ELR \partial \omega} = -\sum -P_i \cdot \left[\frac{\partial \mathrm{G}(\mathrm{x}_t)}{\partial \omega} - \frac{\partial \mathrm{G}(\mathrm{x}_{t-1})}{\partial \omega}\right]$$

$$\frac{\partial^2 l}{\partial ELR \partial \theta} = -\sum -P_i \cdot \left[\frac{\partial \mathrm{G}(\mathrm{x}_t)}{\partial \theta} - \frac{\partial \mathrm{G}(\mathrm{x}_{t-1})}{\partial \theta}\right]$$

$$\frac{\partial^2 l}{\partial \omega^2} = \sum\left\{\left[\frac{-\mathrm{c_{i,t}}}{\left(G(x_t) - G(x_{t-1})\right)^2}\right] \cdot \left[\frac{\partial \mathrm{G}(\mathrm{x}_t)}{\partial \omega} - \frac{\partial \mathrm{G}(\mathrm{x}_{t-1})}{\partial \omega}\right]^2 + \left[\frac{\mathrm{c_{i,t}}}{G(x_t) - G(x_{t-1})} - ELR \cdot P_i\right] \cdot \left[\frac{\partial^2 \mathrm{G}(\mathrm{x}_t)}{\partial \omega^2} - \frac{\partial^2 \mathrm{G}(\mathrm{x}_{t-1})}{\partial \omega^2}\right]\right\}$$

$$\frac{\partial^2 l}{\partial \omega \partial \theta} = \sum\left\{\left[\frac{-\mathrm{c_{i,t}}}{\left(G(x_t) - G(x_{t-1})\right)^2}\right] \cdot \left[\frac{\partial \mathrm{G}(\mathrm{x}_t)}{\partial \omega} - \frac{\partial \mathrm{G}(\mathrm{x}_{t-1})}{\partial \omega}\right] \cdot \left[\frac{\partial \mathrm{G}(\mathrm{x}_t)}{\partial \theta} - \frac{\partial \mathrm{G}(\mathrm{x}_{t-1})}{\partial \theta}\right] + \left[\frac{c}{G(x_t) - G(x_{t-1})} - ELR \cdot P_i\right] \cdot \left[\frac{\partial^2 \mathrm{G}(\mathrm{x}_t)}{\partial \omega \theta} - \frac{\partial^2 \mathrm{G}(\mathrm{x}_{t-1})}{\partial \omega \partial \theta}\right]\right\}$$

$$\frac{\partial^2 l}{\partial \theta^2} = \sum\left\{\left[\frac{-\mathrm{c_{i,t}}}{\left(G(x_t) - G(x_{t-1})\right)^2}\right] \cdot \left[\frac{\partial \mathrm{G}(\mathrm{x}_t)}{\partial \theta} - \frac{\partial \mathrm{G}(\mathrm{x}_{t-1})}{\partial \theta}\right]^2 + \left[\frac{\mathrm{c_{i,t}}}{G(x_t) - G(x_{t-1})} - ELR \cdot P_i\right] \cdot \left[\frac{\partial^2 \mathrm{G}(\mathrm{x}_t)}{\partial\ \theta^2} - \frac{\partial^2 \mathrm{G}(\mathrm{x}_{t-1})}{\partial \theta^2}\right]\right\}$$

Loglogistic Distribution

The following formulas are used in the Derivatives of the Loglikelihood Function

$$G(x) = \frac{x^{\omega}}{x^{\omega} + \theta^{\omega}} = 1 - \left(\frac{1}{1 + \left(\frac{x}{\theta}\right)^{\omega}} \right)$$

$$\frac{\partial G(x)}{\partial \omega} = \left(\frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \left(\frac{\theta^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \ln\left(\frac{x}{\theta}\right)$$

$$\frac{\partial G(x)}{\partial \theta} = \left(\frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \left(\frac{\theta^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \left(\frac{-\omega}{\theta}\right)$$

$$\frac{\partial^2 G(x)}{\partial \omega^2} = \left(\frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \left(\frac{\theta^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \ln\left(\frac{x}{\theta}\right)^2 \cdot \left[1 - 2 \cdot \left(\frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}\right)\right]$$

$$\frac{\partial^2 G(x)}{\partial \omega \partial \theta} = \left(\frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \left(\frac{\theta^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \left(\frac{-1}{\theta}\right) \cdot \left\{1 + \omega \cdot \ln\left(\frac{x}{\theta}\right) \cdot \left[1 - 2 \cdot \left(\frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}\right)\right]\right\}$$

$$\frac{\partial^2 G(x)}{\partial \theta^2} = \left(\frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \left(\frac{\theta^{\omega}}{x^{\omega} + \theta^{\omega}}\right) \cdot \left(\frac{\omega}{\theta^2}\right) \cdot \left\{1 + \omega \cdot \left[1 - 2 \cdot \left(\frac{x^{\omega}}{x^{\omega} + \theta^{\omega}}\right)\right]\right\}$$