Worcester Polytechnic Institute

Major Qualifying Project

Explorations of Sequence Risk

Authors:

Jon Cohen

Jeremy John

Jake Steinberg

Advisor:

Professor Jon Abraham

Professor Barry Posterro

April 25th, 2019

Table of Contents

Abstract

Retirement planning can be thought of as two distinct periods. The accumulation phase during working years, and a decumulation phase after retirement. The accumulation phase is the period an individual invests and accumulates assets that would fund their retirement. The second phase is the decumulation phase, that starts the day an individual retires. During these phases, the individual's investment portfolio experiences varying returns, contributions, and withdrawals. Periodic contributions and varying returns result in the portfolio being impacted by many risks during both phases. We chose to focus on sequence risk, which is the uncertainty created by the order of a specific set of returns.

Our work is based on parts from Clare, Seaton, Smith, and Thomas's paper on sequence risk. Simulating portfolios' accumulation and decumulation phases illustrates how different sequences of the same set of returns resulted in different portfolio values. After creating 100,000 permutations each of the accumulation and decumulation portions, we analyzed each permutation's perfect withdrawal rate (PWR) . The perfect withdrawal rate is the percent of the final accumulated value taken out each year that would exactly exhaust the portfolio at the end of retirement. Since the perfect withdrawal rate can only be calculated in hindsight, we attempted to analyze them to gain insight into an ideal rate that could be used in actuality. A common approach for many retirees is the "4 Percent Rule", popularized by William Bengen in the 1990s, in which 4% of the accumulated value should be taken out each year. By calculating and creating a distribution of PWRs, the evidence suggests that following the 4% Rule is very conservative, often leading to excess money by the end of retirement.

Introduction

There are very few circumstances that are a part of our daily lives that are impervious to uncertainty. Our lives are inherently shaped by whims and unpredictability that we often overlook, from the moment we are born to our final seconds. The moment we begin working, we start a pivotal stage in our lives that plays a role in determining the quality of life we can afford in the future. At the onset of this phase we have a plan that we would like our lives to follow. Most of us disregard the uncertainty associated with our plans, the obstacles or achievements that will shape our futures; we always expect the status quo. This leads to many people being ill-prepared for retirement, expecting average returns and the average longevity when trying to outline their retirements.

Planning the latter stage of our lives is complicated as we face similar expenses, but without the same steady income. The preparation for retirement starts, during the accumulation phase, as individuals accumulate and invest assets that will be used to fund one's retirement. This phase starts when we begin our careers and ends the day we retire.

The second phase, which starts right after the accumulation phase and spans the rest of our lives, is the **decumulation phase**. This phase is when the money is withdrawn during an individual's retirement. With different expenses, priorities, and lifestyles; planning the decumulation phase varies on an individual basis. However, an issue that influences all portfolios is **sequence risk**, or the risk tied to the order of returns that an asset earns. Sequence risk plays a big part in both the accumulation and decumulation phases. While there are multiple ways to calculate the risk to return ratios of investments, there isn't a definitive formula for calculating sequence risk.

Accumulation

Background

As people start saving for retirement, they may be unaware of the impact that the sequence of their investment returns has on their portfolio. The risk of a portfolio's order of returns, known as **sequence risk**, is the uncertainty of the sequence of returns on the final value. The only instance a portfolio is impervious to sequence risk is when its only transaction is a single initial deposit. In that case, the order of returns earned can be disregarded. In the figures below, we worked out all permutations of a portfolio earning -15%, 10%, and 20% annually over a three-year period with a single initial contribution of $1,000.

Table 1.1 : Single-Transaction Portfolios Example

Annual Returns:			
	10%	20%	-15%

Scenario 1: Single Investment of $1,000

Year 1	Year 2	Year 3		
$1,000	$0	$0		
20%	10%	-15%	**TOTAL**	**$1122**

1000*1.20*1.10*0.85 = $1122

Scenario 2: Single Investment of $1,000

Year 1	Year 2	Year 3		
$1,000	$0	$0		
10%	20%	-15%	**TOTAL**	**$1122**

1000*1.10*1.20*0.85 = $1122

Scenario 3: Single Investment of $1,000

Year 1	Year 2	Year 3		
$1,000	$0	$0		
20%	-15%	10%	**TOTAL**	**$1122**

1000*1.20*0.85*1.10 = $1122

Scenario 4: Single Investment of $1,000

Year 1	Year 2	Year 3		
$1,000	$0	$0		
10%	-15%	20%	**TOTAL**	**$1122**

1000*1.10*0.85*1.20 = $1122

Scenario 5: Single Investment of $1,000

Year 1	Year 2	Year 3		
$1,000	$0	$0		
-15%	20%	10%	**TOTAL**	**$1122**

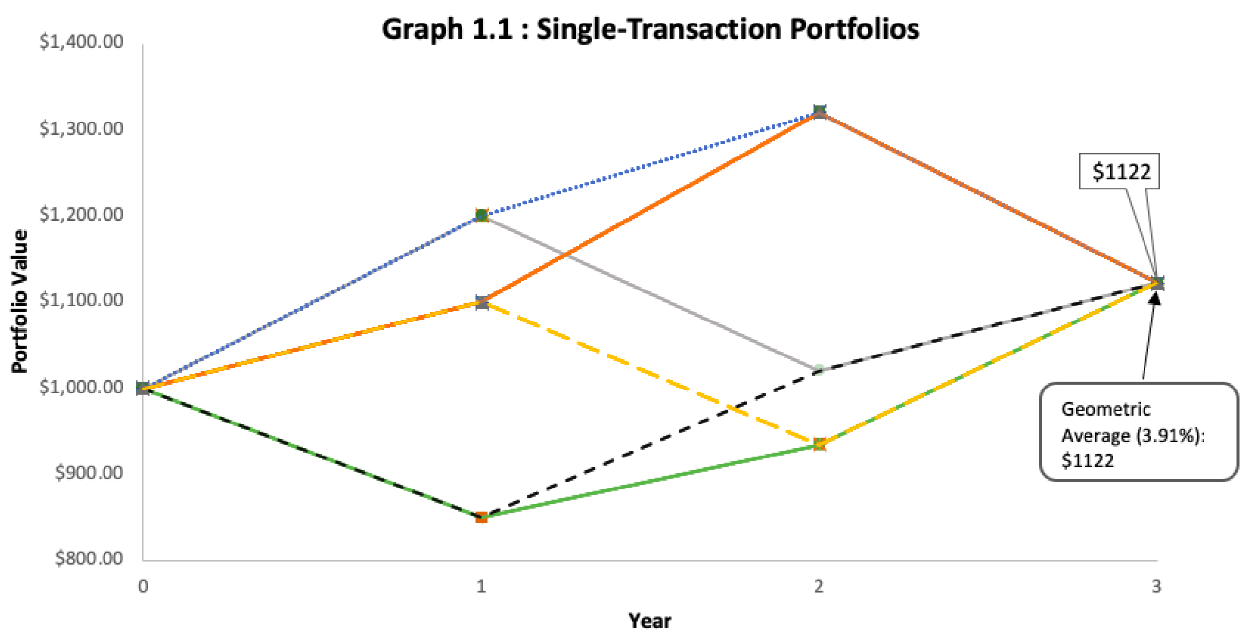
1000*0.85*1.20*1.10 = $1122

Scenario 6: Single Investment of $1,000

Year 1	Year 2	Year 3		
$1,000	$0	$0		
-15%	10%	20%	**TOTAL**	**$1122**

1000*0.85*1.10*1.20 = $1122

Each of the six permutations has an average geometric annual return of 3.9%. Irrespective of their orders, each of the investments in Table 1.1 accumulated to $1,122. Graph 1.1 below illustrates the data from the Table 1.1. Each line represents one of the six scenarios and each point represents the portfolio's values at the end of each year. The graph also depicts the final-amounts converging, as the six-sequences intersect at $1,122 at the end of Year 3. This shows sequence risk does not affect single-transaction portfolios. In these six cases, since the initial contributions experience the same three returns compounded in different orders, they all result in the same outcome. This is due to the commutative property of multiplication, as 1.2* 1.15* 0.85 = 0.85* 1.2* 1.15 (or any of the four other permutations).



However, in most cases, individuals will make periodic contributions into their retirement portfolio during the accumulation phase. They lack the liquid capital necessary to fund their whole retirement using a single-transaction. When analyzing portfolios with multiple deposits, we must also look at the impact sequence risk will have on their end values. In Table 1.2 below, we again worked out each of the permutations of a portfolio earning -15%, 10%, and 20% annual returns. However, this time, contributions of $1000 are made annually. Each portfolio has the same average geometric annual return (3.9%) as in the previous example. But since each portfolio contains multiple deposits, the order of the returns affects the end accumulated amount of the portfolio.

Table 1.2 : Impact of Sequence Risk

Annual Returns:	10%	20%	-15%

Scenario 1: Annual Deposit of $1,000

Year 1	Year 2	Year 3		
$1000	$1000	$1000		
20%	10%	-15%	**TOTAL**	**$2907**

Scenario 2: Annual Deposit of $1,000

Year 1	Year 2	Year 3		
$1000	$1000	$1000		
10%	20%	-15%	**TOTAL**	**$2992**

Scenario 3: Annual Deposit of $1,000

Year 1	Year 2	Year 3		
$1000	$1000	$1000		
20%	-15%	10%	**TOTAL**	**$3157**

Scenario 4: Annual Deposit of $1,000

Year 1	Year 2	Year 3		
$1000	$1000	$1000		
10%	-15%	20%	**TOTAL**	**$3342**

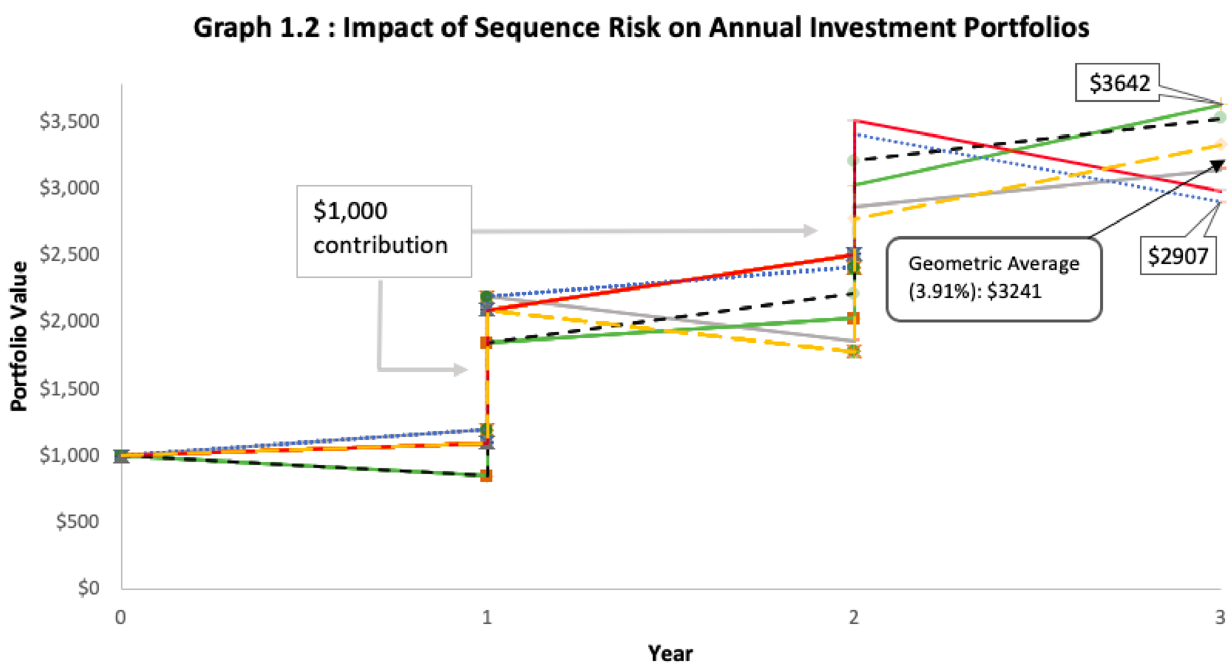
Scenario 5: Annual Deposit of $1,000

Year 1	Year 2	Year 3		
$1000	$1000	$1000		
-15%	20%	10%	**TOTAL**	**$3542**

Scenario 6: Annual Deposit of $1,000

Year 1	Year 2	Year 3		
$1000	$1000	$1000		
-15%	10%	20%	**TOTAL**	**$3642**

Conceptually, to understand the impact of the order of returns in these scenarios, think about each contribution made into the portfolio as an individual investment. The investment in Year 1 will earn all three returns (accumulating to $1122 seen earlier). However, the investment in Year 2 will only earn the last two returns, and the investment in Year 3 will earn only the final return. So, the difference in value of each portfolio results from the return on deposits made in Year 2 and Year 3. Since the initial investment for all six examples earn the same three returns, the rates earned by the second and third deposits leads to the difference in end portfolio values.



Given a set of returns experienced during the accumulation phase, it is more beneficial for a retirement portfolio to have a sequence of returns where the returns towards the end are higher than those at the start. Returns experienced at the end of the horizon are earned by a larger portion of contributions made to the portfolio. For a given set of returns, an ascending sequence of returns results in the highest end value, while a descending sequence of the returns results in the lowest accumulated value.

To reiterate, multiple-deposit portfolios that experience the same average geometric annual returns with differing sequences, will have differing end values. While the end values of the six single-deposit portfolios converged, the multiple-deposit scenarios did not converge. There is a 25% difference ($3642 vs $2907) between the largest (returns in ascending order) and smallest accumulated values (returns in descending order). Furthermore, the effects of sequence risk are magnified as a portfolio's horizon lengthens and as its returns grow more volatile. If it were possible to accurately predict returns, a portfolio would be able to manage sequence risk. By divesting during periods of negative returns it would not incur losses and it would maximize profits by remaining invested during positive returns. Although it is impossible to predict the returns of the market and thus manage sequence risk, investors should still be aware of the risk as they save for retirement.

Methodology: Rate Development

In order to simulate an individual's retirement, we first looked at the accumulation process. One of the goals was to model this phase of retirement and analyze it to illustrate the potential impact of sequence risk. The portfolio value at the end of the accumulation phase will be used to fund the individual's retirement. We assume that an individual builds and invests their retirement portfolio for 40 years, until the day of their retirement. In order to keep calculations simple, clear, and direct; we made two stipulations. First, the retirement portfolios were comprised 100% of equity. Second, since the S&P 500 is the standard market indicator, the simulated portfolio's returns were based solely on 40 annual rates (including dividends) experienced by the index. Typically, investors lean towards having a diversified set of stocks, bonds, options, and other securities. However, the principles concluded from our analysis also apply to individuals that are not fully invested in equity or the S&P 500. We put these two conditions in place to make calculations simple and direct.

We wanted the 40 rates to be randomly chosen, while also being representative of returns experienced during a realistic 40-year period. To simulate the 40-year horizon of the accumulation phase, we first gathered a set of 147 historical S&P 500 rates, including dividends, from 1871 to 2017. From this set, 38 rates were pseudo-randomly chosen and the other 2 rates, highlighted in Table 1.3, were added manually to mimic extremes that stock market could experience. The 38 rates were randomly selected until they roughly matched the distribution of the larger set of 147 historical rates. The extremes were deliberately chosen so that they could exist within the data, as they cancel out. The maximum was 60% and the minimum was -37.5%:

$[1+(60\%)] * [1+(-37.5\%)] = 1$

Table 1.3 : 40 Annual Accumulation Returns

60.00%	49.37%	38.02%	33.67%	32.24%	32.00%	29.07%	28.72%
27.10%	20.84%	19.67%	19.06%	18.69%	11.98%	11.93%	11.59%
11.51%	11.16%	11.03%	7.60%	7.09%	6.38%	5.96%	5.45%
5.44%	5.01%	4.72%	3.66%	3.63%	3.61%	3.52%	-0.80%
-1.06%	-3.39%	-3.42%	-8.63%	-12.05%	-12.32%	-24.21%	**-37.50%**

Comparing the distributions of the 40 chosen rates to the set of historical S&P returns, showed that the 40 random rates have an average yearly return of 9.3%, comparable to the actual average of 9.15%. We also sought to match the standard deviation and skew of the historical set, to assure that our rates were representative of the actual market. The standard deviation of the S&P's annual rates is 18.63% and the standard deviation of our returns is 18.53%. Additionally, the skewness of both were within [-2,2], implying the skew is negligible.

Table 1.4 : Rates Comparison	40 Random Rates	Historical S&P Rates *1871-2017*
Minimum	-37.50%	-44.20%
Maximum	60.00%	56.79%
Geometric Average	9.35%	9.15%
Standard Deviation	18.53%	18.63%

Calculations and Findings

Flat Annual Deposits

After developing the set of 40 returns, we created 100,000 permutations of these 40 rates. In other words, we generated 100,000 random and distinct sequences utilizing the same 40 rates. With these 100,000 scenarios developed for the accumulation phase, we attempted to produce results that would showcase the effects of sequence risk and other ideas presented in the paper by Clare, Seaton, Smith, and Thomas. The 100,000 permutations of the 40 returns were used to calculate respective end accumulated values. In regards to retirement, it would be the lump sum that would be used to fund an individual's retirement. By comparing sequences of the same 40-rates and their accumulated values, we would in essence be analyzing the impact of the order of the returns. Noting the impact that the order of returns has on the final accumulated value shows the effects of sequence risk in portfolio analysis.

In order to compare the embedded sequence risk across scenarios, we computed the end accumulated values for each of the 100,000 scenarios. To fund our portfolio, we chose to make annual $5,000 deposits. To calculate the end values of the portfolio, first we converted each sequence of the returns into factors. The growth-factor of each year, shown in the third row of the table below, is the product of all successive returns from that year forward. Conceptually, this growth-factor would be the return experienced by the deposit made in that corresponding year. In Table 1.5 below, the $5,000 deposited in Year 1 would grow by 3457.4%. The $5,000 invested in Year 2 would experience a 3350.8% change. And the final the $5,000 contributed in year 40 would experience a 11.93% return. To produce each scenario's final portfolio value, we calculated the dot-product of the growth factors and the set of annual deposits.

Table 1.5 : Calculating End Accumulated Values

Years	1	2	...	39	40	
Scenario Return	3.61%	11.03%	...	5.44%	11.93%	
`Growth Factors	$\prod_{i=1}^{40}(1+r_i)$ = 35.7541	$\prod_{i=2}^{40}(1+r_i)$ = 34.5084	...	$\prod_{i=39}^{40}(1+r_i)$ = 1.1802	$\prod_{i=40}^{40}(1+r_i)$ = 1.1193	Flat 9.35% Return: **$2,031,563**
Annual Deposit	$5,000	$5,000	...	$5,000	$5,000	**End Accumulated Value:**
Accumulated Deposit	$178,770	$172,542	...	$5,901	$5,597	**$2,099,978**

In addition to 100,000 randomly generated scenarios, we also constructed two controlled "perfect" scenarios. These perfectly-ordered scenarios were comprised of the same set 40 returns used for the previous 100,000 permutations. However, for the perfect scenarios we ordered the sequence of returns in ascending and descending order. The "perfect" ascending scenario, when rates were strictly increasing annually (i.e. lower rates experienced at the start of the accumulation phase), provided the highest end accumulation of all 100,002 scenarios. The "perfect" descending scenario, when rates were strictly decreasing annually (i.e. lower rates appeared at the end of the accumulation phase), provided the lowest end accumulation value of all 100,002 scenarios.

Table 1.6 : "Perfect" Sequence of Returns

Year	1	2	3	4	...	37	38	39	40
"Perfect" Ascending Returns	-37.50%	-24.21%	-12.32%	-12.05%	...	33.67%	38.02%	49.37%	60.00%
"Perfect" Descending Returns	60.00%	49.37%	38.02%	33.67%	...	-12.05%	-12.32%	-24.21%	-37.50%

Although the average annual returns were the same over the 40-year accumulation period for 100,002 scenarios, the difference in the order of the returns had a bearing on the end accumulation values. The perfect ascending scenario further highlighted the benefit of having the negative returns at the start off the accumulation phase followed by larger returns toward the end. And the perfect descending scenario demonstrated the catastrophe of experiencing positive returns early and lower, negative returns toward later on in the period.

Table 1.7 : Perfect Ascending Sequence End Value

Years	1	2	...	39	40
Perfect Ascending Returns	-37.50%	-24.21%	...	49.37 %	60%
Portfolio Value Start of Year	$5000	$8125	...	$5,007,109	$7,484,118
Portfolio Value End of Year	$3125	$6158	...	$7,479,119	**$11,974,590**

Flat 9.35% Return:

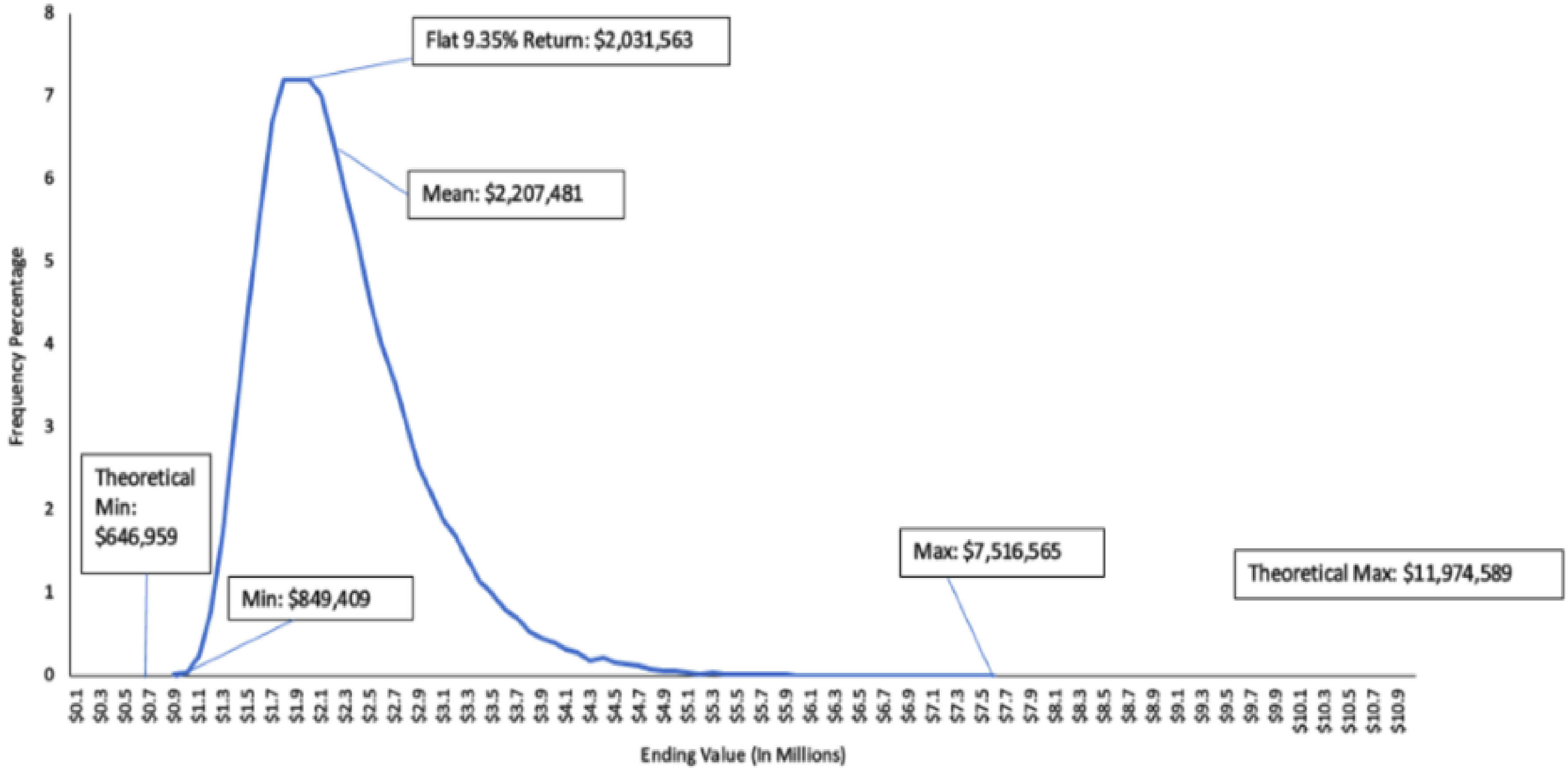
$2,031,563

Table 1.8 : Perfect Descending Sequence End Value

Years	1	2	...	39	40
Perfect Descending Returns	60%	49.37%	...	-24.21%	-37.50%
Portfolio Value Start of Year	$5000	$13,000	...	1,359,196	$1,035,135
Portfolio Value End of Year	$8000	$19,418	...	$1,030,135	**$646,959**

In order to provide more insight into the accumulated values, we calculated the distribution of the accumulated values of the 100,002 scenarios. Graph 1.3 shows the distribution of the dataset, displaying the frequency of the final accumulated values. The diagram also shows the max, min, theoretical max (perfect-ascending), theoretical min (perfect descending), mean of the accumulated values and the geometric average return value of the dataset. While the mean is the arithmetic mean of the 100,000 scenarios (excluding the perfect sequences). The consistent geometric average return is the ending value of a portfolio experiencing the geometric average annual return (9.35%) for all 40 years of the accumulation period.

Graph 1.3 : Distribution of Level-Contribution Accumulated Values



Increasing Annual Deposits

The flat annual investment of $5,000 made the calculations and the presentation of the ideas from the accumulation phase, simple and direct. However, that is not an accurate representation of typical investing behavior. Usually, as time passes an individual starts making larger periodic contributions to their retirement portfolios. So in order to simulate this increase in investments, we introduced six more contribution patterns. Like the initial level pattern, the six new options start with an annual investment of $5,000. However, each subsequent annual deposit increases at six different rates (i.e. 0.25%, 0.5%, 0.75%, 1%, 2%, and 3%). Table 1.9 shows the annual deposits for each distinct contribution pattern. These increasing yearly contributions seek to mirror and account for pay raises and inflation that might be experienced in realistic situations.

Table 1.9: Annual Deposits at Contribution Patterns

Year	1	2	3	...	38	39	40
0%	$5,000	$5,000	$5,000	...	$5,000	$5,000	$5,000
0.25%	$5,000	$5,012.50	$5,025.03	...	$5,483.93	$5497.64	$5,511.39
0.5%	$5,000	$5,025	$5,050.13	...	$6,013.32	$6,043.39	$6,073.60
0.75%	$5,000	$5,037.50	$5,062.69	...	$6,5932.30	$6,641.74	$6,691.56
1%	$5,000	$5,050	$5,100.50	...	$7,225.38	$7,297.64	$7370.61
2%	$5,000	$5,100	$5,202	...	$10,403.42	$10,611.49	$10,823.72
3%	$5,000	$5,150	$5,304.50	...	$14,926.13	$15,373.92	$15,835.13

To gain a better understanding of different situations or factors that may be affected by sequence risk, we calculated the terminal values for 100,000 scenarios using each of the six different contribution patterns. We repeated the earlier calculations and analysis of the flat annual investments, on these new contribution patterns.

Table 1.10 : Accumulated Value at 1% and 3% Contribution Patterns

Years	1	2	…	39	40	
Scenario Return	3.61%	11.03%	…	5.44%	11.93%	
Growth Factors	$\prod_{i=1}^{40}(1+r_i)$ = 35.754	$\prod_{i=2}^{40}(1+r_i)$ = 34.5084	…	$\prod_{i=39}^{40}(1+r_i)$ = 1.1802	$\prod_{i=40}^{40}(1+r_i)$ = 1.1193	
Annual Deposit - 1% Contribution	$5,000	$5,050	…	$7,298	$7371	**1% Accumulated Value:**
Accumulated Deposit	$178,770	$179,317	…	$8,613	$8,250	$2,298,882
Annual Deposit - 3% Contribution	$5,000	$5,150	…	$15,374	$15,835	**3% Accumulated Value:**
Accumulated Deposit	$178,770	$177,718	…	$18,144	$17,724	$2,816,274

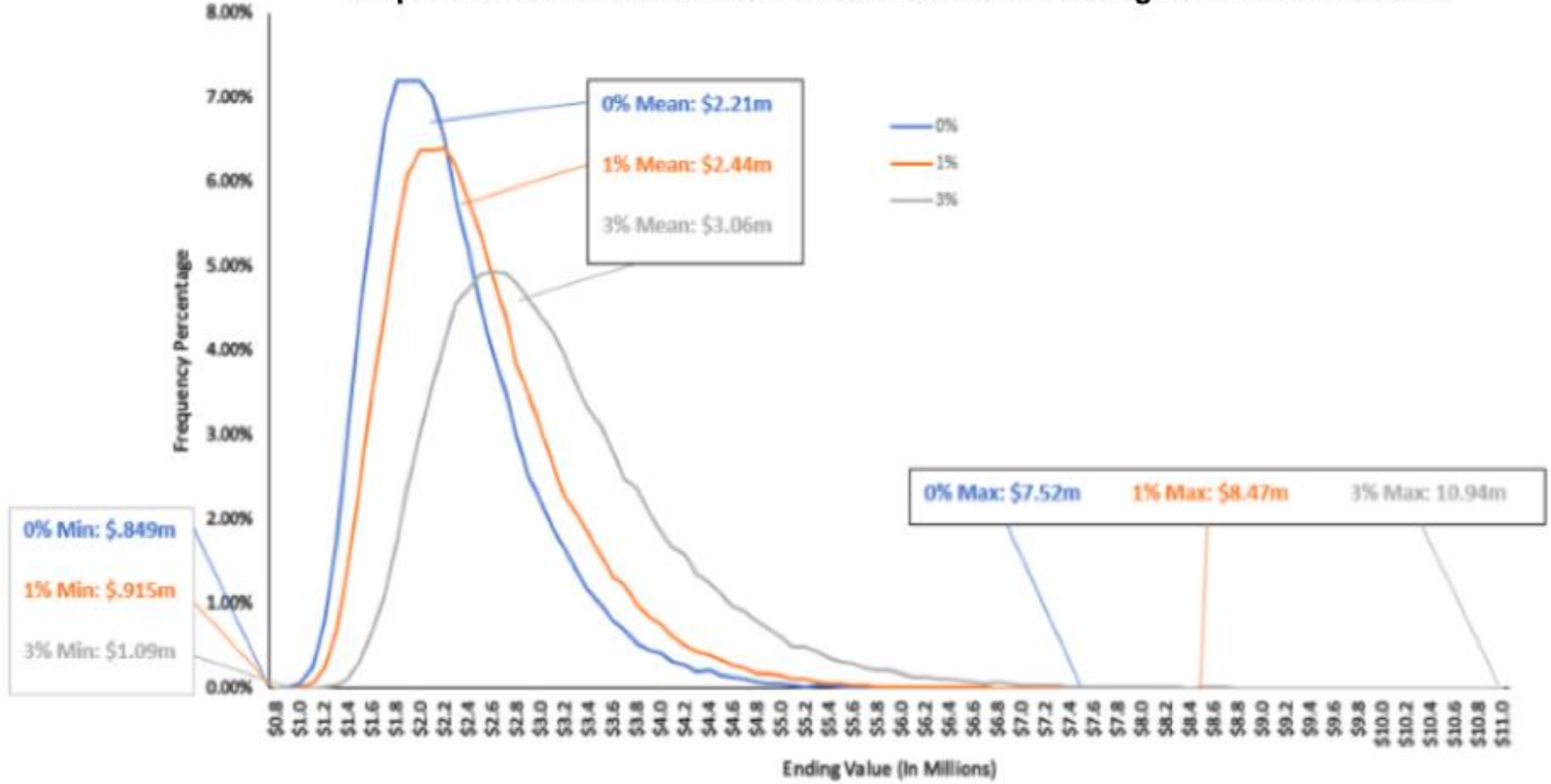
We also created and calculated the accumulated values of the perfectly ordered scenarios for the six new investment patterns. For each contribution pattern, as expected, the perfect ascending scenarios produced the highest accumulated value and the perfect descending scenarios produced the lowest accumulated value. However, as the investment pattern increases, the perfect-descending portfolio value is not increasing proportionally to the perfect-ascending portfolio. The difference in final values of portfolios with contributions growing by 3% and level contributions are drastic when they experience favorable (ascending) sequences ($18M vs. $12M). However, the results in Table 1.11 show that difference is less significant when the portfolios experience an unfortunate (descending) sequence ($766K vs. $647K). Additionally, the portfolio with the level deposits is putting in around $5,000 less annually than the portfolio with the 3% increasing contribution pattern. This suggests that higher annual investment options will lead to higher portfolio values, however, they also could potentially be inefficiently utilizing capital.

Table 1.11 : Perfect Sequences' End Values at Increasing Contribution Patterns

Contribution Pattern	0%	0.25%	0.50%	0.75%	1%	2%	3%
"Perfect" Ascending Value	$11,974,588	$12,377,056	$12,797,244	$13,236,057	$13,694,447	$15,744,805	$18,199,983
"Perfect" Descending Value	$646,959	$654,740	$662,839	$671,276	$680,072	$719,338	$766,547
Flat 9.35% Return - Average Value	$2,031,563	$2,081,045	$2,132,658	$2,186,522	$2,242,761	$2,494,275	$2,796,147

To analyze the distributions of the accumulated values under different contribution patterns, we graphed the distribution of each dataset at the various contribution options. The charts for each contribution-option follow the same format that was used to illustrate the distribution of the $5,000- level annual investment. The perfect ascending (theoretical max) and perfect descending (theoretical min) are not displayed. In Graph 1.4, we overlaid the distribution of the 1% and 3%-increasing investment option to the distribution of the initial level contribution distribution.

Graph 1.4 : Distribution of Accumulated Values at Increasing Contribution Patterns



As expected, the accumulation scenario with 3% annual contribution growth tends to have higher ending values. We notice that the lower terminal values occur more frequently as the contribution pattern decreases and the average ending value is higher as the percent of the contribution pattern increases. However, larger contribution patterns also result in higher standard deviation, pointing to more variability in ending accumulated values. Another comparison, noted in Table 1.12, shows the portfolio's max value gets higher as the contribution patterns get higher. The maximum for the 3% scenario is around $11 million, compared to a much smaller $8.5 million and $7.6 million for the 1% and 0% respectively. This pattern will continue as the rate of contribution increases. While we can compare the overall shape and spread of the distributions, we cannot compare the values because of the different annual contributions.

Table 1.12 : End Values at Increasing Contribution Patterns

Contribution Pattern	0%	0.25%	0.50%	0.75%	1%	2%	3%
Max	$7,516,565	$7,741,386	$7,975,457	$8,219,243	$8,473,237	$9,602,165	$10,941,955
Min	$849,409	$864,631	$880,557	$897,229	$914,695	$993,551	$1,089,818
Average	$2,207,481	$2,262,963	$2,320,838	$2,381,236	$2,444,299	$2,726,294	$3,064,594
Standard Deviation	$655,153	$674,978	$695,648	$717,208	$739,705	$840,098	$960,032

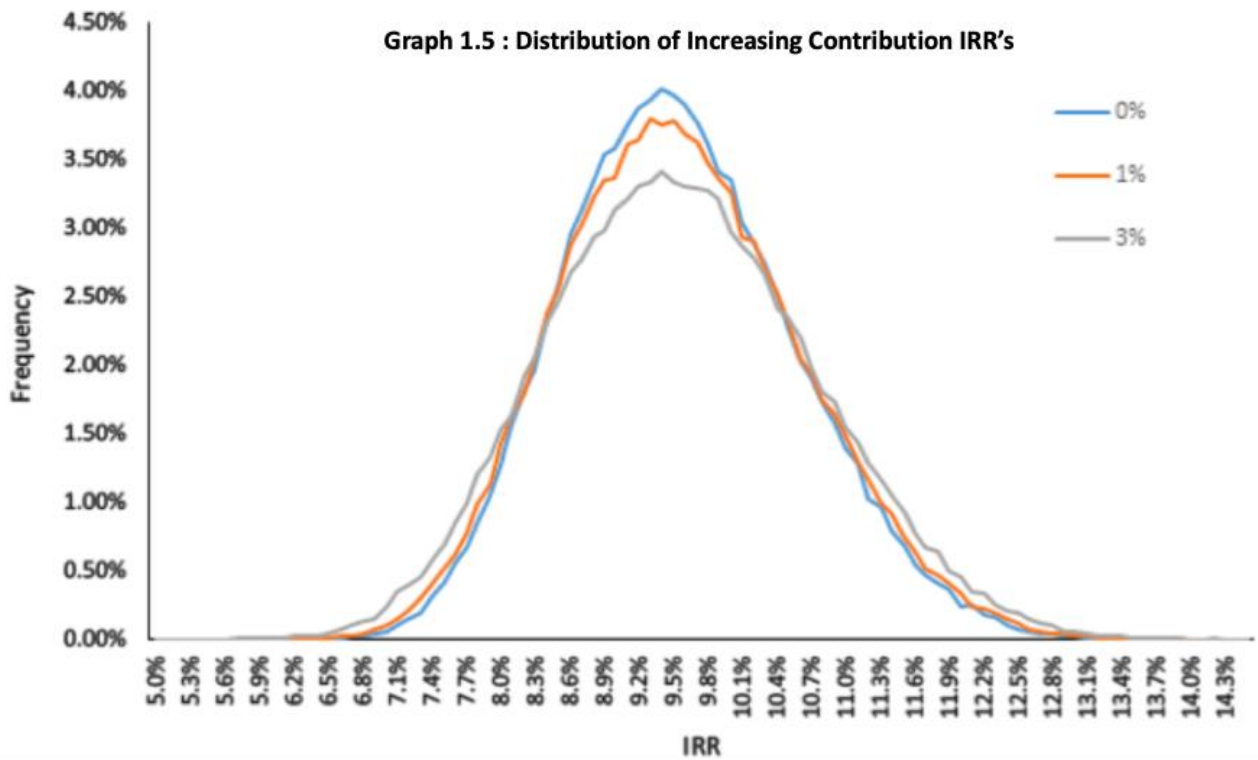
To compare the identical sets of 40 annualized returns at the varying investment options, we calculated the internal rate of return (IRR) for each of the scenarios. The IRR is a measure used to estimate the profitability of potential investments. It would indicate the rate of growth a portfolio would generate under the different investment options, and thus allow us to compare these rates. The IRR is a money-weighted return, meaning that each contribution and when you deposit it affects the IRR. This is different than the 9.35% geometric average return, which is a time-weighted return, meaning that only the time horizon matters for the calculation. The IRR measures how well the portfolios did by taking into account the amount and timings of the contributions. We calculated the average IRR for each of the contribution options and found that increases in the contribution pattern option did not result in a significant change in average IRR's. The portfolios saw similar rates of growth irrespective of how large the contributions were.

Table 1.13 : IRRs at Increasing Contribution Patterns

IRRs							
Contribution Pattern	0%	0.25%	0.50%	0.75%	1%	2%	3%
Max	13.94%	13.98%	14.02%	14.06%	14.11%	14.28%	14.47%
Min	6.13%	6.08%	6.03%	5.98%	5.92%	5.69%	5.43%
Average	9.50%	9.50%	9.50%	9.51%	9.51%	9.51%	9.52%
Standard Deviation	1.00%	1.01%	1.03%	1.04%	1.05%	1.10%	1.16%

We plotted the IRR for each investment option and found the they roughly resemble a normal curve. There was a slight increase in the average of the IRR's as the contribution compounding rate increased. The lowest contribution level (level deposits) had an average of 9.50%, while the highest average IRR was 9.52% corresponding to the 3% compounding rate.

Although the means for each contribution pattern were quite similar, they differed in their variance. As shown in Graph 1.5, as the compounding rate increased, the distributions of the IRR's widened. Portfolios with the 3% accumulation strategy had significantly higher contributions as the accumulation phase progressed. So, when the portfolio experienced negative or positive returns towards the end of the horizon, the effects on the IRR were amplified. This resulted in the portfolios experiencing the higher and lower IRR's more frequently and leads to the larger variance in IRR's.



To further compare the effects of sequence risk between the contribution options, we came up with a "score" that evaluated each sequence of returns. To compute the score, we first normalized each sequence by comparing it to a baseline value. To calculate the score, we divided each scenario's ending accumulated value by the ending accumulated value of the average run. This average run or the "**Flat 9.35% run**" was the final accumulated value of the portfolio experiencing the annual average geometric return (9.35%) for each of the 40 years of the accumulation phase.

$$Scenario\ Score\ = \frac{Scenario's\ End\ Accumulated\ Value}{Flat\ 9.35's\ End\ Accumulated\ Value} \qquad \text{Eq (1.1)}$$

Table 1.14 : Scenario Scores at Increasing Contribution Patterns

Contribution Pattern	0%	0.25%	0.50%	0.75%	1%	2%	3%
Highest Score	3.6999	3.7200	3.7397	3.7590	3.7780	3.8497	3.9132
Lowest Score	0.4181	0.4155	0.4129	0.4103	0.4078	0.3983	0.3898
Average Score	1.0866	1.0874	1.0882	1.0891	1.0899	1.0930	1.0960
Std dev	0.3225	0.3243	0.3262	0.3280	0.3298	0.3368	0.3433
Flat 9.35% Return	$2,031,563	$2,081,045	$2,132,658	$2,186,522	$2,242,761	$2,494,275	$2,796,147

By comparing the standard deviation of the scores from each contribution pattern, we can begin to quantify the risk associated with each one. A higher standard deviation of scores indicates more sequence risk, which can lead to more volatility in terminal values.

During the accumulation phase, individuals make periodic contributions into a portfolio that will be used to fund their retirement. All portfolios with multiple deposits are susceptible to sequence risk. Given a set of returns and deposits, the closer the sequence of those returns is to ascending order, the larger the ending accumulated value. Sequence risk also impacts portfolios whose periodic contributions are irregular or are not level. The contribution pattern of your portfolio does not usually play a significant role in determining its rate of growth, as the average IRR at each contribution pattern was roughly the same. A crucial factor in determining the terminal value of a portfolio is the specific sequence of returns it experiences.

Decumulation

Background

Individuals spend years planning, investing, and saving and are finally rewarded when they retire. However, even during retirement individuals still have many of the same expenses they had in the past. People move into the decumulation, or spend-down phase, during which they withdraw money and deplete the assets accrued during their accumulation phase. The funds in the retirement accounts will continue to be invested and experience returns during the decumulation period.

Retirement portfolios are subject to many risks. One of the biggest concerns is a poorly performing economy. We are not considering that risk here. In our work, each retirement portfolio is subject to the same returns. However, these returns are applied in a different order, or sequence, and this gives rise to a different concern: sequence risk, which can lead to very different outcomes for different scenarios. Due to the annual withdrawals, the portfolio during the spend-down period is susceptible to sequence risk. The end value or even how long a portfolio remains funded will be partially determined by the sequence of the returns it experience. Portfolios are inversely impacted by sequence risk during the spend-down period, relative to the accumulation period. During the decumulation phase, the fund value is at its highest at the start and decreases over time. So, it is beneficial to have the higher rates at the start and the lower rates at the end (i.e., a decreasing scenario will be better for the spend-down phase).

The decumulation phase in our model starts with a lump sum available to provide retirement benefits. The initial portfolio value for the decumulation phase equals the ending portfolio value from the accumulation phase. The accumulation phase's horizon was from around the start of an individual's working life until the day of their retirement (40 years for the purpose of this analysis). The decumulation phase has a new, independent horizon length that spans the individual's retirement or until their funds dry up. We realize 20 years may not match every individual's retirement plans, but is used for illustrative purposes. Individuals who fear they will run out of money or outlive 20 years of retirement can opt to purchase a deferred annuity, ensuring a steady stream of income.

During the accumulation phase, the highest ending portfolio value resulted when the returns were arranged in ascending order and the lowest ending value resulted when they were arranged in descending order. However, during the decumulation phase it is the reverse. As it is more beneficial for a portfolio to experience the sequence of its 20 returns in descending order and it is unfavorable for the portfolio to have the sequence of returns in ascending order. While accumulating, the portfolio had annual contributions and grew larger as time passed. However, during the spend-down period, it experiences yearly withdrawals, and decreases in value as time goes by. Since a portfolio in the decumulation phase would be at is largest at the beginning of the period, it is preferable for it to experience the larger returns of the sequence as early as possible. If the portfolio experiences an early negative return, it could be devastating because it would be difficult to recover without steady contributions.

Due to lack of deposits into the accounts, individuals in their decumulation phase are more likely to be risk-averse. With their livelihoods in their hands, individuals try to make careful, calculated decisions in order to maintain enough money to fund their entire retirement. One common measure implemented among retirees is choosing a withdrawal plan that will cover their expenses and be sustainable over their projected horizon length. One of the most widespread retirement related advice is the 4% Rule, introduced and advocated by William P. Bengen in the early 1990's. The 4% Rule recommends that retirees withdraw 4% of the initial portfolio amount annually, while adjusting only for inflation each year. For example, a retiree with an investment portfolio upon retirement of $1,000,000 would withdraw $40,000 in his or her first year of retirement, and then adjust that $40,000 withdrawal in subsequent years for inflation.,. Bengen stated that on average, the 4% withdrawal rate is the "highest that satisfies the desired portfolio life".

In many cases using the 4% Rule may be too conservative and can leave behind a large surplus at the end of the 20-year decumulation horizon. With different goals for each individual, some seek to use as much if not all their money during their retirement and others hope to leave behind an inheritance for their beneficiaries. Given the foresight of knowing the returns the decumulation portfolio will experience, it is possible to compute a "perfect withdrawal rate" for each scenario. At this perfect withdrawal rate the fund will be exhausted (equal $0) exactly at the end of the decumulation horizon (20 years in our paper). The rate can also be easily adjusted to incorporate any inheritance an individual may plan on leaving behind.

The perfect withdrawal rate (PWR) is the annual percentage of the initial portfolio value such that, when taken out each year, results in the portfolio equaling zero as the horizon expires. The perfect withdrawal amount (PWA) is the corresponding annual withdrawal. It is the perfect withdrawal rate * initial portfolio value. A study of these perfect withdrawal rates and their distribution may validate or make us reconsider using the 4% Rule. These are the formulas to calculate the PWR and PWA:

$$K_{i+1} = (K_i - w)(1 + r_i) \quad \textbf{Eq (2.1)}$$

K_i : balance at the beginning of the year i

w : yearly withdrawal amount = Perfect Withdrawal Amount

r_i : annual rate of return percentage in year i

$$\text{Perfect Withdrawal Amount} = w = [K_S \prod_{i=1}^{n} (1+r_i) - K_E] \,/\, \sum_{j=1}^{n} (\prod_{i=j}^{n} (1+r_i)) \quad \textbf{Eq (2.2)}$$

K_S : balance at the beginning of the decumulation period (20 years in our work)

K_E : balance at the end of the decumulation period ($0 in our work)

$$\text{Perfect Withdrawal Rate} = w/\,K_S \quad \textbf{Eq (2.3)}$$

Methodology: Rate Development

Similarly to how we created the set of 40 accumulation rates, the 20 decumulation rates were pseudo-randomly chosen to represent the S&P 500. Included in the 20 rates were the same extreme values from the accumulation phase. The 60% and -37.5% returns are included in the rates as extremes, but do not affect the average return since they cancel each other out.

Table 2.1: 20 Annual Investment Returns

60.00%	28.72%	27.10%	25.83%	23.06%
21.29%	21.11%	19.06%	18.35%	15.96%
11.93%	11.16%	10.82%	10.17%	8.12%
-3.42%	-4.73%	-9.09%	-32.11%	**-37.50%**

Once again, we wanted to have the 20 rates to closely resemble the S&P 500. The average return of 9.06% for our 20 is very close to the overall average of 9.15%. The standard deviation of our rates is slightly higher than the overall standard deviation at 21.48% and 18.53%, respectively. Our small sample size of 20 leads to the standard deviations not being an exact match, but we still feel the rates are representative.

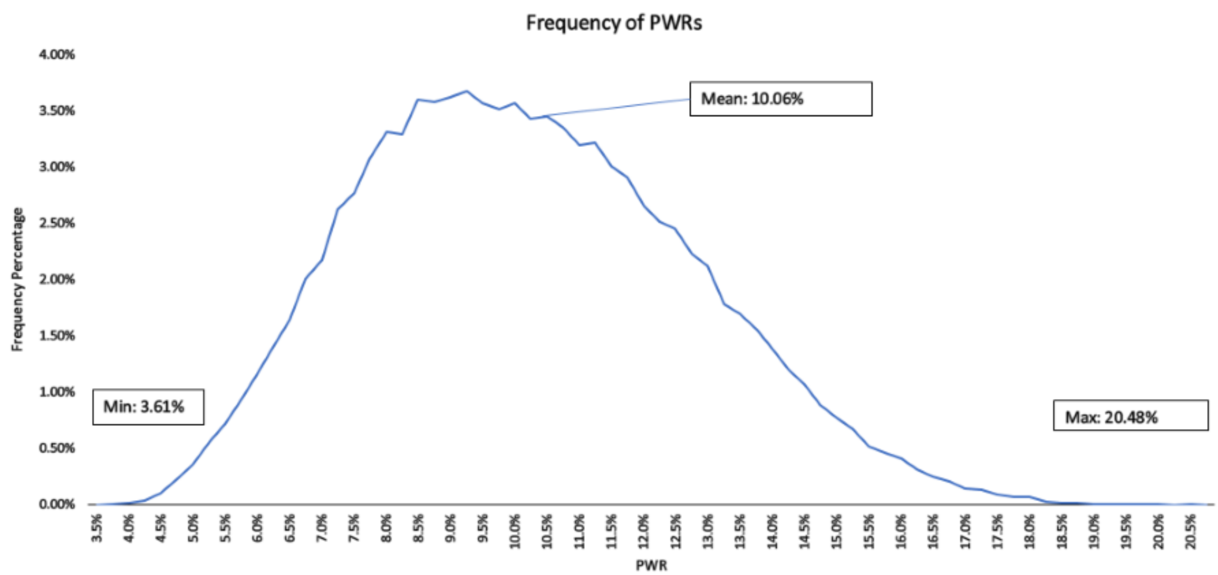
Table 2.2: Comparing 20 rates to the S&P 500

	20 Random Rates	Historical S&P (1871-2017)
Minimum	-37.50%	-44.20%
Maximum	60.00%	56.79%
Geometric Average	9.06%	9.15%
Standard Deviation	18.53%	21.48%

Calculations and Findings

Mirroring the accumulation phase, we generated 100,000 permutations of the 20 rates. For each of the permutations, we assumed a $5,000,000 starting value for consistency. Since we had the foreknowledge of the exact order and magnitude of each return, we were able to calculate the PWR and PWA for each scenario using equations 2.2 and 2.3. We plotted the 100,000 PWRs based on frequency to see the variation in the possible results.

Graph 2.1:



The shape of the curve resembles a normal curve, with most of the PWRs ending up around 10%. There is a large spread of the potential outcomes, from around 3% to around 20%. To put these percentages into more perspective, looking at the PWAs reveals the difference in annual allowance. When looking at the difference between three scenarios in the table below, it is important to keep in mind that each had the same starting value of $5,000,000 and each experienced the same 20 annual returns. The only difference lies in the sequence in which the returns were experienced.

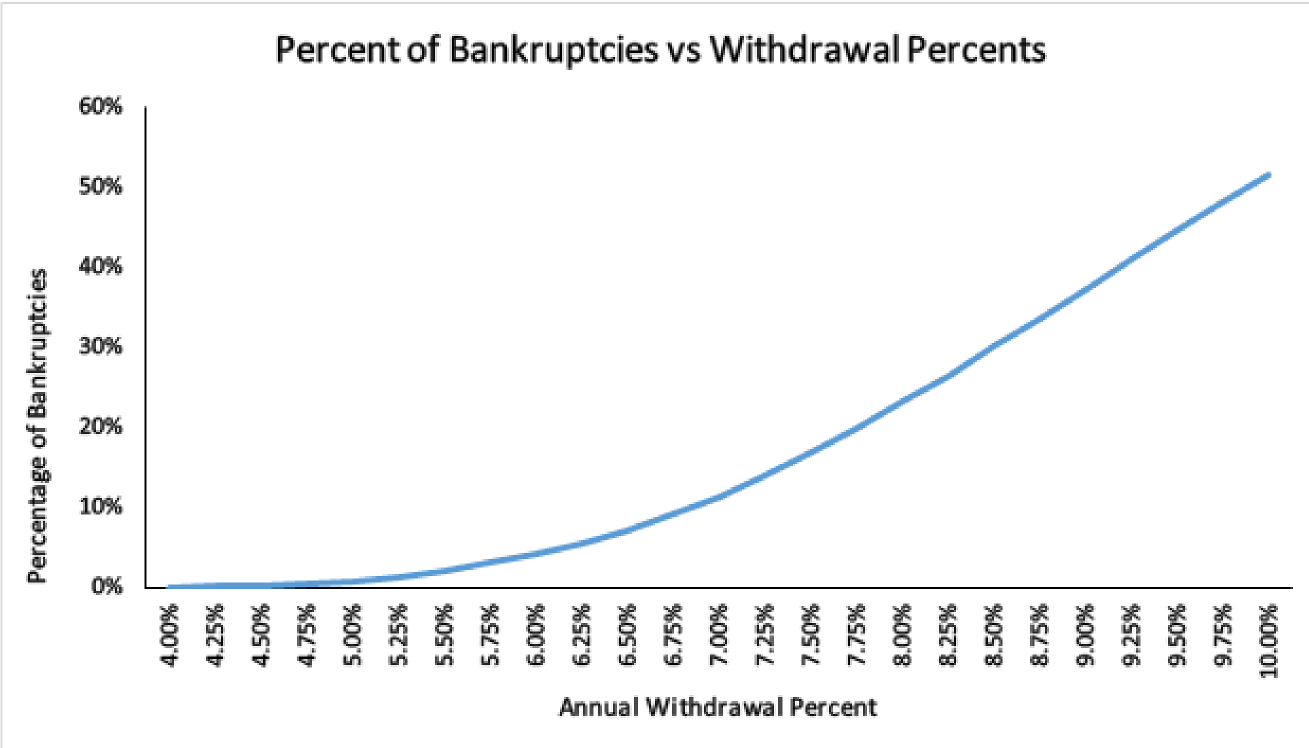
Table 2.3: Perfect Withdrawal Amounts

	PWR	PWA
Min	3.61%	$180,578
Mean	10.06%	$503,058
Max	20.48%	$1,023,855

Comparing to the 4% Rule

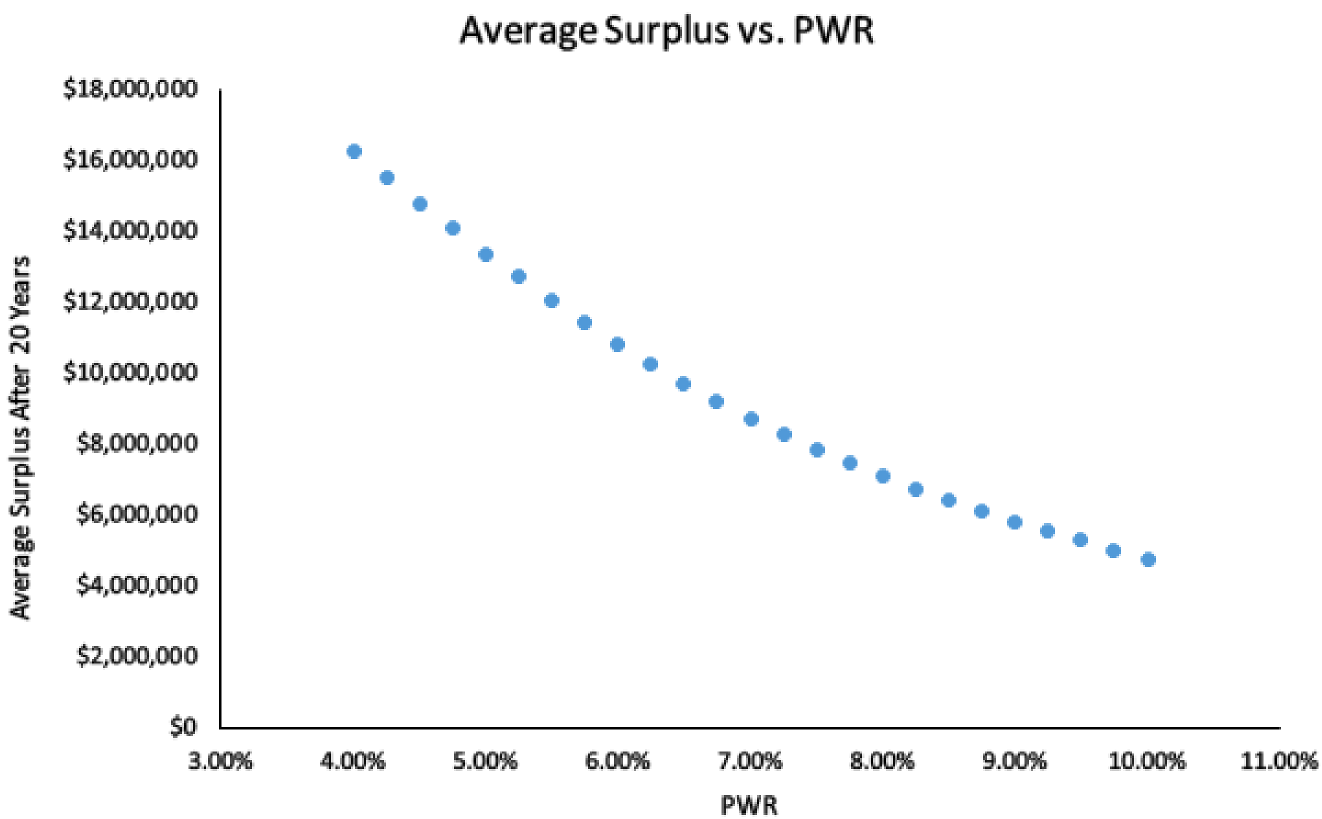
We noticed a surprisingly large difference between the average PWR of 10.06% and the common 4% Rule. We wanted to explore the possibility of establishing a new withdrawal rate that would be higher than 4%, while maintaining a conservative feel. To test this, we took our 100,000 scenarios and implemented the 4% Rule. In other words, we used with the $5,000,000 starting value, with the same 100,000 permutations of returns. But we assumed a 4% withdrawal rate, irrespective of our calculated perfect withdrawal rate. Then one of two things would happen: either the portfolio would go bankrupt before the decumulation horizon (20 years), or there would be excess funds at the end of 20 years. Since the average PWR was 10.06% for the 100,000 permutations, unsurprisingly a withdrawal rate of 4% proved to be too cautious. In fact, of the 100,000 scenarios only 0.016% ended in bankruptcy when withdrawing 4% annually. This low bankruptcy rate led to testing increased percentages to see how high the annual withdrawal rate could go while still limiting the bankruptcy percentage.

Graph 2.2:



The bankruptcy percentage stays small well past the 4% withdrawal rate. Additionally, it is important to note that the portfolios that don't go bankrupt are ending with extra money that may never be spent. In some of these cases "extra money" is an understatement. In fact, the portfolios that didn't go bankrupt with a 4% annual withdrawal rate had an average ending balance of $16.3 million at the end of the 20-year period. It is important to remember that this is because the portfolios are fully invested in equity during the decumulation phase. Since the average annual return of 9% is greater than the 4% withdrawal rate, the portfolios will be growing at a faster rate than the withdrawals being made. A conservative retiree may feel that a 1-2% chance of bankruptcy is too high for them, but they must also consider the cases when they leave too much money behind.

Graph 2.3:



Interestingly, the average amount of "extra money" from non-bankrupt portfolios can still be sizeable for the higher withdrawal rates. For instance, even at the 10% withdrawal rate (bankruptcy = 50%) the average amount of surplus funds at the end of the horizon exceeds $4 million. This shows how drastically different the end values are for the sequences of returns. For the 10% withdrawal rate, which is taking out $500,000 a year for 20 years, there are two very different outcomes. About half end with zero money at some point while the other half provides an annual income of $500,000 a year and average $4 million left over. Based on the high average PWR and average surplus amounts, a risk-inclined investor might conclude that the 4% Rule is too conservative.

Table 2.4:

Annual Withdrawal Rate	Average Year of Bankruptcy (if bankruptcy occurs)
4%	17.5
5%	16.5
6%	15.5
7%	15.0
8%	14.6
9%	14.1
10%	13.5

Table 2.4 shows how the average time of bankruptcy decreases when the withdrawal rates increase. The average year of bankruptcy is based only on the scenarios that experienced a bankruptcy. As expected, the higher withdrawal rates often lead to an earlier bankruptcy. However, this is not a likely scenario in actual practice. Most of the bankruptcies occur after 10 years have elapsed. After such a long period of bad returns, it is not likely an individual would

continue to take out such a large percentage of their initial savings. However, we decided to look at the scenarios as if the withdrawal percent remained constant for simplicity.

Trend Following

It is impossible to accurately predict the future return an investment will experience. In turn, we will never truly be able to control sequence risk. However, there are a few technical trading strategies recommended by experts. These strategies are intended to maximize the benefits of high returns, while limiting the losses of the lower returns. We decided to utilize trend following techniques in order to attempt to harness sequence risk during the decumulation phase. We focused on implementing sequence risk mitigating strategies during the decumulation phase because experiencing negative returns could leave the portfolio irreversibly damaged. A portfolio would have time and additional contributions to recover from bad returns early in the accumulation phase. However, it does not have the same leeway during the decumulation phase.

Trend-following is a trading strategy that quantifies historic data to predict the future growth or decline of an investment. The strategy uses the rolling-average price of an equity as an indicator to see if it will continue to grow. Many experts recommend using a 10-month rolling-average, which is the average stock/equity price for the previous 10 months. If the current price is greater than the rolling average, the strategy proposes that the individual stays invested as the security is expected to see growth. However, if the current price is lower than the rolling average, the individual should divest to cash until equities experience an increase in value. The 10-month moving average is supposed to present the short-term state of the investment. Comparing it to current value of the investment would tell you if it is overvalued and the price will decline or if it is undervalued and will continue to grow.

For the purposes of our project, we used the 12-month moving average as our measure when we utilized trend-following techniques. And instead of looking at the stock price we examined the returns of the S&P 500. We first implemented this strategy on continuous, historic periods. In order to fully be able to study trend-following techniques, we believed that we needed to see it implemented on historic 20-year periods. We felt there was a relation between one month's return and the next month's or previous month's return. So, because we were calculating the 12-month moving average, we looked at historic 20-year blocks. all monthly S&P returns from January of 1950. Since we were looking at continuous periods, we only were able to generate 586 scenarios.

Our returns were based on the historical monthly S&P 500 rates from January 1950 to November 2018. Each sequence was made up of 240 consecutive monthly rates. Our first scenario's first return was January 1950 and its last return was December 1969. Our second scenario's first return was February 1950 and its last return was January 1970. This scenario generating process was repeated 583 more times, until our final sequence which started in December 1998 and ended in November 2018.

After generating the 586 scenarios, we analyzed each sequence by first transforming the 240 monthly rates to 20 annual rates. These annual rates worked both as the portfolio return and the 12-month moving average. For each sequence, we compared each return to the previous year's return. If the current return was higher than the previous, the portfolio would remain invested and in the next year it would experience the S&P 500's return. If the current return was lower than the previous, the portfolio would divest to cash and in the next year it would experience a 2% return. After implementing the trend-following strategies, shown in the graphic below, on each sequence, we applied the same calculation to get each scenario's perfect withdrawal rate and perfect withdrawal amount.

Table 2.5: Comparing returns before and after trend following

Year	1	2	3	4	...	17	18	19	20
Original Return	4.51%	3.40%	3.24%	2.40%	...	1.92%	3.03%	3.42%	0.91%
Return after Trend-Following	4.51%	3.40%	2.00%	2.00%	...	2.00%	2.00%	3.42%	0.91%

These 586 scenarios have different average annual returns because each sequence is made up of a different set of returns. Since they have differing average annual returns and are made up of different sets of returns, we cannot compare each scenario. So, in addition to calculating the PWRs for the 586 scenarios using trend-following strategies, we also calculated their PWRs without trend-following. After comparing each scenario's PWR with trend-following to its respective PWR without trend-following, in all 586 scenarios the former was greater than the latter. In fact, PWR with trend-following had a higher average, max-value, and min-value.

Table 2.6 : Trend Following PWR

	PWR with Trend-Following	PWR without Trend-Following
Average	8.16%	6.58%
Standard Deviation	0.233%	0.153%
Max	8.79%	6.93%
Min	7.60%	6.29%

People work their whole careers in anticipation for retirement. Since it is such an important part of life, people want to be prepared for it. People spend time and money to ensure their hard-earned money will be used fittingly in retirement. However, it is impossible to predict exactly how returns will affect that money. This is the reason why suggestions such as the 4% Rule exist, and why we must keep analyzing returns to better prepare for the uncertainty involved with retirement. In our analysis, a simple trend following approach had a significant impact on the amount of each annual withdrawals. Each investor is different, and other approaches may appeal more to them based on their risk tolerance.

In actuality, the PWR does not affect a retiree. Withdrawal decisions have to be made before returns are experienced. The PWRs are most useful for making predictions, and demonstrating the potential outcomes for a set of returns. Each retiree is in control of their own withdrawal amounts, and should consider different aspects of the economy before choosing an appropriate one. Sequence risk is a crucial deciding factor in a retirement portfolio's success, which can lead to drastically different portfolio values. We recommend that sequence risk be considered when thinking about a retirement portfolio and an ideal withdrawal amount.

Glossary

- Accumulation Phase - The hypothetical 40 years of employment where someone was investing their income.
- Accumulated Value - Final value of the total savings after 40 years of investment
- Decumulation Phase - The hypothetical 20 years of retirement where someone was withdrawing their savings.
- Internal Rates of Returns (IRR) - Measure of an investments rate of return
- Sequence Risk - The analysis of the order in which your investment returns occur.
- Perfectly Ascending Scenario - given a set of returns, the perfect ascending scenarios is the sequence of those returns that are in strictly increasing order.
- Perfectly Descending Scenario - given a set of returns, the perfect descending scenarios is the sequence of those returns that are in strictly decreasing order.
- Flat 9.35% Return - is the 40 year sequence where a portfolio experiences the annual average geometric return (9.35%) each year.

Bibliography

Clare, Andrew, et al. "Reducing Sequence Risk Using Trend Following and the CAPE Ratio." *Financial Analysts Journal*, vol. 73, no. 4, 2017.