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The Radio Number of Biregular Paths

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Abstract

A radio labeling of a graph is function $f : V(G) \to \{0, 1, \ldots, l\}$ such that $|f(u) - f(v)| \geq diam(G) + 1 - d_G(u, v)$ for all $u, v \in V(G)$. The radio number of a graph G, denoted as $rn(G)$, is the minimum span of any radio labeling of G. We provide background on some graphs with known radio numbers. We define a class of trees called biregularized paths which are formed by taking a path P and adding leaves to the vertices of P until each has the same degree m. We give bounds for the radio numbers of both the even and odd biregularized paths and give algorithms that attain each of these bounds respectively. We then discuss extending our results to a more general class of trees.

Executive Summary

The Frequency Assignment Problem, introduced by Hale in 1980 [4], motivates several graph labeling problems. The general idea of the Frequency Assignment Problem is: given the locations of radio transmitters, assign frequencies (labels or colors) to the transmitters that satisfy certain constraints and minimize a certain objective function. In this paper, we study one of the graph labeling problems motivated by the Frequency Assignment Problem called radio labeling.

A radio labeling of a graph is function $f : V(G) \to \{0, 1, \ldots, l\}$ such that $|f(u) - f(v)| \geq diam(G) + 1 - d_G(u, v)$ for all $u, v \in V(G)$. The radio number of a graph G, denoted by $rn(G)$, is the minimum span of any distance labeling of G.

We provide the radio numbers for the few families of graphs for which they are known, including paths, which are of particular interest to our project. The radio number of a path of order n is provided below and was first determined by Liu and Zhu in 2005 [12].

$$rn(P_n) = \begin{cases} 2k^2 + 2 & n = 2k + 1 \quad k \geq 2 \\ 2k(k-1) + 1 & n = 2k \quad k \geq 1 \end{cases}$$

We move on to study the radio numbers of trees which have not yet been completely determined. We make use of several results on measures of centrality, as they are important in bounding the radio number of trees. In particular, we use the status, $s(v)$, of a vertex which is the sum of the distances between v and every other vertex in G. The median of a graph, $M(G)$, consists of the vertices with the minimum status in G. Note that the status of the median of graph G is simply denoted by $s(G)$.

As shown in [9], the radio number of a tree T of order n and diameter d is given by

$$rn(T) \geq (n - 1)(d + 1) + 1 - 2s(T)$$

Moreover, equality holds if and only if for every median vertex w^*, there exists a radio labeling f with $f(u_0) = 0 < f(u_1) < \cdots < f(u_{n-1})$, where all the following hold (for all $0 \leq i \leq n - 2$):

(1) u_i and u_{i+1} belong to different branches (unless one of them is w^*);

(2) $\{u_0, u_{n-1}\} = \{w^*, v\}$, where v is some vertex with $d(v, w^*) = 1$;

(3) $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, w^*) - d(u_{i+1}, w^*)$.

We present the idea of a tightness path, which we use to interpret these constraints.

The main focus of our project is with a family of graphs that we call biregular trees. Biregular trees are formed by taking a tree T and adding leaves to the vertices of T until each has the same degree $m \geq \Delta$ where Δ is the maximum degree of T. This results in a biregular tree with degrees 1 and m. We will call the resulting tree *m-biregular* and the process *m-biregularization*. This paper is concerned with trees created from biregularizing the path P_n of order n. We will call these trees biregular paths with degrees 1 and m. In general, these are denoted by $H_{p,m}$ where p is the order of the original path and the resulting graph has degrees 1 and m.

We prove that

$$rn(H_{2k,m}) = 2k^2m - 2k^2 + 2k + 1$$

We begin with the base case of $H_{4,3}$, where $k = 2$, and induct on m. Then we take $H_{2k,m}$ and induct on k.

We also prove that

$$rn(H_{2k+1,m}) = 2k^2m - 2k^2 + 2mk + m + 1$$

First we prove the radio number is bounded below by this value. Then we provide an algorithm, defined inductively, that attains this lower bound.

We then discuss extending our results to a more general class of trees and discuss other questions and conjectures we have regarding radio labeling in general.

Acknowledgements

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1 Introduction

Finding the radio number of a graph can be regarded as an optimization problem in radio networks where the goal is to assign frequencies to locations so that they do not interfere with each other while minimizing the span of frequencies. In order to ensure there is no interference, we impose a constraint based on the distance between locations because frequencies assigned to closer locations have stronger interference.

We discuss the radio numbers of several types of graphs including a class of trees which we refer to as biregularized paths. Biregularized paths are created by taking a path P and adding leaves to the vertices of P until each has the same degree. We can think of this process as taking a central network, represented by the original path, and adding on remote locations, represented by leaf vertices, connected to each central location. We give bounds for the radio numbers of both the even and odd biregularized paths and give algorithms that attain each of these bounds respectively.

We start off by introducing notation and terminology that we use throughout the paper. We use d to denote the distance metric for graphs. The diameter $diam(G)$ is the maximum distance between any two vertices in G. A geodesic is a path from u to v of length $d(u, v)$. If $d(u, v) = diam(G)$ then u and v are said to be antipodal vertices.

2 Background

2.1 The Frequency Assignment Problem

2.1.1 The General Case

The formalization of the Frequency Assignment Problem is introduced by Hale in [4]. The general idea of the Frequency Assignment Problem is: given the locations of radio transmitters, assign frequencies (labels or colors) to the transmitters that satisfy certain constraints and minimize a certain objective function. To denote the frequency assignment we use $f(v) \in \mathbb{Z}^+$ to indicate the frequency assigned to transmitter v, where $\mathbb{Z}^+$ denotes the nonnegative integers. For the purpose of this paper we are only concerned with one type of constraint that is an instance of a *Frequency-Distance constraint* (F*D), and one type of objective function, the *span*. An F*D constraint is a constraint on the difference between frequencies that depends only on the physical distance between transmitters. The *span* of the assignment is the difference between the largest and smallest frequency. The span is used to identify the needed range of radio frequencies for a given configuration. For more information about other constraints and objective functions consult [4].

Frequency Assignment problems that only involve F*D constraints are called *Frequency-Distance Constrained Assignment Problem* (F*D-CAP). The simplest type of F*D-CAP is the Frequency-Distance Cochannel Assignment Problem (F*D-CCAP), where there is one radius $r > 0$. For any distinct transmitters u and v, if $d(u, v) \leq r$ then $f(v) \neq f(u)$. In other words, if a transmitter v is within a certain radius, r, of another transmitter u, then they cannot have the same frequency. Given this constraint, the goal is to minimize the span of the frequency assignment. When the transmitter configuration is represented by a graph and $r = 1$ we get a famous problem called the Coloring Problem, where two vertices are given different colors when they are adjacent and the goal is to minimize the number of colors used. The optimal number of colors, and so frequencies, for a graph is called the chromatic number $\chi(G)$.

Another version of this problem has two positive radii, $r(0)$ and $r(1)$. This problem is called the *Frequency-Distance Constrained Adjacent Channel Assignment Problem* (F*D-ACAP). This assignment requires that $d(u, v) \leq r(i)$ implies $|f(u) - f(v)| \neq i$ for $i \in \{0, 1\}$. Note that this is equivalent to saying that $d(u, v) = i$ implies $|f(u) - f(v)| \geq a(i)$. An important instance of this problem occurs in graph theory when $r(0) = 2$ and $r(1) = 1$, called the $L(2, 1)$-problem or distance 2 labeling problem. This requires adjacent vertices have frequencies at least two apart and vertices at distance two to have different frequencies. Figure 1 shows and example of a distance 2 labeling.

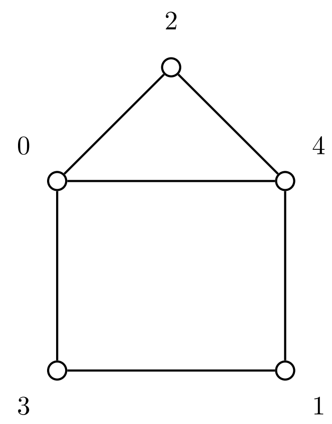

Figure 1: Example of an $L(2,1)$-labeling

These problems generalize to the General Minimum Span Channel Assignment Problem (F*D-CAP) for which we offer a simplification. We first start with $b \in \mathbb{Z}^+$, positive radii $r(0) > r(1) > \cdots > r(b) > 0$, and finite sets $\{0\} = A(0) \subset A(1) \subset \cdots \subset A(b) \subset \mathbb{Z}^+$. The constraints of the problem are: for distinct transmitters u and v, $d(u,v) \leq r(i)$ implies $|f(u) - f(v)| \notin A(i)$. Here it is important to note that the contents of the sets may not be consecutive. For example, the problem could be $b = 2$, $r(0) = 4$, $r(1) = 2$, $r(2) = 1$, $A(0) = \{0\}$, $A(1) = \{0,1,4\}$ and $A(2) = \{0,1,2,3,4\}$. This would mean if $d(u,v) = 2$ then $|f(u) - f(v)| \notin \{0,1,4\}$, and so their difference could be 2 or 3.

In [4] the Frequency Assignment problems are concerned with instances of a finite set of points from a plane which yield a metric with the possibility of non-integer distances. Since we approach this subject from graph theory, the instances that we are concerned with are finite graphs, G, along with the normal integer graph distance metric $d : V(G) \times V(G) \to \mathbb{Z}^+$. For this paper, we consider only the F*D-CAPs where the transmitter configuration is represented by a graph.

2.1.2 Radio Labelings

Definition [5] For a graph G, an *l-labeling* is a function $f : V(G) \to \{0, 1, \ldots, l\}$. We call l the *span* of the labeling.

Definition [7] A *radio r-labeling* is an l-labeling, f, where

$$|f(u) - f(v)| \geq r + 1 - d_G(u,v) \quad \forall u, v \in V(G)$$

There are many radio r-labelings of interest. We have already seen an example, the $L(2,1)$ problem, which is equivalent to finding a minimum span radio $2-labeling$. When $r = diameter - 1$ it is called an *radio antipodal labeling* because this allows antipodal vertices of have the same label. [8] A *multilevel distance labeling*, or *distance labeling*, is a radio r-labeling with $r = diam(G)$.

The results of this paper only concern distance labelings, therefore we use radio labeling and distance labeling interchangeably.

Definition [12] The *radio number* of a graph G, denoted as $rn(G)$, is the minimum span of any distance labeling of G.

Definition When a distance labeling has a span of $rn(G)$ it is called an *optimal*.

There are some authors who use labels from the positive integers and refer to the maximum label as the radio number. Note that our definition of radio number is always one less than the radio number obtained using this alternate definition.

The diameter of a disconnected graph is infinity, so the constraint on the labeling cannnot be satisfied. To fix this issue we define the radio number of a disconnected graph as the maximum radio number of the connected components. As an example, if G is the union of two connected graphs A and B, then $rn(G) = max\{rn(A), rn(B)\}$. Therefore it is only important to determine the radio number of connected graphs.

The Computational Complexity of Determining the Radio Number It is known that solving for the radio number of a graph with diameter 2 is NP-hard. In general, for $l \geq 2diam(G) - 2$ solving for the radio l number is NP-hard. However, the complexity for $l < 2diam(G) - 2$ is unknown. [6]

2.2 Measures of Centrality

Later in this paper it is important to determine the vertices that are most central to a certain graph. There are many different measures of centrality, each having their importance. For this paper the most important measure of centrality is called *status* by Buckley and Harary in [1], which Liu calls *weight* in [9]. Although we make many references to Liu's work, we use the terminology presented by Buckley and Harary, as weight has meaning to a different measure of centrality that we use.

Definition [1] The *status* of a vertex v in graph G, denoted $s_G(v)$, ($s(v)$ when the graph is understood) is the sum of the distances between v and every other vertex in G. Specifically, since $d(u, u) = 0$, then

$$s(v) = \sum_{u \in V(G)} d(u, v)$$

The status is useful to determine locations of buildings, such as post offices, where there is a desire to minimize the average travel distance from a central location. Therefore, the most central vertices from this respect are those with minimum status.

Definition [1] For a graph G, the *median*, $M(G)$, is the set of vertices with minimum status. A vertex v with minimum status is said to be a *median* vertex. The minimum status of a graph is denoted $s(G) = min\{s(v)|v \in V(G)\}$.

Although this paper is only concerned with the median, we make use of another measure of centrality to help determine the median. Before we introduce this measure we need more terminology.

Definition [1] In a tree T, a *branch* at a vertex v is a maximal subtree that has v as an endpoint.

It is important to remember that we consider only maximal subtrees. Therefore, a leaf has only one branch.

Definition [1] The *weight* of a vertex v, denoted $w(v)$, is the maximum number of edges in a branch at v.

Figure 2 is an example of a tree in which each vertex is labeled with its weight. Each end vertex has weight $|E| := |V| - 1$ as the only branch is the entire tree.

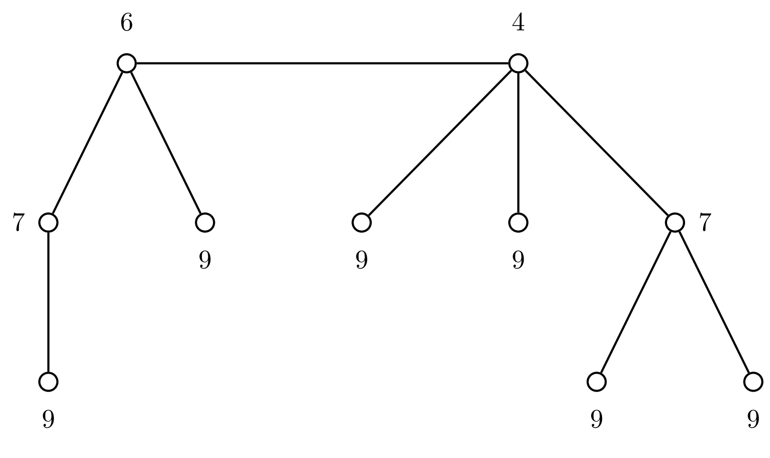

Figure 2: Weights

Definition [1] The *centroid* of a tree is the set of vertices of minimum weight

The following theorem unites the idea of median and centroid.

Theorem 1 *[1] In a tree, a vertex is a median vertex if and only if it is a centroid vertex.*

Theorem 1 allows us to use another theorem about the centroid and apply it to the median.

Theorem 2 *[1] The centroid of a tree consists of either one vertex or a pair of adjacent vertices.*

There are some useful tools to determine the median vertices. One observation concerns the comparison of the statuses of adjacent vertices.

Observation 3 *[1] For adjacent vertices u and v, define $c(u, v)$ to be the number of vertices closer to u than to v. Then, $s(u) - s(v) = c(v, u) - c(u, v)$.*

A simple example to illustrate this point is P_6 as shown in Figure 3. Here we see that $0 = s(u) - s(v) = c(v, u) - c(u, v) = 2 - 2 = 0$. Also $2 = s(w) - s(v) = c(v, w) - c(w, v) = 3 - 1 = 2$.

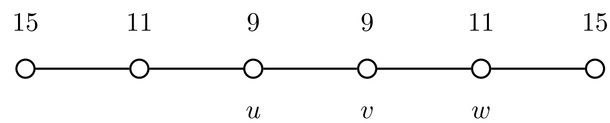

Figure 3: Statuses P_6

Furthermore, status is a vertex property that is constant under isomorphism because the distance between two vertices remains the same under isomorphism. A consequence of this is, if there is an isomorphism that maps a vertex u to a non-adjacent vertex v, then $s(u) = s(v)$. Since the median can only contain adjacent vertices, neither u nor v is a median vertex.

Another interesting observation is worthy of a lemma.

Lemma 4 *Let v be a median vertex of a tree T and let $W = (v = u_0, u_1, u_2, \ldots, u_n = u)$ be a geodesic from v to u where u_1 is not a median vertex. Then $s(v) = s(u_0) < s(u_1) < \cdots < s(u_n)$*

Proof Obviously $s(v) < s(u_1)$. Thus $s(v) - s(u_1) = c(u_1, v) - c(v, u_1) < 0$. Thus v is closer to more vertices than u_1 is. The image below shows that $c(v, u_1) = Q$ and $c(u_1, v) = N_1 + 1 + M_1$, thus $N_1 + 1 + M_1 - Q < 0$.

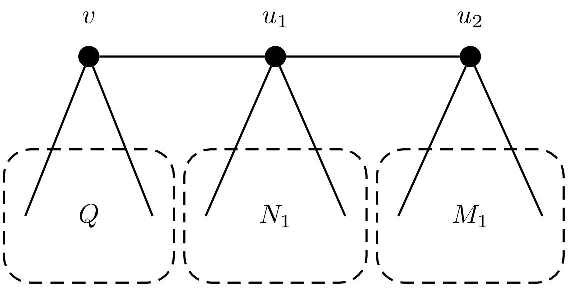

Figure 4: Vertices closer to u_1 than to u_2

This shows that

$$\begin{aligned} s(u_1) - s(u_2) &= c(u_2, u_1) - c(u_1, u_2) \\ &= M_1 - (N_1 + 1 + Q) = M_1 - N_1 - 1 - Q \\ &< M_1 - Q < 0 \end{aligned} \tag{2.1}$$

and so $s(u_1) < s(u_2)$

The image below is a generalization of Figure 4.

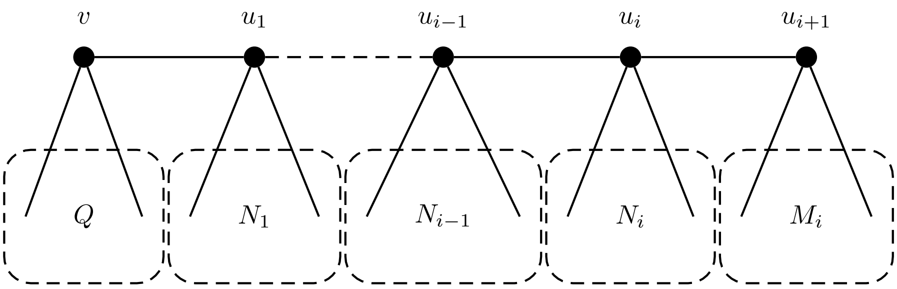

Figure 5: Vertices closer to u_i than to u_{i+1}

We generalize this result to

$$\begin{aligned} s(u_i) - s(u_{i+1}) &= c(u_{i+1}, u_i) - c(u_i, u_{i+1}) \\ &= M_i - \Big[Q + 1 + (N_1 + 1) + (N_2 + 1) + \cdots + (N_{i-1} + 1) + N_i\Big] \\ &< M_1 - Q < 0 \end{aligned} \tag{2.2}$$

and so $s(u_i) < s(u_{i+1})$. $\square$

Corollary 5 *The median vertices are the only vertices having statuses less than or equal to the statuses of their neighbors. More formally, if $s(v) \leq s(u)$ for all u adjacent to v, then v is a median vertex.*

Proof If a vertex u is not a median vertex than there exists a geodesic from u to a median v. Thus the vertex of this geodesic that is adjacent to u has a status smaller than the status of u. □

2.3 Known Results on Radio Numbers

2.3.1 Paths

We begin our survey by discussing the paths, P_n, on n vertices. Liu and Zhu determine the radio number for both even and odd vertex paths by proving a lower bound and then providing an algorithm that attains this bound.

Theorem 6 *[12]*

$$rn(P_n) = \begin{cases} 2k^2 + 2 & n = 2k+1 \quad k \geq 2 \\ 2k(k-1)+1 & n = 2k \quad k \geq 2 \end{cases} \tag{2.3}$$

We examine the algorithm that gives a labeling which attains the radio number. In each case, the algorithm has two steps. The first step orders the vertices and the second assigns labels from smallest to largest based on that ordering. To give meaning to the ordering we must first give a reference to the vertices of the path. For this we define P_n as the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-2}, v_{n-1}\}\}$, as shown in Figure 6.

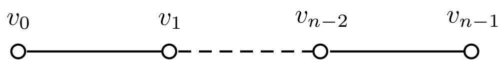

Figure 6: Original ordering of vertices of P_n

For both odd and even paths, we generate the ordering $x_0, \ldots, x_{n-1}$, associating each x_i with a vertex v_j of the path. The sequence is divided into three parts: beginning, iteration, and ending. The beginning and ending are constant, with the iteration depending on the order of the path. For the even case the sequence is:

$$v_{k-1}, v_{2k-1}, (v_1, v_{1+k}, v_2, v_{2+k}, \ldots, v_{k-2}, v_{k-2+k}), v_0, v_k$$

For the odd case the sequence is:

$$v_{k-1}, v_{2k-1}, v_0, v_k, v_{2k}, (v_2, v_{2+k}, v_3, v_{3+k}, v_4, v_{4+k}, \ldots, v_{k-2}, v_{k-2+k}), v_1, v_{1+k}$$

Notice that for the odd case the fixed components of the sequence give 7 vertices, which means the sequence must be modified slightly for $k = 2$. The ordering we use for $n = 5$ is $(v_3, v_0, v_2, v_4, v_1)$, which can be obtained by deleting the first vertex of the beginning and the last vertex of the ending.

We illustrate some examples of sequences for both cases where the black circular nodes represent the beginning of the sequence, the white circular nodes represent the end of the sequence and the square nodes represent the iteration of the sequence.

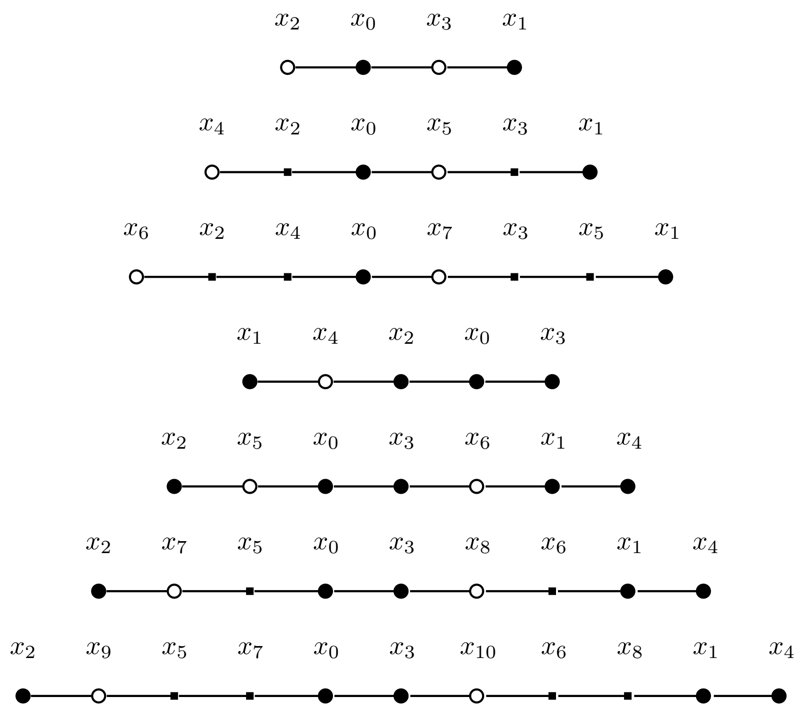

Figure 7: Orderings for P_4, P_6, P_8; P_5, P_7, P_9, P_{11}

After the vertices have been ordered we define the labeling f by $f(x_0) = 0$ and $f(x_i) = f(x_{i-1}) + n - d(x_{i-1}, x_i)$.

We offer an example of this labeling for both an even and an odd path. We include computations used to implement the algorithm as well as figures of the resulting labelings.

Example P_4

$$rn(P_4) = 2(2)(2-1) + 1 = 5$$

$$\begin{aligned}
x_0 &= v_1 & \qquad f(x_0) &= 0 \\
x_1 &= v_3 & \qquad f(x_1) &= 0 + 4 - 2 = 2 \\
x_2 &= v_0 & \qquad f(x_2) &= 2 + 4 - 3 = 3 \\
x_3 &= v_2 & \qquad f(x_3) &= 3 + 4 - 2 = 5
\end{aligned} \tag{2.4}$$

Figure 8: Radio labeling of P_4

Example P_7

$$rn(P_7) = 2(3)^2 + 2 = 20$$

$$\begin{aligned}
x_0 &= v_2 & \qquad f(x_0) &= 0 \\
x_1 &= v_5 & \qquad f(x_1) &= 0 + 7 - 3 = 4 \\
x_2 &= v_0 & \qquad f(x_2) &= 4 + 7 - 5 = 6 \\
x_3 &= v_3 & \qquad f(x_3) &= 6 + 7 - 3 = 10 \\
x_4 &= v_6 & \qquad f(x_4) &= 10 + 7 - 3 = 14 \\
x_5 &= v_1 & \qquad f(x_5) &= 14 + 7 - 5 = 16 \\
x_6 &= v_4 & \qquad f(x_6) &= 16 + 7 - 3 = 20
\end{aligned} \tag{2.5}$$

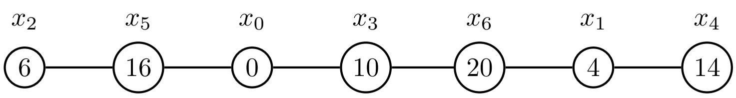

Figure 9: Radio labeling of P_7

We finish the section with Figure 10, which displays the labelings that correspond to the orderings in Figure 7.

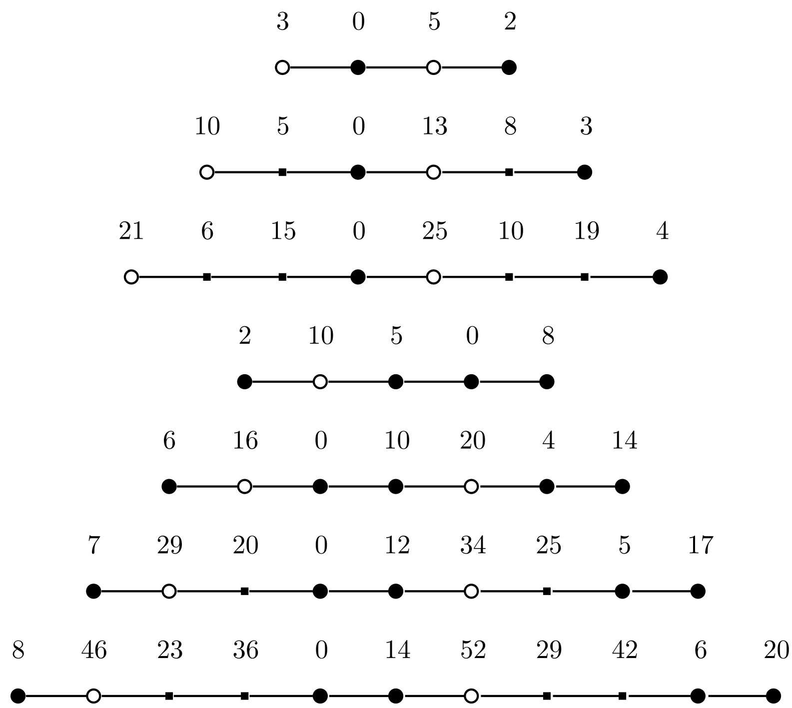

Figure 10: Labelings for P_4, P_6, P_8; P_5, P_7, P_9, P_{11}

2.3.2 Cycles

Liu and Zhu also present an algorithm for cycles in [12] that attains the lower bound they prove in the same text.

Theorem 7 *[12]*

$$rn(C_n) = \begin{cases} \frac{n-2}{2}\phi(n) + 1 & n \equiv 0, 2 \pmod 4 \\ \frac{n-1}{2}\phi(n) & n \equiv 1, 3 \pmod 4 \end{cases} \tag{2.6}$$

where

$$\phi(n) = \begin{cases} k+1 & n = 4k+1 \\ k+2 & n = 4k+r \text{ for } r = 0, 2, 3 \end{cases} \tag{2.7}$$

We have translated the formulas from Theorem 7 to appear explicitly as

$$rn(C_n) = \begin{cases} \frac{n^2+6n-8}{8} & n \equiv 0 \pmod 4 \\ \frac{n^2+2n-3}{8} & n \equiv 1 \pmod 4 \\ \frac{n^2+4n-4}{8} & n \equiv 2 \pmod 4 \\ \frac{n^2+4n-5}{8} & n \equiv 3 \pmod 4 \end{cases} \tag{2.8}$$

The process of finding the radio labeling is the same as for paths: order the vertices then determine the labels based on that ordering. To generate an ordering and subsequently a labeling of the vertices of C_n, we require the use of four functions: d_i, f_i, τ, and f.

Liu and Zhu define d_i and f_i for $0 \leq i \leq n-2$, which they call the distance gap sequence and the color gap sequence respectively. They use d_i and f_i to define τ and f which are used to order and label the vertices. The definitions for these functions are separated into four cases:
Case 1: n=4k

$$d_i = \begin{cases} 2k & i \equiv 0,2 \pmod 4 \\ k & i \equiv 1 \pmod 4 \\ k+1 & i \equiv 3 \pmod 4 \end{cases} \tag{2.9}$$

$$f_i = \begin{cases} 1 & \text{if i is even} \\ k+1 & \text{if i is odd} \end{cases} \tag{2.10}$$

Case 2: n=4k+2

$$d_i = \begin{cases} 2k+1 & \text{if i is even} \\ k+1 & \text{if i is odd} \end{cases} \tag{2.11}$$

$$f_i = \begin{cases} 1 & \text{if i is even} \\ k+1 & \text{if i is odd} \end{cases} \tag{2.12}$$

Case 3: n=4k+1

$$d_{4i} = d_{4i+2} = 2k - i \tag{2.13}$$

$$d_{4i+1} = d_{4i+3} = k + 1 + i \tag{2.14}$$

$$f_i = 2k - d_i + 1 \tag{2.15}$$

Case 4: n=4k+3

$$\begin{aligned} d_{4i} = d_{4i+2} &= 2k+1-i \\ d_{4i+1} &= k+1+i \\ d_{4i+3} &= k+2+i \end{aligned} \tag{2.16}$$

We now define τ by $\tau(0) = 0$ and

$$\tau(i+1) = \tau(i) + d_i \pmod n \tag{2.17}$$

Let $x_i = v_{\tau(i)}$ where the graph is already labeled by v_i for $i = 0, \ldots, n-1$ s.t. $v_i \sim v_{i+1}$ and $v_0 \sim v_{n-1}$ as shown in Figure 11.

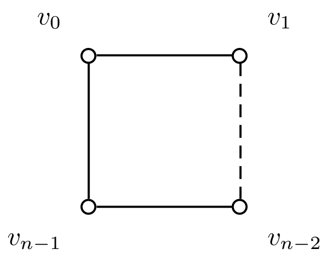

Figure 11: Original ordering of C_n

Now we use a function f which is the labeling defined by $f(x_0) = 0$ and $f(x_{i+1}) = f(x_i) + f_i$.

We provide example of each of these 4 cases.

Example C_4

$$rn(C_4) = \frac{4-2}{2}(3) + 1 = 4$$

$$\begin{aligned} &\tau(0) = 0 && f(x_0) = 0 \\ &\tau(1) = 0+2 \pmod 4 = 2 && f(x_1) = 0+1 = 1 \\ &\tau(2) = 2+1 \pmod 4 = 3 && f(x_2) = 1+2 = 3 \\ &\tau(3) = 3+2 \pmod 4 = 1 && f(x_3) = 3+1 = 4 \end{aligned} \tag{2.18}$$

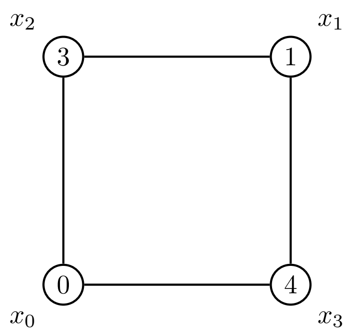

Figure 12: Radio labeling of C_4

Example C_5

$$rn(C_5) = \frac{5-1}{2}(2) = 4$$

$$\begin{aligned}
\tau(0) &= 0 & f(x_0) &= 0 \\
\tau(1) &= 0+2 \pmod 5 = 2 & f(x_1) &= 0+1 = 1 \\
\tau(2) &= 2+2 \pmod 5 = 4 & f(x_2) &= 1+1 = 2 \\
\tau(3) &= 4+2 \pmod 5 = 1 & f(x_3) &= 2+1 = 3 \\
\tau(4) &= 1+2 \pmod 5 = 3 & f(x_4) &= 3+1 = 4
\end{aligned} \tag{2.19}$$

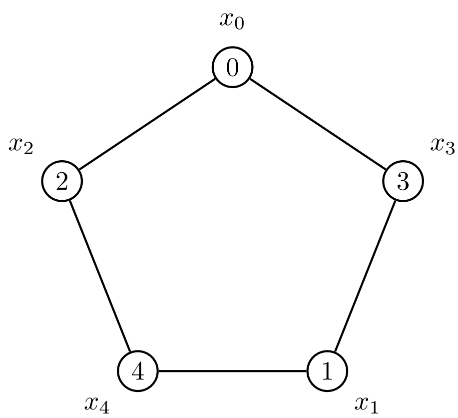

Figure 13: Radio labeling of C_5

Example C_6

$$rn(C_6) = \frac{6-2}{2}(3) + 1 = 7$$

$$
\begin{aligned}
&\tau(0) = 0 && f(x_0) = 0 \\
&\tau(1) = 0 + 3 \pmod 6 = 3 && f(x_1) = 0 + 1 = 1 \\
&\tau(2) = 3 + 2 \pmod 6 = 5 && f(x_2) = 1 + 2 = 3 \\
&\tau(3) = 5 + 3 \pmod 6 = 2 && f(x_3) = 3 + 1 = 4 \\
&\tau(4) = 2 + 2 \pmod 6 = 4 && f(x_4) = 4 + 2 = 6 \\
&\tau(5) = 4 + 3 \pmod 6 = 1 && f(x_5) = 6 + 1 = 7
\end{aligned} \tag{2.20}
$$

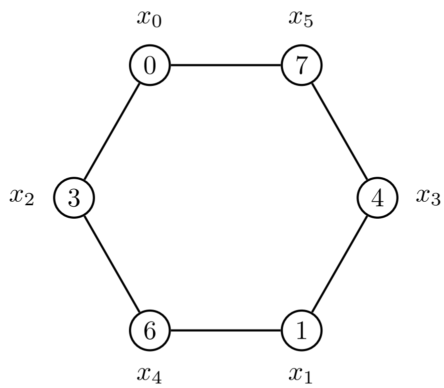

Figure 14: Radio labeling of C_6

Example C_7

$$rn(C_7) = \frac{7-1}{2}(3) = 9$$

$$
\begin{aligned}
&\tau(0) = 0 && f(x_0) = 0 \\
&\tau(1) = 0 + 3 \pmod 7 = 3 && f(x_1) = 0 + (4 - 3) = 1 \\
&\tau(2) = 3 + 2 \pmod 7 = 5 && f(x_2) = 1 + (4 - 2) = 3 \\
&\tau(3) = 5 + 3 \pmod 7 = 1 && f(x_3) = 3 + (4 - 3) = 4 \\
&\tau(4) = 1 + 3 \pmod 7 = 4 && f(x_4) = 4 + (5 - 3) = 6 \\
&\tau(5) = 4 + 2 \pmod 7 = 6 && f(x_5) = 6 + (4 - 2) = 8 \\
&\tau(6) = 6 + 3 \pmod 7 = 2 && f(x_6) = 8 + (4 - 3) = 9
\end{aligned} \tag{2.21}
$$

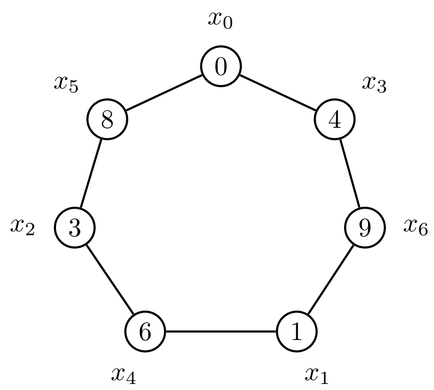

Figure 15: Radio labeling of C_7

2.3.3 Trees

Before we begin this section, we introduce some terminology for trees.

Definition For a tree T rooted at vertex w, if a and b are distinct vertices, we say a is an *ancestor* of b if every path from w to b includes a.

In [9], Liu finds a lower bound for the radio number of trees using its median.

Definition For a tree T rooted at vertex w and distinct vertices $u, v \in T$, define

$$\phi_w(u,v) = \max\{d(w,t) : \text{t is a common ancestor of } u \text{ and } v\}$$

By examining the radio labeling constraint on consecutively labeled vertices u_i and u_{i+1} in a labeling $0 = f(u_0) < \ldots < f(u_{n-1})$ and considering the tree rooted at one of it's median vertices, w^*, Liu finds the lower bound for its radio number

$$\begin{aligned} rn(T) = f(u_{n-1}) &\geq (n-1)(diam(T)+1) - \sum_{i=0}^{n-2} d(u_{i+1}, u_i) \\ &= (n-1)(diam(T)+1) - \Big\{ 2\sum_{u \in V(T)} d(u, w^*) - d(u_0, w^*) - d(u_{n-1}, w^*) \\ &\quad - 2\sum_{i=0}^{n-2} \phi_{w^*}(u_{i+1}, u_i) \Big\} \\ &\geq (n-1)(diam(T)+1) - (2s(T)-1) \qquad (2.22) \end{aligned}$$

Liu finds that the second inequality of 2.22 has equality if and only if $\phi_{w^*}(u_{i+1}, u_i)\} = 0$ for all $i = 0 \dots n-2$ and $\{u_0, u_{n-2}\} = \{w^*, v\}$ for some v with $d(w^*, v) = 1$. In other words, each pair of consecutive vertices u_i and u_{i+1} are on different branches and the median w^* and a vertex in the neighborhood of (w^*) are first and last to be labeled. The first inequality is tight only when all of the distance one constraints are tight. From this analysis we arrive at Theorem 8.

Theorem 8 *[9] Let T be a an n-vertex tree with diameter $diam(T)$. Then*

$$rn(T) \geq (n-1)(diam(T)+1) + 1 - 2s(T)$$

Moreover, the equality holds if and only if for each median vertex, w^*, there exists a radio labeling f with $0 = f(u_0) < f(u_1) < \cdots < f(u_{n-1})$, where all the following hold (for all $0 \leq i \leq n-2$):

(1) $\{u_0, u_{n-1}\} = \{w^*, v\}$, where v is some vertex with $d(v, w^*) = 1$;

(2) u_i and u_{i+1} belong to different branches (unless one of them is w^*);

(3) $f(u_{i+1}) = f(u_i) + diam(T) + 1 - d(u_i, w^*) - d(u_{i+1}, w^*)$.

Notice that every optimal radio labeling must follow the properties $(1)-(3)$ in Theorem 8. Furthermore, for every tree with two median vertices, the optimal labeling must begin at one median vertex and terminate at the other median vertex.

As observed in [9], it is interesting to note that even paths achieve this lower bound, but odd paths do not. We provide an example of each below.

Example $rn(P_6) = (6-1)(5+1) + 1 - 2(9) = 13$

Figure 16: Radio labeling of P_6

Example $rn(P_5) \geq (5-1)(4+1) + 1 - 2(6) = 9$

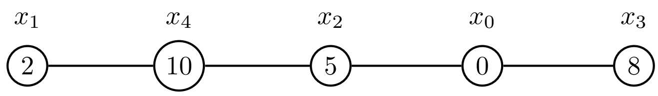

Figure 17: Radio labeling of P_5

2.3.4 Other Graphs of Interest

There are some other simple graphs that have known radio numbers.

The star graph S_n, is the tree on $n+1$ vertices having one vertex of degree n and n leaves. As stated in [9], $rn(S_n) = n+1$, which is the bound given in the same document. This can be easily seen in Figure 18. Furthermore, the labeling follows the three properties in Theorem 8 for which trees attain this bound.

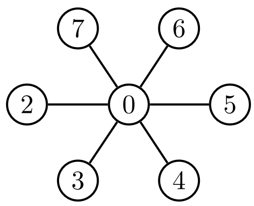

Figure 18: Radio labeling of S_6

Spiders are trees with only one vertex of degree larger than 2 and can be viewed as a generalization of stars. Liu determines a lower bound for the radio number of spiders and characterizes the graphs that reach this bound [9]. A spider is defined as a tree rooted at vertex $v_{0,0}$ and having $n \geq 3$ legs, with leg i having l_i vertices, such that $l_1 \geq l_2 \geq \cdots \geq l_n \geq 1$. This is denoted by $S_{l_1,l_2,l_3,\ldots,l_n}$ having $|V| = l_1+l_2+\cdots+l_n+1$. Furthermore we denote the vertices of the graph by $v_{i,j}$ with i denoting the i^{th}-leg and j denoting the distance from $v_{0,0}$. Figure 19 illustrates this notation.

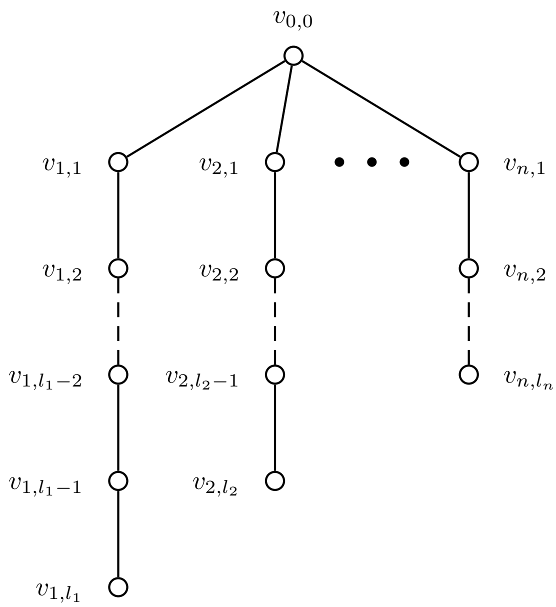

Figure 19: Vertex labeling of $S_{l_1,l_2,l_3,\dots,l_n}$

Theorem 9 *[9] Let $G = S_{l_1,l_2,\dots,l_n}$. Then*

$$rn(G) \geq \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + \lceil \frac{l_1 - l_2}{2} \rceil \lfloor \frac{l_1 - l_2}{2} \rfloor + 1$$

The next theorem characterizes the spiders whose radio number achieves the bound in the previous theorem. Here we define $\bar{l_1}$ as the sum of the vertices in the legs other than l_1, so $\bar{l_1} := l_2 + l_3 + \cdots + l_n = |V| - (l_1 + 1)$.

Theorem 10 *[9] Let $G = S_{l_1,l_2,\dots,l_n}$ with $n \geq 3$. Then*

$$rn(G) = \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + \lceil \frac{l_1 - l_2}{2} \rceil \lfloor \frac{l_1 - l_2}{2} \rfloor + 1$$

if and only if $\bar{l_1} \geq \frac{l_1+l_2-1}{2}$.

To prove this Liu presents algorithms for several cases based on the value of $l_1 - l_2$. We do not discuss these, but they are presented in [9].

The square of a graph is created by adding edges $\{u, v\}$ if u and v are distance two apart in original graph. Liu and Xie have determined the radio number for the squares of paths and cycles in [11] and [10] respectively.

Cartesian Products

Definition [13] The *Cartesian Product* of two graphs G and H, denoted $G\Box H$, is defined by vertex set

$$V(G\Box H) := V(G) \times V(H) := \{(g,h)\big| g \in V(G) \text{ and } h \in V(H)\}, \tag{2.23}$$

and by edge set

$$E(G\Box H) := \Big\{\{(g,h),(g',h')\}|g = g' \text{ and } \{h,h'\} \in E(H) \\ \text{OR } h = h' \text{ and } \{g,g'\} \in E(G)\Big\} \tag{2.24}$$

A way to see this construction is by making $|V(H)|$ copies of graph G, indexing the vertices by (g,h), and adding an edge between (g,h) and (g',h') if g and g' reference the same vertex of G and $h \sim h'$ in H. In [13] Morris-Rivera, Tomova, Wyels, and Yeager compute the radio number for $C_n\Box C_n$ to be

$$rn(C_n\Box C_n) = \begin{cases} 2k^3 + 4k^2 - k & n = 2k \\ 2k^3 + 4k^2 + 2k + 1 & n = 2k+1 \end{cases} \tag{2.25}$$

An interesting consequence of this result is that it proves $rn(G\Box H)$ is not always equal to $rn(G)rn(H)$. [13]

The Cartesian product of two paths is known as a grid graph. The radio number for an $n \times n$ grid graph has not yet been completely determined, but there has been work done on upper and lower bounds for this number. We present an example of the radio labeling for the case where $n = 3$, which we obtained using an exhaustive method.

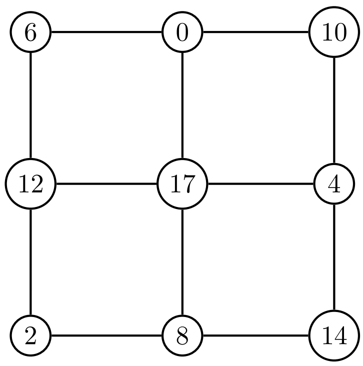

Figure 20: Radio labeling of 3×3 grid graph

The Hypercubes Q_n, are an instance of Cartesian products where Q_n is created by the product of n copies of K_2. More explicitly, we define $Q_n := K_2 \square K_2 \square \cdots \square K_2$.

Theorem 11 *[8] For any integer $n \geq 1$,*

$$rn(Q_n) = (2^{n-1} - 1)\lceil \frac{n+3}{2} \rceil + 1 \tag{2.26}$$

Graphs of diameter 2 have been studied extensively, mainly as instances of distance two labelings. The *complete n-partite* graph, $K_{n_1,n_2,\cdots,n_k}$, is an example which has been determined to be

$$rn(K_{n_1,n_2,\cdots,n_k}) = (k-2) + \sum_{i=1}^{k} n_i \tag{2.27}$$

In Figure 21 we offer an optimal labeling for the complete bipartite graph $K_{5,3}$, which has a span of $(2-2)+(5+3) = 8$.

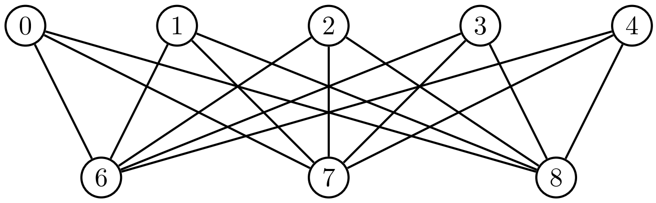

Figure 21: Optimal labeling for $K_{5,3}$

2.3.5 Related Theorems

Theorem 12 *[2] If H is a connected subgraph of a connected graph G, such that $diam(H) < diam(G)$, then $rn(H) < rn(G)$.*

Similarly, we have the following lemma.

Lemma 13 *If H is a subgraph of G with $diam(H) = diam(G)$ then $rn(H) \leq rn(G)$*

Proof Suppose f is a radio labeling of G with span $rn(G)$. Define the distance labeling g of H as $f(H)$. This labeling is valid because $d_H(u, v) \geq d_G(u, v)$ $\forall u, v \in V(H)$. Thus

$$d_H(u, v) + |f(u) - f(v)| \geq d_G(u, v) + |f(u) - f(v)| \geq diam(G) + 1$$

So $rn(H) \leq rn(G)$.$\square$

The shortest path between two antipodal vertices is a natural candidate to consider for bounding the radio number of G.

Theorem 14 *[2] If G is a connected graph of order n and diameter $diam(G)$, then*

$$rn(P_{diam(G)+1}) \leq rn(G) \leq rn(P_{diam(G)+1}) + diam(G)(n - diam(G) - 1).$$

and so explicitly

$$rn(G) \geq \begin{cases} 2k^2 + 2, & diam(G) = 2k+1 \\ 2k^2 - 2k + 1, & diam(G) = 2k \end{cases}$$

and

$$rn(G) \leq \begin{cases} 2kn + n - 2k^2 - 2k + 2, & diam(G) = 2k+1 \\ 2kn - 2k^2 - 4k + 1, & diam(G) = 2k \end{cases}$$

Definition A radio labeling is called *radio graceful* if the span of the labeling is $n-1$, so every label $0, 1, \ldots, n-1$ is used.

If a radio graceful labeling exists for a graph G, then $rn(G) = n-1$, so any optimal labeling of G is a radio graceful labeling. The simplest example of a radio graceful labeling is an optimal labeling of K_n. A more interesting case is an optimal labeling for the Petersen graph, which is displayed in Figure 22.

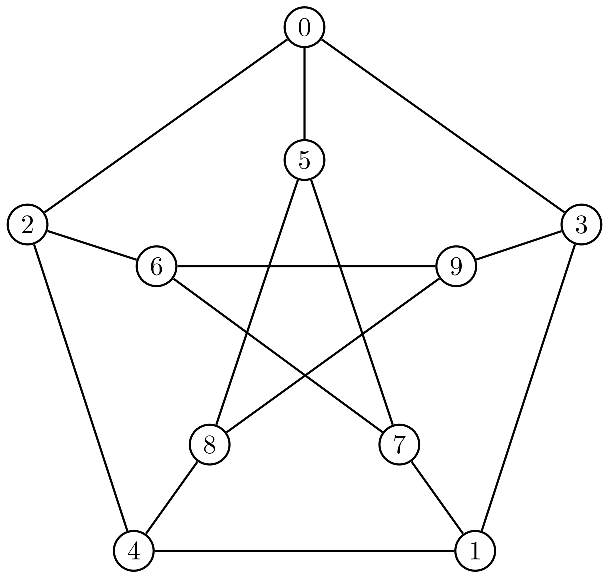

Figure 22: Optimal radio labeling of the Petersen graph

3 New Results

3.1 The Tightness Digraph

The algorithms presented for paths and for trees that attain the lower bound in Theorem 8 both require that the constraint between x_i and x_{i+1} has equality under the labeling f, meaning

$$|f(x_{i+1}) - f(x_i)| = diam(G) + 1 - d(x_i, x_{i+1}) \tag{3.1}$$

We say that x_i is *tight* with x_{i+1} to mean their constraint has equality. Without an explicit order it is important to know the direction of the equality, since for a graph G and $u, v \in V(G)$

$$f(v) = f(u) + diam(G) + 1 - d(v, u) \tag{3.2}$$

and

$$f(u) = f(v) + diam(G) + 1 - d(v, u) \tag{3.3}$$

may result in a very different labeling. As a convention, saying that vertex u is *tight* with vertex v means $f(v) - f(u) > 0$. Furthermore, a digraph can be constructed from a graph G and radio labeling f by having an arc directed from a vertex u to vertex v if u is *tight* with v. This digraph is referred to as the *Tightness Digraph* corresponding to the graph G and labeling f, and is denoted by $TD(G, f)$. More formally, for graph $G = (V, E)$ and labeling $f : V \to \mathbb{Z}^+$.

$$TD(G, f) = \left\{(u, v)\middle| u, v \in V \text{ and } f(v) = f(u) + diam(G) + 1 - d(v, u)\right\} \tag{3.4}$$

Our main motivation for constructing the tightness digraph is to study the tightness paths of a graph G with labeling f.

Definition A *tightness path* of a graph G with n vertices and labeling f is a directed path in $TD(G, f)$ from x_0 to x_{n-1}.

Observation 15 *Every optimal radio labeling must have at least one tightness path.*

Proof Suppose there is a labeling, f, of graph G, such that there is no tightness path in the tightness digraph. Let A be the maximal connected subdigraph of $TD(G, f)$ that contains x_0. Since there is no directed path from x_0 to x_{n-1}, x_{n-1} is not in A. Let i be the smallest subscript such that $x_i \notin A$. Then for all $j < i$, $f(x_i) > f(x_j) + diam(G) + 1 - d(x_j, x_i)$, and so we can reduce the label of x_i until there exists a tightness amongst an earlier vertex of the ordering. Now x_i is in the new maximal connected subdigraph A'. This process can be repeated until we reach x_{n-1}. Therefore we result in a new labeling with span less than the span of f and so f is not optimal. □

Note that one graph can have multiple tightness paths for the same labeling. For a given graph G and labeling f, we are interested in whether there is a tightness path that goes through every vertex in G because this occurs if and only if x_i is tight with x_{i+1} for each $0 \leq i \leq n-1$.

Observation 16 *Statement (3) from 8 is equivalent to saying a tightness path goes through every vertex.*

We offer several examples of the tightness digraph. In Figures 23, 24 and 25 the tightness digraphs for C_4, P_4, and P_7 are given for the labelings given in Section 2.3. The tightness digraph maintains the vertices of the original graph, but deletes the edges. For the tightness digraph we must maintain the same vertex labeling as in the original graph. By convention the vertices of the digraph are always displayed in a cycle using the ordering to simplify the appearance and to easily determine the existence of a tightness path through every vertex.

The idea of a tightness path helps us interpret property (3) from Theorem 8. This property is equivalent to the existence of a tightness path from x_0 to x_{n-1} that goes through every vertex.

Notice that in the labelings for $P4$, $P7$ and $C4$, there is a tightness path that goes through every vertex, however in Figure 26 we see that not every optimal labeling has this property.

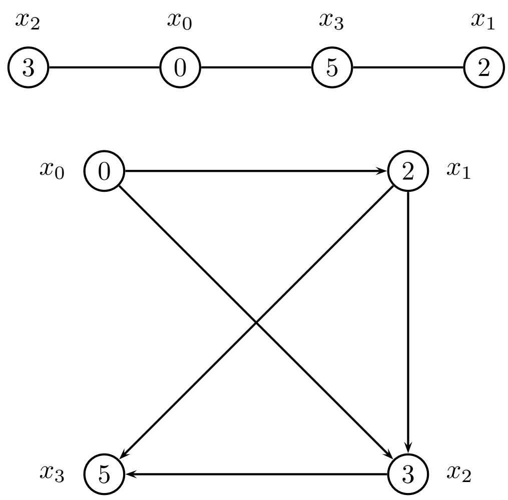

Figure 23: Tightness digraph for optimal radio labeling of P_4

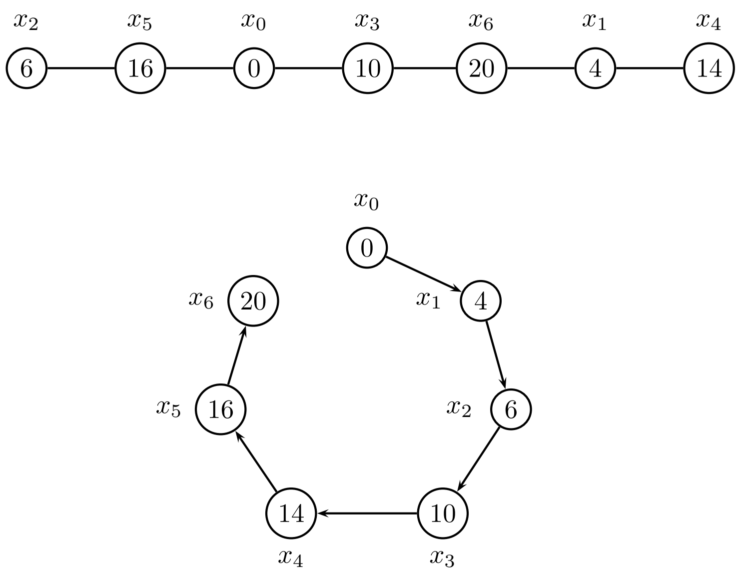

Figure 24: Tightness digraph for optimal radio labeling of P_7

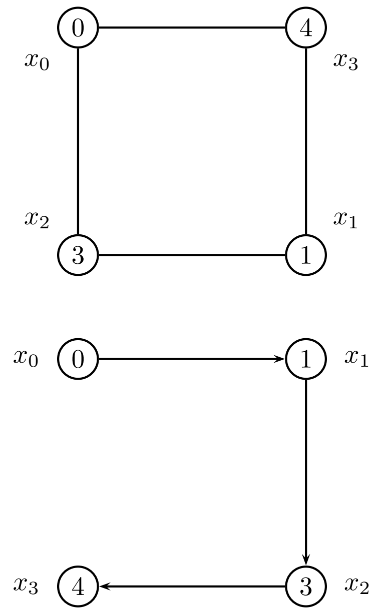

Figure 25: Tightness digraph for optimal radio labeling of C_4

Definition [2] For a radio labeling f, the *complementary labeling* $\bar{f}$ is defined by $\bar{f}(u) := span(f) - f(u)$ for all $u \in V$.

The complementary labeling is a useful concept, but a more useful concept is the reverse ordering.

Definition For an ordering of the vertices $X = (x_0, x_1, \ldots, x_{n-1})$, the *reverse ordering* is defined by $\bar{X} := (x_{n-1}, x_{n-2}, \ldots, x_1, x_0)$.

For both the complementary labeling and the reverse ordering the span is the same as the original labeling. The importance of these two concepts is seen by looking at (1) from Theorem 8, which says:

(1) $\{u_0, u_{n-1}\} = \{w^*, v\}$, where v is some vertex with $d(v, w^*) = 1$;

Thus an optimal ordering can occur as either $(x_0 = w^*, x_1, \ldots, x_{n-1} = v)$ or
$(x_0 = v, x_1, \ldots, x_{n-1} = w^*)$. In the second case we could reverse the ordering to have w^* occur as the first vertex. Thus, without loss of generality we can assume $x_0 = w^*$.

There is one important distinction between the complementary labeling and reverse ordering, which we illustrate in the following example using the tightness digraph. Figure 26 displays an alternative ordering, and subsequent labeling, for P_5. Unlike the labeling presented in [12], this labeling does not have a tightness path through every vertex. By comparison of the three digraphs in Figures 26 - 28 we see that taking the complementary labeling only reverses the direction of the edges of the digraph. The digraph of the reverse ordering is created by taking the digraph of the complementary labeling and adding that each vertex must be tight with a previous vertex. If a tightness path goes through every vertex, then the complementary labeling is equivalent to the reverse ordering.

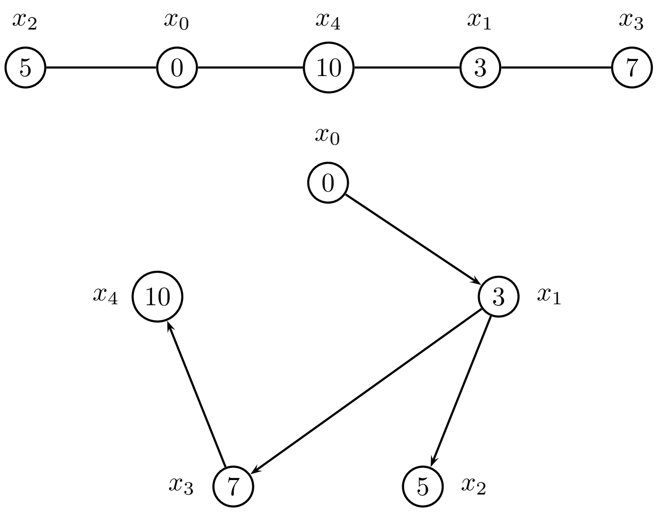

Figure 26: Tightness digraph for P_5

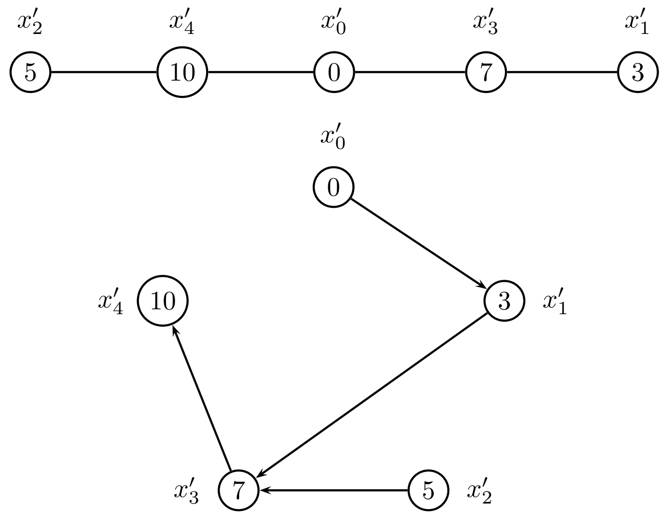

Figure 27: Tightness digraph for complementary labeling

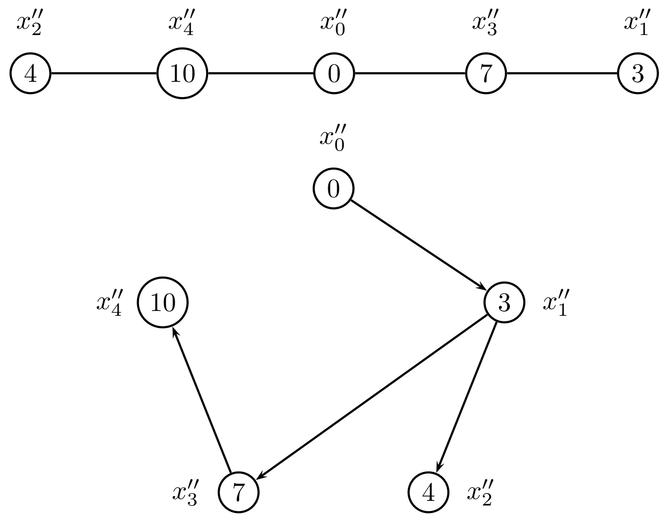

Figure 28: Tightness digraph for reverse ordering

3.2 Biregular Paths

3.2.1 Introduction

Definition A graph is said to be *(a,b)-biregular* if its vertex degrees assume exactly two values, a and b where $1 \leq a < b$. [3]

This paper is mainly concerned with the creation of biregular trees by taking a tree T and adding leaves to the vertices of T until each has the same degree $m \geq \Delta$ where Δ is the maximum degree of T. This results in a 1,m-biregular tree. We call the resulting tree *m-biregular* and the process *m-biregularization.*

Lemma 17 *The median vertices of an m-biregularized tree, T^*, are the same as those of the original tree, T.*

Proof Recall that $c_T(u,v)$ is the number of vertices in T closer to u than to v. We first prove that for two adjacent vertices u and v, the number of vertices in $J = c_{T^*}(v,u) - c_T(v,u)$ is dependent only on, and is proportional to, $c_T(v,u)$. This shows that if $c_T(v,u) \geq c_T(u,v)$ then $c_{T^*}(v,u) \geq c_{T^*}(u,v)$ and so $s_{T^*}(u) \geq s_{T^*}(v)$ by Observation 3.

If we root the tree T at a vertex v, then for a vertex u adjacent to v, let $B_v(u)$ denote the branch of T rooted at v that contains u. The vertices of the branches other than $B_v(u)$ are the vertices counted in $c_T(v,u)$, noting that v is not counted. Let m be an integer greater than or equal to the maximum degree in T, and T^* is the m-biregularization of T. Now, let x be a leaf vertex of T counted in $c_T(v,u)$, thus x is closer to v than to u. Denote y_0 as the vertex adjacent to x. Let us create a new tree T_1 from T by moving x to be a leaf vertex adjacent to a vertex y_1, with $deg(y_1) < m$, that is not in $B_v(u)$. Notice that one of y_0 or y_1 may be v. This gives that

$$\begin{aligned} deg_{T_1}(y_0) &= deg_T(y_0) - 1 \\ deg_{T_1}(y_1) &= deg_T(y_1) + 1 \end{aligned} \tag{3.5}$$

The degrees of the other vertices remain the same. Therefore, when T_1 is m-biregularized, the vertices other than y and y_0 are given the same number of vertices as in the process of biregularizing T; y_1 receives one more and y_0 receives one less. This means that for both $c_T(v,u)$ and $c_{T_1}(v,u)$, J vertices were added. This process can be used inductively to create any set of branches rooted at v, required that the maximum degree does not exceed m, showing that J is dependent only on the size of $c_T(v,u)$. This allows us to use the path on $c_T(v,u) + 1$ vertices as a means to calculate J. Figure 3.2.1 shows that we add $m - 2$ leaves to each of $c_T(v,u)$ vertices and $m - 1$ leaves to one vertex.

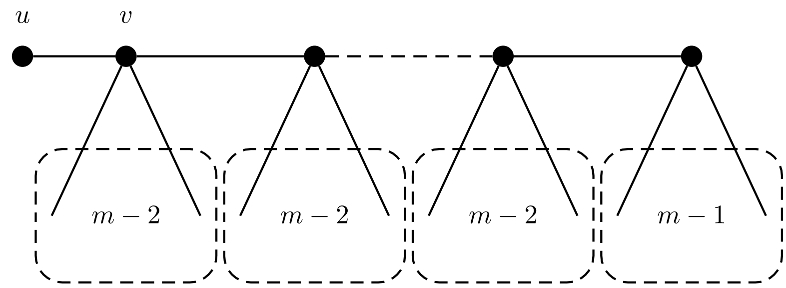

Figure 29: Leaves added when m-biregularizing T

This can be used to calculate the number of vertices added to $c_T(u,v)$ to give

$$\begin{aligned}
c_{T^*}(u,v) &= c_T(u,v)(m-2)+m-1 \\
c_{T^*}(v,u) &= c_T(v,u)(m-2)+m-1 \\
s_{T^*}(v)-s_{T^*}(u) &= c_{T^*}(u,v)-c_{T^*}(u,v) \\
&= \big(c_T(u,v)-c_T(v,u)\big)(m-2) \\
&= \big(s_T(v)-s_T(u)\big)(m-2) \qquad (3.6)
\end{aligned}$$

Which proves that if $s_T(v) \leq s_T(u)$ then $s_{T^*}(v) \leq s_{T^*}(u)$. Using Corollary 5, the median is the only vertex with status smaller than, or equal to, each adjacent vertex in T so it is also has the smallest status in T^*. Thus if v is a median of T it is a median of T^*. The last step is to show that if v is the only median, no new median is created. Since $s_T(v) < s_T(u)$ for any adjacent u the equation 3.6 gives for $m > 3$ that the inequality remains strict. Thus v is the only median. $\square$

This paper is concerned with trees created from m-biregularizing paths. We call these trees biregular paths. In general, these are denoted as $H_{p,m}$ where p is the order of the original path and having degrees 1 and m. We call the case when $p = 2k$ the even biregular path and the case when $p = 2k+1$ the odd biregular path. Obviously $m \geq 2$ and when $m = 2$ the result is just a path on $p+2$ vertices. An interesting family of graphs occurs when $m = 4$ which is a class of hydrocarbons. In this section we present two separate algorithms for radio labelings of biregular paths, one for the even path and one for the odd path.

3.2.2 The Radio Number of $H_{2k,m}$

Theorem 18

$$rn(H_{2k,m}) = 2k^2m - 2k^2 + 2k + 1$$

We have divided the proof of Theorem 18 into two sections. In order to show this is the radio number for all $m \geq 3$, we begin with the base case of $H_{4,3}$, where $k = 2$, and induct on m. Then we take $H_{4,m}$ and induct on k.

Induction on m In this section, we present the base case for our induction and induct on m for $k = 4$. We begin with $H_{4,3}$ for the purpose of illustrating the algorithm, but the algorithm and equation still work for $H_{2,3}$. The ordering and radio labeling of $H_{4,3}$ are presented in Figure 30, where the sequence $x_0, x_1, \ldots, x_9$ denotes the ordering and numbers inside of the vertices denote the labels of the vertices.

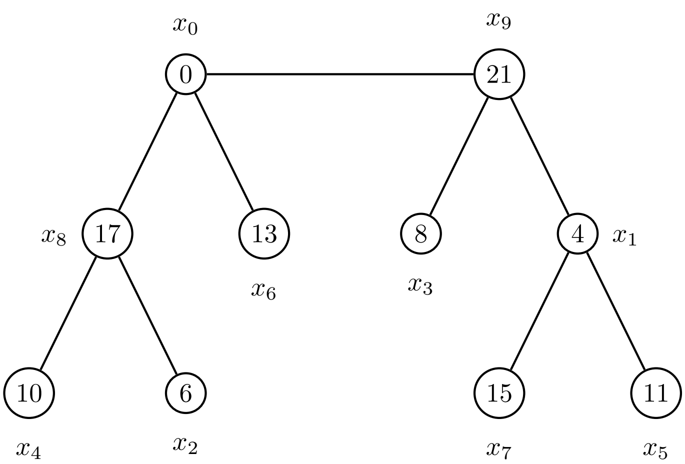

Figure 30: Radio labeling of $H_{4,3}$

It is easy to verify that the labeling is a valid labeling. Simple calculation using Theorem 8 shows that the labeling of $H_{4,3}$ in Figure 30 is optimal. Firstly, note that the vertex x_0 is a median vertex.

$$s(H_{4,3}) = s(x_0) = 3(1) + 4(2) + 2(3) = 17 \tag{3.7}$$

Using this together with Theorem 8 we get

$$\begin{aligned} rn(H_{4,3}) &\geq \big(diam(H_{4,3})+1\big)(n-1)+1-2s(H_{4,3}) \\ &= (6)(9)+1-2(17)=21 \end{aligned} \tag{3.8}$$

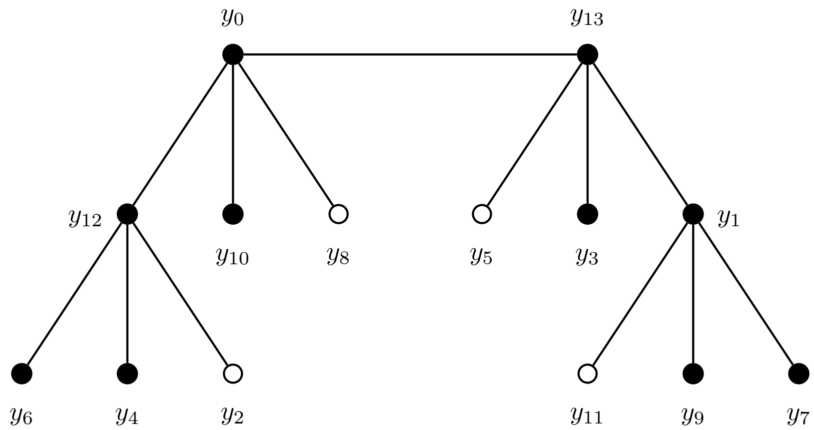

Figure 31: Ordering of $H_{4,4}$

In general, we can calculate the status of the median vertex.

$$s(H_{4,m}) = (m)(1)+(2m-2)(2)+(m-1)(3) = 8m-7 \tag{3.9}$$

So by Theorem 8

$$\begin{aligned} rn(H_{4,m}) &\geq \big(diam(H_{4,m})+1\big)(n-1)+1-2s(H_{4,m}) \\ &= (5+1)(4m-3)+1-2(8m-7) \\ &= 24m-18+1-16m+14 \\ &= 8m-3 \end{aligned}$$

Now we show inductively that equality holds in 3.10 for all m.

Induction on the ordering from $H_{4,m-1}$ to $H_{4,m}$ is done by extending the order to include the 4 new vertices. We denote the original ordering by $X = (x_0, x_1, \ldots, x_{4m-8}, x_{4m-7})$ with the original labeling f_0. The new ordering is denoted by $Y = (y_0, y_1, \ldots, y_{4m-4}, y_{4m-3})$ with the new labeling f. As an example we show how the ordering is extended from $H_{4,3}$ to $H_{4,4}$ in Figure 31. The black vertices indicate the vertices that remain from $H_{4,3}$ and the white vertices are the new vertices unique to $H_{4,4}$. Just as for the algorithm presented for paths, the induction separates the vertices into three sections: the beginning, the iteration, and the end. The beginning and the ending sections remain constant, with the iteration section changing due to the new vertices. An explanation of the induced order is explained in detail soon.

We have already presented our optimal radio labeling for $H_{4,3}$ as our base case. We show that our radio labeling for $H_{4,m-1}$ being optimal implies that our radio labeling for $H_{4,m}$ is optimal. From the radio labeling of $H_{4,m-1}$ we calculate the new radio labeling by making adjustments based on the new vertices added. For example, in Figure 31 the first new vertex is assigned to be y_5 and so the ordering and therefore labeling for vertices $y_0, y_1, \ldots, y_4$ are the same as for $H_{4,3}$. In general, the first new vertex in $H_{4,m}$ is y_{2m-3}. The full ordering of $H_{4,m}$ from the ordering of $H_{4,m-1}$ is presented in Figure 32.

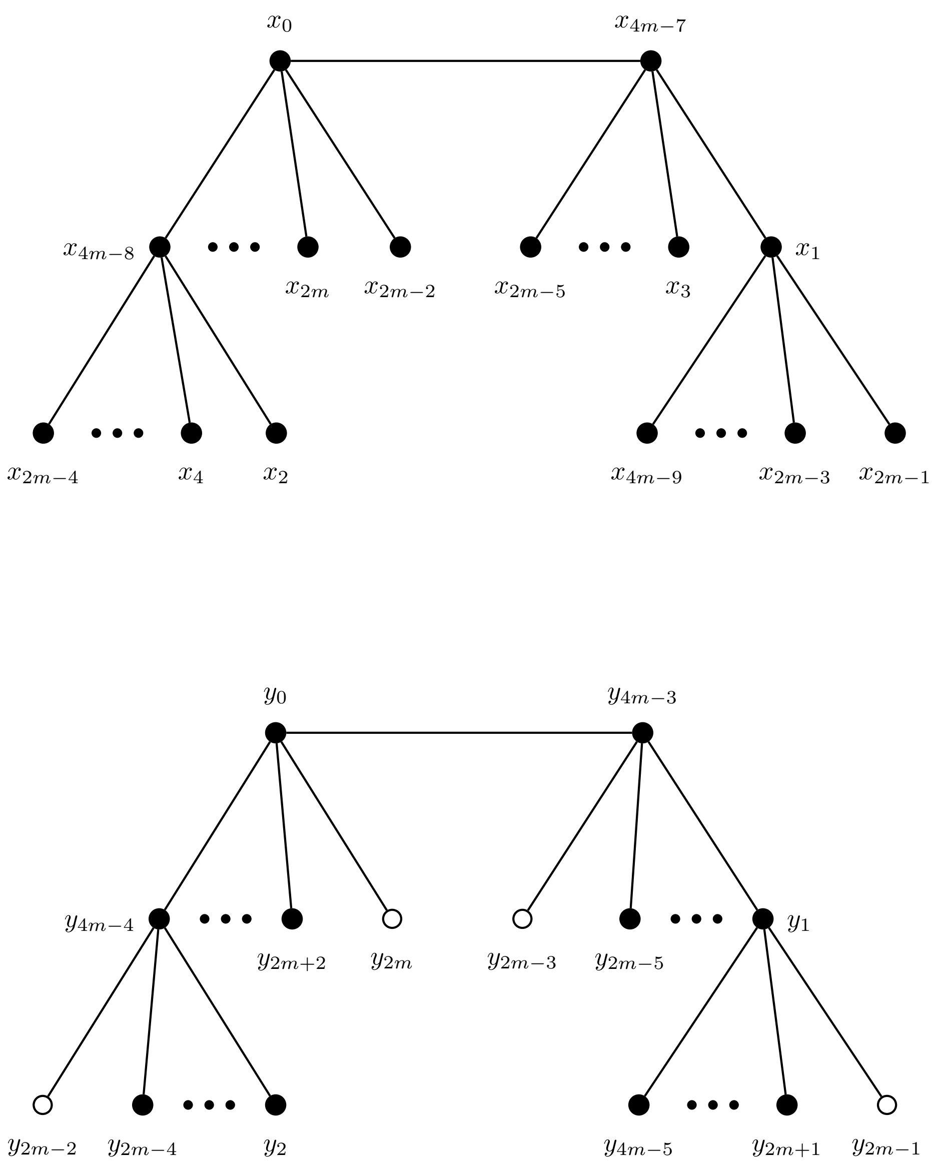

Figure 32: Induced ordering for $H_{4,m}$

As shown in Figure 32, the new ordering is defined with $x_i = y_i$ for $i \in$

$\{0, 1, \ldots, 2m-4\}$ and $x_i = y_{i+4}$ for $i \in \{2m-3, 2m-2, \ldots, 4m-7\}$. The new vertices are grouped together as $y_{2m-3}, y_{2m-2}, y_{2m-1}, y_{2m}$, as shown in Figure 32.

Assuming that Theorem 18 holds for $H_{4,m-1}$, we show that it also holds for $H_{4,m}$. So we must show

$$\begin{aligned} rn(H_{4,m}) &= 8m - 3 \\ &= 8(m-1) - 3 + 8 \\ &= rn(H_{4,m-1}) + 8 \end{aligned} \tag{3.10}$$

In calculation of the new labeling f we see that for $i \in \{1, \ldots, 2m-4\}$

$$f(y_0) = f_0(x_0) = 0 \tag{3.11}$$

$$f(y_i) - f(y_{i-1}) = f_0(x_i) - f_0(x_{i-1}) \tag{3.12}$$

which gives

$$f(y_i) = f_0(x_i) \tag{3.13}$$

Similarly for $i \in \{2m-2, 2m-1 \ldots, 4m-7\}$,

$$f(y_{i+4}) - f(y_{i+3}) = f_0(x_i) - f_0(x_{i-1}) \tag{3.14}$$

But because we add new vertices the labels are not the same for f as for f_0. We must calculate the labels of the new vertices based upon the labels of the old vertices.

$$\begin{aligned} f(y_{2m-3}) &= diam(H_{4,m}) + 1 - d(y_{2m-3}, y_{2m-4}) + f(y_{2m-4}) \\ &= 6 - 4 + f_0(x_{2m-4}) \\ &= 2 + f_0(x_{2m-4}) \\ f(y_{2m-2}) &= 6 - d(y_{2m-2}, y_{2m-3}) + f(y_{2m-3}) \\ &= 2 + (2 + f_0(x_{2m-4})) \\ &= 4 + f_0(x_{2m-4}) \\ f(y_{2m-1}) &= 6 - d(y_{2m-1}, y_{2m-2}) + f(y_{2m-2}) \\ &= 1 + (4 + f_0(x_{2m-4})) \\ &= 5 + f_0(x_{2m-4}) \\ f(y_{2m}) &= 6 - d(y_{2m}, y_{2m-1}) + f(y_{2m-1}) \\ &= 2 + (5 + f_0(x_{2m-4})) \\ &= 7 + f_0(x_{2m-4}) \\ f(y_{2m+1}) &= 6 - d(y_{2m+1}, y_{2m}) + f(y_{2m}) \\ &= 2 + (7 + f_0(x_{2m-4})) \\ &= 9 + f_0(x_{2m-4}) \end{aligned} \tag{3.15}$$

Note that since $y_{2m+1} = x_{2m-3}$ we compare the label in f to that in f_0. From the labeling of $H_{4,m-1}$, we have

$$\begin{aligned} f_0(x_{2m-4}) &= f_0(x_{2m-3}) - \Big(diam(H_{4,m-1}) + 1 - d(x_{2m-3}, x_{2m-4})\Big) \\ &= f_0(x_{2m-3}) - 1 \end{aligned} \tag{3.16}$$

Combining 3.15 and 3.16 we have

$$\begin{aligned} f(y_{2m+1}) &= 9 + f_0(x_{2m-4}) \\ &= 9 - 1 + f_0(x_{2m-3}) \\ &= 8 + f_0(x_{2m-3}) \end{aligned} \tag{3.17}$$

Since $f(y_{2m+2}) - f(y_{2m+1}) = f_0(x_{2m-2}) - f_0(x_{2m-3})$, then

$$\begin{aligned} f(y_{2m+2}) &= 8 + f_0(x_{2m-3}) + f_0(x_{2m-2}) - f_0(x_{2m-3}) \\ &= 8 + f_0(x_{2m-2}) \end{aligned} \tag{3.18}$$

Using 3.14, for $i \in \{2m-2, 2m-1\ldots, 4m-7\}$ we define $f(y_{i+4})$ by

$$\begin{aligned} f(y_{i+4}) &= f(y_{i+3}) + f_0(x_i) - f_0(x_{i-1}) \\ &= 8 + f_0(x_{i-1}) + f_0(x_i) - f_0(x_{i-1}) \\ &= 8 + f_0(x_i) \end{aligned}$$

And so for $i \in \{2m-2, 2m-1\ldots, 4m-7\}$ we add 8 to each label of X to get the label of the corresponding vertex from Y. Consequently we see that the highest label in this labeling is

$$f(y_{4m-3}) = 8 + f_0(x_{4m-7}) \tag{3.19}$$

Since f_0 is a valid labeling for $H_{4,m-1}$, it is easy to show that f is a valid labeling for $H_{4,m}$ because we need only be concerned with the new vertices. Clearly all labelings of y_0 through y_{2m-4} are valid in $H_{4,m}$ because they remain unchanged from $H_{3,m-1}$.

By definition, for $i \in \{1, \ldots, 4m-3\}$ the constraint

$$f(y_i) \geq f(y_{i-1}) + diam(H_{4,m}) + 1 - d(y_{i-1}, y_i) \tag{3.20}$$

holds. Next we check the constraint between y_i and y_{i-2}. In our calculation of this constraint we use the distances between y_i and y_{i-2} for $i \in \{2m-3, \ldots, 2m+2\}$. For ease we list these in the table below.

	$2m-3$	$2m-2$	$2m-1$	$2m$	$2m+1$	$2m+2$
$d(y_{i-1}, y_i)$	4	4	5	4	4	4
$d(y_{i-2}, y_{i-1})$	4	4	4	5	4	4
$d(y_{i-2}, y_i)$	2	2	3	3	2	2

We know that equality holds for the constraint in 3.20, so

$$\begin{aligned} f(y_i) - f(y_{i-2}) &= \big(f(y_i) - f(y_{i-1})\big) + \big(f(y_{i-1}) - f(y_{i-2})\big) \\ &= 2(diam(H_{4,m}) + 1) - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) \\ &= 12 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) \end{aligned} \tag{3.21}$$

We want to show that

$$\begin{aligned} f(y_i) - f(y_{i-2}) &\geq diam(H_{4,m}) + 1 - d(y_{i-2}, y_i) \\ &= 6 - d(y_{i-2}, y_i) \end{aligned} \tag{3.22}$$

Using 3.21, this constraint is equivalent to

$$\begin{aligned} 12 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) &\geq 6 - d(y_{i-2}, y_i) \\ 6 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) + d(y_{i-2}, y_i) &\geq 0 \end{aligned} \tag{3.23}$$

We refer to the left hand side of this inequality as $I_2(i)$. So we require that $I_2(i) \geq 0$.

A simple calculation shows that this constraint holds for each of our y_i and y_{i-2} for $i \in \{2m-3, \ldots, 2m+2\}$. In actuality, each of these constraints has equality. For $i \geq 2m+3$, $I_2(i) \geq 0$ because the vertices remain at the same distance in $H_{4,m}$ as the equivalent vertices in $H_{4,m-1}$.

Next is to show the constraint between y_{i-3} and y_i holds.

We want to show that

$$\begin{aligned} f(y_i) - f(y_{i-3}) &\geq diam(H_{4,m}) + 1 - d(y_{i-3}, y_i) \\ f(y_i) - f(y_{i-3}) + d(y_{i-3}, y_i) &\geq 6 \end{aligned} \tag{3.24}$$

Since we always alternate between branches, y_{i-3} and y_i are always on different branches. This gives that $d(y_{i-3}, y_i) \geq 3$. Also, since $f(y_i) \geq 1 + f(y_{i-1})$ then clearly $f(y_i) - f(y_{i-3}) \geq 3$. Putting these two facts together, we can easily see that we have satisfied the constraint in 3.24.

The case for the constraint between y_{i-4} and y_i shows that all subsequent constraints also hold because we can remove the use of the distance between the two vertices. The uniqueness of the labels gives that $f(y_i) \geq 4 + f(y_{i-4})$. Examination of the ordering shows that it is always true that $d(y_i, y_{i-4}) \geq 2$ and so the constraint always holds regardless of i. Thus this radio labeling is a valid labeling.

Induction on k In this section, we fix m and induct on k to show the radio number from Theorem 18 holds for any k:

$$rn(H_{2k,m}) = \big(diam(H_{2k,m}) + 1\big)(n-1) + 1 - 2s(H_{2k,m})$$

We start with the base case where $k = 2$.

$$rn(H_{4,m}) = (5 + 1)(4m - 3) + 1 - 2(8m - 7) = 8m - 3 \tag{3.25}$$

Note that in this section, our figures show the case when $m = 4$ but our calculations do not assume this.

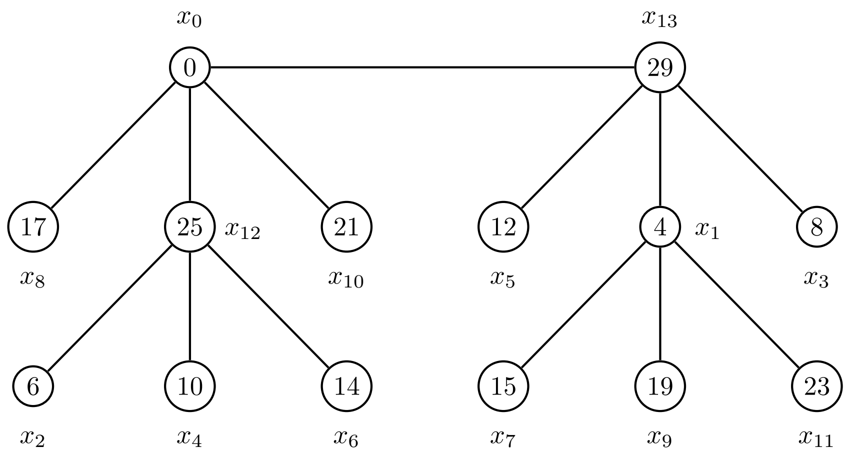

Figure 33: Radio labeling of $H_{4,4}$

In general, $diam(H_{2k,m}) = 2k - 1$, $n = (m - 1)(2k) + 2$ and

$$\begin{aligned} s(H_{2k,m}) &= m + \sum_{i=2}^{k} 2(m-1)i + (m-1)(k+1) \\ &= k^2m - k^2 + 2km - 2k + 1 \end{aligned} \tag{3.26}$$

so the radio number is

$$\begin{aligned} rn(H_{2k,m}) &= \big(diam(H_{2k,m}) + 1\big)(n - 1) + 1 - 2s(H_{2k,m}) \\ &= (2k + 2)(2km - 2k + 1) + 1 - 2(k^2m - k^2 + 2km - 2k + 1) \\ &= 2k^2m - 2k^2 + 2k + 1 \end{aligned} \tag{3.27}$$

To create an ordering on $H_{2k,m}$ from an ordering on $H_{2(k-1),m}$, we add $2(m-1)$ vertices in the manner illustrated in Figure 34, where the black vertices indicate the vertices in $H_{2(k-1),m}$ and the white vertices indicated the vertices in $H_{2k,m}$.

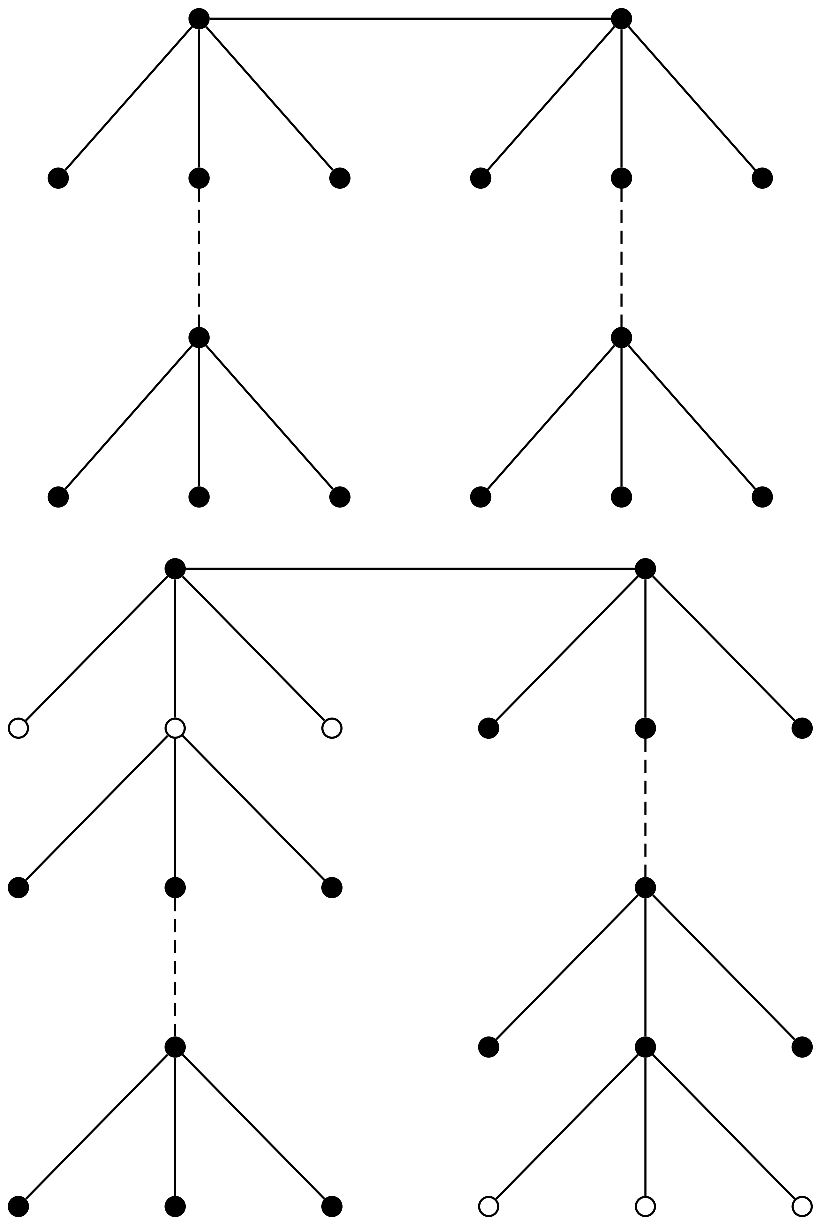

Figure 34: $H_{2(k-1),4}$ and $H_{2k,4}$

We order these new vertices from x_0 to x_M by the process illustrated in Figure 35, where $i = M_0$ indicates the last ordering in $H_{2(k-1),m}$ and $i = M$

indicates the last ordering in $H_{2k,m}$. We can easily see that $M_0 = (m-1)(2k-2)+2-1$ and $M = (m-1)(2k)+2-1$ and $M_0 - 1 = M - 1 - 2(m-1)$, $M_0 - 2 = M - 2 - 2(m-1)$, etc.

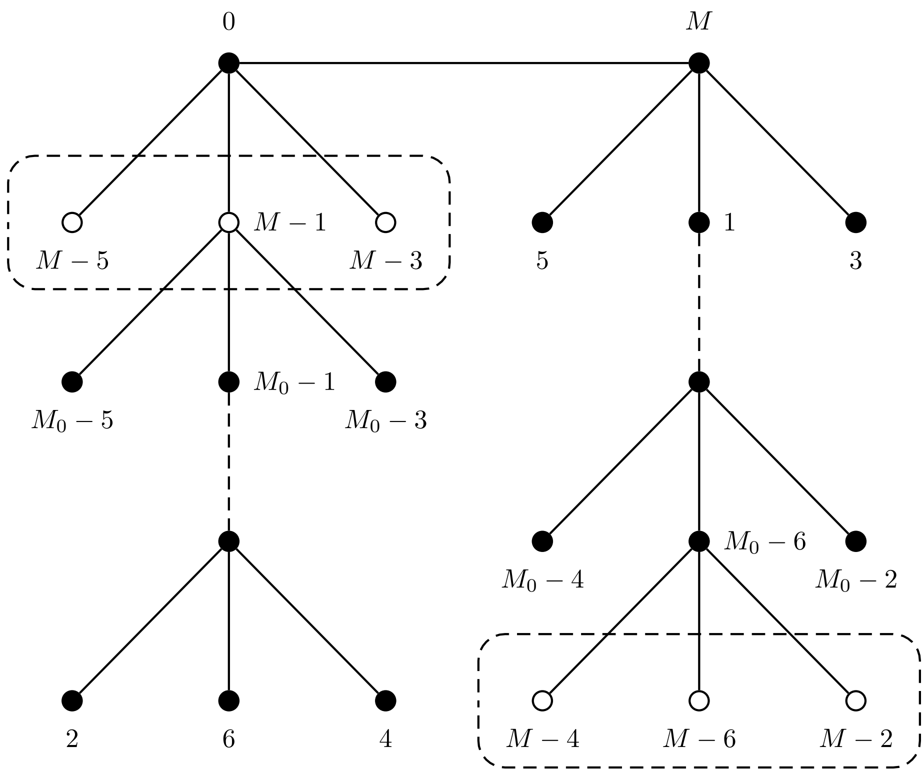

Figure 35: Ordering of $H_{2k,4}$

First we check that our algorithm produces a labeling that gives the radio number for $H_{2k,m}$ given in 3.27; for now we assume that our labeling is a valid radio labeling. We assume that our labeling achieved the desired bound for$H_{2(k-1),m}$, producing the radio number in 3.27:

$$\begin{aligned} rn(H_{2(k-1),m}) &= 2(k-1)^2m - 2(k-1)^2 + 2(k-1) + 1 \\ &= 2k^2m - 2k^2 - 4km + 6k + 2m - 3 \qquad (3.28) \end{aligned}$$

We wish to show that this implies that 3.27 holds for $H_{2k,m}$.

Thus, we solve for the difference between $rn(H_{2(k-1),m})$ and $rn(H_{2k,m})$

$$\begin{aligned} rn(H_{2(k-1),m}) &= 2k^2m - 2k^2 - 4km + 6k + 2m - 3 \\ &= 2k^2m - 2k^2 + 2k + 1 + (-4km + 4k + 2m - 4) \\ &= rn(H_{2k,m}) - (4km - 4k - 2m + 4) \end{aligned} \tag{3.29}$$

So we wish to show that $rn(H_{2k,m}) = rn(H_{2(k-1),m}) + (4km - 4k - 2m + 4)$.

The difference in the radio numbers between $H_{2k,m}$ and $H_{2(k-1),m}$ can be thought of as the effects of two causes: the difference between new and old labels of $x_0 \ldots x_{M_0-1}$, the vertices common to $H_{2k,m}$ and $H_{2(k-1),m}$; and the difference between each consecutive new vertex, $x_{M-(2m-2)}, \ldots, x_M$.

Since $H_{2k,m}$ has a diameter 2 greater than that of $H_{2(k-1),m}$, if two consecutively labeled vertices, x_{i-1} and x_i, from $H_{2(k-1),m}$ remain the same distance in $H_{2k,m}$, the difference between their labels increases by 2. So $\big(f(x_i) - f(x_{i-1})\big) - \big(f_0(x_i) - f_0(x_{i-1})\big) = 2$. Similarly, if two consecutively vertices, x_{i-1} and x_i, from $H_{2(k-1),m}$ have a distance of one greater in $H_{2k,m}$, the difference between their labels only increases by 1. So $\big(f(x_i) - f(x_{i-1})\big) - \big(f_0(x_i) - f_0(x_{i-1})\big) = 1$.

Figure 36 shows the changes from $rn(H_{2(k-1),m})$ to $rn(H_{2k,m})$. The circular vertices are common to $H_{2k,m}$ and $H_{2(k-1),m}$, so they are labeled in Figure 36 with the difference between their labels from $H_{2(k-1),m}$ to $H_{2k,m}$, $\big(f(x_i) - f(x_{i-1})\big) - \big(f_0(x_i) - f_0(x_{i-1})\big)$ where $i = 1, \ldots, M - (2m-2) - 1$. Figure 36 shows the new vertices in $H_{2k,m}$ as ovals, labeled with $f(x_i) - f(x_{i-1})$ where $i = M - (2m-2), \ldots, M$. For simplicity, in Figure 36, we label the outside of the vertices with i for vertex x_i.

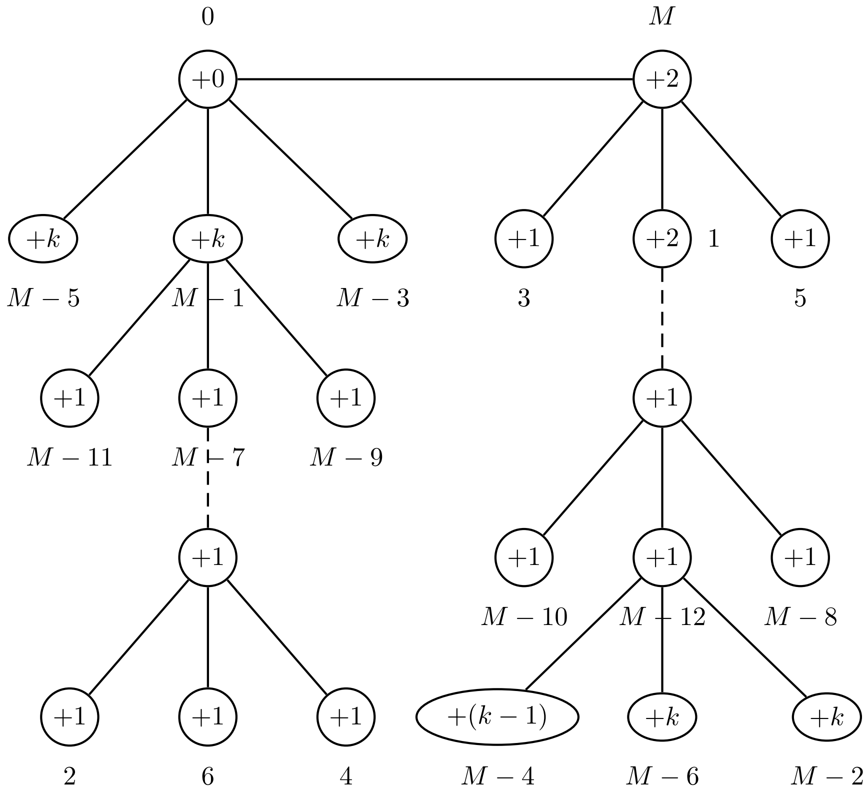

Figure 36: Changes from $H_{2(k-1),4}$ to $H_{2k,4}$

The start of the algorithm has the distance between x_0 and x_1 stay constant, thus, as stated before, 2 is added to $f_0(x_1)$ and all subsequent labels so $f(x_1) = f_0(x_1) + 2$. For $2 \leq i \leq M_0 - 1$ the distance between x_{i-1} and x_i increased by 1 due to the insertion of vertex $M - 1$, and so the difference between their labels increases by 1. Note that the diameter of the graph is $2k + 1$ and so the distance between $M_0 - 1 = M - (2m - 2) - 1$ and $M - (2m - 2)$ is $k + 3$ and so the difference of their labels is

$$\begin{aligned} f(x_{M-(2m-2)}) - f(x_{M-(2m-2)-1}) &= (2k + 1) + 1 - (k + 3) \\ &= k - 1 \end{aligned} \tag{3.30}$$

By similar argument, for $M - (2m - 2) + 1 \leq i \leq M - 1$

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= (2k + 1) + 1 - (k + 2) \\ &= k \end{aligned} \tag{3.31}$$

Lastly, the distance between M and its predecessor stays the same, and so adds 2 to its label.

To tally up these increases we see that we add 1 to the radio number for each of $2(k-1)(m-1)-1$ vertices,2 for 2 vertices, $(k-1)$ for 1 vertex, and k for $2(m-1)-1$ vertices; so in total

$$\begin{aligned} rn(H_{2k,m}) - rn(H_{2(k-1),m}) &= 2(k-1)(m-1) - 1 + 2(2) + (k-1) \\ &\quad + k(2m-3) \\ &= 4km - 4k - 2m + 4 \end{aligned} \tag{3.32}$$

Thus we have shown 3.29 holds so our labeling reaches the desired radio number. Now we check that given that our labeling of $H_{2(k-1),m}$ is valid then so is the labeling of $H_{2k,m}$.

Recall that once we establish the ordering $x_0, x_1, \ldots, x_{n-1}$ the labeling is given by making the constraint tight between consecutively labeled vertices. So $f(x_0) = 0$ and $f(x_i) = f(x_{i-1}) + 2k + 2 - d(x_{i-1}, x_i)$. In order to show this is a valid radio labeling, we must show that $f(x_i) \geq f(x_{i-j}) + 2k + 2 - d(x_i, x_{i-j})$ for $2 \leq i \leq n$ and $2 \leq j \leq diam(H_{2k,m})$ where $j \leq i$. Since, by our algorithm, vertices from $H_{2(k-1),m}$ only get further away in $H_{2k,m}$, we can assume that $f(x_i) \geq f_0(x_i) \geq f_0(x_{i-j}) + 2k + 2 - d_0(x_i, x_{i-j})$ where d_0 is the distance function for $H_{2(k-1),m}$.

For each vertex from $H_{2(k-1),m}$ we can assume that all labelings $f_0(x_i)$ are valid, so we need only show that the the difference in the new labels compensates for the increase of 2 in the diameter:

$$f(x_i) - f(x_{i-j}) \geq f_0(x_i) - f_0(x_{i-j}) + 2 \tag{3.33}$$

For $i = 2$, we need only check the constraint for $j = 2$. Since $f(x_1) - f(x_0) = f_0(x_1) - f_0(x_0) + 2$, then clearly $f(x_2) - f(x_0) \geq f_0(x_2) - f_0(x_0) + 2$. So 3.33 holds for $i = 2$, $j = 2$.

Similarly for each $i = 2, \ldots, M_0 - 1$

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f_0(x_i) - f_0(x_{i-1}) + 1 \\ f(x_{i-1}) - f(x_{i-2}) &= f_0(x_{i-1}) - f_0(x_{i-2}) + 1 \end{aligned} \tag{3.34}$$

Therefore we can expand the equation to show for $2 \leq j \leq diam(H_{2k,m})$ where $j \leq i$ that:

$$f(x_i) - f(x_{i-j}) \geq f_0(x_i) - f_0(x_{i-j}) + 2 \tag{3.35}$$

so 3.33 holds for $i = 1, \ldots, M_0 - 1$.

Now we must check the validity of the labels on x_i for $i = M - (2m - 2), \ldots, M$. Recall that the diameter of $H_{2k.m}$ is $2k + 1$, so the constraint we must follow is:

$$f(x_i) - f(x_{i-j}) \geq 2k + 2 - d(x_i, x_{i-j}) \tag{3.36}$$

First we check the $j = 2$ constraint. By assumption,

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= 2k - d(x_{i-1}, x_i) \quad \text{and} \\ f(x_{i-1}) - f(x_{i-2}) &= 2k - d(x_{i-2}, x_{i-1}), \quad \text{so} \\ f(x_i) - f(x_{i-2}) &= 4k - d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}) \end{aligned} \tag{3.37}$$

Generalizing the notation from the previous section, we define

$$I_2(i) := 2k + 2 - d(x_i, x_{i-1}) - d(x_{i-1}, x_{i-2}) + d(x_i, x_{i-2}). \tag{3.38}$$

Thus to satisfy $f(x_i) \geq f(x_{i-2}) + 2k - d(x_{i-2}, x_i)$ we need to show

$$I_2(i) = 2k + 2 - d(x_i, x_{i-1}) - d(x_{i-1}, x_{i-2}) + d(x_i, x_{i-2}) \geq 0 \tag{3.39}$$

Similarly, for j=3 we have

$$I_3(i) := 4k + 4 - d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}) - d(x_{i-3}, x_{i-2}) + d(x_{i-3}, x_i) \geq 0 \tag{3.40}$$

	M-6	M-5	M-4	M-3	M-2	M-1	M
$d(x_{i-1}, x_i)$	k+3	k+2	k+2	k+2	k+2	k+2	2
$d(x_{i-2}, x_{i-1})$	k+2	k+3	k+2	k+2	k+2	k+2	k+2
$d(x_{i-3}, x_{i-2})$	k+2	k+2	k+2	k+2	k+2	k+2	k+2
$d(x_{i-2}, x_i)$	3	3	2	2	2	2	k
$d(x_{i-3}, x_i)$	k+3	k+2	k+3	k+2	k+2	k+2	2

So for each x_i for $i = M - (2m - 2), \ldots, M$, we can easily verify that 3.39 and 3.40 hold:

	M-6	M-5	M-4	M-3	M-2	M-1	M
$I_2(i)$	0	0	0	0	0	0	2k-2
$I_3(i)$	2k	2k-1	2k+1	2k	2k	2k	2k

For $j \geq 4$, we can show that the constraint is satisfied based on the previous constraints holding.

For $j = 4$, we need

$$I_4(i) = 6k+6-d(x_{i-1}, x_i)-d(x_{i-2}, x_{i-1}), -d(x_{i-3}, x_{i-2})-d(x_{i-4}, x_{i-3})+d(x_{i-4}, x_i) \tag{3.41}$$

We know from the $j = 2$ case that

$$\begin{aligned} &2k + 2 - d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}) + d(x_{i-2}, x_i) \geq 0 \quad \text{and} \\ &2k + 2 - d(x_{i-3}, x_{i-2}) - d(x_{i-4}, x_{i-3}) + d(x_{i-4}, x_{i-2}) \geq 0, \quad \text{so} \\ &4k + 4 - d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}) - d(x_{i-3}, x_{i-2}) - d(x_{i-4}, x_{i-3}) \\ &\qquad + d(x_i, x_{i-2}) + d(x_{i-4}, x_{i-2}) \geq 0 \end{aligned} \tag{3.42}$$

Since x_i and x_{i-2} are always on the same branch,

$$\begin{aligned} d(x_{i-2}, x_i) &\leq k+1 \quad \text{and} \\ d(x_{i-4}, x_{i-2}) &\leq k+1, \quad \text{so} \\ d(x_{i-2}, x_{i-4}) + d(x_{i-4}, x_{i-2}) &\leq 2k+2 \end{aligned} \tag{3.43}$$

So now we have

$$\begin{aligned} 6k+6 &- d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}) - d(x_{i-3}, x_{i-2}) - d(x_{i-4}, x_{i-3}) \\ &+ d(x_i, x_{i-2}) + d(x_{i-4}, x_{i-2}) \geq 0 \end{aligned}$$

and since $d(x_{i-4}, x_i) \geq 0$, this shows that $I_4(i) \geq 0$.

For each case $j > 4$, we have $I_{j-1}(i) - d(x_{i-(j-1)}, x_i) \geq 0$. We know $2k+2 = diam(H_{2k,n}) + 1 > d(x, y)$ for any x and y so $[2k+2 - d(x_{i-j}, x_{i-(j-1)})] \geq 0$. We also know $d(x_{i-j}, x_i) \geq 0$, so

$$\begin{aligned} I_j(i) &= I_{j-1}(i) + 2k + 2 - d(x_{i-j}, x_{i-(j-1)}) + d(x_{i-j}, x_i) - d(x_{i-(j-1)}, x_i) \\ &= [I_{j-1}(i) - d(x_{i-(j-1)}, x_i)] + [2k+2 - d(x_{i-j}, x_{i-(j-1)})] + d(x_{i-j}, x_i) \geq 0 \end{aligned} \tag{3.44}$$

Therefore, we have shown that all of the labels are valid and thus the labeling is a valid radio labeling.

3.2.3 The Radio Number of $H_{2k+1,m}$

Theorem 19

$$rn(H_{2k+1,m}) = 2k^2m - 2k^2 + 2mk + m + 2$$

We have divided the proof of this theorem into three sections, proof of lower bound, induction on m and induction on k. First we prove the lower bound

$$rn(H_{2k+1,m}) \geq 2k^2m - 2k^2 + 2mk + m + 2$$

Then we provide an algorithm, defined inductively, that attains this lower bound. In order to show 0.1 for all $m \geq 3$, we begin with the base case of $H_{3,3}$, where $k = 2$, and induct on m. Then we take $H_{3,m}$ and induct on k.

Proof of Lower Bound In this section we seek to prove that the lower bound given by Theorem 8 for $H_{2k+1,m}$ is unattainable. To show this we prove that the three requirements for equality in Theorem 8 cannot all be satisfied. Therefore, we make the assumption that two of the requirements hold and show the third cannot. We assume that:

(2) x_i and x_{i+1} are on different branches (unless one is the median)

(3) A tightness path goes through every vertex.

Thus we wish to show that (1) does not hold, i.e. if x_0 is the median then x_{n-1} must not be adjacent to x_0. For this section we make use of more terminology. Since we are working with $H_{2k+1,m}$ there is only one median vertex, so we define $M := \{w^*\}$. We define C to be the set of vertices adjacent to the median that have degree 1, and define $C^* := C \cup M$. We also use the level function $L(v) := d(v, w^*)$ along with the ϕ function from [9] and an observation that relates them.

Observation 20 *[9] For a tree T rooted at vertex w and distinct vertices $u, v \in T$,*

(1) $\phi_w(u, v) = 0$ if and only if u and v belong to different branches (unless one is w), and

(2) $d(u, v) = L(v) + L(u) - 2\phi(u, v)$

Observation 21 *Assume that (2) and (3) from 8 are true then, for $H_{2k+1,m}$, $L(x_i) = k + 1$ implies either x_{i-1} or x_{i+1} is in C^*.*

Proof Assume that $L(x_i) = k + 1$ and $x_{i-1} \notin C^*$. Without loss of generality, let x_i be on the left branch and x_{i-1} on the right branch. Since x_i and x_{i-1} are on different branches, by Observation 21 $\phi(x_i, x_{i-1}) = 0$. This yields the following,

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= (2k + 3) - L(x_i) - L(x_{i-1}) + 2(0) \\ &= (2k + 3) - (k + 1) - L(x_{i-1}) \\ &= k + 2 - L(x_{i-1}) \end{aligned} \tag{3.45}$$

Since x_i and x_{i+1} are on different branches we have

$$\begin{aligned} f(x_{i+1}) - f(x_i) &= (2k + 3) - L(x_{i+1}) - L(x_i) + 2(0) \\ &= (2k + 3) - (k + 1) - L(x_{i+1}) \\ &= k + 2 - L(x_{i+1}) \end{aligned} \tag{3.46}$$

And so

$$\begin{aligned} f(x_{i+1}) - f(x_{i-1}) &= (2k + 4) - L(x_{i+1}) - L(x_{i-1}) \\ &\geq (2k + 3) - d(x_{i-1}, x_{i+1}) \\ &= (2k + 3) - \big(L(x_{i-1}) + L(x_{i+1}) - 2\phi(x_{i-1}, x_{i+1})\big) \\ &= (2k + 3) - L(x_{i+1}) - L(x_{i-1}) + 2\phi(x_{i-1}, x_{i+1}) \end{aligned} \tag{3.47}$$

This is equivalent to the following inequality

$$1 \geq 2\phi(x_{i-1}, x_{i+1}) \tag{3.48}$$

Therefore since $\phi(x_{i-1}, x_{i+1})$ must be an integer, $\phi(x_{i-1}, x_{i+1}) = 0$. This means x_{i-1} and x_{i+1} are on different branches or one of them is w^*. Since we assumed x_{i-1} is on the right branch, and x_i is on the left, then x_{i+1} must be a member of C. The argument is similar to show that if $L(x_i) = k+1$ and $x_{i+1} \notin C^*$ then $x_{i-1} \in C^*$. The only difference to note is that x_{i-1} may be the median, while x_{i+1} cannot. □

Theorem 22

$$rn(H_{2k+1,m}) > (n-1)(diam(H_{2k+1,m}) + 1) + 1 - 2s(H_{2k+1,m})$$

Proof Observation 21 shows us that at best each vertex of C accounts for exactly two vertices on level $k+1$ through having $x_i \in C^*$ with x_{i-1} and x_{i+1} on level $k+1$. Since $|C| = m-2$ this can account for at most $2m-4$ vertices on level $k+1$. Furthermore, using the median as x_0, one more vertex on level $k+1$ can be used as x_1. This accounts for $2m-3$ vertices. There are, however, $2m-2$ vertices on level $k+1$. This means that at least one vertex on level $k+1$ is not preceded or followed by a vertex of C^* in the ordering. This has one of two consequences. The first is that the tightness path is violated. The second is that the final vertex of level $k+1$ is the last vertex of the ordering, violating (1) of Theorem **??** which states that $x_0 \sim x_{n-1}$. Therefore, equality in Theorem **??** cannot be satisfied. □

We can restate the inequality from Theorem 22 as follows.

$$\begin{aligned} n &= 2km - 2k + m + 1 \\ diam(H_{2k+1,m}) &= 2k + 2 \quad \text{and} \\ s(H_{2k+1,m}) &= k^2m - k^2 + 3km - 3k + m \end{aligned}$$

$$\begin{aligned} rn(H_{2k+1,m}) \geq &(n-1)(diam(H_{2k+1,m}) + 1) + 1 - 2s(H_{2k+1,m}) + 1 \\ = &(2km - 2k + m)(2k+3) + 1 \\ &- 2(k^2m - k^2 + 3km - 3k + m) + 1 \\ = &2k^2m - 2k^2 + 2mk + m + 2 \end{aligned} \tag{3.49}$$

Induction on m We begin our induction on m with the base case of $H_{3,3}$. From Theorem 22, we have a lower bound for the radio number of $H_{3,3}$.

$$rn(H_{3,3}) \geq (5)(7) + 1 - 2\big(1(3) + 2(4)\big) + 1 = 15 \tag{3.50}$$

In Figure 37 we present the ordering and labeling for $H_{3,3}$ that attains this bound. It is simple to calculate that this is a valid labeling.

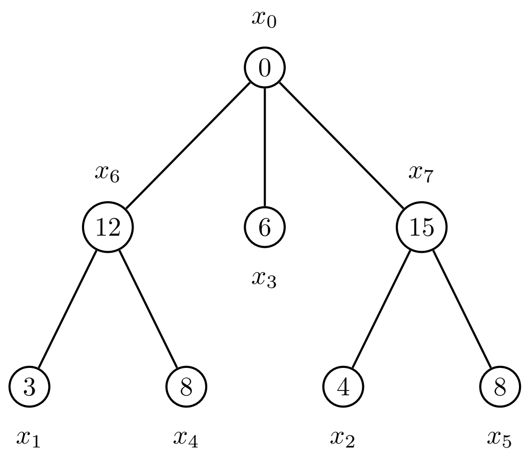

Figure 37: Radio labeling of $H_{3,3}$

In general the lower bound for $rn(H_{3,m})$ is

$$\begin{aligned} rn(H_{3,m}) &\geq (5)(3m-2)+1-2\big(1(m)+2(2(m-1))\big)+1 \\ &= (15m-10)+1-(10m-8)+1 \\ &= 5m \end{aligned} \tag{3.51}$$

Our method for extending the ordering from $H3, m-1$ to $H_{3,m}$ parallels the method we used for the even case. To begin our analysis of the induction on m we look at $H_{3,m-1}$. For this we use the sequence $x_0, x_1, \ldots, x_{3(m-1)-2}$, and the labeling f_0. We add three vertices to $H_{3,m-1}$ as seen in Figure 39, with the white vertices depicting those added in this step. Just as for the the even case, the induction separates the vertices into three sections: the beginning, the iteration, and the end. The beginning only consists of the median, $x_0 = w^*$. The ending consists of the last three vertices which are always the same.

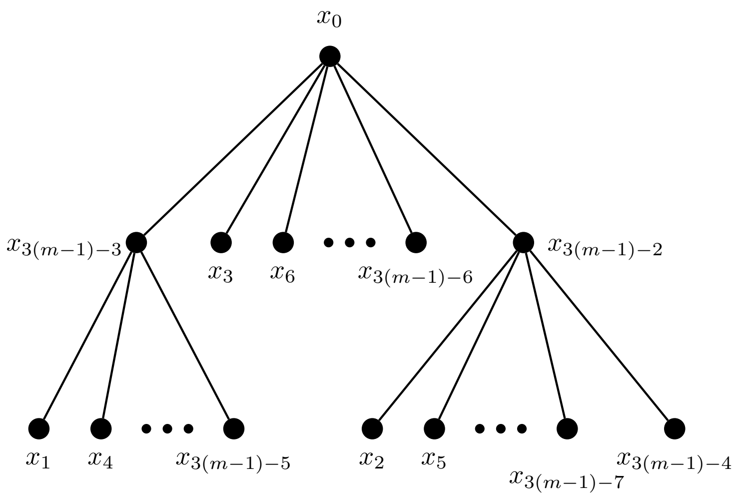

Figure 38: Radio labeling of $H_{3,m-1}$

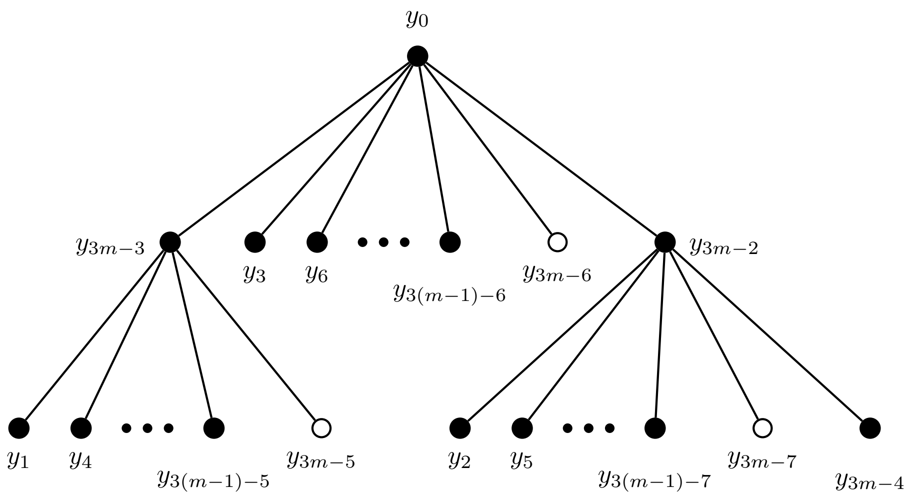

Figure 39: Radio labeling of $H_{3,m}$

Assuming that Theorem 19 holds for $H_{3,m-1}$, we wish to show that it also holds for $H_{3,m}$. So we must show

$$\begin{aligned} rn(H_{3,m}) &= 5m \\ &= 5(m-1)+5 \\ &= rn(H_{3,m-1})+5 \end{aligned} \tag{3.52}$$

We accomplish this by comparing the labels of $x_{3(m-1)-2}$ and y_{3m-2} because they are the radio numbers of $H_{3,m-1}$ and $H_{3,m}$ respectively.

Since $y_i = x_i$ for $i = 0, 1, \ldots, 3(m-1)-7$, the labeling for these vertices are the same for f as for f_0. Starting with y_{3m-7} we see that

$$\begin{aligned} f(y_{3m-7}) &= f(y_{3(m-1)-5}) + (5) - d(y_{3(m-1)-5}, y_{3m-7}) \\ &= f(y_{3(m-1)-5}) + 5 - 4 \\ &= f(y_{3(m-1)-5}) + 1 && (3.53) \\ f(y_{3m-6}) &= f(y_{3m-7}) + (5) - d(y_{3m-7}, y_{3m-6}) \\ &= f(y_{3m-7}) + 2 && (3.54) \\ f(y_{3m-5}) &= f(y_{3m-6}) + (5) - d(y_{3m-6}, y_{3m-5}) \\ &= f(y_{3m-6}) + 2 && (3.55) \end{aligned}$$

For $f(y_{3m-3})$ we have that the constraint with y_{3m-4} is not tight, but the constraint with y_{3m-5} is. This results in

$$\begin{aligned} f(y_{3m-3}) &= f(y_{3m-5}) + (5) - d(y_{3m-5}, y_{3m-3}) \\ &= f(y_{3m-5}) + 4 \end{aligned} \tag{3.56}$$

Finally,

$$\begin{aligned} f(y_{3m-2}) &= f(y_{3m-3}) + (5) - d(y_{3m-3}, y_{3m-2}) \\ &= f(y_{3m-3}) + 3 \end{aligned} \tag{3.57}$$

Combining 3.53-3.57 we find $rn(H_{3,m})$ to be

$$f(y_{3m-2}) = f(y_{3(m-1)-5}) + 12 \tag{3.58}$$

By comparison to $rn(H_{3,m-1})$

$$\begin{aligned} f_0(x_{3(m-1)-3}) &= f_0(x_{3(m-1)-5}) + (5) - d(x_{3(m-1)-5}, x_{3(m-1)-3}) \\ &= f_0(x_{3(m-1)-5}) + 4 && (3.59) \\ f_0(x_{3(m-1)-2}) &= f_0(x_{3(m-1)-3}) + (5) - d(x_{3(m-1)-3}, x_{3(m-1)-2}) \\ &= f_0(x_{3(m-1)-3}) + 3 \\ &= f_0(x_{3(m-1)-5}) + 7 && (3.60) \end{aligned}$$

Since $f_0(x_{3(m-1)-5}) = f(y_{3(m-1)-5})$ we use **??** to get that

$$\begin{aligned} f(y_{3m-2}) &= f(y_{3(m-1)-5}) + 12 \\ &= f_0(x_{3(m-1)-5}) + 12 \\ &= f_0(x_{3(m-1)-2} - 7) + 12 \\ &= f_0(x_{3(m-1)-2}) + 5 \end{aligned} \tag{3.61}$$

Thus we have shown that

$$rn(H_{3,m}) = rn(H_{3,m-1}) + 5$$

Now we must show that f is a valid labeling. Since we are assuming that the labeling f_0 of $H_{3,m-1}$ is valid, the labeling f of $H_{3,m}$) for vertices y_0 through $y_{3(m-1)-5}$ is still valid because the labels remain the same.

By definition, for $i \in \{1, \dots, 3m-2\}$ the constraint

$$f(y_i) \geq f(y_{i-1}) + diam(H_{3,m}) + 1 - d(y_{i-1}, y_i) \tag{3.62}$$

holds. Next we check the constraint between y_i and y_{i-2} and the constraint between y_i and y_{i-3}. In computing these constraints we use the distances between y_i and y_{i-3} for $i \in \{3m-7, \dots, 3m-2\}$. For ease we list these in the table below.

	$3m-7$	$3m-6$	$3m-5$	$3m-4$	$3m-3$	$3m-2$
$d(y_{i-1}, y_i)$	4	3	3	4	3	3
$d(y_{i-2}, y_{i-1})$	3	4	3	3	4	3
$d(y_{i-3}, y_{i-2})$	3	3	4	3	3	4
$d(y_{i-2}, y_i)$	3	3	4	3	1	1
$d(y_{i-3}, y_i)$	2	2	2	2	2	3

We know that equality holds for the constraint in 3.62, so

$$\begin{aligned} f(y_i) - f(y_{i-2}) &= \big(f(y_i) - f(y_{i-1})\big) + \big(f(y_{i-1}) - f(y_{i-2})\big) \\ &= 2(diam(H_{3,m}) + 1) - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) \\ &= 10 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) \end{aligned} \tag{3.63}$$

For the constraint between y_i and y_{i-2}, we want to show that

$$\begin{aligned} f(y_i) - f(y_{i-2}) &\geq diam(H_{4,m}) + 1 - d(y_{i-2}, y_i) \\ &= 5 - d(y_{i-2}, y_i) \end{aligned} \tag{3.64}$$

Using 3.63, this constraint is equivalent to

$$\begin{aligned} 10 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) &\geq 5 - d(y_{i-2}, y_i) \\ 5 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) + d(y_{i-2}, y_i) &\geq 0 \end{aligned} \tag{3.65}$$

As in the even case, we refer to the left hand side of this inequality as $I_2(i)$. So we require that $I_2(i) \geq 0$. A simple calculation shows that this constraint holds for $i \in \{3m-7, \dots, 3m-2\}$. Now we must check the constraint between y_i and y_{i-3}

$$\begin{aligned} f(y_i) - f(y_{i-3}) &= \big(f(y_i) - f(y_{i-1})\big) + \big(f(y_{i-1}) - f(y_{i-2})\big) + \big(f(y_{i-2}) - f(y_{i-3})\big) \\ &= 3(diam(H_{3,m}) + 1) - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) - d(y_{i-3}, y_{i-2}) \\ &= 15 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) \end{aligned} \tag{3.66}$$

We want to show that

$$\begin{aligned} f(y_i) - f(y_{i-3}) &\geq diam(H_{3,m}) + 1 - d(y_{i-3}, y_i) \\ &= 5 - d(y_{i-3}, y_i) \end{aligned} \tag{3.67}$$

Using 3.66, this constraint is equivalent to

$$\begin{aligned} 15 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) - d(y_{i-3}, y_{i-2}) &\geq 5 - d(y_{i-3}, y_i) \\ 10 - d(y_{i-1}, y_i) - d(y_{i-2}, y_{i-1}) - d(y_{i-3}, y_{i-2}) + d(y_{i-3}, y_i) &\geq 0 \end{aligned} \tag{3.68}$$

We refer to the left hand side of this inequality as $I_3(i)$. So we require that $I_3(i) \geq 0$. A simple calculation shows that this constraint holds for $i \in \{3m-7, \dots, 3m-2\}$.

To verify the constraint between y_i and y_{i-j} where $j \geq 4$, we only need the fact that the labels are distinct. So $f(y_i) - f(y_{i-j}) \geq 4$. Since $d(y_{i-j}, y_i) \geq 1$, we see that

$$\begin{aligned} f(y_i) - f(y_{i-j}) \geq 4 &\geq 4 + 1 - d(y_{i-j}, y_i) \\ &= diam(H_{3,m}) + 1 - d(y_{i-3}, y_i) \end{aligned} \tag{3.69}$$

Thus our valid labeling of $H_{3,m-1}$ extends to a valid labeling of $H_{3,m}$.

Induction on k In this section, we present an algorithm for ordering and subsequently labeling the vertices of $H_{2k+1,m}$ so that the labeling is a radio labeling achieving the lower bound for the radio number in Theorem 22.

We start with the base case where $k = 1$.

$$\begin{aligned} rn(H_{3,m}) &\geq (5)(3m-2) + 1 - 2\big(1(m) + 2(2(m-1))\big) + 1 \\ &= 5m \end{aligned} \tag{3.70}$$

As in the even case, our figures show the case when $m = 4$ but our calculations do not assume this.

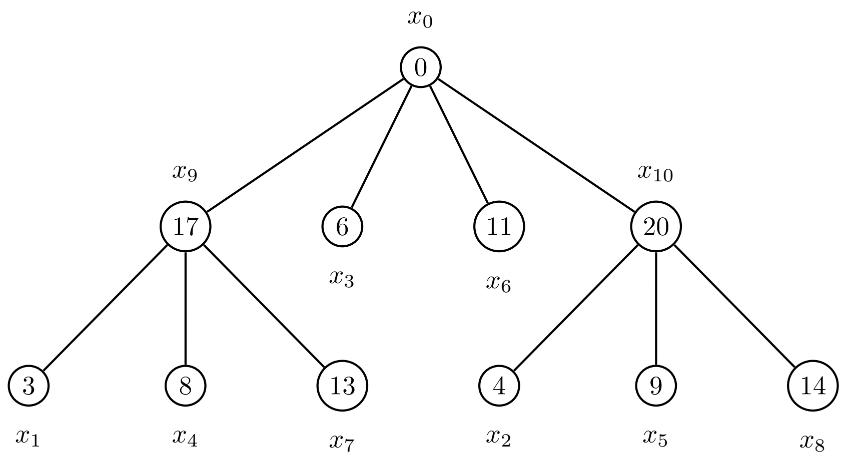

Figure 40: Radio labeling of $H_{3,4}$

To make the algorithm clear, we first show how to extend the ordering of $H_{3,m}$ to $H_{5,m}$.

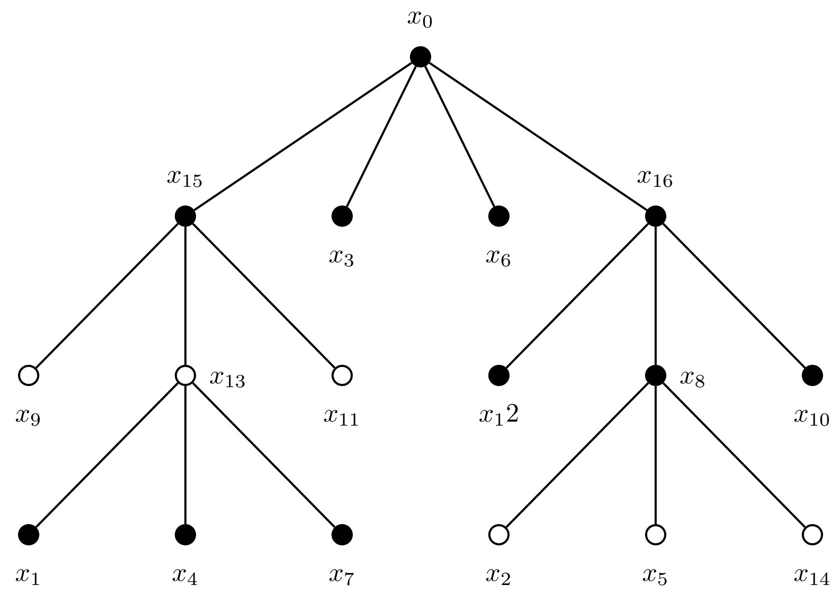

Figure 41: Ordering of $H_{5,4}$

In order to create an ordering on $H_{2k+1,m}$ from an ordering on $H_{2(k-1)+1,m}$, we add $2(m-1)$ vertices. This is shown in Figures 42 and 43 where the black vertices indicate the vertices in $H_{2(k-1)+1,m}$ and the white vertices indicate those added by the induction. We order these new vertices with M_0 indicating the last ordering in $H_{2(k-1)+1,m}$ and M indicating the last ordering in $H_{2k+1,m}$. We can easily see that $M_0 = (2(k-1)+1)(m-1)+2-1$ and $M = (2k+1)(m-1)+2-1$.

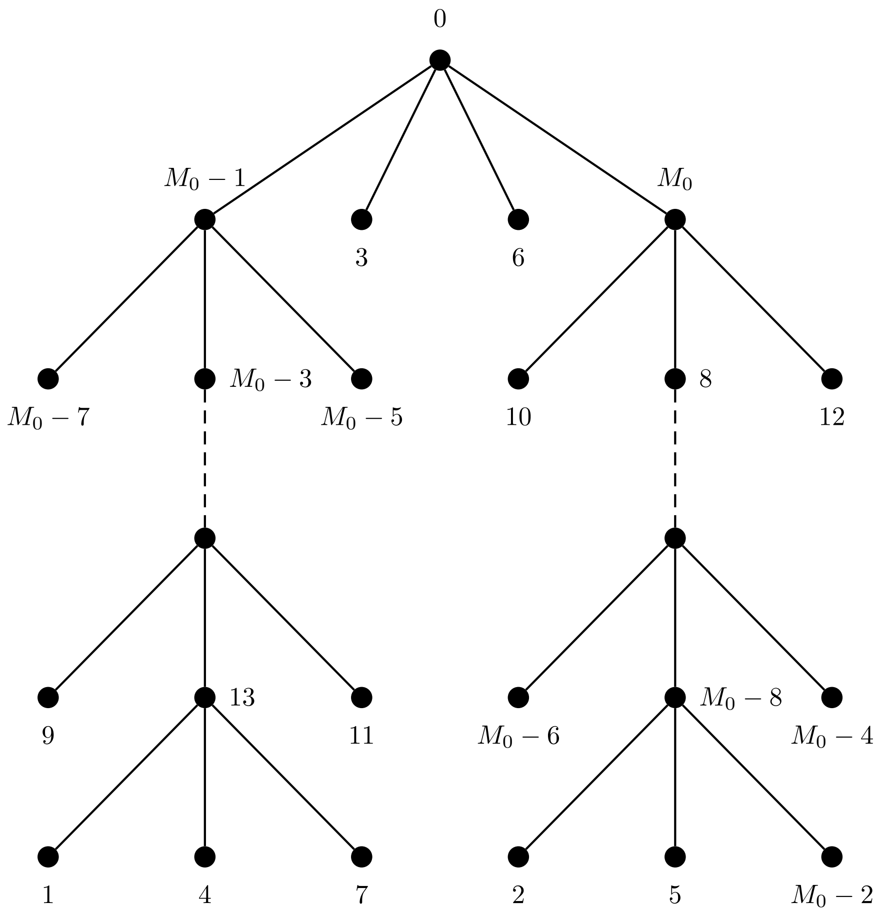

Figure 42: Vertices of $H_{2k-1,4}$

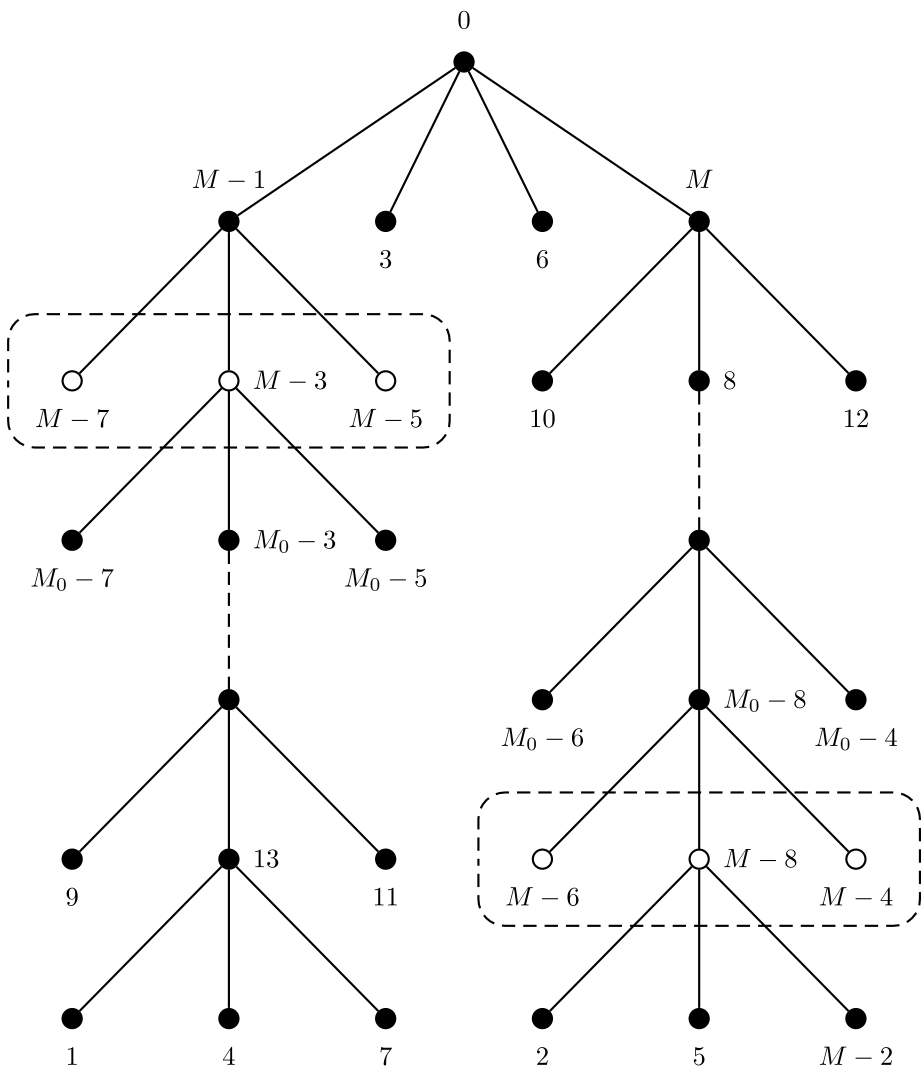

Figure 43: New and old vertices of $H_{2k+1,4}$

First we check that our algorithm produces a labeling that gives the correct radio number. The radio number for the biregularized path on $2(k-1)+1$

vertices is

$$\begin{aligned}
rn(H_{2(k-1)+1,m}) &= 2(k-1)^2m - 2(k-1)^2 + 2(k-1)m + m + 2 \\
&= 2k^2m - 2k^2 - 2km + 4k + m \\
&= 2k^2m - 2k^2 + 2km + m + 2 - 4km + 4k - 2 \\
&= rn(H_{2k+1,m}) - (4km - 4k + 2) \qquad (3.71)
\end{aligned}$$

so we wish to show that $rn(H_{2k,m}) = rn(H_{2(k-1),m}) + (4km - 4k + 2)$.

The difference in the radio numbers in $H_{2k+1,m}$ and $H_{2(k-1)+1,m}$ can be thought of as the effects of two causes: the difference between new and old labels of $x_0 \dots x_{M_0-3}$, the vertices common to $H_{2k+1,m}$ and $H_{2(k-1)+1,m}$; and the difference between each consecutive new vertex, $x_{M-2m}, \dots, x_M$.

Since $H_{2k+1,m}$ has a diameter 2 greater than that of $H_{2(k-1)+1,m}$, if two consecutively labeled vertices from $H_{2(k-1)+1,m}$ remain the same distance in the new graph, the difference between their labels grows by 2. Similarly, if two consecutively vertices from $H_{2(k-1)+1,m}$ have a distance of one greater in the new graph, the difference between their labels grows by 1.

Figure 44 shows the changes from $rn(H_{2(k-1)+1,m}$ to $rn(H_{2k+1,m})$ that each vertex accounts for. The black vertices are common to $H_{2k+1,m}$ and $H_{2(k-1)+1,m}$, so they are labeled with the difference in the distance between their labels from $H_{2(k-1)+1,m}$ to $H_{2k+1,m}$. The white vertices are the new vertices in $H_{2k+1,m}$, so each is labeled with the difference between its label and that of the vertex labeled directly before it. For simplicity, in Figure 44, we label the outside of the vertices with i for vertex x_i.

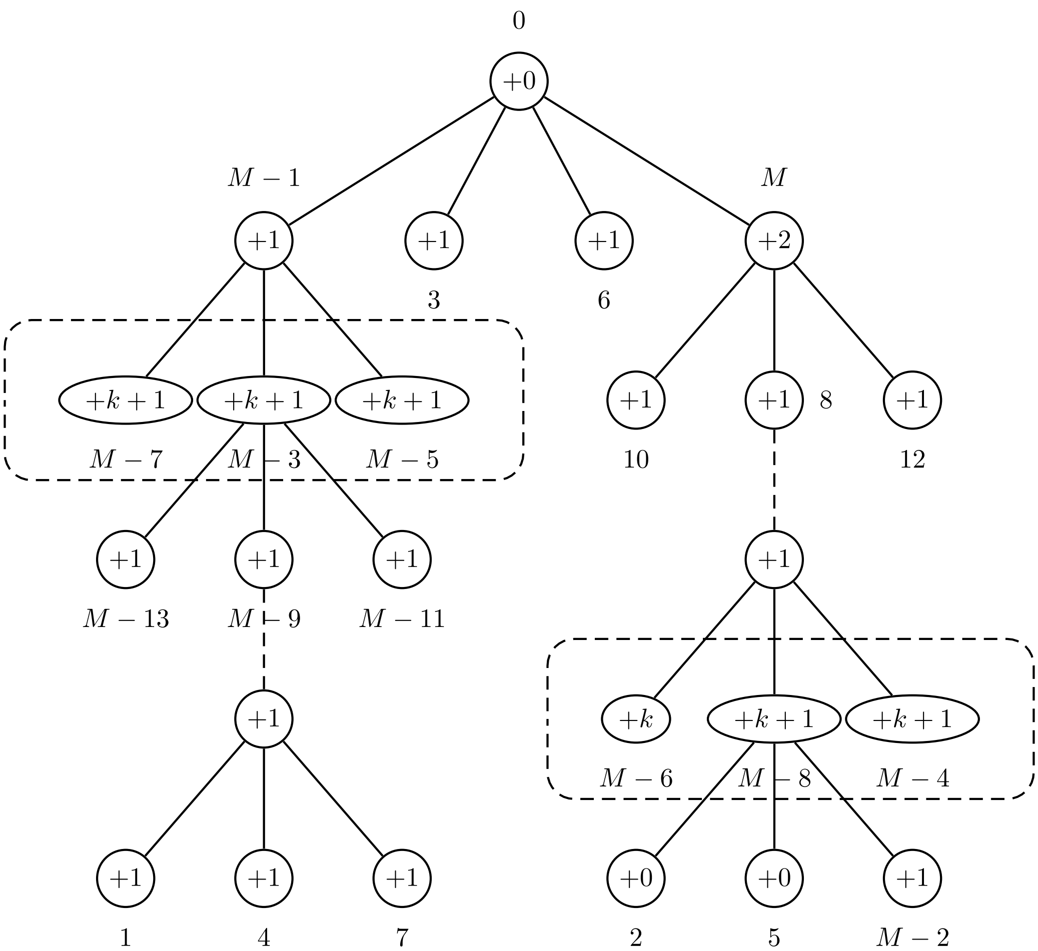

Figure 44: Changes from $H_{2(k-1)+1,4}$ to $H_{2k+1,4}$

We add 1 to the radio number for each of $2(k-1)(m-1)+1$ vertices, 2 for 1 vertex, k for 1 vertex and $k+1$ for each of $2(m-1)-1$ vertices, so in total

$$\begin{aligned} rn(H_{2k+1,m}) - rn(H_{2(k-1)+1,m}) &= 2(k-1)(m-1)+1+2+k+(k+1)(2(m-1)-1) \\ &= 4km - 4k + 2 \end{aligned} \tag{3.72}$$

Now we check that given that our labeling on $H_{2(k-1)+1,m}$ is a valid radio labeling, so is our labeling on $H_{2k+1,m}$.

First we must check this constraint for $i = 2, \ldots, 3m-3$. We begin with the constraint for $j = 2$.

Recall that the $j = 2$ constraint is equivalent to 3.39 Thus to satisfy

$$f(x_i) \geq f(x_{i-2}) + 2k + 3 - d(x_{i-2}, x_i) \tag{3.73}$$

we need to show

$$I_2(i) := 2k + 3 - d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}) + d(x_{i-2}, x_i) \geq 0 \tag{3.74}$$

Below we give a table that lists the important distances for $i = 2, \ldots, 3m-5$. First we give these for $i = 2$ and $i = 3$ and then we separate the indices by their equivalence class mod 3. These indices begin with $q = 2$, and end with the last index $q = m - 2$.

	2	3	4	3q-1	3q	3q+1
$d(x_{i-1}, x_i)$	2k+2	k+2	k+2	2k+2	k+2	k+2
$d(x_{i-2}, x_{i-1})$	k+1	2k+2	k+2	k+2	2k+2	k+2
$d(x_{i-2}, x_i)$	k+1	k+2	2k+2	k+2	k+2	2k+2

So for each x_i for $i = 2, \ldots, 3m - 5$, we can easily verify that 3.39 holds, as shown in the table below.

	2	3	4	3q-1	3q	3q+1
$I_2(i)$	0	1	2k+1	1	1	2k+1

For $i = 2, \ldots, 3m-5$, we can see that the constraints for $j \geq 3$ always holds by looking at the differences between labels of consecutive vertices. First we note that x_0 and x_1 are a distance of $k+1$ apart so $f(x_1) - f(x_0) = k + 2$. For $q = 0 \ldots m - 2$, since x_{3q+1} and $x_{3(q+1)-1} = x_{3q+2}$ are antipodal,

$$f(x_{3q+2}) - f(x_{3q+1}) = 1 \tag{3.75}$$

The distance between x_{3q+1} and x_{3q+2} is $k + 2$, so

$$f(x_{3q+3}) - f(x_{3q+2}) = k + 1 \tag{3.76}$$

Finally, the distance between x_{3q+3} and $x_{3(q+1)+1} = x_{3q+4}$ is also $k + 2$ so

$$f(x_{3q+4}) - f(x_{3q+3}) = k + 1 \tag{3.77}$$

Thus the difference between any two vertices x_i and x_{i-j} with $i = 2, \ldots, 3m-5$ and $j \geq 3$ must be at least $2k + 3$:

$$\begin{aligned} f(x_i) - f(x_{i-j}) &\geq f(x_i) - f(x_{i-3}) \\ &= \big(f(x_i) - f(xi-1)\big) + \big(f(x_{i-1}) - f(xi-2)\big) + \big(f(x_{i-2}) - f(xi-3)\big) \\ &\geq 1 + k + 1 + k + 1 \\ &= 2k + 3 \end{aligned} \tag{3.78}$$

So clearly,

$$\begin{aligned} f(x_i) - f(x_{i-j}) &\geq 2k+3 \\ &= diam(H_{2k+1,m}) + 1 \\ &\geq diam(H_{2k+1,m}) + 1 - d(x_{i-j}, x_i) \end{aligned} \tag{3.79}$$

Therefore we have shown that the labeling is valid through vertex x_{3m-5}.

For $i = 3m-4, \ldots, M_0 - 3$, the vertex x_i is from $H_{2(k-1)+1,m}$ so we can assume these labelings were valid in the original graph, $H_{2(k-1)+1,m}$. Since the diameter increased by 2 from $H_{2(k-1)+1,m}$ to $H_{2k+1,m}$, we know that

$$\begin{aligned} f(x_i) - f(x_{i-j}) &\geq diam(H_{2k+1,m}) + 1 - d(x_{i-j}, x_i) \\ &= 2k+3 - d(x_{i-j}, x_i) \\ &= diam(H_{2(k-1)+1,m}) + 3 - d(x_{i-j}, x_i) \end{aligned} \tag{3.80}$$

We also see that

$$\begin{aligned} f_0(x_i) - f_0(x_{i-j}) + 2 &\geq diam(H_{2(k-1)+1,m}) + 1 - d(x_{i-j}, x_i) + 2 \\ &= (2k+2) + 1 - d(x_{i-j}, x_i) \\ &= 2k+3 - d(x_{i-j}, x_i) \end{aligned} \tag{3.81}$$

Thus we need only show that

$$f(x_i) - f(x_{i-j}) \geq f_0(x_i) - f_0(x_{i-j}) + 2 \tag{3.82}$$

Recalling Figure 44, we added 1 to $f_0(x_i) - f_0(x_{i-1})$ for $i = 3m-4, \ldots, M_0 - 3$. Thus,

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f_0(x_i) - f_0(x_{i-1}) + 1 \\ f(x_{i-1}) - f(x_{i-2}) &= f_0(x_{i-1}) - f_0(x_{i-2}) + 1 \\ &\vdots \\ f(x_{i-j}) - f(x_{i-j-1}) &= f_0(x_{i-j}) - f_0(x_{i-j-1}) + 1 \end{aligned} \tag{3.83}$$

And so we see that

$$\begin{aligned} f(x_i) - f(x_{i-j}) =& \big(f(x_i) - f(x_{i-1})\big) + \big(f(x_{i-1}) - f(x_{i-2})\big) + \cdots \\ & + \big(f(x_{i-j}) - f(x_{i-j-1})\big) \\ =& \big(f_0(x_i) - f_0(x_{i-1}) + 1\big) + \big(f_0(x_{i-1}) - f_0(x_{i-2}) + 1\big) + \cdots \\ & + \big(f_0(x_{i-j}) - f_0(x_{i-j-1}) + 1\big) \\ =& f_0(x_i) - f_0(x_{i-j}) + j \\ \geq& f_0(x_i) - f_0(x_{i-j}) + 2 \end{aligned}$$

Thus we have shown 3.82 to be true and so the labels for $i = 3m - 4, \ldots, M_0 - 3$ are valid.

Now we need to check the validity of labelings for $i = M - 2(m-1) - 2 = M - 2m, \ldots, M - 2$. First we check the $j = 2$ constraint. By the construction of the labeling from the ordering, for $i = M - 2m, \ldots, M - 2$

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= 2k + 3 - d(x_{i-1}, x_i) \quad \text{and} \\ f(x_{i-1}) - f(x_{i-2}) &= 2k + 3 - d(x_{i-2}, x_{i-1}) \quad \text{so} \\ f(x_i) - f(x_{i-2}) &= 4k + 6 - d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}). \end{aligned} \tag{3.84}$$

Thus to satisfy $f(x_i) - f(x_{i-2}) \geq +2k + 3 - d(x_{i-2}, x_i)$ we need to show that the inequality $I_2(i) \geq 0$ holds

$$I_2(i) = 2k + 3 - d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}) + d(x_{i-2}, x_i) \geq 0 \tag{3.85}$$

Similarly, for j=3,

$$I_3(i) = 4k + 6 - d(x_{i-1}, x_i) - d(x_{i-2}, x_{i-1}) - d(x_{i-3}, x_{i-2}) + d(x_{i-3}, x_i) \geq 0 \tag{3.86}$$

	M-2m	M-2m+1	M-2m+2	M-2q-1	M-2q
$d(x_{i-1}, x_i)$	k+3	k+2	k+2	k+2	k+2
$d(x_{i-2}, x_{i-1})$	k+2	k+3	k+2	k+2	k+2
$d(x_{i-2}, x_i)$	3	3	2	2	2
I_2	1	1	1	1	1
$d(x_{i-3}, x_{i-2})$	k+2	k+2	k+3	k+2	k+2
$d(x_{i-3}, x_i)$	k+3	k+1	k+3	k+2	k+2
I_3	2k+2	2k	2k+2	2k	2k

	M-5	M-4	M-3	M-2
$d(x_{i-1}, x_i)$	k+2	k+2	k+2	k+3
$d(x_{i-2}, x_{i-1})$	k+2	k+2	k+2	k+2
$d(x_{i-2}, x_i)$	2	2	2	3
I_2	1	1	1	1
$d(x_{i-3}, x_{i-2})$	k+2	k+2	k+2	k+2
$d(x_{i-3}, x_i)$	k+2	k+2	k+2	k+3
I_3	2k	2k	2k	2k+2

Now we must show these labels are valid for the constraints where $j \geq 4$.

We notice that for $i = 2m, \ldots, M - 2$,

$$\begin{aligned} d(x_{i-1}, x_i) &\leq k + 3 \text{ and so} \\ f(x_i) - f(x_{i-1}) &\geq k \end{aligned} \tag{3.87}$$

Therefore

$$\begin{aligned}f(x_i) - f(x_{i-j}) &\geq jk \\ &\geq 2k + 3 + (j-2)(k) - 3\end{aligned}$$

Since $k \geq 2$, we have, for $j \geq 4$,

$$\begin{aligned}f(x_i) - f(x_{i-j}) &\geq 2k + 3 + (2)(2) - 3 \\ &\geq 2k + 3 - d(x_{i-j}, x_i)\end{aligned}$$

This shows that for $i = M - 2m, \ldots, M - 2$ the constraint between x_i and x_{i-j} holds for $j \geq 4$.

Lastly, we need to check the constraints for $i = M - 1, M$. We know that x_{M-1} is tight with x_{M-3} so the constraint holds with equality for $j = 2$

We need to start with checking the $j = 1$ constraint for $i = M - 1$.

$$\begin{aligned}f(x_{M-1}) &\geq f(x_{M-2}) + (2k+3) - d(x_{M-2}, x_{M-1}) \\ f(x_{M-3}) + 2k + 2 &\geq f(x_{M-2}) + (2k+3) - (k+2)\end{aligned}$$

This is equivalent to

$$\begin{aligned}k + 3 - (f(x_{M-2}) - f(x_{M-3})) &\geq 0 \\ k + 3 - (2k + 3 - (k+3)) &\geq 0 \\ 3 &\geq 0\end{aligned}$$

For $j \geq 3$

$$\begin{aligned}f(x_{M-1}) &= f(x_{M-3}) + (2k+3) - d(x_{M-3}, x_{M-1}) \\ &= f(x_{M-3}) + (2k+3) - 1 \\ &\geq f(x_{M-1-j}) + (2k+3) \\ &> f(x_{M-1-j}) + (2k+3) - d(x_{M-1-j}, x_{M-1})\end{aligned}$$

Thus the constraint between x_{M-1} and x_{M-1-j} holds for $j \geq 2$. Similarly for $i = M$ and $j \geq 2$,

$$\begin{aligned}f(x_M) &= f(x_{M-1}) + (2k+3) - d(x_{M-3}, x_{M-1}) \\ &= f(x_{M-1}) + (2k+3) - 2 \\ &\geq f(x_{M-j}) + (2k+3) - 1 \\ &\geq f(x_{M-j}) + (2k+3) - d(x_{M-1-j}, x_{M-1})\end{aligned}$$

Therefore, the constraint between x_M and x_{M-j} holds for $j \geq 2$, which concludes the proof that this labeling is valid.

3.2.4 Extensions to the Biregularized Path

Extending the Algorithm for Even Biregular Paths The algorithm presented for the even biregular paths can be used for trees that are not biregular. Notice that for the induction on k, we could have added a different number of leaves at each level, instead of keeping a constant m. Any number of vertices can be added to a level the left branch as long as the same number of vertices are added on the corresponding level on the right branch. In Figure 45 we give an example. The only requirement is that the tree is created from $H_{2k,3}$ so that it prevents x_i and x_{i+2} from being adjacent.

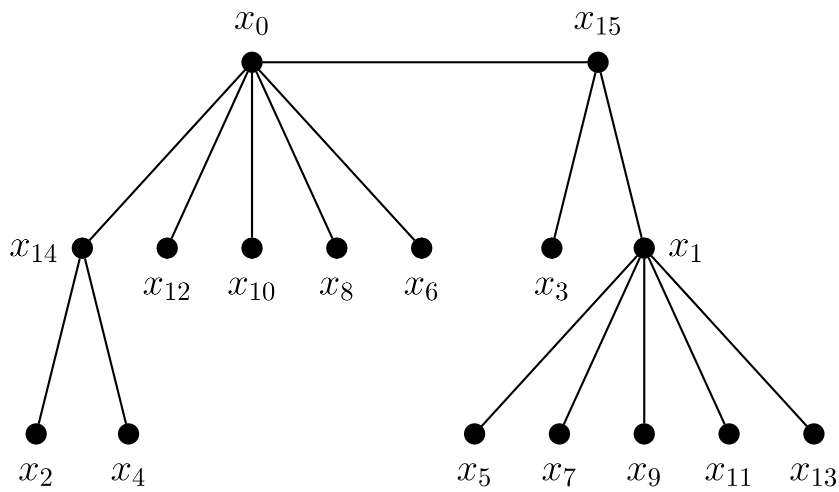

Figure 45: Extended biregular graph

A Note on Odd Biregular Paths For the proof that $H_{2k+1,m}$ cannot reach the lower bound of Theorem 8 we used Observation 21 to show that only $2m-3$ of the $2m-2$ vertices of level $k+1$ can be used in an algorithm that would attain the lower bound. If one more leaf is added to the median to give $deg(w^*) = m+1$, then the radio number is equal to the lower bound from Theorem 8. We illustrate this in Figure 46 with a modification of $H_{5,3}$. The lower bound is $rn(T) \geq (7)(12) + 1 - 2(24) = 37$.

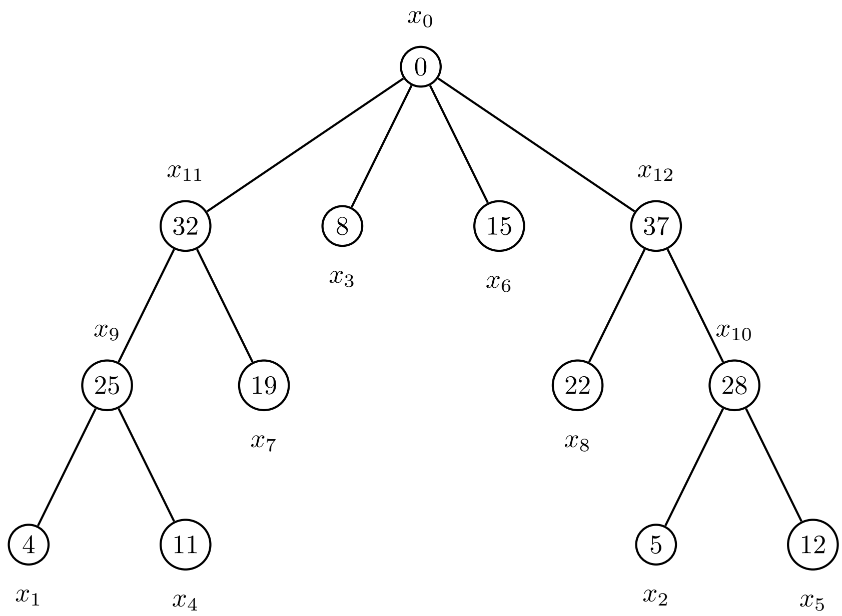

Figure 46: Extended $H_{2k+1,m}$

Biregularized Star As stated earlier, the star S_n has a radio number that attains the lower bound from Theorem 8, and any biregularized star also attains the bound given in Theorem 8. Since biregularization always increases the diameter of a tree by 2, the diameter of a biregularized S_n has diameter 4. In Figure 47 we show the biregularization of S_3 and the ordering that produces a labeling that attains this bound. It is not too difficult to see that the labeling is valid, and since it abides by the three conditions of Theorem 8, it attains the bound given in the same theorem. This ordering can easily be expanded to any star S_n.

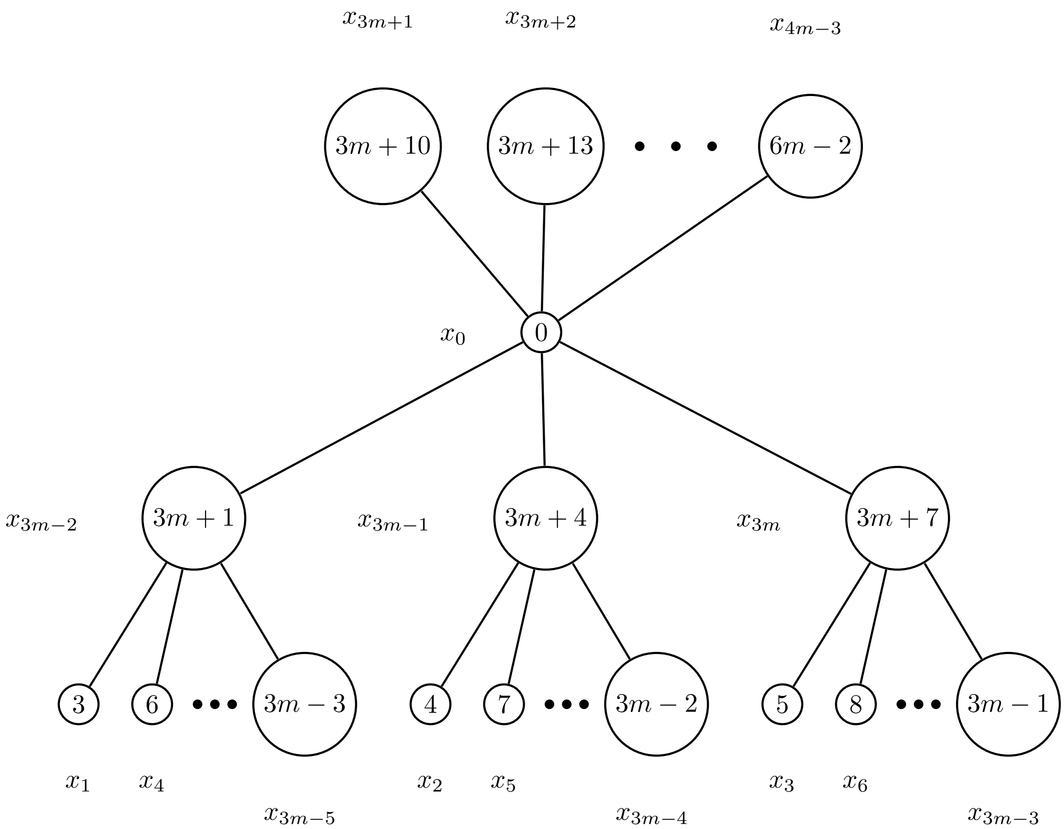

Figure 47: Radio labeling of the general biregularization of S_3

3.3 Open Questions

Question 23 *For what number of vertices, n, and what choice of diameter, $diam(G)$, or the the pair $(n, diam(G))$, is there a radio graceful labeling?*

Notice that the only graph of diameter 1 is K_n so for the pair (n,1) there is always a graceful labeling. The Petersen graph, on 10 vertices with diameter 2, has a graceful labeling so it corresponds to the pair (10,2).

Lemma 24 *For every integer $n \geq 5$ there is a graph on n vertices with diameter 2 that has a radio graceful labeling.*

Proof Let G be the graph of K_n with a cycle C_n removed. Thus for vertices $v_0, v_1, \ldots, v_{n-1}$, the vertex v_i is adjacent to every vertex except for x_{i-1} and x_{i+1}, with vertex v_0 is not adjacent to v_{n-1}. This gives diameter 2. Furthermore, a radio graceful labeling of G has the ordering $x_i = v_i$. Notice that $n \geq 4$ is required to have G be connected.

We now present another lemma without proof.

Lemma 25 *For $n \geq 16$ and even there exists a graph on n vertices with diameter 3 that has a radio graceful labeling.*

To show this we only present a radio graceful labeling for the case when $n = 16$, which is shown in Figure 48. To generalize this graph we create a graph G_k from K_k, where for each i we remove the edges $\{x_{i-2}, x_i\}, \{x_{i-1}, x_i\}, \{x_i, x_{i+1}\}, \{x_i, x_{i+2}\}$, with the understanding that the addition and subtraction is performed $\pmod k$. Then we label the graph $G_k \Box K_2$. If we were to draw the tightness digraph corresponding to the labeling in Figure 48 it would have an interesting property. Each vertex would be tight with the three preceding vertices.

Question 26 *For what graphs does there exist an optimal radio labeling with a tightness path that passes through every vertex?*

From Section 3.1, we have labelings for P_4, P_7 and C_4 which have this property. In fact,the labelings provided in Sections 2.3.1 and 2.3.2 give such labelings for all paths and cycles respectively. We have now shown that we can find such a labeling for even biregular paths, but we do not know whether such a path exists for every odd biregular path.

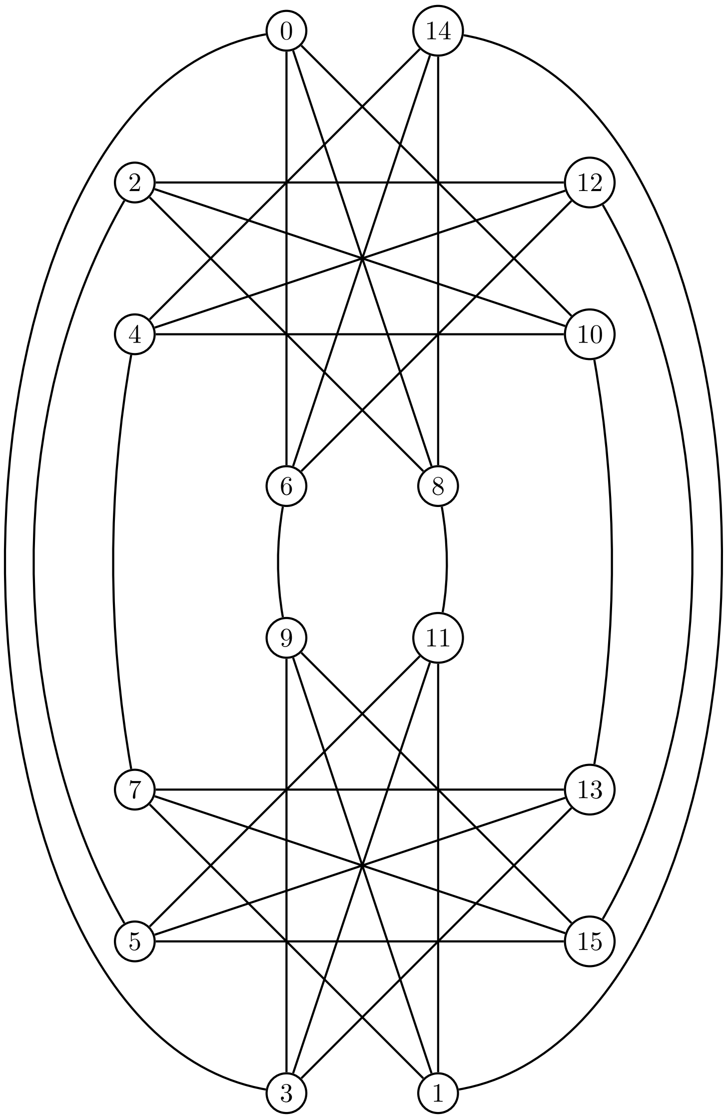

Figure 48: Radio graceful labeling of $G_k \square K_2$

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