In the Wake of the Financial Crisis
—
Regulators' and Investors' Perspectives

by

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A Dissertation

Submitted to the Faculty

of the

WORCESTER POLYTECHNIC INSTITUTE

in partial fulfillment of the requirements for the

Degree of Doctor of Philosophy

in

Mathematical Sciences

by

April 2019

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Acknowledgements

Over the past six years, I have received support and encouragement from a great number of individuals. First and foremost, I would like to express my deepest appreciation to my advisor, Dr. Stephan Sturm. My work would not have been possible without his nurturing and guiding. His advices on both research as well as on my career have been priceless. Besides my advisor, I would also like to extend my sincere thanks to my dissertation committee members, Dr. Agostino Capponi, Dr. Igor Cialenco, Dr. Randy Paffenroth and Dr. Gu Wang, for their great support and invaluable comments.

A special thanks to the Mathematical Science department at Worcester Polytechnic Institute, especially Dr. Luca Capogna, Dr. Christopher J. Larsen, Dr. Marcel Y. Blais and Dr. Michael Richard Johnson for all their supports during my Ph.D. program. Thanks are also due to the National Science Foundation (NSF) under grant 1321794 and Worcester Polytechnic Institute for their financial support. Without these supports, I would not have been able to discuss my scientific development with other coauthors and peers.

I also had great pleasure of working with Dr. Zach Feinstein, Dr. Birgit Rudloff, Dr. Eric Schaanning, and Dr. Mackenzie Wildman. I would like to recognize the assistance that I received from the Mathematical Science office, especially Ms. Rhonda Podell, Ms. Ellen Mackin and others. I am also grateful to all of those whom educated me in my whole life.

Last but not least, I appreciate my parents very much for their love and guidance, which is the fundamental and beginning for me to pursue my dreams.

Abstract

Before the 2008 financial crisis, most research in financial mathematics focused on risk management and pricing of options without considering effects of counterparties' default, illiquidity problems, systemic risk and the role of the repurchase agreement (Repo). During the 2008 financial crisis, a frozen Repo market led to a shutdown of short sales in the stock market. Cyclical interdependencies among financial corporations caused that a default of one firm seriously affected other firms and even the whole financial network.

In this dissertation, we will consider financial markets which are shaped by financial crises. This will be done from two distinct perspectives, an investor's and a regulator's. From an investor's perspective, recently models were proposed to compute the total valuation adjustment (XVA) of derivatives without considering a potential crisis in the market. In our research, we include a possible crisis by apply an alternating renewal process to describe the switching between a normal financial status and a financial crisis status. We develop a framework for pricing the XVA of a European claim in this state-dependent framework. We represent the price as a solution to a backward stochastic differential equation and prove the existence and uniqueness of the solution. To study financial networks from a regulator's perspective, one popular method is the fixed point based approach by L. Eisenberg and T. Noe. However, in practice, there is no accurate record of the interbank liabilities and thus one has to estimate them to use Eisenberg – Noe type models. In our research, we conduct a sensitivity analysis of the Eisenberg – Noe framework, and quantify the effect of the estimation errors to the clearing payments. We show that the effect of the missing specification of interbank connection to clearing payments can be described via directional derivatives that can be represented as solutions of fixed point equations. We also compute the probability of observing clearing payment deviations of a certain magnitude.

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Chapter 1

Introduction

1.1 Motivation

While the most important cause of the 2008 financial crisis was unhealthy mortgage market, the interconnections of the financial market was crucial for the spread of the contagion (crisis). The percentage of sub-prime mortgage, compared with all mortgages, doubled in 2004 and continued to increase until 2006. In 2006, the Federal Reserve increased interest rates. Because many mortgages were given to people without enough financial capacity, many households experienced high financial pressure. Some sold their house, others had to default on their mortgage loans. Because the supply of houses was larger than the demand, prices of real estate dropped sharply. Meanwhile many investment banks also suffered, because of losses in their investments in sub-prime mortgages and loans. Some financial companies even went bankrupt during that time, such as Lehman Brothers. The 2008 sub-prime financial crisis resulted out from this situation.

The crisis of sub-prime mortgages spread out significantly. Two major reasons are their interconnections among financial companies and roles in sale and repurchase agreement (Repo) markets. Before the 2008 financial crisis, it is more popular to study models with pairwise structure — a pair of an investor and its direct counterparty. Based on these models, an effect of a company's default could be controlled within its direct counterparty. However, all counterparties have further counterparties on their own. The interconnections among financial companies extended effects of several defaults to the global financial markets. Moreover, severe bankruptcies of companies led to low confidence in many securities. Because the trades in the Repo market use these securities as collateral, low confidence in these securities caused adverse effects to the Repo market. As the Repo market froze, companies could not finance themselves through the Repo market, which caused liquidity issues. The Liquidity risk aggravate the financial market.

Overall, this adverse loop of market risk, credit risk, liquidity risk and systemic risk led the whole financial environment collapse. To deal with the decreasing price of houses, stocks and financial derivatives, we need to include market risk. To deal with the defaults of counterparties, we need to take credit risk into account. To include market illiquidity problems, we need to describe liquidity risk. In order to model the interconnections among companies in the whole financial market, we need to consider systemic risk.

However, it is hard to include all risks in one model. In order to practically tackle the loop, we divide it into two aspects: an investor's aspect and a regulator's aspect. From the investor's perspective, we are interested in the pricing of options while taking credit risk, liquidity risk and funding spread into consideration. From the regulator's perspective, we are interested in systemic risk, the effect of one default event to the whole financial market.

1.2 Literature Review

Both options trading and interbank loans have a long history. The 20^{th} century witnessed the birth and development of financial mathematics, but the 2008 financial crisis changed it dramatically. An overview of the effects of the financial crisis and their modeling in financial mathematics can be found in Bullard et al. (2009), Bijlsma et al. (2010), and Hördahl and King (2008).

Many different indicators have been proposed to delliate a financial crisis from regular market behavior. Hollo et al. (2012) introduce a Composite Indicator of Systemic Stress (CISS), which puts more weight on the stress shared by several markets at the same time. Whaley (2008) argues that the CBOE's Market Volatility Index (VIX), introduced in Whaley (1993), is a good investor fear gauge of the expected return volatility of the S&P 500 index over the next 30 days. Bekaert et al. (2013), Bekaert and Hoerova (2014) decompose the squared VIX index into two parts, the conditional variance of stock returns and the equity variance premium. Adrian and Brunnermeier (2011) introduce the value at risk of financial institutions conditional on other institutions being in distress (abbreviated CoVaR) as a new measure for systemic risk. The difference between the three month London Interbank Offered Rate (LIBOR) and the government's interest rate for a three-month period (called the Ted spread) changed alot during the financial crisis. As result, Heider et al. (2009) and Acharya and Skeie (2011) argue that the Ted spread is a good indicator for the liquidity and counterparty risk in the interbank system. Mancini et al. (2013), Coffey et al. (2009) and Gorton and Metrick (2012), Gorton et al. (2010) use the Ted spread to measure the capital constraints in a secured lending system. Boudt et al. (2013) confirm the existence of a two-regime Ted spread over the period between 2006 and 2011, describing stable and unstable situation by the analysis of historical data. However, these difference regimes have so far not been considered in pricing derivatives.

Many risky rates changed a lot during the financial crisis, some research focus on the performance of the banking system during the financial crisis. After review the definition of the Sale and Repurchase Agreement (Repo) market, the general collateral and different Repo rates, Sundaresan (2009) compares the Repo rates with other short-term interest rates. Ivashina and Scharfstein (2010) state that bank lending is affected more by the credit-line drawdown than the short-term debt. Hördahl and King (2008) point out that the Repo market froze during the financial crisis and compare its performances in the US, Euro countries and UK. Gorton (2009) studies the contagion of subprime mortgages to other securities and the effects of asymmetric information to the spreading of the risk. Gorton and Metrick (2012) compare the securitized banking (i.e. the Repo market) in the 2008 financial crisis with the traditional banking in the banking panics in the 19^{th} century. They find that the LIBOR-OIS spread is strongly correlated with changes in the credit spread and the Repo rate.

After the financial crisis, Basel regulations have required to take the effect of default risk and cost of collateralization strategies into account in the replication framework (Basel Committee on Banking Supervision (2010)). From the derivative traders' perspectives, research focused on the valuation adjustment (XVA) of a fair price due to credit risk and funding risk. Before the 2008 financial crisis, it was common to apply LIBOR as the risk-free interest rate. But the LIBOR is not a risk-free rate, as can be seen by the large Ted spread during the financial crisis. Research need to include asymmetric interest rates into their pricing models. Cvitanić and Karatzas (1993) introduce the setting of different interest rates for borrowing and lending in a stochastic control problem for hedging. Korn (1995) prices European call and put options with a higher interest rate for borrowing than lending. El Karoui et al. (1997) study super-hedging under asymmetric interest rates by nonlinear backward stochastic differential equations. Piterbarg (2010) calculates the adjustment for non-collateralised derivatives in a setting including a Repo account. Laughton and Vaisbrot (2012) emphasize the necessity of a funding valuation adjustment (FVA) in an incomplete market. Siadat and Hammarlid (2017) discuss the hedging, collateral optimization and reverse stress testing with funding cost and funding benefit adjustment. Bielecki et al. (2018) study the nonlinear arbitrage-free pricing of derivatives, considering differential funding cost, collaterlization, counterparty credit risk, and capital requirements.

Besides the FVA, the credit valuation adjustment (CVA) is the difference between derivatives' prices with and without credit risk. Since it is a measure of counterparties' credit risk, some banks already studied it before the financial crisis. At first, research focuses on the default risk of the counterparty (unilaterally), which lead to a price asymmetry. To solve this problem, bilateral models were introduced. The debt valuation adjustment (DVA) is the difference between the price of a derivative with and without the benefit from the investor's default risk. Bielecki et al. (2008) and Crépey et al. (2010) study the valuation and hedging of credit default swap (CDS) including counterparty credit risk. Crépey et al. (2013) model the total valuation adjustment using a Markovian pre-default backward stochastic differential equation and give numerical valuation results of several derivatives. Brigo et al. (2013) derive a risk-neutral pricing formula, using a backward stochastic differential equation considering counterparty credit risk, funding and collateral service cost. Burgard and Kjaer (2011a,b) generalize Piterbarg's model by considering bilateral credit risk. Nie and Rutkowski (2014) prove the existence of fair bilateral prices. Bielecki and Rutkowski (2013) introduce a general semimartingale market framework for an arbitrage-free valuation. Bichuch et al. (2015, 2016, 2018a) introduce a backward stochastic differential equation representation of European call and put option prices with bilateral credit risk, asymmetric funding, Repo, and collateral rates. Bichuch et al. (2018b) extend the valuation of the XVA with considering the uncertainty bond rates.

From the regulator's perspective, research focuses on the network models to quantify the effect of one default event to the whole financial system. An important stream of the literature on contagion in networks has focused on interbank contagion, building on the network model of Eisenberg and Noe (2001). Central banks and regulators have applied the model to study default cascades in their jurisdictions' banking systems (Anand et al. (2014), Hałaj and Kok (2015), Boss et al. (2004), Elsinger et al. (2013), Upper (2011), Gai et al. (2011)). Hüser (2015) provides a comprehensive and detailed review of the interbank contagion literature. Hurd (2016) presents a unified mathematical framework for modeling these contagion channels. Recently, the Bank of England has extended this model to analyze insolvency contagion in the UK financial system (Bardoscia et al. (2017)). Multiple, extensions of this model have been developed to include effects such as

- *Bankruptcy costs*: Elsinger (2009), Rogers and Veraart (2013), Elliott et al. (2014), Glasserman and Young (2015), Weber and Weske (2017),

- *Cross-ownershi*p: Elsinger (2009), Elliott et al. (2014), Weber and Weske (2017)

- *Fire sales*: Cifuentes et al. (2005), Nier et al. (2007), Gai and Kapadia (2010), Chen et al. (2016), Amini et al. (2015, 2016b), Weber and Weske (2017), Feinstein (2017a), Feinstein and

El-Masri (2017), Feinstein (2017b), Di Gangi et al. (2015), and

- *Multiple maturities*: Capponi and Chen (2015), Kusnetsov and Veraart (2016), Banerjee et al. (2018).

Moreover, a number of papers analyze the implications of network topology on systemic risk in greater detail. Amini et al. (2016a) derive rigorous asymptotic results for the magnitude of the default cascade in terms of network characteristics and find that institutions that have large connectivity and a high number of "contagious links" contribute most to contagion. Detering et al. (2016) show that if the degree distribution of the network does not have a second moment, local shocks can propagate through the entire network. This is relevant as realistic financial networks typically display a core-periphery structure with inhomogeneous degree distribution (Cont et al. (2013)). Chong and Klüppelberg (2018) characterize the joint default distribution of a financial system for all possible network structures and show how Bayesian network theory can be applied to detect contagious channels.

Regulators have identified the inclusion of such contagion mechanisms in stress tests as a key priority (Basel Committee on Banking Supervision (2015), Anderson (2016)). Furthermore, recent research illustrates that accounting for feedback effects and contagion can change the pass/fail result in stress tests for individual institutions (Cont and Schaanning (2017)).

A key ingredient required to estimate contagion in these models is the so-called liabilities matrix L, where L_{ij} is the nominal liability of bank i to bank j. Often, the exact bilateral exposures are not known and thus need to be estimated (Hałaj and Kok (2013), Anand et al. (2015), Elsinger et al. (2013), Hałaj and Kok (2015)). Despite considerable efforts after the crisis to improve data collection, data gaps have not been closed yet. Beyond logistical issues like the standardization of reporting formats and the creation of unique and universal institution identifiers, further hurdles remain, such as legal restrictions that limit regulators' access only to data pertinent to their respective jurisdictions. Therefore, the estimation of specific bilateral exposures remains an important issue (Langfield et al. (2014), Anand et al. (2015, 2018), Financial Stability Board and International Monetary Fund (2015)). The early literature often used entropy maximizing techniques to "fill in the blanks" in the liabilities matrix given the total assets and liabilities of banks (viz. the row and column sums of L). However, a growing empirical literature has shown that real-world interbank networks look quite different from the homogeneous networks that are obtained with such techniques (Bech and Atalay (2010), Mistrulli (2011), Cont et al. (2013), Soramäki et al. (2007)). A recent Bayesian method to estimate the bilateral liabilities, given the total liabilities and potential other prior information, is proposed in Gandy and Veraart (2016) and applied to reconstruct CDS markets in Gandy and Veraart (2017). In particular, Mistrulli (2011), Gandy and Veraart (2016) show how wide estimates of systemic risk may fluctuate when estimating contagion on real-world and heterogeneous networks versus uniform networks. This highlights the pivotal role that the matrix of bilateral exposures plays in quantifying the extent of contagion when computing default cascades. Beyond the above-mentioned legal hurdles that restrict regulator's access to data outside their jurisdiction, another important example of uncertainty in the interbank exposures arises due to time gaps between data collection and the run of the stress test: For some regulatory stress tests (e.g. Dodd-Frank stress tests) data is collected annually, which can both give rise to window-dressing behavior by banks, as well as exposures naturally changing over time. In this case the existence or non-existence of an exposure between two banks will be known, and the uncertainty mainly surrounds its magnitude. Capponi et al. (2016) study the effects of the network topology on systemic risk through the use of majorization-based tools. Birge et al. (2018) discuss the risk analysis and policy of a financial network with limited information by several optimization models. To the best of our knowledge, Liu and Staum (2010) is so far the only paper that performs a sensitivity analysis of the Eisenberg–Noe model. Their analysis focuses on the sensitivity of the clearing vector with respect to the initial net worth of each bank.

1.3 Structure

The organization of this dissertation is as follows. The first part of this dissertation focuses on the total valuation adjustment for one European option with investor's and its counterparty's defaults, different financial statuses, and funding liquidity problems. The second part focuses on a sensitivity analysis of the Eisenberg-Noe financial network model with respect to the error in liabilities.

In Chapter 2, we study an alternating renewal processes without independent increments property. Then, we define a corresponding jump counting process in Section 2.2. In Section 2.3, we define a stochastic integral with respect to the jump counting process and prove a martingale decomposition theorem.

In Chapter 3, we prove most theoretical results about backward stochastic differential equations (BSDEs). We construct a general BSDE containing stochastic integrals including nonindependent increments process in Section 3.1. Under some necessary assumptions, we prove the existence and uniqueness of its solutions in Section 3.2. In Section 3.3, we reduce the original BSDE with a jump terminal condition to a BSDE in a smaller filtration with a continuous terminal condition.

In Chapter 4, we price European options and compute the total valuation adjustment (XVA), considering credit risk, asymmetric interest rate, and different financial states. In Section 4.1, we review several topics about the Repo market. In Section 4.2, we apply an alternating renewal process to describe the switching between financial regimes. After the discussion of several financial accounts in Section 4.3, we create a hedging portfolio for European options in Section 4.4. In Section 4.5, we construct a BSDE to evaluate the arbitrage-free price of a European option and prove the existence and uniqueness of the solution. In Section 4.6, we construct a BSDE of the XVA and derive a corresponding reduced BSDEs to a smaller filtration. In Section 4.7, we estimate the parameters of the alternating renewal process by the Ted spread historical data and analyze the sensitivity of the XVA with respect to the financial states, volatilities and funding rates.

In Chapter 5, we present the Eisenberg–Noe framework and provide initial continuity results of that model. We then study directional derivatives and the Taylor series of the Eisenberg–Noe clearing payments with respect to the relative liability matrix. These results allow us to consider the sensitivity of the clearing payments. In Section 5.2 we use the directional derivatives in order to determine the perturbations to the relative liabilities matrix that present the "worst" errors in terms of misspecification of the clearing payments and impact to society. These results are extended to also consider the probability of the various estimation errors. In Section 5.3 we implement our sensitivity analysis on data calibrated to a network of European banks. The main results of this chapter are already published in Feinstein et al. (2018).

Chapter 2

Poisson Processes With Not-independent Increments Property

In order to describe the switching between a normal financial status and a financial crisis status, we study an alternating renewal process and its corresponding jump counting process. With the stochastic integral with respect to this process, we prove a martingale decomposition theorem, including nonindependent increments processes.

2.1 Alternating Renewal Processes

In this section, we will discuss cumulative distribution functions and properties of the alternating renewal process. An alternating renewal process is a nonhomogeneous Poisson process without independent increments property. This process is well known in engineering field, where it is used to describe a service period and a shutdown period of a machine. For a machine, we assume that the service time follows an exponential distribution with a constant parameter. The shut down time, or the time to fix this machine, follows an exponential distribution with another constant intensity. This process is also called an On-Off process.

An alternating renewal process is a stochastic process switching between 0 and 1. We denote $U_i \sim \exp(\lambda_U)$ as the i^{th} inter-arrival (holding) time in "0" status and $V_i \sim \exp(\lambda_V)$ as the i^{th} holding time in "1" status, where $i \in \mathbb{N}$. Let T_i be the arrival (alternating) time for $i \in \mathbb{N}$,

$$\begin{aligned}
&T_1 = U_1, \\
&T_2 = U_1 + V_1, \\
&T_3 = U_1 + V_1 + U_2, \\
&\cdots\cdots
\end{aligned}$$

Definition 2.1.1. *An alternating renewal process is defined as*

$$\beta_t = \sum_{i=1}^{\infty} (-1)^{i+1} \mathbb{1}_{\{T_i \le t\}},$$

where odd inter-arrival times $T_{2n+1} - T_{2n}$ follow an exponential distribution with a constant parameter $\lambda_U > 0$, and even inter-arrival times $T_{2n} - T_{2n-1}$ follow an exponential distribution with a constant parameter $\lambda_V > 0$.

The graph of one path of the stochastic process β is given in Figure 2.1.

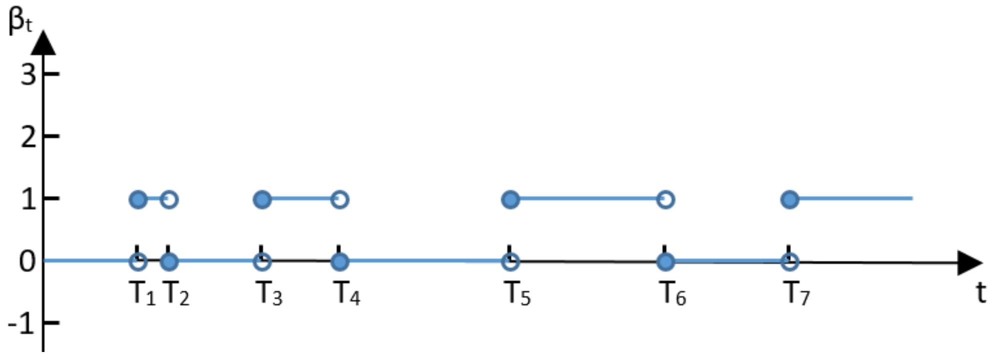


Figure 2.1: One path of the process β.

Distributions

By its definition, the stochastic process β is a right continuous Markov process with left limits (*càdlàg*) switching between status "0" and status "1". At each alternating time T_n, the jump direction of the process β depends on the status of β_{T_n-}. When $\beta_{T_n-} = 0$, then $\beta_{T_n} = 1$ as a result of an upward jump. When $\beta_{T_n-} = 1$, then $\beta_{T_n} = 0$ as a result of a downward jump. Therefore, the alternating renewal process β does not have an independent increments property. Because β has only countable many jumps almost surely, it has one left-continuous with right limits modification (*càglàd*). We decompose β as the sum of two processes β^+ and β^-:

$$\beta_t^+ = \sum_{s \le t} \mathbb{1}_{\{\beta_{s-} < \beta_s\}}, \qquad \beta_t^- = -\sum_{s \le t} \mathbb{1}_{\{\beta_{s-} > \beta_s\}}. \tag{2.1.1}$$

The process β^+ is a jump counting process of the upward jumps and the process β^- is a jump counting process of a downward jumps.

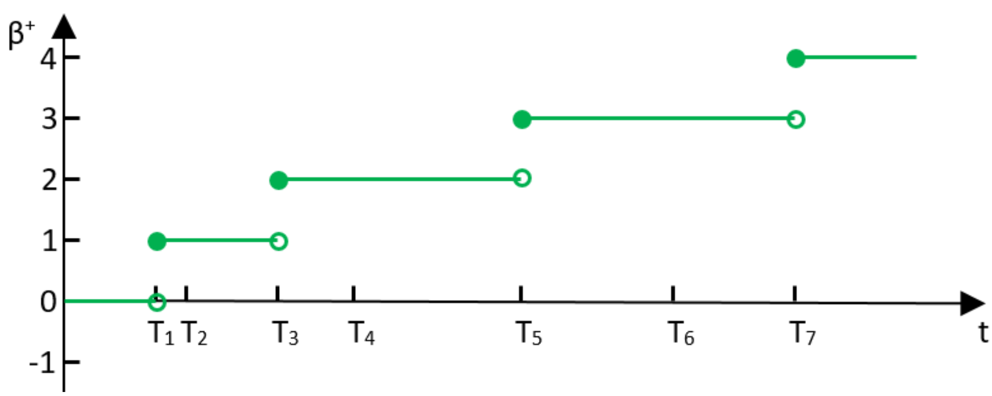


Figure 2.2: One path of the process β^+.

Since β^- is a non-increasing *càdlàg* process, it is a supermartingale. The supermartingale β^- is a negative Poisson process with inter-arrival times (holding time) following independent identically

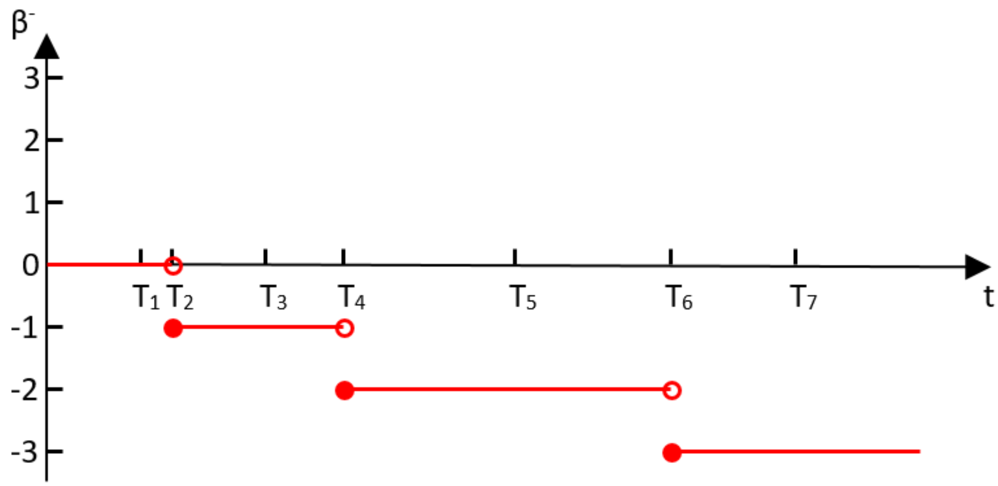


Figure 2.3: One path of the process β^-.

distributed (i.i.d.) exponential distributions. Since the inter-arrival times of the process β^- are the sum of inter-arrival times of processes U_i and V_i, we have that the relation $\frac{1}{\lambda} = \frac{1}{\lambda_U} + \frac{1}{\lambda_V}$, so the inter-arrival times of β^- are exponential distributed random variables with parameter λ, where $\lambda = \frac{\lambda_U \lambda_V}{\lambda_U + \lambda_V}$.

In the same way, we know β^+ is a submartingale. Contrary to β^-, it does not have the independent increments property. We define $Y_i = V_i + U_{i+1}$, which is an exponential distributed random variable with parameter λ. The alternating time is given by $T_1^+ = U_1 = T_1 \sim \exp(\lambda_U)$ and $T_{n+1}^+ = U_1 + \sum_{i=1}^n Y_i = T_{2n+1}$ for $n \geq 1$. By the properties of the sum of i.i.d. random variables, we have $\sum_{i=1}^n Y_i \sim Gamma(n, \lambda)$. By the convolution, we know $T_{n+1}^+ = T_{2n+1}$ follows a sum of one exponential distribution $\exp(\lambda_U)$ and a Gamma distribution $Gamma(n, \lambda)$, where $\lambda = \frac{\lambda_U \lambda_V}{\lambda_U + \lambda_V}$.

So the probability density function of $T_{2n+1} = U_1 + \sum_{i=1}^n Y_i$ is

$$
\begin{aligned}
f_{T_{2n+1}}(z) &= f_{U_1 + \sum_{i=1}^n Y_i}(z) \\
&= \int_{-\infty}^{\infty} f_{U_1}(z-y) f_{\sum_{i=1}^n Y_i}(y) dy \\
&= \int_0^z \lambda_U \exp\big(-\lambda_U(z-y)\big) \frac{\lambda^n y^{n-1}}{(n-1)!} \exp(-\lambda y) dy \\
&= \frac{\lambda_U \lambda^n \exp(-\lambda_U z)}{(n-1)!} \int_0^z y^{n-1} \exp\big((\lambda_U - \lambda)y\big) dy \\
&= \frac{\lambda_U \lambda^n \exp(-\lambda_U z)}{(n-1)!} \exp\big((\lambda_U - \lambda)y\big) \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{(n-1)! y^k}{k! (\lambda_U - \lambda)^{n-k}} \Big|_0^z \\
&= \frac{\lambda_U \lambda^n \exp(-\lambda_U z)}{(n-1)!} \exp\big((\lambda_U - \lambda)y\big) \sum_{k=0}^{n-1} \frac{(-1)(n-1)! y^k}{k! (\lambda - \lambda_U)^{n-k}} \Big|_0^z
\end{aligned}
$$

$$= \frac{\lambda_U \lambda^n \exp(-\lambda_U z)}{(n-1)!} \exp\left((\lambda_U - \lambda)z\right) \sum_{k=0}^{n-1} \frac{(-1)(n-1)!z^k}{k!(\lambda-\lambda_U)^{n-k}} - \frac{\lambda_U \lambda^n \exp(-\lambda_U z)}{(n-1)!} \sum_{k=0}^{n-1} \frac{(-1)(n-1)!0^k}{k!(\lambda-\lambda_U)^{n-k}}$$
$$= \frac{\lambda_U \lambda^n \exp(-\lambda_U z)}{(\lambda-\lambda_U)^n} - \lambda_U \lambda^n \exp(-\lambda z) \sum_{k=0}^{n-1} \frac{z^k}{k!(\lambda-\lambda_U)^{n-k}}.$$

Here we apply the formula

$$\int x^n \exp(ax)dx = \exp(ax) \sum_{k=0}^{n} (-1)^k \frac{n! x^{n-k}}{(n-k)! a^{k+1}} = \exp(ax) \sum_{k=0}^{n} (-1)^{n-k} \frac{n! x^k}{k! a^{n-k+1}}.$$

Thus, we compute the cumulative distribution function of T_{2n+1}. For $n = 0$, we have

$$\mathbb{P}(T_1 \le t) = \int_0^t \lambda_U \exp(-\lambda_U u)du = 1 - \exp(-\lambda_U t).$$

For $n \ge 1$, by integration by parts, we have

$$\begin{aligned}
\mathbb{P}(T_{2n+1} \le t) &= \int_0^t f_{T_{2n+1}}(z)dz \\
&= \int_0^t \frac{\lambda_U \lambda^n \exp(-\lambda_U z)}{(\lambda-\lambda_U)^n} dz - \int_0^t \lambda_U \lambda^n \exp(-\lambda z) \sum_{k=0}^{n-1} \frac{z^k}{k!(\lambda-\lambda_U)^{n-k}} dz \\
&= \frac{\lambda^n}{(\lambda-\lambda_U)^n} \int_0^t \lambda_U \exp(-\lambda_U z)dz - \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n}{k!(\lambda-\lambda_U)^{n-k}} \int_0^t z^k \exp(-\lambda z)dz \\
&= -\frac{\lambda^n \exp(-\lambda_U z)}{(\lambda-\lambda_U)^n}\Big|_0^t - \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n \exp(-\lambda z)}{k!(\lambda-\lambda_U)^{n-k}} \sum_{j=0}^{k} \frac{(-1)^{k-j} k! z^j}{j!(-\lambda)^{k-j+1}}\Big|_0^t \\
&= \frac{\lambda^n}{(\lambda-\lambda_U)^n} - \frac{\lambda^n \exp(-\lambda_U t)}{(\lambda-\lambda_U)^n} + \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n \exp(-\lambda z)}{(\lambda-\lambda_U)^{n-k}} \sum_{j=0}^{k} \frac{z^j}{j! \lambda^{k-j+1}}\Big|_0^t \\
&= \frac{\lambda^n}{(\lambda-\lambda_U)^n} - \frac{\lambda^n \exp(-\lambda_U t)}{(\lambda-\lambda_U)^n} + \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n \exp(-\lambda t)}{(\lambda-\lambda_U)^{n-k}} \sum_{j=0}^{k} \frac{t^j}{j! \lambda^{k-j+1}} \\
&\quad - \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n}{(\lambda-\lambda_U)^{n-k}} \sum_{j=0}^{k} \frac{0^j}{j! \lambda^{k-j+1}} \\
&= \frac{\lambda^n}{(\lambda-\lambda_U)^n} - \frac{\lambda^n \exp(-\lambda_U t)}{(\lambda-\lambda_U)^n} + \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n \exp(-\lambda t)}{(\lambda-\lambda_U)^{n-k}} \sum_{j=0}^{k} \frac{t^j}{j! \lambda^{k-j+1}} \\
&\quad - \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n}{(\lambda-\lambda_U)^{n-k}} \frac{1}{\lambda^{k+1}} \\
&= 1 - \frac{\lambda^n \exp(-\lambda_U t)}{(\lambda-\lambda_U)^n} + \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n \exp(-\lambda t)}{(\lambda-\lambda_U)^{n-k}} \sum_{j=0}^{k} \frac{t^j}{j! \lambda^{k-j+1}}.
\end{aligned}$$

We get the probability distribution function of β_t^+ as follows:

$$\begin{aligned}\mathbb{P}(\beta_t^+ = 0) &= \mathbb{P}(T_1 > t) \\ &= \int_t^\infty \lambda_U \exp(-\lambda_U z) dz \\ &= e^{-\lambda_U t}.\end{aligned} \tag{2.1.2}$$

$$\begin{aligned}\mathbb{P}(\beta_t^+ = 1) &= \mathbb{P}(T_1 \le t) - \mathbb{P}(T_3 \le t, T_1 \le t) \\ &= \mathbb{P}(T_1 \le t) - \mathbb{P}(T_3 \le t) \\ &= 1 - \exp(-\lambda_U t) - (1 - \frac{\lambda \exp(-\lambda_U t)}{\lambda - \lambda_U} + \frac{\lambda_U \exp(-\lambda t)}{\lambda - \lambda_U}) \\ &= \frac{\lambda_U}{\lambda - \lambda_U}\big(\exp(-\lambda_U t) - \exp(-\lambda t)\big).\end{aligned} \tag{2.1.3}$$

For $n \ge 1$

$$\begin{aligned}\mathbb{P}(\beta_t^+ = n+1) &= \mathbb{P}(T_{2n+1} \le t) - \mathbb{P}(T_{2n+3} \le t, T_{2n+1} \le t) \\ &= \mathbb{P}(T_{2n+1} \le t) - \mathbb{P}(T_{2n+3} \le t) \\ &= \frac{\lambda^{n+1} \exp(-\lambda_U t)}{(\lambda - \lambda_U)^{n+1}} - \frac{\lambda^n \exp(-\lambda_U t)}{(\lambda - \lambda_U)^n} \\ &\quad + \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n \exp(-\lambda t)}{(\lambda - \lambda_U)^{n-k}} \sum_{j=0}^{k} \frac{t^j}{j!\lambda^{k-j+1}} - \sum_{k=0}^{n} \frac{\lambda_U \lambda^{n+1} \exp(-\lambda t)}{(\lambda - \lambda_U)^{n-k+1}} \sum_{j=0}^{k} \frac{t^j}{j!\lambda^{k-j+1}} \\ &= \frac{\lambda_U \lambda^n \exp(-\lambda_U t)}{(\lambda - \lambda_U)^{n+1}} - \sum_{k=0}^{n-1} \frac{\lambda_U^2 \lambda^n \exp(-\lambda t)}{(\lambda - \lambda_U)^{n-k+1}} \sum_{j=0}^{k} \frac{t^j}{j!\lambda^{k-j+1}} - \frac{\lambda_U \exp(-\lambda t)}{\lambda - \lambda_U} \sum_{k=0}^{n} \frac{t^k \lambda^k}{k!}.\end{aligned} \tag{2.1.4}$$

Remark 2.1.2. *Note that Equation (2.1.3) can be recovered from Equation (2.1.4) by setting $n = 0$ and eliminate the second term $\sum_{k=0}^{n-1}$.*

2.1.1 Properties

Proposition 2.1.3. *The stochastic process β^- is square integrable on any finite time horizon.*

Proof. By the definition of a homogeneous Poisson process, we have $\mathbb{P}(-\beta^- = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$. For any given $t < \infty$, $\mathbb{E}[(\beta_t^-)^2] = \lambda t + (\lambda t)^2 < \infty$.

□

Proposition 2.1.4. *The stochastic process β^+ is square integrable on any finite time horizon.*

Proof. If $t < T_1$, then $\beta_t^+ = 0$ and $\mathbb{E}[(\beta_t^+)^2] = 0$.

If $t \ge T_1$, then $\hat{\beta}_t^+ = \beta_t^+ - \beta_{T_1-}^+$ is a homogeneous Poisson process with parameter λ. So

$$\mathbb{E}[(\hat{\beta}_t^+)^2] = (\lambda t)^2 + \lambda t < \infty.$$

Since $\beta^+_{T_1-} = 0$, we have

$$\mathbb{E}[(\beta_t^+)^2] = \mathbb{E}[(\beta_t^+ - \beta^+_{T_1-})^2] = \mathbb{E}[(\hat{\beta}_t^+)^2] < \infty.$$

Overall, we proved β_t^+ is square integrable for $0 \le t < \infty$.

□

Proposition 2.1.5. *For the stochastic process β^+, there exist a finite variation stochastic process Λ^+ such that $\tilde{\beta}_t^+ := \beta_t^+ - \Lambda_t^+ = \beta_t^+ - \int_0^t \lambda_s^+ ds$, $\tilde{\beta}^+$ is a martingale with respect to the natural filtration $(\mathscr{F}_t^\beta)_{t\ge 0}$, $\mathscr{F}_t^\beta = \sigma(\beta_s : s \le t)$.*

Proof. By Proposition 2.1.4, the process β^+ is square integrable. Since β_t^+ is a nondecreasing process, it is a submartingale. By the Doob–Meyer Decomposition Theorem, there exist a finite variation process Λ_t^+ such that $\tilde{\beta}_t^+ = \beta_t^+ - \Lambda_t^+$, $\tilde{\beta}^+$ is a square integrable martingale with respect to the filtration $(\mathscr{F}_t^\beta)_{t\ge 0}$.

Then, by the intensity of Poisson processes $\lambda_t^+ = \lim_{h\to 0} \frac{\mathbb{P}(\beta^+_{t+h} - \beta^+_t = 1)}{h}$ and $\Lambda_t^+ := \int_0^t \lambda_s^+ ds$, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{\mathbb{P}(\beta^+_{t+h} - \beta^+_t = 1)}{h} &= \lim_{h\to 0} \frac{\mathbb{P}(\beta^+_{t+h} - \beta^+_t = 1 | t < T_1)}{h} \mathbb{1}_{\{t<T_1\}} + \lim_{h\to 0} \frac{\mathbb{P}(\beta^+_{t+h} - \beta^+_t = 1 | T_1 \le t)}{h} \mathbb{1}_{\{T_1 \le t\}} \\
&= \lim_{h\to 0} \frac{\mathbb{P}(\beta^+_h = 1 | t < T_1)}{h} \mathbb{1}_{\{t<T_1\}} + \lim_{h\to 0} \frac{\mathbb{P}(\beta^+_{t+h} - \beta^+_t = 1 | T_1 \le t)}{h} \mathbb{1}_{\{T_1 \le t\}}.
\end{aligned}$$

For the second term, when $T_1 < t < t+h$, the process $(\beta_{t+h} - \beta_t)_{t\ge T_1}$ is a Poisson process with the intensity λ. For the first term,

$$\begin{aligned}
\lim_{h\to 0} \frac{\mathbb{P}(\beta^+_h = 1 | t < T_1)}{h} &= \lim_{h\to 0} \frac{\frac{\lambda_U}{\lambda - \lambda_U}\big(\exp(-\lambda_U h) - \exp(-\lambda h)\big)}{h} \\
&= \frac{\lambda_U}{\lambda - \lambda_U} \lim_{h\to 0} \frac{\exp(-\lambda_U h) - \exp(-\lambda h)}{h} \\
&= \frac{\lambda_U}{\lambda - \lambda_U} \lim_{h\to 0} \frac{-\lambda_U h \exp(-\lambda_U h) + \lambda h \exp(-\lambda h)}{h} \\
&= \frac{\lambda_U}{\lambda - \lambda_U} \lim_{h\to 0} \big(\lambda \exp(-\lambda h) - \lambda_U \exp(-\lambda_U h)\big) \\
&= \frac{\lambda_U}{\lambda - \lambda_U} (\lambda - \lambda_U) \\
&= \lambda_U.
\end{aligned}$$

Therefore, $\lambda_t^+ = \lambda_U \mathbb{1}_{\{t<T_1\}} + \lambda \mathbb{1}_{\{T_1 \le t\}}$, the proposition is proved.

□

Lemma 2.1.6. *The process β has countably infinite many jumps and $T_\infty := \lim_{n\to\infty} T_n = \infty$ almost surely.*

Proof. Based on the definition, β_t has countably infinite many jumps. For any given time $t \ge 0$, a sequence of events $\{T_{2n+1} \le t\}$ is decreasing events as $n \to \infty$. By the equation $\mathbb{P}(T_{2n+1} \le t)$ and

Fubini's theorem, we have

$$\mathbb{P}(T_\infty \le t) = \lim_{n\to\infty} \mathbb{P}(T_{2n+1} \le t)$$
$$= \lim_{n\to\infty} \Big\{1 - \frac{\lambda^n \exp(-\lambda_U t)}{(\lambda - \lambda_U)^n} + \sum_{k=0}^{n-1} \frac{\lambda_U \lambda^n \exp(-\lambda t)}{(\lambda-\lambda_U)^{n-k}} \sum_{j=0}^{k} \frac{t^j}{j!\lambda^{k-j+1}}\Big\}$$

$$= 1 - \exp(-\lambda_U t) \lim_{n\to\infty} \Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n + \frac{\lambda_U}{\lambda}\exp(-\lambda t) \lim_{n\to\infty}\Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n \sum_{k=0}^{n-1}\sum_{j=0}^{k} \frac{(\lambda t)^j}{j!}\Big(\frac{\lambda-\lambda_U}{\lambda}\Big)^k$$
$$= 1 - \exp(-\lambda_U t) \lim_{n\to\infty} \Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n + \frac{\lambda_U}{\lambda}\exp(-\lambda t) \lim_{n\to\infty}\Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n \sum_{j=0}^{n-1}\frac{(\lambda t)^j}{j!}\sum_{k=j}^{n-1} \Big(\frac{\lambda-\lambda_U}{\lambda}\Big)^k$$
$$= 1 - \exp(-\lambda_U t) \lim_{n\to\infty} \Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n$$
$$+ \frac{\lambda_U}{\lambda}\exp(-\lambda t) \lim_{n\to\infty}\Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n \sum_{j=0}^{n-1}\frac{(\lambda t)^j}{j!}\frac{\Big(\frac{\lambda-\lambda_U}{\lambda}\Big)^j\Big(1-\Big(\frac{\lambda-\lambda_U}{\lambda}\Big)^{n-j}\Big)}{1-\frac{\lambda-\lambda_U}{\lambda}}$$
$$= 1 - \exp(-\lambda_U t) \lim_{n\to\infty} \Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n$$
$$+ \frac{\lambda_U}{\lambda}\exp(-\lambda t) \lim_{n\to\infty}\Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n \sum_{j=0}^{n-1}\frac{(\lambda t)^j}{j!}\frac{\lambda}{\lambda_U}\Big(\Big(\frac{\lambda-\lambda_U}{\lambda}\Big)^j - \Big(\frac{\lambda-\lambda_U}{\lambda}\Big)^n\Big)$$
$$= 1 - \exp(-\lambda_U t) \lim_{n\to\infty} \Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n + \exp(-\lambda t) \lim_{n\to\infty}\Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n \sum_{j=0}^{n-1}\frac{((\lambda-\lambda_U)t)^j}{j!}$$
$$- \exp(-\lambda t)\lim_{n\to\infty}\sum_{j=0}^{n-1}\frac{(\lambda t)^j}{j!}$$
$$= 1 - \exp(-\lambda_U t) \lim_{n\to\infty} \Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n \Big(1 - \exp\big(-(\lambda-\lambda_U)t\big)\sum_{j=0}^{n-1}\frac{\big((\lambda-\lambda_U)t\big)^j}{j!}\Big)$$
$$- \exp(-\lambda t)\lim_{n\to\infty}\sum_{j=0}^{n-1}\frac{(\lambda t)^j}{j!},$$
$$= 1 - I_1 - I_2,$$

where the term $1-\exp\big(-(\lambda-\lambda_U)t\big)\sum_{j=0}^{n-1}\frac{\big((\lambda-\lambda_U)t\big)^j}{j!}$ in I_1 is the tail probability of a Poisson process with parameter $\lambda-\lambda_U$. By the Proposition 1 in Glynn (1987), we have

$$1-\exp\big(-(\lambda-\lambda_U)t\big)\sum_{j=0}^{n-1}\frac{\big((\lambda-\lambda_U)t\big)^j}{j!} \le \Big(1-\frac{\lambda-\lambda_U}{n+1}\Big)^{-1}\exp\big(-(\lambda-\lambda_u)t\big)\frac{\big((\lambda-\lambda_U)t\big)^n}{n!}.$$

Thus, the term I_1

$$\begin{aligned} I_1 \leq& \exp(-\lambda_U t) \lim_{n\to\infty} \Big(\frac{\lambda}{\lambda-\lambda_U}\Big)^n \Big(1-\frac{\lambda-\lambda_U}{n+1}\Big)^{-1} \exp\big(-(\lambda-\lambda_u)t\big) \frac{\big((\lambda-\lambda_U)t\big)^n}{n!} \\ =& \exp(-\lambda t) \lim_{n\to\infty} \Big(1-\frac{\lambda-\lambda_U}{n+1}\Big)^{-1} \frac{(\lambda t)^n}{n!} \end{aligned}$$

where it is nonnegative when n is even. So, we have $\lim_{n\to\infty} I_1 = 0$. For the term I_2, we get

$$I_2 = \exp(-\lambda t)\exp(\lambda t) = 1.$$

Therefore $\mathbb{P}(T_\infty \leq t) = 1 - 0 - 1 = 0$. Since t is arbitrary, we proved $T_\infty = \infty$ almost surely. □

Theorem 2.1.7. *For an alternating renewal process β, there exists a finite variation stochastic process Λ^β such that $\tilde{\beta}_t := \beta_t - \Lambda^\beta_t = \beta_t - \int_0^t \lambda^\beta_s ds$, $\tilde{\beta}$ is a martingale with respect to the filtration $(\mathscr{F}^\beta_t)_{t\geq 0}$.*

Proof. By Proposition 2.1.3 and 2.1.4, the integrability is trivial. Since $-\beta^-$ is a Poisson process with parameter λ, we have that process $\tilde{\beta}^-_t := \beta^-_t + \lambda t, \tilde{\beta}^-$ is a martingale with respect to the filtration $(\mathscr{F}^\beta_t)_{t\geq 0}$. By Proposition 2.1.5, $\tilde{\beta}^+$ is a martingale with respect to the filtration $(\mathscr{F}^\beta_t)_{t\geq 0}$. Since $\beta_t = \beta^+_t + \beta^-_t$, we define $\Lambda_t = \Lambda^+_t + \lambda t$ and $\lambda^\beta_t = \lambda^+_t + \lambda$, then the expectation is as follows

$$\begin{aligned} \mathbb{E}\Big[\beta_t - \int_0^t \lambda^\beta_u du|\mathscr{F}^\beta_s\Big] &= \mathbb{E}\Big[\beta^+_t + \beta^-_t - \int_0^t \lambda^+_u - \lambda du|\mathscr{F}^\beta_s\Big] \\ &= \mathbb{E}\Big[\beta^+_t - \int_0^t \lambda^+_u du|\mathscr{F}^\beta_s\Big] + \mathbb{E}\Big[\beta^-_t + \int_0^t \lambda du|\mathscr{F}^\beta_s\Big] \\ &= \beta^+_s - \int_0^s \lambda^+_u ds + \beta^-_s + \lambda s \\ &= \beta_s - \int_0^s \lambda^\beta_u du. \end{aligned}$$

So $\tilde{\beta}_t := \beta_t - \Lambda^\beta_t = \beta_t - \int_0^t \lambda^\beta_s ds, \tilde{\beta}$ is a martingale with respect to the filtration $(\mathscr{F}^\beta_t)_{t\geq 0}$. □

2.2 Jump Counting Processes

In this section, we define a nondecreasing processes corresponding to alternating renewal processes β, called jump counting processes, and discuss its property.

Definition 2.2.1. *For the alternating renewal process β with parameters λ_U and λ_V, we define a jump counting process J as*

$$J_t = \sum_{s\leq t} \mathbb{1}_{\{\beta_s - \beta_{s-}\neq 0\}}(s). \tag{2.2.1}$$

Remark 2.2.2. *The value of the stochastic process J at time t is the number of jumps of the process β until time t. By Equation* (2.1.1)*, we have $J_t = \beta^+_t - \beta^-_t$.*

Proposition 2.2.3. *The jump counting process J has countably infinite many jumps and $T_\infty = \infty$.*

Proof. By Equation (2.2.1) and the Lemma 2.1.6, we have this result. □

Remark 2.2.4. *Similar to the process β, the jump counting process J is a stochastic process without independent increments property when $\lambda_U \neq \lambda_V$.*

Proposition 2.2.5. *The jump counting process J is a square integrable semimartingale with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$.*

Proof. By Equation (2.2.1), J is a nondecreasing stochastic process, so it is a finite variation process. Thus, J_t is a semimartingale with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$.

Based on Remark 2.2.2, Proposition 2.1.3 and 2.1.4, we have $\mathbb{E}[(J_t)^2] = \mathbb{E}[(\beta_t^+ - \beta_t^-)^2] \leq 2\mathbb{E}[(\beta_t^+)^2] + 2\mathbb{E}[(\beta_t^-)^2] < \infty$. □

Proposition 2.2.6. *For the jump counting process J, there exists a finite variation stochastic process Λ^J such that $\tilde{J}_t := J_t - \Lambda_t^J = J_t - \int_0^t \lambda_s^J ds, \tilde{J}$ is a square integrable martingale with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$.*

Proof. Since $J = \beta^+ - \beta^-$, we define $\Lambda_t^J := \Lambda_t^+ + \lambda t, \lambda^J = \lambda^+ + \lambda^-$. Similar to the proof of Theorem 2.1.7, $\tilde{J}_t := J_t - \Lambda_t^J = J_t - \int_0^t \lambda_s^J ds$ is a martingale with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$ by Proposition 2.1.5.

By Proposition 2.2.5, we have $\mathbb{E}[(\tilde{J}_t)^2] = \mathbb{E}[(J_t - \int_0^t \lambda_s^J ds)^2] \leq 2\mathbb{E}[(J_t)^2] + 2\mathbb{E}[(\int_0^t \lambda_s^J ds)^2] < \infty$. So $\tilde{J}_t$ is a square integrable martingale with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$.

□

We call the stochastic process $\tilde{J}$ as a compensated jump counting process of J.

2.3 Stochastic Calculus With Respect To Compensated Jump Counting Processes

In general, we can rewrite a corresponding martingale problem with respect to each BSDE, since the stochastic integral with respect to a Brownian motion is a local martingale. Then, the existence and uniqueness of a solution to a BSDE is a direct result of the martingale representation theoremKaratzas and Shreve (1998). For preparation of the proof in Chapter 3, we discuss the stochastic integral with respect to the process $\tilde{J}$ and a theorem similar to the martingale representation theorem. We first prove that the space of all square integrable martingales with a stochastic integer representation is a Banach space and then prove the martingale decomposition theorem.

2.3.1 Stochastic Integrals

Since the compensated jump counting process $\tilde{J}$ is a martingale with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$, we can define the stochastic integral with respect to $\tilde{J}$; see (Métivier 1982, Chapter 4) for details. Let X be a predictable process with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$, and we denote the stochastic integral with respect to the compensated jump counting process $\tilde{J}$ as

$$\int_0^t X_s d\tilde{J}_t.$$

Proposition 2.3.1 (Isometry)**.** *Let X be a predictable process with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$ and $[\tilde{J}]_t$ be the quadratic variation of the compensated jump counting process $\tilde{J}$. We have the following isometry property*

$$\mathbb{E}\left[\left(\int_0^t X_s d\tilde{J}_s\right)^2\right] = \mathbb{E}\left[\int_0^t (X_s)^2 d[\tilde{J}]_s\right] = \mathbb{E}\left[\int_0^t (X_s)^2 \lambda_s^J ds\right].$$

Proof. By Proposition 2.2.6, we know that $\tilde{J}$ is a square integrable martingale with respect to the filtration $(\mathscr{F}_t^\beta)_{t\geq 0}$. By Proposition 18.13 in Métivier (1982) and Proposition 2.2.6, we have this result. $\square$

The Space of Square Integrable Martingales

To prove that a square integrable martingale has a representation of stochastic integrals with nonindependent increments processes, we introduce several assumptions and notations at first.

Assumption 2.3.2. *Let W be a Brownian motion, β be an alternating renewal process, ϖ^I, ϖ^C be two compensated processes with a single exponential distributed jump, which are independent and strongly orthogonal.*

Notation 2.3.3.

- $(\Omega, \mathscr{F}, \mathbb{P})$ *is a probability space.*
- $\mathscr{F}_t = \sigma(W_s, \beta_s, \varpi_s^I, \varpi_s^C : s \leq t)$.
- $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ *is a filtered probability space.*
- $\mathscr{H}^{\beta,2} = \{X | X$ *is* $\mathscr{B}([0,t]) \otimes \mathscr{F}_t^\beta$ *predictable process with* $\|X\|_{\mathscr{H}_t^2} < \infty,$ *for* $\forall t \leq T\}$, *where* $\|\cdot\|_{\mathscr{H}_T^2} = \mathbb{E}[\int_0^t X_s^2 ds]$.
- $\mathscr{H}^2 = \{(X, X^I, X^C, X^\beta) | X, X^I, X^C, X^\beta$ *are* $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$ *predictable process with* $\|X\|_{\mathscr{H}_T^2} < \infty, \|X^I\|_{\mathscr{H}_T^2} < \infty, \|X^C\|_{\mathscr{H}_T^2} < \infty$ *and* $\|X^\beta\|_{\mathscr{H}_T^2} < \infty,$ *for* $\forall t \leq T\}$.
- $\mathscr{M}^\beta = \{M | M$ *is a* $(\mathscr{F}_t^\beta)_{t\geq 0}$ *martingale with* $\sup_{t\leq T} \mathbb{E}[M_t^2] < \infty,$ *for* $\forall t \leq T\}$.
- $\mathscr{M} = \{M | M$ *is a* $(\mathscr{F}_t)_{t\geq 0}$ *martingale with* $\sup_{t\leq T} \mathbb{E}[M_t^2] < \infty,$ *for* $\forall t \leq T\}$.
- $\mathscr{M}_T^{\beta,*} = \{M_T | M_T$ *is a* $\mathscr{F}_T^\beta$ *measurable random variable with* $M_T := I_T^\beta(X) = \int_0^T X_s d\tilde{J}_s$, *for* $\sup_{t\leq T} \mathbb{E}[M_t^2] < \infty, X \in \mathscr{H}^{\beta,2}\}$.
- $\mathscr{M}^{\beta,*} = \{M | M \in \mathscr{M}^\beta$ *and* $M_t := I_t^\beta(X) = \int_0^t X_s d\tilde{J}_s$ *with* $X \in \mathscr{H}^{\beta,2},$ *for* $\forall t \leq T\}$.
- $\mathscr{M}_T^* = \{M_T | M_T$ *is a* $\mathscr{F}_T$ *measurable random variable with* $M_T =: I_T(X) = \int_0^T X_s dW_s + \int_0^T X_s^I d\varpi_t^I + \int_0^T X_s^C d\varpi_t^C + \int_0^T X_s^\beta d\tilde{J}_s,$ *for* $\sup_{t\leq T} \mathbb{E}[M_t^2] < \infty, (X, X^I, X^C, X^\beta) \in \mathscr{H}_T^2\}$.
- $\mathscr{M}^* = \{M | M \in \mathscr{M}$ *and* $M_t =: I_t(X) = \int_0^t X_s dW_s + \int_0^t X_s^I d\varpi_s^I + \int_0^t X_s^C d\varpi_s^C + \int_0^t X_s^\beta d\tilde{J}_s$ *with* $(X, X^I, X^C, X^\beta) \in \mathscr{H}^2,$ *for* $\forall t \leq T\}$.

Proposition 2.3.4. *Given* $(\Omega, \mathscr{F}, (\mathscr{F}_t^\beta)_{t\geq 0}, \mathbb{P})$, $(\mathscr{M}^{\beta,*}, \|\ \|)$ *is a Banach space with the norm* $\|\cdot\|^2 = \mathbb{E}[M_t^2]$ *for* $M \in \mathscr{M}^{\beta,*}$.

Proof. I. We prove first that $(\mathscr{M}^{\beta,*}, \| \ \|)$ is a vector space. Let $M_t^{(1)}, M_t^{(2)} \in \mathscr{M}^{\beta,*}$, then we have two predictable square integrable processes $X^{(1)}$ and $X^{(2)}$ such that $M_t^{(1)} = \int_0^t X_s^{(1)} d\tilde{J}_s, M_t^{(2)} = \int_0^t X_s^{(2)} d\tilde{J}_s$. Let $\mu_1, \mu_2 \in \mathbb{R}$ and $X = \mu_1 X^{(1)} + \mu_2 X^{(2)}$, then X is still predictable and square integrable. So $\mu_1 M_t^{(1)} + \mu_2 M_t^{(2)} = \int_0^t X_s d\tilde{J}_s \in \mathscr{M}^{\beta,*}$.

II. Let $M_t^{(n)} \in \mathscr{M}^{\beta,*}$ be a Cauchy sequence. Since this vector space is a topological normed space, the limit of $M_t^{(n)}$ exists, denoted by $M_t = \lim_{n\to\infty} M_t^{(n)}$. Then we need to prove that $M_t \in \mathscr{M}^{\beta,*}$. Since $M_t^{(n)} \in \mathscr{M}^{\beta,*}$, there exists a sequence of adapted square integrable processes $X^{(n)}$ such that $M_t^{(n)} = I_t^{\beta}(X^{(n)}) = \int_0^t X_s^{(n)} d\tilde{J}_s$.

i) Let $X = \lim_{n\to\infty} X^{(n)}$, we need to prove it exist and X is predictable and square integrable. Since $M_t^{(n)}$ is a Cauchy sequence, for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|M_t^{(n)} - M_t^{(m)}\| < \epsilon$ for any $n, m > N$. Since $M_t^{(n)}, M_t^{(m)} \in \mathscr{M}^{\beta,*}$, there exist $X^{(n)}, X^{(m)}$ such that $M_t^{(n)} = \int_0^t X_s^{(n)} d\tilde{J}_s, M_t^{(m)} = \int_0^t X_s^{(m)} d\tilde{J}_s$. Based on the definition of $\mathscr{H}^{\beta,2}$ and the isometry property, we have

$$
\begin{aligned}
\|X^{(n)} - X^{(m)}\|_{\mathscr{H}_t^2} =& \mathbb{E}\left[\int_0^t |X_s^{(n)} - X_s^{(m)}|^2 ds\right] \\
=& \mathbb{E}\left[\int_0^t |X_s^{(n)} - X_s^{(m)}|^2 \frac{1}{\lambda_s^J} \lambda_s^J ds\right] \\
\leq& \frac{1}{\lambda} \mathbb{E}\left[\int_0^t |X_s^{(n)} - X_s^{(m)}|^2 \lambda_s^J ds\right] \\
=& \frac{1}{\lambda} \mathbb{E}\left[\int_0^t |X_s^{(n)} - X_s^{(m)}|^2 d[\tilde{J}]_s\right] \\
=& \frac{1}{\lambda} \mathbb{E}\Big[\Big(\int_0^t |X_s^{(n)} - X_s^{(m)}| d\tilde{J}_s\Big)^2\Big] \\
=& \frac{1}{\lambda} \|M^{(n)} - M^{(m)}\|_t^2 \\
<& \epsilon.
\end{aligned}
$$

So the sequence $X^{(n)}$ is a Cauchy sequence. Since $\mathscr{H}^{\beta,2}$ is complete, the limit $X = \lim_{n\to\infty} X^{(n)}$ exists. By the definition of X, the predictability and square integrability properties are trivial.

ii) For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|M_t^{(n)} - M_t\|^2 < \epsilon$ for any $n > N$. We need to prove the square integrability. For any $n > N$ be given, we have $\|M\|^2 \leq (\|M_t - M_t^{(n)}\| + \|M_t^{(n)}\|)^2 \leq (\epsilon + \|M_t^{(n)}\|)^2 < \infty$, which proved the square integrability. By the square integrability, $M_t = \lim_{n\to\infty} M_t^{(n)} = \lim_{n\to\infty} \int_0^t X_s^{(n)} d\tilde{J}_s = \int_0^t \lim_{n\to\infty} X_s^{(n)} d\tilde{J}_s = \int_0^t X_s d\tilde{J}_s$, which proved $M_t \in \mathscr{M}^{\beta,*}$.

So any Cauchy sequence $M_t^{(n)} \in \mathscr{M}^{\beta,*}$ is convergent in the space $(\mathscr{M}^{\beta,*}, \| \ \|)$, which means $(\mathscr{M}^{\beta,*}, \| \ \|)$ is close.

Overall, $(\mathscr{M}^{\beta,*}, \| \ \|)$ is a Banach space. □

Remark 2.3.5. *By the square integrability in this Banach space, it is also a Hilbert space for a inner produce* $< M_t^1, M_t^2 >= \mathbb{E}[M_t^1 M_t^2]$.

Proposition 2.3.6. $\mathscr{M}_t^{\beta,*} \in \mathscr{M}^{\beta}$, *which means for any* $M_t := I_t^{\beta}(X) \in \mathscr{M}^{\beta,*}$, $I^{\beta}(X)$ *is a square integrable martingale with respect to the filtration* $(\mathscr{F}_t^{\beta})_{t\geq 0}$.

Proof. I. By Proposition 2.3.4, the square integrability of $I_t^\beta(X)$ follows the square integrability of X.

II. For a given $t \geq 0$, we need to prove the martingale property of $I_t^\beta(X)$. By the definition of process $\tilde{J}$, We have

$$\begin{aligned}
\mathbb{E}[I_T^\beta(X)|\mathscr{F}_t^\beta] &= \mathbb{E}\Big[\int_0^T X_s d\tilde{J}_s|\mathscr{F}_t^\beta\Big] \\
&= \mathbb{E}\Big[\int_0^t X_s d\tilde{J}_s|\mathscr{F}_t^\beta\Big] + \mathbb{E}\Big[\int_t^T X_s d\tilde{J}_s|\mathscr{F}_t^\beta\Big] \\
&= \int_0^t X_s d\tilde{J}_s + \mathbb{E}\Big[\int_t^T X_s d\tilde{J}_s|\mathscr{F}_t^\beta\Big] \\
&= I_t^\beta(X) + \mathbb{E}\Big[\int_t^T X_s d\tilde{J}_s|\mathscr{F}_t^\beta\Big] \qquad (2.3.1)
\end{aligned}$$

Next, We show that the second term in the above equation is equal 0 in two steps.

i) Assume that X is an elementary process, i.e. $X = \sum_{i=1}^N a_{t_i} \mathbb{1}_{\{t_i \leq s < t_{i+1}\}}$ with a_{t_i} are $\mathscr{F}_{t_i}^\beta$ measurable square integrable random variables and $0 = t_1 < \cdots < t_{k-1} < t < t_k < \cdots < t_N = T$. Then the second term in Equation (2.3.1) becomes

$$\begin{aligned}
\mathbb{E}\Big[\int_t^T X_s d\tilde{J}_s|\mathscr{F}_t^\beta\Big] &= \mathbb{E}\Big[\int_t^T \sum_{i=1}^N a_{t_i} \mathbb{1}_{\{t_i \leq s < t_{i+1}\}} d\tilde{J}_s|\mathscr{F}_t^\beta\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^N \int_t^T a_{t_i} \mathbb{1}_{\{t_i \leq s < t_{i+1}\}} d\tilde{J}_s|\mathscr{F}_t^\beta\Big] \\
&= \sum_{i=1}^N \mathbb{E}\Big[a_{t_i} \int_t^T \mathbb{1}_{\{t_i \leq s < t_{i+1}\}} d\tilde{J}_s|\mathscr{F}_t^\beta\Big] \\
&= \mathbb{E}[a_{t_{k-1}}(\tilde{J}_{t_k} - \tilde{J}_t)|\mathscr{F}_t^\beta] + \sum_{i=k}^N \mathbb{E}[a_{t_i}(\tilde{J}_{t_{i+1}} - \tilde{J}_{t_i})|\mathscr{F}_t^\beta] \\
&= \mathbb{E}\Big[\mathbb{E}[a_{t_{k-1}}(\tilde{J}_{t_k} - \tilde{J}_t)|\mathscr{F}_{t_i}^\beta]\Big|\mathscr{F}_t^\beta\Big] + \sum_{i=k}^N \mathbb{E}\Big[\mathbb{E}[a_{t_i}(\tilde{J}_{t_{i+1}} - \tilde{J}_{t_i})|\mathscr{F}_{t_i}^\beta]\Big|\mathscr{F}_t^\beta\Big] \\
&= \mathbb{E}\Big[a_{t_{k-1}}\mathbb{E}[\tilde{J}_{t_k} - \tilde{J}_t|\mathscr{F}_{t_i}^\beta]\Big|\mathscr{F}_t^\beta\Big] + \sum_{i=k}^N \mathbb{E}\Big[a_{t_i}\mathbb{E}[\tilde{J}_{t_{i+1}} - \tilde{J}_{t_i}|\mathscr{F}_{t_i}^\beta]\Big|\mathscr{F}_t^\beta\Big]. \qquad (2.3.2)
\end{aligned}$$

By the definition of $\tilde{J}_t = \sum_{s \leq t} \mathbb{1}_{\{\beta_s - \beta_{s-} \neq 0\}}(s) - \int_0^t \lambda_s^J ds$, we have

$$\begin{aligned}
\tilde{J}_{t_{i+1}} - \tilde{J}_{t_i} &= \sum_{s \leq t_{i+1}} \mathbb{1}_{\{\beta_s - \beta_{s-} \neq 0\}}(s) - \int_0^{t_{i+1}} \lambda_s^J ds - \Big(\sum_{s \leq t_i} \mathbb{1}_{\{\beta_s - \beta_{s-} \neq 0\}}(s) - \int_0^{t_i} \lambda_s^J ds\Big) \\
&= \sum_{t_i < s \leq t_{i+1}} \mathbb{1}_{\{\beta_s - \beta_{s-} \neq 0\}}(s) - \int_{t_i}^{t_{i+1}} \lambda_s^J ds.
\end{aligned}$$

Let $s' = s - t_i$ for $s \geq t_i$, we have $s = s' + t_i$, then

$$\begin{aligned}
\tilde{J}_{t_{i+1}} - \tilde{J}_{t_i} &= \sum_{t_i \leq s'+t_i \leq t_{i+1}} \mathbb{1}_{\{\beta_{s'+t_i} - \beta_{(s'+t_i)-}\}}(s') - \int_{t_i}^{t_{i+1}} \lambda^J_{s'+t_i} d(s'+t_i) \\
&= \sum_{0 < s' \leq t_{i+1}-t_i} \mathbb{1}_{\{\beta_{s'+t_i} - \beta_{(s'+t_i)-} \neq 0\}}(s') - \int_0^{t_{i+1}-t_i} \lambda^J_{s'+t_i} d(s').
\end{aligned}$$

Let $\beta'_t = \beta_{t'+t_i}$ and $\lambda^{J'}_t = \lambda^J_{t'+t_i}$, we have

$$\tilde{J}_{t_{i+1}} - \tilde{J}_{t_i} = \sum_{0 \leq s \leq t} \mathbb{1}_{\{\beta'_s - \beta'_{s-} \neq 0\}}(s) - \int_0^t \lambda^{J'}_s ds,$$

denoted by $\tilde{J}'_t$. By the definition of compensated jump counting processes, $\tilde{J}'$ is a compensated jump counting process by the memoryless property of exponential processes. When $\beta_{t_i} = 0$, $\tilde{J}'$ has same distribution as $\tilde{J}$. When $\beta_{t_i} = 1$, $\tilde{J}'$ is a compensated jump counting process starting with the initial value 1. In summary, we have $\mathbb{E}[\tilde{J}'_t \mathbb{1}_{\{\beta_{t_i}=0\}}] = 0$ and $\mathbb{E}[\tilde{J}'_t \mathbb{1}_{\{\beta_{t_i}=1\}}] = 0$ for $t \geq t_i$. By the Markov property of a Poisson process, we have

$$\begin{aligned}
\mathbb{E}[\tilde{J}_{t_{i+1}} - \tilde{J}_{t_i} | \mathscr{F}^\beta_{t_i}] &= \mathbb{E}[\tilde{J}'_{t_{i+1}} | \sigma(\beta_{t_i})] \\
&= \frac{\mathbb{E}[\tilde{J}'_{t_{i+1}} \mathbb{1}_{\{\beta_{t_i}=1\}}]}{\mathbb{P}(\beta_{t_i} = 1)} \mathbb{1}_{\{\beta_{t_i}=1\}} + \frac{\mathbb{E}[\tilde{J}'_{t_{i+1}} \mathbb{1}_{\{\beta_{t_i}=0\}}]}{\mathbb{P}(\beta_{t_i} = 0)} \mathbb{1}_{\{\beta_{t_i}=0\}} \\
&= 0\mathbb{1}_{\{\beta_{t_i}=1\}} + 0\mathbb{1}_{\{\beta_{t_i}=0\}} \\
&= 0.
\end{aligned}$$

Similarly, the first term in Equation (2.3.2) becomes $\mathbb{E}\Big[a_{t_{k-1}} \mathbb{E}[\tilde{J}_{t_k} - \tilde{J}_t | \mathscr{F}^\beta_{t_i}] \Big| \mathscr{F}^\beta_t\Big] = 0$. Thus, the second term in Equation (2.3.1) becomes

$$\mathbb{E}\Big[a_{t_{k-1}} \cdot 0 \Big| \mathscr{F}^\beta_t\Big] + \sum_{i=k}^N \mathbb{E}\Big[a_{t_i} \cdot 0 \Big| \mathscr{F}^\beta_t\Big] = 0.$$

Therefore $\mathbb{E}[I^\beta_T(X) | \mathscr{F}^\beta_t] = I^\beta_t(X)$. Since t is any arbitrage time, we proved the martingale property of $I^\beta(X)$ when X is an elementary process.

ii) For any predictable square integrable stochastic process X, there exists a sequence of square integrable elementary processes $X^{(n)} \to X$, as $n \to \infty$. By the square integrability, the second term in Equation (2.3.1) becomes

$$\begin{aligned}
\mathbb{E}\Big[\int_t^T X_s d\tilde{J}_s | \mathscr{F}^\beta_t\Big] &= \mathbb{E}\Big[\int_t^T \lim_{n\to\infty} X^{(n)}_s d\tilde{J}_s | \mathscr{F}^\beta_t\Big] \\
&= \lim_{n\to\infty} \mathbb{E}\Big[\int_t^T X^{(n)}_s d\tilde{J}_s | \mathscr{F}^\beta_t\Big] \\
&= \lim_{n\to\infty} 0 \\
&= 0.
\end{aligned}$$

So we proved the martingale property of $I^\beta(X)$. $\square$

Remark 2.3.7. *In Proposition 2.3.6, we apply the square integrability to switch the sequence of a integral and limit in the proof. Without the square integrability property, $I^\beta(X)$ still has the martingale property with a L^1 integrability condition, by adding a condition to satisfy the requirement to switch the integral and the limit.*

Assume that the stochastic processes $W, \beta, \varpi^I, \varpi^C$ are independent and strongly orthogonal, Proposition 2.3.4 and Proposition 2.3.6 can be extended to the martingale M with respect to the natural filtration $(\mathscr{F}_t)_{t\geq 0}$.

Proposition 2.3.8. *Given $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, $(\mathscr{M}^*, \|\ \|)$ is a Banach space with the norm $\|\cdot\|^2 = \mathbb{E}[M_t^2]$ for $M_t \in M^*$.*

Proof. I. First we prove that $(\mathscr{M}^*, \|\ \|)$ is a vector space. Let $M_t^1, M_t^2 \in \mathscr{M}^*$ and $\mu_1, \mu_2 \in \mathbb{R}$. By the linearity of stochastic integrals, we have

$$\begin{aligned}\mu_1 M_t^1 + \mu_2 M_t^2 = &\int_0^t (\mu_1 X_s^1 + \mu X_s^2) dW_s + \int_0^t (\mu_1 X_s^{1,I} + \mu_2 X_s^{2,I}) d\varpi_s^I \\ &+ \int_0^t (\mu_1 X_s^{1,C} + \mu_2 X_s^{2,C}) d\varpi_s^C + \int_0^t (\mu_1 X_s^{1,\beta} + \mu_2 X_s^{2,\beta}) d\tilde{J}_s \\ &\in \mathscr{M}_t^*.\end{aligned}$$

II. Since $(\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a normed space, we want to prove the completeness of this space. By the definition of $\|\cdot\|^2$, Proposition 2.3.4 and the strongly orthogonality of the processes $W, \beta, \varpi^I, \varpi^C$, we have the completeness of $(\mathscr{M}^*, \|\ \|)$.

Overall, we proved that $(\mathscr{M}^*, \|\ \|)$ is a Banach space. □

Proposition 2.3.9. *$\mathscr{M}^* \in \mathscr{M}$, which means for any $M_t = I_t(X) \in \mathscr{M}^*$, $I(X)$ is a square integrable martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$.*

Proof. I. For the square integrability, since $(X, X^I, X^C, X^\beta) \in \mathscr{H}^2$, we have this property.

II. Then we want to prove the martingale property of $I(X)$. By the linearity of the expectation, independence and Proposition 2.3.6, we have

$$\begin{aligned}\mathbb{E}\Big[I_T(X)|\mathscr{F}_t\Big] &= \mathbb{E}\Big[\int_0^T X_s dW_s + \int_0^T X_s^I d\varpi_s^I + \int_0^T X_s^C d\varpi_s^C + \int_0^T X_s^\beta d\tilde{J}_s|\mathscr{F}_t\Big] \\ &= \mathbb{E}\Big[\int_0^T X_s dW_s|\mathscr{F}_t\Big] + \mathbb{E}\Big[\int_0^T X_s^I d\varpi_s^I|\mathscr{F}_t\Big] + \mathbb{E}\Big[\int_0^T X_s^C d\varpi_s^C|\mathscr{F}_t\Big] + \mathbb{E}\Big[\int_0^T X_s^\beta d\tilde{J}_s|\mathscr{F}_t\Big] \\ &= \mathbb{E}\Big[\int_0^T X_s dW_s|\mathscr{F}_t^W\Big] + \mathbb{E}\Big[\int_0^T X_s^I d\varpi_s^I|\mathscr{F}_t^I\Big] + \mathbb{E}\Big[\int_0^T X_s^C d\varpi_s^C|\mathscr{F}_t^C\Big] + \mathbb{E}\Big[\int_0^T X_s^\beta d\tilde{J}_s|\mathscr{F}_t^\beta\Big] \\ &= \int_0^t X_s dW_s + \int_0^t X_s^I d\varpi_s^I + \int_0^t X_s^C d\varpi_s^C + \int_0^t X_s^\beta d\tilde{J}_s \\ &= I_t(X).\end{aligned}$$

Thus, $\mathscr{M}^*$ is a subspace of $\mathscr{M}$. □

2.3.2 Martingale Decomposition Theorem

In this section, we will prove the existence and uniqueness of a decomposition of a square integrable martingale. Any square integrable martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ can be write in a form as the sum of stochastic integrals with respect to $W, \tilde{J}, \varpi^I, \varpi^C$ and a orthogonal term. Similar to Proposition 4.1 in Kunita and Watanabe (1967), we will prove a decomposition for a random variable M_T and then extend the decomposition ot any martingale $M \in M_t$.

Proposition 2.3.10. *Let M_T be an $\mathscr{F}_T^\beta$ measurable random variable with $\sup_{t\leq T}\mathbb{E}[M_t^2] < \infty$, $(\mathscr{M}_T^*)^\perp$ be the orthogonal space with respect to $\mathscr{M}_T^*$. Then, there exists a unique pair $I_T(X) \in \mathscr{M}_T^*$ and $Y \in (\mathscr{M}_T^*)^\perp$ such that $M_T = I_T(X) + Y_T$.*

Proof. I. First we want to prove the existence of the decomposition $M_T = I_T(X)+Y_T$. Because $\mathscr{M}_T^*$ is complete subspace of $\mathscr{M}_T$, there exists an orthogonal space of $\mathscr{M}_T^*$ in $\mathscr{M}_T$, denoted as $(\mathscr{M}_T^*)^\perp$.

Let $I_T(X) = proj_{\mathscr{M}_T^*} M_T, Y = M_T - I_T(X)$. We have $M_T = I_T(X) + Y$ with

$$\|I_T(X)Y\|^2 = \|I_T(X)(M_T - I_T(X))\|^2 = \|(proj_{\mathscr{M}_T^*} M_T)^2 - (proj_{\mathscr{M}_T^*} M_T)^2\|^2 = 0.$$

So we have $Y \in (\mathscr{M}_T^*)^\perp$. We proved the existence of the decomposition.

II. Then, we want to prove the uniqueness of the decomposition. Assume that there exists two decompositions, $I_T^1(X), I_T^2(X) \in \mathscr{M}_T^*$ and $Y_1, Y_2 \in (\mathscr{M}_T^*)^\perp$ such that $M_T = I_T^1(X)+Y_1 = I_T^2(X)+Y_2$. Thus, $I_T^1(X)-I_T^2(X) = Y_2-Y_1$. Since $\mathscr{M}_T^*$ and $(\mathscr{M}_T^*)^\perp$ are linear spaces and $I_T^1(X)-I_T^2(X) \in \mathscr{M}_T^*$ and $Y_2 - Y_1 \in (\mathscr{M}_T^*)^\perp$, we have $I_T^1(X) - I_T^2(X) = Y_2 - Y_1 \in \mathscr{M}_T^* \cap (\mathscr{M}_T^*)^\perp$. By the property of an orthogonal space, $I_T^1(X) - I_T^2(X) = Y_2 - Y_1 = 0$. Therefore, we get $I_T^1(X) = I_T^2(X) \in \mathscr{M}_T^*$ and $Y_1 = Y_2 \in (\mathscr{M}_T^*)^\perp$. So, we proved the uniqueness. □

By the unique decomposition of $M_T \in \mathscr{M}_T$, we have the following decomposition of a martingale $M \in \mathscr{M}$.

Theorem 2.3.11. *Let $M \in \mathscr{M}_t$. Then, there exists a unique decomposition $M = I(X) + Y$ within the space $(\mathscr{M}, \|\ \|)$, where $I(X) \in \mathscr{M}^*$ and $Y \in (\mathscr{M}^*)^\perp$.*

Proof. I. We want to prove the existence of a decomposition of $M \in \mathscr{M}$.

By Proposition 2.3.10, for any $M_T \in \mathscr{M}_T$, we have $M_T = Y_T + I_T(X)$, where

$$I_T(X) = \int_0^T X_s dW_s + \int_0^T X_s^I d\varpi_t^I + \int_0^T X_s^C d\varpi_t^C + \int_0^T X_s^\beta d\tilde{J}_s \in M_T^*,$$

and $Y_T \in (M_T^*)^\perp$.

Since M is a martingale, by Proposition 2.3.9, we have

$$M_t = \mathbb{E}[M_T|\mathscr{F}_t] = \mathbb{E}[Y_T + I_T(X)|\mathscr{F}_t] = \mathbb{E}[Y_T|\mathscr{F}_t] + I_t(X).$$

Define a stochastic process Y by $Y_t := \mathbb{E}[Y_T|\mathscr{F}_t]$. So we have a decomposition of $M_t = Y_t + I_t(X)$. Then, we want to prove that $Y \in (\mathscr{M}^*)^\perp$. Since $I(X) \in \mathscr{M}^*$, we denote $I_t(X) = \int_0^t X_s dW_s +$

$\int_0^t X_s^I d\varpi_t^I + \int_0^t X_s^C d\varpi_t^C + \int_0^t X_s^\beta d\tilde{J}_s \in \mathscr{M}^*$. Then

$$\begin{aligned}
\mathbb{E}[Y_t I_t(X)] &= \mathbb{E}\Big[Y_t\Big(\int_0^t X_s dW_s + \int_0^t X_s^I d\varpi_s^I + \int_0^t X_s^C d\varpi_s^C + \int_0^t X_s^\beta d\tilde{J}_s\Big)\Big] \\
&= \mathbb{E}\Big[\mathbb{E}[Y_T|\mathscr{F}_t]\Big(\int_0^t X_s dW_s + \int_0^t X_s^I d\varpi_s^I + \int_0^t X_s^C d\varpi_s^C + \int_0^t X_s^\beta d\tilde{J}_s\Big)\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[Y_T\Big(\int_0^t X_s dW_s + \int_0^t X_s^I d\varpi_s^I + \int_0^t X_s^C d\varpi_s^C + \int_0^t X_s^\beta d\tilde{J}_s\Big)|\mathscr{F}_t\Big]\Big] \\
&= \mathbb{E}\Big[Y_T\Big(\int_0^t X_s dW_s + \int_0^t X_s^I d\varpi_s^I + \int_0^t X_s^C d\varpi_s^C + \int_0^t X_s^\beta d\tilde{J}_s\Big)\Big].
\end{aligned}$$

Define $X_s' = \mathbb{1}_{\{s\le t\}} X_s, X_s'^I = \mathbb{1}_{\{s\le t\}} X_s^I, X_s'^C = \mathbb{1}_{\{s\le t\}} X_s^C, X_s'^\beta = \mathbb{1}_{\{s\le t\}} X_s^\beta$, $\forall s \in [0,T]$, $(X_s', X_s'^I, X_s'^C, X_s'^\beta) \in \mathscr{H}^2$, we have $I_t(X) = I_T(X') \in \mathscr{M}_T^*$. Hence

$$\begin{aligned}
\mathbb{E}\Big[Y_t I_t(X)\Big] =& \mathbb{E}\Big[Y_T\Big(\int_0^t X_s dW_s + \int_0^t X_s^I d\varpi_s^I + \int_0^t X_s^C d\varpi_s^C + \int_0^t X_s^\beta d\tilde{J}_s\Big)\Big] \\
=& \mathbb{E}\Big[Y_T\Big(\int_0^T X_s' dW_s + \int_0^T X_s'^I d\varpi_s^I + \int_0^T X_s'^C d\varpi_s^C + \int_0^T X_s'^\beta d\tilde{J}_s\Big)\Big] \\
=& 0.
\end{aligned}$$

Since $Y \perp I(X)$ for any $I(X) \in \mathscr{M}^*$, we proved $Y \in (\mathscr{M}^*)^\perp$.

II. Next, we want to prove the uniqueness of this decomposition. For any given $M \in \mathscr{M}$, we assume that $I^1(X), I^2(X) \in \mathscr{M}^*$ and $Y^1, Y^2 \in (\mathscr{M}^*)^\perp$ such that $M_t = Y_t^1 + I_t^1(X) = Y_t^2 + I_t^2(X)$, for $0 \le t \le T < \infty$. Then $0 = M_t - M_t$ becomes

$$\begin{aligned}
0 &= Y_t^1 - Y_t^2 + I_t^1(X) - I_t^2(X) \\
&= Y_t^1 - Y_t^2 + \int_0^t (X_s^1 - X_s^2) dW_s + \int_0^t (X_s^{I,1} - X_s^{I,2}) d\varpi_s^I + \int_0^t (X_s^{C,1} - X_s^{C,2}) d\varpi_s^C \\
&\qquad + \int_0^t (X_s^{\beta,1} - X_s^{\beta,2}) d\tilde{J}_s,
\end{aligned}$$

which means that

$$Y_t^2 - Y_t^1 = \int_0^t (X_s^1 - X_s^2) dW_s + \int_0^t (X_s^{I,1} - X_s^{I,2}) d\varpi_s^I + \int_0^t (X_s^{C,1} - X_s^{C,2}) d\varpi_s^C + \int_0^t (X_s^{\beta,1} - X_s^{\beta,2}) d\tilde{J}_s \in \mathscr{M}^*.$$

Since $Y^2 - Y^1 \in (\mathscr{M}^*)^\perp \cap \mathscr{M}^*$, we have $Y^2 - Y^1 = 0$ within the space $(\mathscr{M}, \|\ \|)$. And $I^1(X) - I^2(X) = 0$. So the decomposition is unique in the space $(\mathscr{M}, \|\ \|)$. $\square$

Chapter 3

Backward Stochastic Differential Equations

In this chapter, we will define a general backward stochastic differential equation (BSDE), including a stochastic integral with respect to the process $\tilde{J}$. Because $\tilde{J}$ does not have the independent increments property, we cannot apply any general martingale representation theorem for Brownian motions and Lévy processes. Therefore, we will apply the martingale decomposition in Chapter 2 and some results of a fixed point problem to prove the existence and uniqueness of the solutions. Then we will rewrite the original BSDEs with the filtration $(\mathscr{F}_t)_{t\geq 0}$ to a smaller filtration and show their equivalence.

For convenience, we define several notations here.

Notation 3.0.1.

- $\mathbb{H}^2 = \{X | X : \Omega \times [0,T] \to \mathbb{R}$ *is a predictable process with* $\mathbb{E}\big[\int_0^T |X_s|^2 ds\big] < \infty\}$.
- $\mathbb{S}^2 = \{X | X : \Omega \times [0,T] \to \mathbb{R}$ *is a càdlàg adapted processes with* $\mathbb{E}\big[\sup_{s\in[0,T]} |X_s|^2\big] < \infty\}$.
- $\mathbb{M}^2 = \{M | M$ *is a martingale with respect to* $(\mathscr{F}_t)_{t\geq 0}$ *in* $\mathbb{S}^2\}$.

3.1 Construction of the General BSDEs

Let W be a Brownian Motion, N^I, N^C be two Poisson processes with parameters λ^I, λ^C and β be an alternating renewal process. Given indicator processes $H_t^i := \mathbb{1}_{\{N_t^i \geq 1\}}, i \in \{I, C\}$, then ϖ^i are the compensated processes with respect to $H^i, i \in \{I, C\}$. Define $\tau^I = \inf\{t : N_t^I = 1\}, \tau^C = \inf\{t : N_t^C = 1\}$ and $\tau = \tau^I \wedge \tau^C$. Let $\tilde{J}$ a corresponding compensated jump counting process with parameter λ^J. Assume that $W, \tilde{J}, \varpi^I$ and ϖ^C are independent and strongly orthogonal. In this section, we study a general BSDE on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathscr{F}_t = \sigma(W_s, \beta_s, N_s^I, N_s^C : s \leq t)$ as following form

$$\begin{cases} dV_t = f(\omega, t, V_{t-}, Z_t, Z_t^I, Z_t^C, Z_t^\beta)dt - Z_t dW_t - Z_t^I d\varpi_t^I - Z_t^C d\varpi_t^C - Z_t^\beta d\tilde{J}_t, \\ V_T = \theta, \end{cases} \tag{3.1.1}$$

where $0 \le t \le T < \infty, (Z, Z^I, Z^C, Z^\beta) \in \mathscr{H}^2$.

To simplify notations, we define a martingale M of a specific form and a generator $F_t(V, M)$ as follows.

Definition 3.1.1. *Define* $M_t := \int_0^t Z_s dW_s + \int_0^t Z_s^I d\varpi_t^I + \int_0^t Z_s^C d\varpi_t^C + \int_0^t Z_s^\beta d\tilde{J}_s$, *for* $(Z, Z^I, Z^C, Z^\beta) \in \mathscr{H}^2$ *and* $0 \le t \le T < \infty$.

Proposition 3.1.2. *M is a martingale with respect to* $(\mathscr{F}_t)_{t \ge 0}$.

Proof. Since $(Z, Z^I, Z^C, Z^\beta) \in \mathscr{H}^2$, we have that $Z_t, Z_t^I, Z_t^C, Z_t^\beta$ are $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$ predictable and square integrable. By Proposition 2.3.6, we have that M is a martingale. □

Definition 3.1.3. *Define a generator as a function* $F_t(V, M) : \mathscr{H}^2 \times \mathscr{M}^* \to \mathbb{S}^2$ *with* $F_t(V, M) = \int_0^t f(\omega, s, V, Z, Z^I, Z^C, Z^\beta) ds$.

For general BSDE (3.1.1), we have the following assumptions:

Assumption 3.1.4.

(i) (Lipschitz Conditon) The generator $f : \Omega \times [0, T] \times \mathbb{R}^5 \to \mathbb{R}, (\omega, t, v, z, z^I, z^C, z^\beta) \mapsto (\omega, t)$ *is predictable and Lipschitz continuous in* v, z, z^I, z^C, z^β, *i.e. for* $(v_1, z_1, z_1^I, z_1^C, z_1^\beta) \in \mathbb{R}^5$ *and* $(v_2, z_2, z_2^I, z_2^C, z_2^\beta) \in \mathbb{R}^5$, *we have*

$$\begin{aligned}&|f(\omega, t, v_1, z_1, z_1^I, z_1^C, z_1^\beta) - f(\omega, t, v_2, z_2, z_2^I, z_2^C, z_2^\beta)| \\ <&\frac{1}{5T}\left(\frac{1}{\sqrt{T}}|v_1 - v_2| + |z_1 - z_2| + \sqrt{\lambda^I}|z_1^I - z_2^I| + \sqrt{\lambda^C}|z_1^C - z_2^C| + \sqrt{\lambda^J}|z_1^\beta - z_2^\beta|\right).\end{aligned}$$

(ii) (Terminal Condition) The terminal value satisfies $\theta \in L^2(\Omega, \mathscr{F}_T, \mathbb{P})$.

(iii) (Integrability Condition) $f(\omega, t, 0, 0, 0, 0, 0) \in \mathbb{H}^2$.

Proposition 3.1.5. *The generator $F_t(V, M)$ satisfies the following inequality*

$$\|F_t(V_1, M_1) - F_t(V_2, M_2)\|_{\mathbb{S}^2} < \frac{1}{5}\Big(\|V_1 - V_2\|_{\mathbb{S}^2} + \|Z_1 - Z_2\|_{\mathbb{S}^2} + \|Z_1^I - Z_2^I\|_{\mathbb{S}^2} + \|Z_1^C - Z_2^C\|_{\mathbb{S}^2} + \|Z_1^\beta - Z_2^\beta\|_{\mathbb{S}^2}\Big).$$

Proof. Since the function f satisfies the Lipschitz condition in Assumptions 3.1.4 and $W, \tilde{J}, \varpi^I, \varpi^C$ are orthogonal, we have

$$\begin{aligned}
&\|F_t(V_1, M_1) - F_t(V_2, M_2)\|_{\mathbb{S}^2} \\
=&\mathbb{E}\Big[\sup_{0 \le t \le T}\Big|\int_0^t f(\omega, s, V_1, Z_1, Z_1^I, Z_1^C, Z_1^\beta) ds - \int_0^t f(\omega, s, V_1, Z_1, Z_1^I, Z_1^C, Z_1^\beta) ds\Big|^2\Big] \\
=&\mathbb{E}\Big[\sup_{0 \le t \le T}\Big|\int_0^t f(\omega, s, V_1, Z_1, Z_1^I, Z_1^C, Z_1^\beta) - f(\omega, s, V_1, Z_1, Z_1^I, Z_1^C, z_1^\beta) ds\Big|^2\Big] \\
\le&\mathbb{E}\Big[\sup_{0 \le t \le T} t\int_0^t \Big|f(\omega, s, V_1, Z_1, Z_1^I, Z_1^C, Z_1^\beta) - f(\omega, s, V_1, Z_1, Z_1^I, Z_1^C, Z_1^\beta)\Big|^2 ds\Big] \\
<&\mathbb{E}\Big[\sup_{0 \le t \le T}\int_0^t \Big\{\frac{1}{5}\Big(\frac{1}{\sqrt{T}}|V_1 - V_2| + |Z_1 - Z_2| + \sqrt{\lambda^I}|Z_1^I - Z_2^I| + \sqrt{\lambda^C}|Z_1^C - Z_2^C| + \sqrt{\lambda^J}|Z_1^\beta - Z_2^\beta|\Big)\Big\}^2 ds\Big] \\
\le&\mathbb{E}\Big[\sup_{0 \le t \le T}\int_0^t 5 \cdot \frac{1}{25}\Big(\frac{1}{T}|V_1 - V_2|^2 + |Z_1 - Z_2|^2 + \lambda^I|Z_1^I - Z_2^I|^2 + \lambda^C|Z_1^C - Z_2^C|^2 + \lambda^J|Z_1^\beta - Z_2^\beta|^2\Big) ds\Big] \\
=&\mathbb{E}\Big[\sup_{0 \le t \le T}\int_0^t \frac{1}{5}\Big(\frac{1}{T}|V_1 - V_2|^2 + |Z_1 - Z_2|^2 + \lambda^I|Z_1^I - Z_2^I|^2 + \lambda^C|Z_1^C - Z_2^C|^2 + \lambda^J|Z_1^\beta - Z_2^\beta|^2\Big) ds\Big]
\end{aligned}$$

$$
\begin{aligned}
\leq &\frac{1}{5T}\mathbb{E}\Big[\sup_{0\leq t\leq T}\int_0^t |V_1-V_2|^2 ds\Big]\\
&+\frac{1}{5}\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\int_0^t |Z_1-Z_2|^2 ds+\int_0^t \lambda^I|Z_1^I-Z_2^I|^2 ds+\int_0^t \lambda^C|Z_1^C-Z_2^C|^2 ds+\int_0^t \lambda^J|Z_1^\beta-Z_2^\beta|^2 ds\Big)\Big].
\end{aligned}
$$

By the properties of isometry and orthogonality, the second term is

$$
\begin{aligned}
&\frac{1}{5}\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\int_0^t |Z_1-Z_2|^2 ds+\int_0^t \lambda^I|Z_1^I-Z_2^I|^2 ds+\int_0^t \lambda^C|Z_1^C-Z_2^C|^2 ds+\int_0^t \lambda^J|Z_1^\beta-Z_2^\beta|^2 ds\Big)\Big]\\
=&\frac{1}{5}\mathbb{E}\Big[\int_0^T |Z_1-Z_2|^2 ds+\int_0^T \lambda^I|Z_1^I-Z_2^I|^2 ds+\int_0^T \lambda^C|Z_1^C-Z_2^C|^2 ds+\int_0^T \lambda^J|Z_1^\beta-Z_2^\beta|^2 ds\Big]\\
=&\frac{1}{5}\mathbb{E}\Big[\Big(\int_0^T |Z_1-Z_2|dW_s+\int_0^T |Z_1^I-Z_2^I|d\varpi_s^I+\int_0^T |Z_1^C-Z_2^C|d\varpi_s^C+\int_0^T |Z_1^\beta-Z_2^\beta|d\tilde{J}_s\Big)^2\Big]\\
\leq&\frac{1}{5}\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\int_0^t |Z_1-Z_2|dW_s+\int_0^t |Z_1^I-Z_2^I|d\varpi_s^I+\int_0^t |Z_1^C-Z_2^C|d\varpi_s^C+\int_0^t |Z_1^\beta-Z_2^\beta|d\tilde{J}_s\Big)^2\Big]\\
\leq&\frac{1}{5}\|M_1-M_2\|_{\mathbb{S}^2}
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|F_t(V_1,M_1)-F_t(V_2,M_2)\|_{\mathbb{S}^2}\leq&\frac{1}{5T}\mathbb{E}\Big[\int_0^T |V_1-V_2|^2 ds\Big]+\frac{1}{5}\|M_1-M_2\|_{\mathbb{S}^2}\\
\leq&\frac{1}{5T}\mathbb{E}\Big[T\sup_{0\leq t\leq T}|V_1-V_2|^2\Big]+\frac{1}{5}\|M_1-M_2\|_{\mathbb{S}^2}\\
=&\frac{1}{5}\mathbb{E}\Big[\sup_{0\leq t\leq T}|V_1-V_2|^2\Big]+\frac{1}{5}\|M_1-M_2\|_{\mathbb{S}^2}\\
=&\frac{1}{5}\Big(\|V_1-V_2\|_{\mathbb{S}^2}+\|M_1-M_2\|_{\mathbb{S}^2}\Big).
\end{aligned}
$$

So, we proved this proposition. □

3.2 Existence and Uniqueness of Solution

In this section, we will prove the existence and uniqueness of the solutions of the general BSDE. For BSDEs, we can prove the existences of solutions of a BSDE by proving the existence of the solutions for a corresponding fixed point problem, more details in Cheridito and Nam (2017). Since we have a stochastic process with nonindependent increments property, we also need the martingale decomposition theorem to prove the existence and uniqueness.

Theorem 3.2.1. *If the BSDE* (3.1.1) *satisfies the Assumption 3.1.4, then the BSDE* (3.1.1) *admits a unique solutions* $(V,Z,Z^I,Z^C,Z^\beta)\in\mathbb{S}^2\times\mathscr{M}$.

Proof. By Proposition 3.1.5, the generator $F_t(V,M)$ satisfies

$$
\|F_t(V_1,M_1)-F_t(V_2,M_2)\|_{\mathbb{S}^2}<\frac{1}{5}\Big(\|V_1-V_2\|_{\mathbb{S}^2}+\|M_1-M_2\|_{\mathbb{S}^2}\Big).
$$

Since the generator satisfies the terminal condition and integrability condition, by the theorem 3.1 in Cheridito and Nam (2017), the BSDE (3.1.1) admits a unique solution $(V, M) \in \mathbb{S}^2 \times \mathscr{M}$. (Compared with the notation $F_t(k)(V, M)$ in Cheridito and Nam (2017), we only need $F_t(k)(V, M) :\equiv F_t(V, M)$ in our case.)

Since $M \in \mathscr{M}$ is a martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$, by Theorem 2.3.11, we can rewrite $M \in \mathscr{M}$ as

$$M_T - M_t = \int_t^T Z_s dW_s + \int_t^T Z_s^I d\varpi_t^I + \int_t^T Z_s^C d\varpi_t^C + \int_t^T Z_s^\beta d\tilde{J}_s + Y_T - Y_t.$$

With the definition of the generator $F_t(V, M)$, the unique solution $(V, Z, Z^I, Z^C, Z^\beta, Y) \in \mathbb{S}^2 \times \mathscr{M}$ is represented as the form

$$V_t = \xi + \int_t^T f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta) ds + \int_t^T Z_s dW_s + \int_t^T Z_s^I d\varpi_t^I + \int_t^T Z_s^C d\varpi_t^C + \int_t^T Z_s^\beta d\tilde{J}_s + Y_T - Y_t.$$

Based on the form of the general BSDE (3.1.1) and the uniqueness of its solution, the orthogonal term $Y_T - Y_t \equiv 0$. Therefore, the unique solution of the general BSDE (3.1.1) is

$$V_t = \xi + \int_t^T f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta) ds + \int_t^T Z_s dW_s + \int_t^T Z_s^I d\varpi_t^I + \int_t^T Z_s^C d\varpi_t^C + \int_t^T Z_s^\beta d\tilde{J}_s.$$

□

3.3 Reduced Backward Stochastic Differential Equations

Based on BSDE (3.1.1), we assume that the terminal condition V_τ depends on the two stopping time τ^I and τ^C are

$$\begin{cases} -dV_t = f(\omega, t, V_{t-}, Z_t, Z_t^I, Z_t^C, Z_t^\beta) dt - Z_t dW_t - Z_t^I d\varpi_t^I - Z_t^C d\varpi_t^C - Z_t^\beta d\tilde{J}_t, \\ \quad V_\tau = \theta_\tau^I \mathbb{1}_{\{\tau < \tau^C \wedge T\}} + \theta_\tau^C \mathbb{1}_{\{\tau < \tau^I \wedge T\}}, \end{cases} \tag{3.3.1}$$

where $0 \leq T < \infty, (Z, Z^I, Z^C, Z^\beta) \in \mathscr{H}^2$ with Assumptions 3.1.4, $\theta_t^I \in \mathscr{F}_t$ and $\theta_t^C \in \mathscr{F}_t$. In this section, we want to rewrite the general BSDE (3.1.1) with a terminal condition at τ to a BSDE in a smaller filtration $(\mathscr{F}^{W,\beta})_{t\geq 0}, \mathscr{F}_t^{W,\beta} = \sigma(W_s, \beta_s : s \leq t)$.

To prove the existence and uniqueness of the solution to a BSDE, we can consider a corresponding martingale problem, which means the sum of stochastic integrals with respect to Brownian motion, ϖ^I, ϖ^C and $\tilde{J}$ is a local martingale. Crépey and Song (2015) reduce a BSDE to a BSDE within a smaller filtration. Based on the same method, we want to simplify BSDE (3.3.1) in the filtration $(\mathscr{F}_t)_{t\geq 0}$ with a jump terminal condition to a reduced BSDEs within the filtration $(\mathscr{F}_t^{W,\beta})_{t\geq 0}$ with a continuous terminal condition.

First, we consider a corresponding martingale problem to the BSDE (3.3.1)

$$\begin{cases} M_t^V = V_{t\wedge\tau} + \displaystyle\int_0^{t\wedge\tau} f(\omega, s, V_{t-}, Z_t, Z_t^I, Z_t^C, Z_t^\beta) ds \text{ is an } (\mathscr{F}_t)_{t\geq 0} \text{ local martingale}, \\ V_\tau = \theta_\tau^I \mathbb{1}_{\{\tau < \tau^C \wedge T\}} + \theta_\tau^C \mathbb{1}_{\{\tau < \tau^I \wedge T\}}. \end{cases} \tag{3.3.2}$$

We want to show that if V is a solution of BSDE (3.3.1), then M_t^V is a local martingale. Assume that V is the solution to BSDE (3.3.1), then M_t^V is

$$
\begin{aligned}
M_t^V &= V_{t\wedge\tau} + \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta)ds \\
&= \Big(V_\tau + \int_{t\wedge\tau}^{\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta)ds - \int_{t\wedge\tau}^{\tau} Z_s dW_t - \int_{t\wedge\tau}^{\tau} Z_s^I d\varpi_s^I - \int_{t\wedge\tau}^{\tau} Z_s^C d\varpi_s^C - \int_{t\wedge\tau}^{\tau} Z_s^\beta d\tilde{J}_s\Big) \\
&\quad + \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta)ds \\
&= V_\tau + \int_0^{\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta)ds - \int_{t\wedge\tau}^{\tau} Z_s dW_t - \int_{t\wedge\tau}^{\tau} Z_s^I d\varpi_s^I - \int_{t\wedge\tau}^{\tau} Z_s^C d\varpi_s^C - \int_{t\wedge\tau}^{\tau} Z_s^\beta d\tilde{J}_s \\
&= \Big(V_0 - \int_0^{\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta)ds + \int_0^{\tau} Z_s dW_s + \int_0^{\tau} Z_s^I d\varpi_s^I + \int_0^{\tau} Z_s^C d\varpi_s^C + \int_0^{\tau} Z_s^\beta d\tilde{J}_s\Big) \\
&\quad + \int_{0\wedge\tau}^{\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta)ds - \int_{t\wedge\tau}^{\tau} Z_s dW_t - \int_{t\wedge\tau}^{\tau} Z_s^I d\varpi_s^I - \int_{t\wedge\tau}^{\tau} Z_s^C d\varpi_s^C - \int_{t\wedge\tau}^{\tau} Z_s^\beta d\tilde{J}_s \\
&= V_0 + \int_0^{t\wedge\tau} Z_s dW_s + \int_0^{t\wedge\tau} Z_s^I d\varpi_s^I + \int_0^{t\wedge\tau} Z_s^C d\varpi_s^C + \int_0^{t\wedge\tau} Z_s^\beta d\tilde{J}_s,
\end{aligned}
$$

is a local martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. To prove that M_t^V is an $(\mathscr{F}_t)_{t\geq 0}$ local martingale gives V is a solution to BSDE (3.3.1), we need Theorem 2.3.11. We define

$$
\begin{aligned}
M_t^\bullet = &\Big(\theta_\tau^I \mathbb{1}_{\{\tau<\tau^C\wedge T\}} + \theta_\tau^C \mathbb{1}_{\{\tau<\tau^I\wedge T\}} - V_{\tau-}\Big)\mathbb{1}_{\{t\geq\tau\}} \\
&- \int_0^{t-} \Big(\theta_s^I \mathbb{1}_{\{s<\tau^C\wedge T\}} - V_{s-}\Big)(\lambda^I \mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) \\
&- \int_0^{t-} \Big(\theta_s^C \mathbb{1}_{\{s<\tau^I\wedge T\}} - V_{s-}\Big)(\lambda^C \mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}}) \\
= &\Big(\theta_\tau^I \mathbb{1}_{\{\tau<\tau^C\wedge T\}} + \theta_\tau^C \mathbb{1}_{\{\tau<\tau^I\wedge T\}} - V_{\tau-}\Big)\mathbb{1}_{\{t\geq\tau\}} - \int_0^t \Big(\theta_s^I \mathbb{1}_{\{s<\tau^C\wedge T\}} - V_{s-}\Big)\lambda^I \mathbb{1}_{\{s<\tau^I\}}ds \\
&- \int_0^t \Big(\theta_s^C \mathbb{1}_{\{s<\tau^I\wedge T\}} - V_{s-}\Big)\lambda^C \mathbb{1}_{\{s<\tau^C\}}ds + V_{t-}\mathbb{1}_{\{t\geq T\}}.
\end{aligned}
$$

$$M_t^\circ = M_t^V - M_t^\bullet.$$

For $M^\bullet$ we have

$$
\begin{aligned}
M^\bullet = &\Big(\theta_\tau^I \mathbb{1}_{\{\tau<\tau^C\wedge T\}}\mathbb{1}_{\{t\geq\tau\}} - \int_0^t \theta_s^I \mathbb{1}_{\{s<\tau^C\wedge T\}}\lambda^I \mathbb{1}_{\{s<\tau^I\}}ds\Big) \\
&+ \Big(\theta_\tau^C \mathbb{1}_{\{\tau<\tau^I\wedge T\}}\mathbb{1}_{\{t\geq\tau\}} - \int_0^t \theta_s^C \mathbb{1}_{\{s<\tau^I\wedge T\}}\lambda^C \mathbb{1}_{\{s<\tau^C\}}ds\Big) \\
&- \Big(V_{\tau-}\mathbb{1}_{\{T>t\geq\tau\}} - \int_0^t V_{s-}\lambda^I \mathbb{1}_{\{s<\tau^I\}}ds - \int_0^t V_{s-}\lambda^C \mathbb{1}_{\{s<\tau^C\}}ds\Big).
\end{aligned}
$$

Here the intensity of $\mathbb{1}_{\{T>t\geq\tau\}}$ is $\lambda^I \mathbb{1}_{\{s<\tau^I\}} + \lambda^C \mathbb{1}_{\{s<\tau^C\}}$, so $V_{\tau-}\mathbb{1}_{\{T>t\geq\tau\}} - \int_0^t V_{s-}\lambda^I \mathbb{1}_{\{s<\tau^I\}}ds - \int_0^t V_{s-}\lambda^C \mathbb{1}_{\{s<\tau^C\}}ds$ is a local martingale. The other two terms are all local martingales. Thus, $M^\bullet$ is a local martingale. Because M° is a difference of two local martingales, M° is also a local martingale.

We develop a martingale problem and its corresponding BSDEs with a terminal condition 0.

$$\begin{cases} M_t^U = U_{t\wedge\tau} + \int_0^{t\wedge\tau} f(\omega,s,V_{s-},Z_s,Z_s^I,Z_s^C,Z_s^\beta)ds + \int_0^{t\wedge\tau}\theta_s^I\mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) \\ \qquad + \int_0^{t\wedge\tau}\theta_s^C\mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}}) \text{ is an } (\mathscr{F}_t)_{t\geq0} \text{ local martingale,} \\ U_\tau = 0. \end{cases} \tag{3.3.3}$$

Its corresponding BSDEs is

$$\begin{cases} -dU_t = f(\omega,t,V_{t-},Z_t,Z_t^I,Z_t^C,Z_t^\beta)dt - Z_tdW_t - Z_t^Id\varpi_t^I - Z_t^Cd\varpi_t^C - Z_t^\beta d\tilde{J}_t \\ \qquad + \theta_t^I\mathbb{1}_{\{t<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{t<\tau^I\}}dt + d\mathbb{1}_{\{s\geq T>\tau^I\}}) + \theta_t^C\mathbb{1}_{\{t<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{t<\tau^C\}}dt + d\mathbb{1}_{\{s\geq T>\tau^C\}}), \\ U_\tau = 0. \end{cases} \tag{3.3.4}$$

Theorem 3.3.1. *If V is a solution to the martingale problem* (3.3.2)*, then $U_t := V_t\mathbb{1}_{\{t<\tau\}}$ and U is a solution to the martingale problem* (3.3.3)*. Conversely, if U is a solution to the martingale problem* (3.3.3)*, then*

$$V_t := U_t\mathbb{1}_{\{t<\tau\}} + \theta_\tau^I\mathbb{1}_{\{\tau<\tau^C\wedge T\}}\mathbb{1}_{\{t\geq\tau\}} + \theta_\tau^C\mathbb{1}_{\{\tau<\tau^I\wedge T\}}\mathbb{1}_{\{t\geq\tau\}},$$

and V is a solution to the martingale problem (3.3.2)*.*

Proof. Assume that V is a solution to the martingale problem (3.3.2). Since $(\mathbb{1}_{\{t<\tau\}})_- := \lim_{s\uparrow t}\mathbb{1}_{\{s<\tau\}} = \mathbb{1}_{\{t\leq\tau\}}\mathbb{1}_{\{0<\tau\}}$, we have

$$\begin{aligned} V_t\mathbb{1}_{\{t<\tau\}} &= V_0\mathbb{1}_{\{0<\tau\}} + \int_0^t V_{s-}d\mathbb{1}_{\{s<\tau\}} + \int_0^t(\mathbb{1}_{\{t<\tau\}})_-dV_s \\ &= V_0\mathbb{1}_{\{0<\tau\}} + \int_0^t V_{s-}d\mathbb{1}_{\{s<\tau\}} + \int_0^t\mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}dV_s \\ &= V_0\mathbb{1}_{\{0<\tau\}} + \int_0^t V_{s-}d\mathbb{1}_{\{s<\tau\}} + \int_0^t\mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}\Big(dM_s^V - f(\omega,s,V_{s-},Z_s,Z_s^I,Z_s^C,Z_s^\beta)ds\Big) \\ &= V_0\mathbb{1}_{\{0<\tau\}} + \int_0^t V_{s-}d\mathbb{1}_{\{s<\tau\}} + \int_0^t\mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}dM_s^V \\ &\quad - \int_0^t\mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}f(\omega,s,V_{s-},Z_s,Z_s^I,Z_s^C,Z_s^\beta)ds \\ &= V_0\mathbb{1}_{\{0<\tau\}} + \int_0^t V_{s-}d\mathbb{1}_{\{s<\tau\}} + (M_{t\wedge\tau}^V - M_0^V) - \int_0^t\mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}f(\omega,s,V_{s-},Z_s,Z_s^I,Z_s^C,Z_s^\beta)ds \\ &= V_0\mathbb{1}_{\{0<\tau\}} + \int_0^t V_{s-}d\mathbb{1}_{\{s<\tau\}} + (M_{t\wedge\tau}^\circ - M_0^\circ) + (M_{t\wedge\tau}^\bullet - M_0^\bullet) \\ &\quad - \int_0^t\mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}f(\omega,s,V_{s-},Z_s,Z_s^I,Z_s^C,Z_s^\beta)ds \end{aligned}$$

$$
\begin{aligned}
&= V_0\mathbb{1}_{\{0<\tau\}} + \int_0^t V_{s-}d\mathbb{1}_{\{s<\tau\}} + (M^\circ_{t\wedge\tau} - M^\circ_0) + \Big(\theta^I_\tau\mathbb{1}_{\{\tau<\tau^C\wedge T\}} + \theta^C_\tau\mathbb{1}_{\{\tau<\tau^I\wedge T\}} - V_{\tau-}\Big)\mathbb{1}_{\{t\geq\tau\}}\\
&\quad - \int_0^t \Big(\theta^I_s\mathbb{1}_{\{s<\tau^C\wedge T\}} - V_{s-}\Big)(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}})\\
&\quad - \int_0^t \Big(\theta^C_s\mathbb{1}_{\{s<\tau^I\wedge T\}} - V_{s-}\Big)(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\\
&\quad - \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds\\
&= V_0\mathbb{1}_{\{0<\tau\}} + (M^\circ_{t\wedge\tau} - M^\circ_0) - \int_0^{t\wedge\tau} V_{s-}(d\mathbb{1}_{\{s\geq\tau\}} - \lambda^I\mathbb{1}_{\{s<\tau^I\}}ds - \lambda^C\mathbb{1}_{\{s<\tau^C\}}ds - d\mathbb{1}_{\{s\geq T\}})\\
&\quad - \int_0^t \theta^I_s\mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) - \int_0^t \theta^C_s\mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\\
&\quad - \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds.
\end{aligned}
$$

Let $U_t = V_t\mathbb{1}_{\{t<\tau\}}$, plugging the above equation into (3.3.3), we have

$$
\begin{aligned}
M^U_t &= U_{t\wedge\tau} + \int_0^{t\wedge\tau} f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds + \int_0^{t\wedge\tau}\theta^I_s\mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}})\\
&\quad + \int_0^{t\wedge\tau}\theta^C_s\mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\\
&= V_0\mathbb{1}_{\{0<\tau\}} + (M^\circ_{t\wedge\tau} - M^\circ_0) - \int_0^{t\wedge\tau} V_{s-}(d\mathbb{1}_{\{s\geq\tau\}} - \lambda^I\mathbb{1}_{\{s<\tau^I\}}ds - \lambda^C\mathbb{1}_{\{s<\tau^C\}}ds - d\mathbb{1}_{\{s\geq T\}})\\
&\quad - \int_0^{t\wedge\tau}\theta^I_s\mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}})\\
&\quad - \int_0^{t\wedge\tau}\theta^C_s\mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\\
&\quad - \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds + \int_0^{t\wedge\tau} f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds\\
&\quad + \int_0^{t\wedge\tau}\theta^I_s\mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}})\\
&\quad + \int_0^{t\wedge\tau}\theta^C_s\mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\\
&= V_0\mathbb{1}_{\{0<\tau\}} + (M^\circ_{t\wedge\tau} - M^\circ_0) - \int_0^{t\wedge\tau} V_{s-}(d\mathbb{1}_{\{T>s\geq\tau\}} - \lambda^I\mathbb{1}_{\{s<\tau^I\}}ds - \lambda^C\mathbb{1}_{\{s<\tau^C\}}ds),
\end{aligned}
$$

is a local martingale. And the terminal condition $U_\tau = V_\tau\mathbb{1}_{\{\tau<\tau\}} = 0$. So we proved that V is a solution to the martingale problem (3.3.2), implies that U is a solution to the martingale problem (3.3.3).

Conversely, we denote

$$
\begin{aligned}
M^*_t &= \Big(\theta^I_\tau\mathbb{1}_{\{\tau<\tau^C\wedge T\}} + \theta^C_\tau\mathbb{1}_{\{\tau<\tau^I\wedge T\}}\Big)\mathbb{1}_{\{t\geq\tau\}} - \int_0^{t-}\theta^I_s\mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}})\\
&\quad - \int_0^{t-}\theta^C_s\mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}}),
\end{aligned}
$$

which is a local martingale. Assume that U is a solution to the martingale problem (3.3.3), we have

$$
\begin{aligned}
&U_t\mathbb{1}_{\{t<\tau\}} + \theta^I_\tau \mathbb{1}_{\{\tau<\tau^C\wedge T\}}\mathbb{1}_{\{t\geq\tau\}} + \theta^C_\tau \mathbb{1}_{\{\tau<\tau^I\wedge T\}}\mathbb{1}_{\{t\geq\tau\}}\\
=&U_0\mathbb{1}_{\{0<\tau\}} + \int_0^t (\mathbb{1}_{\{s<\tau\}})_- dU_s + \int_0^t U_{s-}d\mathbb{1}_{\{s<\tau\}} + \theta^I_\tau \mathbb{1}_{\{\tau<\tau^C\wedge T\}}\mathbb{1}_{\{t\geq\tau\}} + \theta^C_\tau \mathbb{1}_{\{\tau<\tau^I\wedge T\}}\mathbb{1}_{\{t\geq\tau\}}\\
=&U_0\mathbb{1}_{\{0<\tau\}} + \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}dU_s - \int_0^t U_{s-}d\mathbb{1}_{\{s\geq\tau\}} + M^*_t\\
&\quad + \int_0^{t-} \theta^I_s \mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) + \int_0^{t-} \theta^C_s \mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\\
=&U_0\mathbb{1}_{\{0<\tau\}} + \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}\Big(dM^U_s - f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds\\
&\quad - \theta^I_s \mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) - \theta^C_s \mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\Big)\\
&\quad - \int_0^t U_{s-}d\mathbb{1}_{\{s\geq\tau\}} + M^*_t + \int_0^{t-} \xi^I_s \mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds\\
&\quad + d\mathbb{1}_{\{s\geq T>\tau^I\}}) + \int_0^{t-} \theta^C_s \mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\\
=&U_0\mathbb{1}_{\{0<\tau\}} + \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}dM^U_s - \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}}f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds - U_\tau d\mathbb{1}_{\{t\geq\tau\}} + M^*_t\\
=&U_0\mathbb{1}_{\{0<\tau\}} + M^U_{t\wedge\tau} - \int_0^{t\wedge\tau} f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds + M^*_t.
\end{aligned}
$$

Let $V_t = U_t\mathbb{1}_{\{t<\tau\}} + \theta^I_\tau \mathbb{1}_{\{\tau<\tau^C\wedge T\}}\mathbb{1}_{\{t\geq\tau\}} + \theta^C_\tau \mathbb{1}_{\{\tau<\tau^I\wedge T\}}\mathbb{1}_{\{t\geq\tau\}}$, plugging into (3.3.2). Since M^U and M^* are local martingales, we proved that V_t is a local martingale. We proved that U is a solution to the martingale problem (3.3.3) implies that V is a solution to the martingale problem (3.3.2). □

Next we want to rewrite this BSDE to a BSDE with terminal condition without jumps. The corresponding martingale problem is the following:

$$
\begin{cases}
\quad M^{\bar{U}}_t = \bar{U}_{t\wedge\tau-} + \int_0^{t\wedge\tau} f(\omega,s,V_{s-},Z_s,Z^I_s,Z^C_s,Z^\beta_s)ds\\
\qquad + \int_0^{t\wedge\tau}\Big(\theta^I_s\mathbb{1}_{\{s<\tau^C\wedge T\}} - \bar{U}_{s-}\Big)(\lambda^I\mathbb{1}_{\{s<\tau^I\}}ds + d\mathbb{1}_{\{s\geq T>\tau^I\}})\\
\qquad + \int_0^{t\wedge\tau}\Big(\theta^C_s\mathbb{1}_{\{s<\tau^I\wedge T\}} - \bar{U}_{s-}\Big)(\lambda^C\mathbb{1}_{\{s<\tau^C\}}ds + d\mathbb{1}_{\{s\geq T>\tau^C\}})\\
\qquad\qquad \text{is an } (\mathscr{F}_t)_{t\geq 0} \text{ local martingale,}\\
\bar{U}_{T-}\mathbb{1}_{\{T=\tau\}} = 0
\end{cases}
\tag{3.3.5}
$$

The corresponding BSDE is

$$
\begin{cases}
\quad -d\bar{U}_t = f(\omega,t,\bar{U}_{t-},Z_t,Z^I_t,Z^C_t,Z^\beta_t)dt - Z_tdW_t - Z^I_td\varpi^I_t - Z^C_td\varpi^C_t - Z^\beta_td\tilde{J}_t\\
\qquad + \Big(\theta^I_t\mathbb{1}_{\{t<\tau^C\wedge T\}} - \bar{U}_{t-}\Big)(\lambda^I\mathbb{1}_{\{t<\tau^I\}}dt + d\mathbb{1}_{\{s\geq T>\tau^I\}})\\
\qquad + \Big(\theta^C_t\mathbb{1}_{\{t<\tau^I\wedge T\}} - \bar{U}_{t-}\Big)(\lambda^C\mathbb{1}_{\{t<\tau^C\}}dt + d\mathbb{1}_{\{s\geq T>\tau^C\}}),\\
\bar{U}_{T-}\mathbb{1}_{\{T=\tau\}} = 0
\end{cases}
\tag{3.3.6}
$$

Theorem 3.3.2. *If U is a solution to the martingale problem (3.3.3), then $\bar{U}_t := U_t \mathbb{1}_{\{t<\tau\}} + 2U_{t-}\mathbb{1}_{\{t\geq\tau\}}$ and $\bar{U}_t$ is a solution to the martingale problem (3.3.5). Conversely, if $\bar{U}$ is a solution to the martingale problem (3.3.5), then $U_t := \bar{U}_t \mathbb{1}_{\{t<\tau\}}$ and U_t is a solution to the martingale problem (3.3.3).*

Proof. Assume that U is a solution to the martingale problem (3.3.3), then

$$\begin{aligned}
\bar{U}_t =& U_t \mathbb{1}_{\{t<\tau\}} + 2U_{\tau-}\mathbb{1}_{\{t\geq\tau\}} \\
=& U_0 \mathbb{1}_{\{0<\tau\}} + \int_0^t U_{s-} d\mathbb{1}_{\{s<\tau\}} + \int_0^t (\mathbb{1}_{\{s\leq\tau\}})_- dU_s + 2U_{\tau-}\mathbb{1}_{\{t\geq\tau\}} \\
=& U_0 \mathbb{1}_{\{0<\tau\}} + \int_0^t U_{s-} d\mathbb{1}_{\{s<\tau\}} + \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}} dU_s + 2U_{\tau-}\mathbb{1}_{\{t\geq\tau\}} \\
=& -\int_0^t U_{s-} d\mathbb{1}_{\{s\geq\tau\}} + U_{t\wedge\tau} + 2U_{\tau-}\mathbb{1}_{\{t\geq\tau\}} \\
=& \int_0^t U_{s-} d\mathbb{1}_{\{s\geq\tau\}} + U_{t\wedge\tau} \\
=& \int_0^t U_{s-} d\mathbb{1}_{\{s\geq\tau\}} + M_t^U - \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta) ds \\
& - \int_0^{t\wedge\tau} \theta_s^I \mathbb{1}_{\{s<\tau^C\wedge T\}} (\lambda^I \mathbb{1}_{\{s<\tau^I\}} ds \\
& + d\mathbb{1}_{\{s\geq T>\tau^I\}}) - \int_0^{t\wedge\tau} \theta_s^C \mathbb{1}_{\{s<\tau^I\wedge T\}} (\lambda^C \mathbb{1}_{\{s<\tau^C\}} ds + d\mathbb{1}_{\{s\geq T>\tau^C\}}) \\
=& M_t^U - \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta) ds \\
& - \int_0^{t\wedge\tau} \Big(\theta_s^I \mathbb{1}_{\{s<\tau^C\wedge T\}} - U_{s-}\mathbb{1}_{\{s-<\tau\}}\Big)(\lambda^I \mathbb{1}_{\{s<\tau^I\}} ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) \\
& - \int_0^{t\wedge\tau} \Big(\theta_s^C \mathbb{1}_{\{s<\tau^I\wedge T\}} - U_{s-}\mathbb{1}_{\{s-<\tau\}}\Big)(\lambda^C \mathbb{1}_{\{s<\tau^C\}} ds + d\mathbb{1}_{\{s\geq T>\tau^C\}}) \\
& + \int_0^t U_{s-}(d\mathbb{1}_{\{T>s\geq\tau\}} - \lambda^I \mathbb{1}_{\{s<\tau^I\}} ds - \lambda^C \mathbb{1}_{\{s<\tau^C\}} ds),
\end{aligned}$$

where M^U and the last term are local martingales. Plugging $\bar{U}_t$ into (3.3.5), we have that $M_t^{\bar{U}}$ is a local martingale. By the martingale property, we get $\mathbb{E}[M_T^{\bar{U}} - M_{T-}^{\bar{U}} | \mathscr{F}_{\tau-}] = 0$. So, $\bar{U}_{T-}\mathbb{1}_{\{T=\tau\}} = 0$. So we proved that if U is a solution to the martingale problem (3.3.3), then $\bar{U}_t$ is a solution to the martingale problem (3.3.5).

Next, we assume that $\bar{U}$ is a solution to the martingale problem (3.3.5), then

$$\begin{aligned}
\bar{U}_t \mathbb{1}_{\{t<\tau\}} =& \bar{U}_{t\wedge\tau-} \mathbb{1}_{\{t<\tau\}} \\
=& \bar{U}_{0\wedge\tau-} \mathbb{1}_{\{0<\tau\}} + \int_0^t \bar{U}_{s-\wedge\tau} d\mathbb{1}_{\{s<\tau\}} + \int_0^t (\mathbb{1}_{\{s\leq\tau\}})_- d\bar{U}_{s\wedge\tau-} \\
=& \bar{U}_{0\wedge\tau-} \mathbb{1}_{\{0<\tau\}} + \int_0^t \bar{U}_{s-\wedge\tau} d\mathbb{1}_{\{s<\tau\}} + \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{1}_{\{0<\tau\}} d\bar{U}_{s\wedge\tau-} \\
=& -\int_0^t \bar{U}_{s-\wedge\tau} d\mathbb{1}_{\{s\geq\tau\}} + \bar{U}_{t\wedge\tau-}
\end{aligned}$$

$$
\begin{aligned}
=& -\int_0^t \bar{U}_{s-\wedge\tau} d\mathbb{1}_{\{s\geq\tau\}} + M_t^{\bar{U}} - \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta) ds \\
&\quad - \int_0^{t\wedge\tau} \Big(\theta_s^I \mathbb{1}_{\{s<\tau^C\wedge T\}} - \bar{U}_{s-}\Big)(\lambda^I \mathbb{1}_{\{s<\tau^I\}} ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) \\
&\quad - \int_0^{t\wedge\tau} \Big(\theta_s^C \mathbb{1}_{\{s<\tau^I\wedge T\}} - \bar{U}_{s-}\Big)(\lambda^C \mathbb{1}_{\{s<\tau^C\}} ds + d\mathbb{1}_{\{s\geq T>\tau^C\}}) \\
=& M_t^{\bar{U}} - \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta) ds - \int_0^{t\wedge\tau} \theta_s^I \mathbb{1}_{\{s<\tau^C\wedge T\}}(\lambda^I \mathbb{1}_{\{s<\tau^I\}} ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) \\
&\quad - \int_0^{t\wedge\tau} \theta_s^C \mathbb{1}_{\{s<\tau^I\wedge T\}}(\lambda^C \mathbb{1}_{\{s<\tau^C\}} ds + d\mathbb{1}_{\{s\geq T>\tau^C\}}) \\
&\quad - \int_0^t \bar{U}_{s-\wedge\tau}(d\mathbb{1}_{\{T>s\geq\tau\}} - \lambda^I \mathbb{1}_{\{s<\tau^I\}} ds - \lambda^C \mathbb{1}_{\{s<\tau^C\}} ds),
\end{aligned}
$$

where $M^{\bar{U}}$ and the last term are local martingales. Let $U_t = \bar{U}_t \mathbb{1}_{\{t<\tau\}}$, plugging into (3.3.3), we have that M_t^U is a local martingale. Hence, this converse direction is proved. □

Now we show how to rewrite a BSDE to a smaller filtration $(\mathscr{F}_t^{W,\beta})_{0\leq t\leq T}$. The corresponding martingale problem is the following:

$$
\begin{cases}
M_t^{\breve{U}} = \breve{U}_{t\wedge\tau-} + \displaystyle\int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, \theta_s^I - \breve{U}_s, \theta_s^C - \breve{U}_s, Z_s^\beta) ds + \int_0^{t\wedge\tau} \Big(\theta_s^I - \bar{U}_{s-}\Big)\lambda^I ds \\
\qquad + \displaystyle\int_0^{t\wedge\tau} \Big(\theta_s^C - \bar{U}_{s-}\Big)\lambda^C ds \qquad \text{is an } (\mathscr{F}_t^{W,\beta})_{t\geq 0} \text{ local martingale}, \\
\breve{U}_{T-} = 0.
\end{cases} \tag{3.3.7}
$$

The corresponding BSDE is

$$
\begin{cases}
-d\breve{U}_t = f(\omega, t, \breve{U}_{t-}, Z_t, \xi_t^I - \breve{U}_t, \theta_t^C - \breve{U}_t, Z_t^\beta) dt + \Big(\xi_t^I - \bar{U}_{t-}\Big)\lambda^I dt + \Big(\theta_t^C - \bar{U}_{t-}\Big)\lambda^C dt \\
\qquad\qquad - Z_t dW_t - Z_t^\beta d\tilde{J}_t, \\
\breve{U}_{T-} = 0.
\end{cases} \tag{3.3.8}
$$

Theorem 3.3.3. *If $\bar{U}$ is a solution to the martingale problem* (3.3.5)*, then $\breve{U}_t := \mathbb{E}[\bar{U}_{t\wedge\tau-}|\mathscr{F}_\tau]$ and $\breve{U}$ is a solution to the martingale problem* (3.3.7)*. Conversely, if $\breve{U}$ is a solution to the martingale problem* (3.3.7)*, then $\bar{U}_t := \breve{U}_{t\wedge\tau-}$ and $\bar{U}$ is a solution to the martingale problem* (3.3.5)*.*

Proof. Assume that $\bar{U}$ is a solution to the martingale problem (3.3.5). For $t < \tau \wedge T$, we have

$$
\begin{aligned}
\bar{U}_{t\wedge\tau-} =& M_t^{\bar{U}} - \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta) ds \\
&- \int_0^{t\wedge\tau} \Big(\theta_s^I \mathbb{1}_{\{s<\tau^C\wedge T\}} - \bar{U}_{s-}\Big)(\lambda^I \mathbb{1}_{\{s<\tau^I\}} ds + d\mathbb{1}_{\{s\geq T>\tau^I\}}) \\
&- \int_0^{t\wedge\tau} \Big(\theta_s^C \mathbb{1}_{\{s<\tau^I\wedge T\}} - \bar{U}_{s-}\Big)(\lambda^C \mathbb{1}_{\{s<\tau^C\}} ds + d\mathbb{1}_{\{s\geq T>\tau^C\}}) \\
=& M_t^{\bar{U}} - \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z_s^I, Z_s^C, Z_s^\beta) ds - \int_0^{t\wedge\tau} \Big(\theta_s^I - \bar{U}_{s-}\Big)\lambda^I ds \\
&- \int_0^{t\wedge\tau} \Big(\theta_s^C - \bar{U}_{s-}\Big)\lambda^C ds.
\end{aligned}
$$

Let $\breve{U}_t = \mathbb{E}[\bar{U}_{t\wedge\tau-}|\mathscr{F}_\tau] = \bar{U}_{t\wedge\tau-}$, $Z^I_t = (\xi^I_t - \breve{U}_t)$ and $Z^C_t = (\xi^C_t - \breve{U}_t)$, plugging the above equation into $M^{\breve{U}}_t$, we have

$$
\begin{aligned}
M^{\breve{U}}_t &= \breve{U}_{t\wedge\tau-} + \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, \theta^I_s - \breve{U}_s, \theta^C_s - \breve{U}_s, Z^\beta_s)ds + \int_0^{t\wedge\tau} \Big(\theta^I_s - \bar{U}_{s-}\Big)\lambda^I ds \\
&\quad + \int_0^{t\wedge\tau} \Big(\theta^C_s - \bar{U}_{s-}\Big)\lambda^C ds \\
&= M^{\bar{U}}_t - \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, Z^I_s, Z^C_s, Z^\beta_s)ds - \int_0^{t\wedge\tau} \Big(\theta^I_s - \bar{U}_{s-}\Big)\lambda^I ds \\
&\quad - \int_0^{t\wedge\tau} \Big(\theta^C_s - \bar{U}_{s-}\Big)\lambda^C ds + \int_0^{t\wedge\tau} f(\omega, s, V_{s-}, Z_s, \xi^I_s - \breve{U}_s, \xi^C_s - \breve{U}_s, Z^\beta_s)ds \\
&\quad + \int_0^{t\wedge\tau} \Big(\theta^I_s - \bar{U}_{s-}\Big)\lambda^I ds + \int_0^{t\wedge\tau} \Big(\theta^C_s - \bar{U}_{s-}\Big)\lambda^C ds \\
&= M^{\bar{U}}_t,
\end{aligned}
$$

which is $(\mathscr{F}^{W,\beta}_t)_{t\geq 0}$ measurable. So $M^{\breve{U}}$ is an $(\mathscr{F}^{W,\beta}_t)_{t\geq 0}$ martingale.

Conversely, assume that $\breve{U}$ is a solution to the martingale problem (3.3.7). Since the filtration $(\mathscr{F}^{W,\beta}_t)_{t\geq 0} \subset (\mathscr{F}_t)_{t\geq 0}$ and W, β, N^I, N^C are independent and strongly orthogonal, we have $\bar{U}_t = \breve{U}_{t\wedge\tau-}$ is a solution to the martingale problem (3.3.5). $\square$

Chapter 4

Arbitrage-free Pricing

In this chapter, we price a European option of a stock and calculate the total valuation adjustment (XVA), considering credit risk, asymmetric interest rates, and differential financial states.

4.1 Sale and Repurchase Agreement (Repo) Market

Before we study the switching between differential financial statuses, we review the Sale and Repurchase Agreement (Repo) market and its history. In this section, we analyze different performances during a calm financial market and a financial crisis period as well as its influence to the stock market.

4.1.1 Background

A Sale and Repurchase Agreement (Repo) is the sale of a security combined with an agreement to repurchase the same security at a specified price at the end of the contract. This transaction can be also viewed as a collateralized loan. Over the last 40 years, the size of the Repo market increased dramatically. From 2002 to 2007, its capital size even doubled. To better understand reasons of the popularity and developments of the Repo market in the U.S., we look back at the Great Depression.

The traditional banking system attracts depositors, keeps deposits on their balance sheets and lend loans to the other commercial debtors. The depositors save their money in the traditional banks and get interest rates in return. At the same time, the depositors have a right to withdraw their money at any time. This right is underwritten by the government and applicable to all depositors, including cash-rich depositors. During the Great Depression, some banks failed on their promises. At that time, the public did not have enough information to be able to judge the financial health of many banks. One isolated default cases of one bank led to a terrible panic nationally. All depositors ran to banks and withdrew every cent from their accounts. The more withdrawals, the less cash on the banks balance sheets. The depositors' panics and low beliefs in the financial system led to more bankruptcies. The deteriorating situation of the banking system led to an even more panic and even poorer confidence among depositors. As a result of this adverse loop, Americans experienced the Great Depression in 1930s. More details in Gorton et al. (2010), Gorton and Metrick (2012).

Learning from the Great Depression, the government legislated that companies and banks make public reports on their profits, loans and essential economic information. To protect depositors, the government provides deposition insurances to retail depositors. However, this insurance does not cover the loss of non-retail depositors, including mutual funds, cash-rich companies and sovereign wealth funds. But cash-rich companies need to lend or invest their wealth with some protections in short term. After the Great Depression, more and more companies started to lend and borrow their money in a securitized banking system, i.e. the Repo markets.

4.1.2 Structure of Repo markets

One of the important secularized banking system is the sale and repurchase agreement (Repo) market. A contract in the Repo market specifies two transactions. At the beginning, one party sells a specific security to the counterparty at a given price. At the end of the contract, the party repurchases the same security from its counterparty at the agreed price, which was decided by two parties during the contract's negotiation. Here, the specific security can be seen as a collateral in a collateralized borrowing transaction. The collateral provider is also a cash receiver, and the collateral receiver is also a cash lender. In this terminology, the above transactions can also be explained in another way. At the initial time, the cash provider (collateral receiver) lends m dollar to its counterparty (cash receiver, collateral provider). At the same time, the collateral provider (cash receiver) gives a security as collateral to the cash provider (collateral receiver). At the maturity time, the cash receiver (collateral provider) returns the $m+r$ dollars to the cash provider (collateral receiver). At the same time, the collateral receiver (cash provider) returns the collateral to the collateral provider (cash receiver).

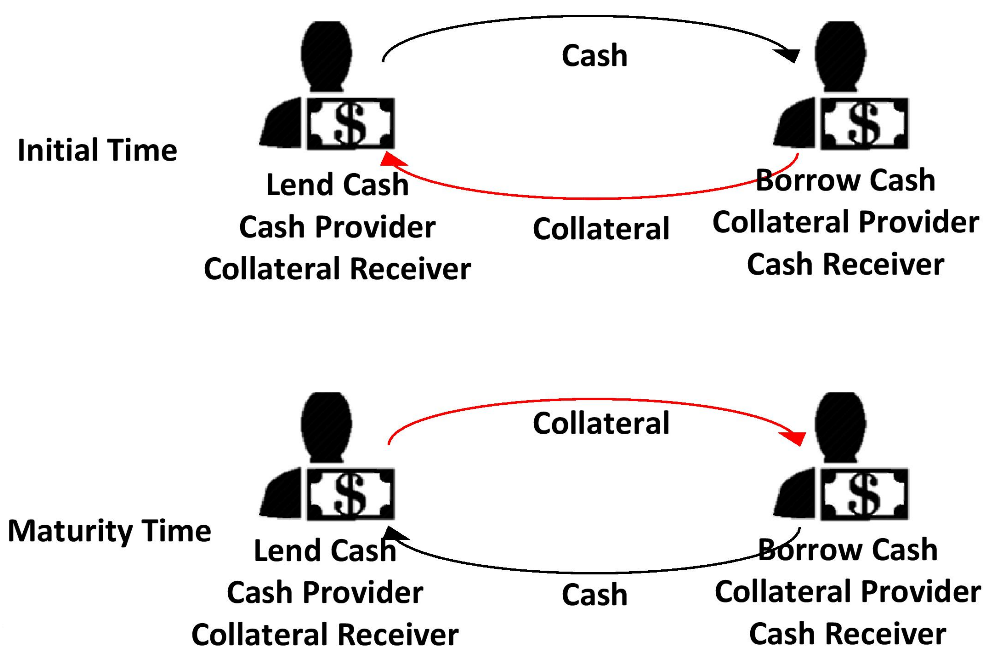


Figure 4.1: Transactions in Repo Market.

In this transaction, a relative difference between the two cash flows $\frac{(m+r-m)}{m} = \frac{r}{m}$ is called a Repo rate. Usually, the market value of the collateral is larger than the cash transaction. For example, when borrowing m dollars, one needs to provide a collateral with a market price as $m+h$ dollars. The relative difference between the market price of the collateral and the cash lent is called the haircut, $\frac{m+h-m}{m} = \frac{h}{m}$. Based on different confidences played in the collateral, the haircut varies from 0.5% to over 8%. In the U.S. Repo market, a group of safe collaterals is called the general collateral, it includes, i.e. 10 years U.S. treasury bonds.

There are three types of Repo markets – Bilateral Repo, Triparty Repo and Hold-in-custody Repo. The Bilateral Repo has been already introduced in the beginning of this section. In the Triparty Repo, there is an agent between two parties in the transactions in the Repo market. The contract is prepared by the agent and the collateral is held by the agent during the lifetime of the contract. A Hold-in-custody Repo can be a Bilateral Repo or a Triparty Repo, but the collateral is held on the balance sheet of the collateral provider during the lifetime of the contract.

As we mentioned before, many companies use the Repo market as a source to borrow money. This is called a cash driven Repo activity. On the other hand, there are many companies that use the Repo market as a source to borrow a specific security to meet their liquidity requirements. This is called a security driven Repo activity. To attract collateral providers of some special securities, the Repo rate can be even a negative value, which means the collateral providers will earn a profit by this Repo activity.

4.1.3 Performance in Sub-prime Financial Crisis

During the 2008 financial crisis, most traders lost their confidence on all collateral except the general collateral. As a result, only contracts with the general collaterals was traded at that time. Although using the general collateral was welcome, traders holding the general collateral were unwilling to lend them to other traders. Some general collateral receivers defaulted and rejected to return the general collateral. As a result, the trading in contracts based on the general collateral also froze. As a result, we assume that there was no contracts at Repo markets during a financial crisis. More details in Gorton and Metrick (2012).

Since the Repo market is an important source of illiquid securities, many traders conduct their short sells of stocks by borrowing the stocks at the Repo market and then selling them in the stock market. Based on the frozen situation of the U.S. Repo markets, the short-selling of stocks in the stock market was affected. Without this source of stocks, short-selling trades also disappeared. In fact, more than 90% of the contracts in the Repo market are short-term contracts, normally for only one day. Thus, most short stock trades ceased during the financial crisis. More information about liquidity problem can be found at Brunnermeier and Pedersen (2008).

4.2 The Status Process

There are many indicators to measure or forecast an incoming financial crisis, such as CISS, VIX, conditional Value-at-Risk (CoVaR), CDS index, and the Ted spread, where proposed. Boudt et al. (2013) used a two-status model to describe the market and funding liquidity problem. In their paper, they use statistical methods to divide the performance of the financial market into two periods, a calm financial period and a financial crisis. They propose a threshold of 48 basis points of the TED spread. In Chapter 2, we already defined an alternating renewal process. In this dissertation, we apply an alternating renewal process to describe this switching between a normal financial status and a financial crisis status.

When $\beta_t = 0$, financial markets are in a calm period. When $\beta_t = 1$, financial markets are in a financial crisis period. Based on the definition of the alternating renewal processes, the average holding time of a calm status follows an exponential distribution with parameter λ_U and the average holding time of a financial crisis follows an exponential distribution with parameter λ_V. Then we use this alternating renewal process to distinguish the performances of the Repo market and the stock market in a calm financial period and a financial crisis.

4.3 Financial Assets

We consider a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, which is rich enough to provide all necessary information, containing information about stock prices, default of risky bonds and switching between a normal financial status and a financial crisis status. The probability $\mathbb{P}$ is the physical probability measure. The basic model follows Bichuch et al. (2018a). The main difference is that we consider different performances of the Repo account and stock account during different financial statuses.

4.3.1 Repo Account

The Repo account is the main source of cash for stock purchase by investors. We assume that Repo rates are different for cash lending and cash borrowing. For cash lenders, they receive constant interest rate r_r^+ from Repo markets. For cash borrowers, they pay constant interest rate r_r^- to Repo markets, with general collateral and implementing long positions.

Let ψ_t be a number of shares of a Repo account, then the Repo rate is

$$r_r(\psi) = r_r^- \mathbb{1}_{\{\psi<0\}} + r_r^+ \mathbb{1}_{\{\psi>0\}}.$$

In a normal financial status, we represent the Repo accounts as $B^{r_r^-}$ and $B^{r_r^+}$ for the borrowers and lenders, respectively. The dynamics of the Repo account is $dB_t^{r_r^\pm} = r_r^\pm B_t^{r_r^\pm} dt$. So, the value of the Repo account is

$$B_t^{r_r} = B_t^{r_r}(\psi) = \exp\Big(\int_0^t r_r(\psi_s) ds \Big).$$

But, Repo markets froze during the financial crisis. Thus, we represent the Repo account

$$(1-\beta_t) B_t^{r_r} = (1-\beta_t) \exp\left(\int_0^t r_r(\psi_s) ds \right),$$

reflecting there is no activity of the Repo market during the financial crisis. The dynamics of the Repo account is

$$(1-\beta_t) dB_t^{r_r^\pm} = (1-\beta_t) r_r^\pm B_t^{r_r^\pm} dt.$$

4.3.2 Stock Security

We assume that $W^{\mathbb{P}}$ is a standard Brownian motion under the physical measure $\mathbb{P}$. A stock price follows a geometric Brownian motion with the dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}},$$

where μ is a constant drift rate and σ is a constant volatility. We assume that the initial price of the stock is S_0.

Let ξ_t be a number of shares of stocks in the stock account. In a normal financial status ($\beta_t = 0$), we have that the value of the stock account is $\xi_t S_t$.

In this dissertation, all short trades of stocks are done by borrowing a specific stock from Repo markets, and then sell them in the stock markets. During the financial crisis, the U.S. Repo markets froze. So no short-selling transactions can be conducted during a financial crisis. In the financial crisis status ($\beta_t = 1$), the stock account is $\mathbb{1}_{\{\xi \geq 0\}} \xi_t S_t$.

We summarize the value of the stock account as

$$(1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}}) \xi_t S_t.$$

Here the term $(1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}})$ is the adjustment, corresponding to the frozen short trades in a financial crisis.

4.3.3 Risky Bond Securities

We denote an investor as "I" and its counterparty as "C". Let $\tau_i, i \in \{I, C\}$ be the default times of an investor and a counterparty, respectively. We assume that default times follow exponential distributions with constant intensities $h_i^{\mathbb{P}}, i \in \{I, C\}$. We use $H_t^i = \mathbb{1}_{\{\tau_i \leq t\}}, t \geq 0, i \in \{I, C\}$ to denote the default indicator processes of risky bonds, underwritten by the investor and the counterparty, respectively. The default times follow an exponential distribution with parameters $r^i + h_i, i \in \{I, C\}$, respectively. So, the two risky bond prices P_t^I, P_t^C have the following dynamics

$$dP_t^i = (r^i + h_i^{\mathbb{P}}) P_t^i dt - P_{t-}^i dH_t^i, \quad P_0^i = \exp(-(r^i + h_i^{\mathbb{P}})T),$$

where $r^i + h_i^{\mathbb{P}}, i \in \{I, C\}$ are constant return rates, respectively.

4.3.4 Collateral Account

The collateral account is to protect one party from the counterparty's default. The collateral process $C := (C_t : t \geq 0)$ is a stochastic process with constant collateral rates. We assume that collateral providers and receivers' interest rates are different. For collateral receivers ($C_t < 0$), they pay a constant interest rate r_c^-. For collateral providers ($C_t > 0$), they receive a constant interest rate r_c^+. We represent the collateral cash account as $B^{r_c^{\pm}}$ by representing collateral interest rates as

$$r_c(c) = r_c^- \mathbb{1}_{\{c<0\}} + r_c^+ \mathbb{1}_{\{0<c\}}.$$

The collateral cash account is

$$B_t^{r_c} := B_t^{r_c}(C) = \exp\left(\int_0^t r_c(C_s) ds \right).$$

Based on this definition, its dynamics is

$$dB_t^{r_c^{\pm}} = r_c^{\pm} B_t^{r_c^{\pm}} dt.$$

Denote a number of shares of the collateral account $B_t^{r_c}$ by ψ_t^c, we have

$$\psi_t^c B_t^{r_c} = -C_t.$$

This means, a collateral receiver ($C_t < 0$) should purchase shares of the collateral account, and vice versa. Assume that $\hat{V}$ be a third party valuation of the European claim, the collateral account is assigned different collateralization levels based on the safe levels of the counterparty. This is modeled as

$$C_t := \alpha \hat{V},$$

where α is the collateralization level. When the counterparty is very reliable, for example the U.S. government, the collateralization level $\alpha = 0$. When the counterparty has low credit rating, the collateralization level could be $\alpha = 1$.

4.3.5 Funding Account

We assume that the lenders' and the borrowers' interest rates are different. For cash lenders, they receive constant interest rates r_f^+ from the treasury desk. For cash borrowers, they pay constant interest rates r_f^- to funding desks. We represent the funding cash accounts as $B^{r_f^{\pm}}$ by representing the funding interest rates as

$$r_f := r_f(\xi) = r_f^- \mathbb{1}_{\{\xi<0\}} + r_f^+ \mathbb{1}_{\{\xi>0\}}.$$

Let ξ_t^f be a number of shares in the funding account at time t. The funding account is

$$B_t^{r_f} := B_t^{r_f}(\xi^f) = \exp\left(\int_0^t r_f(\xi_s^f) ds\right),$$

with dynamics

$$dB_t^{r_f^{\pm}} = r_f^{\pm} B_t^{r_f^{\pm}} dt.$$

4.4 Hedging Portfolio and Arbitrage-Free Pricing

In our model, we want to price an option, considering an investor and its counterparty's default, liquidity problems in the Repo market and the stock markets, and asymmetric interest rates. We assume that the whole Repo market and short stock trades freeze during the financial crisis. We take the default of an investor and its counterparty into account, by including the default of the investor's and a counterparty's bonds in our hedging portfolio.

Assumption 4.4.1. *In a normal financial status, we have that*

$$\psi_t^r B_t^{r_r} = -\xi_t S_t,$$

which means the stock account is financed by the Repo market.

This is a reasonable assumption. Because borrowing in the Repo market is a collateralized borrowing trade, the borrowing Repo rate r_r^- is less than the uncolleralized funding rate r_f^- in general.

However, in the financial crisis status ($\beta_t = 1$), Repo markets freeze. Without a source to borrow stocks, all short trades of stocks freeze too. Meanwhile, an investor has to borrow money from the funding desk to buy stocks at a more expensive funding rate. Considering both: the normal financial status and the financial crisis status, we summarize the relationship of all asset accounts as

$$(1-\beta_t)\psi_t^r B_t^{r_r} + \beta_t \xi_t^f B_t^{r_f} = \beta_t V_t + \beta_t \psi_t^c B_t^{r_c} - (1-\beta_t \mathbb{1}_{\{\xi_t<0\}})\xi_t S_t - \beta_t \xi_t^I P_t^I - \beta_t \xi_t^C P_t^C. \quad (4.4.1)$$

4.4.1 Valuation Measure

For two default indicator processes $H_i^{\mathbb{P}}, i \in \{I, C\}$ with constant parameters $h_i^{\mathbb{P}}, i \in \{I, C\}$, we have

$$\varpi_t^{i,\mathbb{P}} = H_t^i - \int_0^t (1 - H_u^i) h_i^{\mathbb{P}} du,$$

is a $(\mathscr{F}_t)_{t\geq 0}$ martingale.

By the Radon–Nikodym derivative, we define the valuation measure $\mathbb{Q}$ with respect to a discount rate r_D as

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = &\exp\left(\frac{r_D - \mu}{\sigma} W_t^{\mathbb{P}} - \frac{(r_D - \mu)^2}{2\sigma^2} t\right) \\ &\left(1 + \frac{r^I - r_D}{h_I^{\mathbb{P}}}\right)^{H_t^I} \exp((r_D - r^I)t) \left(1 + \frac{r^C - r_D}{h_C^{\mathbb{P}}}\right)^{H_t^C} \exp((r_D - r^C)t).\end{aligned}$$

We denote $\mu_I = r^I + h_I^{\mathbb{P}}$ and $\mu_C = r^C + h_C^{\mathbb{P}}$, which are return rates of risky bonds, underwritten by the investor and counterparty, respectively. The above Radon-Nikodym derivative becomes

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = &\exp\left(\frac{r_D - \mu}{\sigma} W_t^{\mathbb{P}} - \frac{(r_D - \mu)^2}{2\sigma^2} t\right) \left(\frac{\mu^I - r_D}{h_I^{\mathbb{P}}}\right)^{H_t^I} \\ &\exp((r_D - \mu_I + h_I^{\mathbb{P}})t) \left(\frac{\mu^C - r_D}{h_C^{\mathbb{P}}}\right)^{H_t^C} \exp((r_D - \mu^C + h_C^{\mathbb{P}})t).\end{aligned}$$

Under the valuation measure $\mathbb{Q}$, the dynamics of three risky assets are

$$\begin{aligned} dS_t &= r_D S_t dt + \sigma S_t dW_t^{\mathbb{Q}}, \\ dP_t^I &= r_D P_t^I dt - P_{t-}^I d\varpi_t^{I,\mathbb{Q}}, \\ dP_t^C &= r_D P_t^C dt - P_{t-}^C d\varpi_t^{C,\mathbb{Q}}, \end{aligned} \tag{4.4.2}$$

where $W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} - \frac{r_D - \mu}{\sigma} t$ is a Brownian Motion under $\mathbb{Q}$ and $\varpi_t^{i,\mathbb{Q}} = \varpi_t^{i,\mathbb{P}} + \int_0^t (1 - H_u^i)(h_i^{\mathbb{P}} - h_i^{\mathbb{Q}}) du, i \in \{I, C\}$ are $((\mathscr{F}_t)_{t\geq 0}, \mathbb{Q})$ martingales. Here $h_i^{\mathbb{Q}} = \mu_i - r_D \geq 0$ are default intensities of default indicator processes under $\mathbb{Q}$.

4.4.2 Wealth Process

Let $\varphi := (\xi_t, \xi_t^f, \xi_t^I, \xi_t^C, \psi_t^r, \psi_t^c; t \geq 0)$ be our investment strategy, where ξ_t denotes shares of stocks, ξ_t^f denotes shares of the funding account, $\xi_t^i, i \in \{I, C\}$ denotes shares of risky bonds, underwritten by the investor an its counterparty, respectively, ψ_t^r denotes shares of the Repo account, and ψ_t^c denotes shares of the collateral account. Based on previous definitions of all assets, we have the hedging portfolio as

$$V_t(\varphi) = (1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}}) \xi_t S_t + \xi_t^I P_t^I + \xi_t^C P_t^C + \xi_t^f B_t^{r_f} + (1 - \beta_t) \psi_t^r B_t^{r_r} - \psi_t^c B_t^{r_c}. \tag{4.4.3}$$

Its dynamics is

$$dV_t(\varphi) = (1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}}) \xi_t dS_t + \xi_t^I dP_t^I + \xi_t^C dP_t^C + \xi_t^f dB_t^{r_f} + (1 - \beta_t) \psi_t^r dB_t^{r_r} - \psi_t^c dB_t^{r_c}.$$

Definition 4.4.2. *An investment strategy is self-financing if for any $t \in [0,T]$, the following identity*

$$\begin{aligned} V_t(\varphi) = V_0(\varphi) + \int_0^t (1-\beta_t \mathbb{1}_{\{\xi_t<0\}})\xi_t dS_t + \int_0^t \xi_t^I dP_t^I + \int_0^t \xi_t^C dP_t^C \\ + \int_0^t \xi_t^f dB_t^{r_f} + \int_0^t (1-\beta_t)\psi_t^r dB_t^{r_r} - \int_0^t \psi_t^c dB_t^{r_c}, \end{aligned}$$

where $V_0(\varphi)$ is the initial capital.

4.4.3 Closeout Valuation

In this dissertation, we assume that the collateral account is liquid at the default time of both the investor and its counterparty. We do not consider the fire sales and liquidity problems of the collateral account. Let $\tau = \tau^I \wedge \tau^C \wedge T$, where $\tau^I = \inf\{t : H_t^I = 1\}$ is a default stopping time for investors and $\tau^C = \inf\{t : H_t^C = 1\}$ is a default stopping time for the counterparty. Let $0 \leq L_I, L_C \leq 1$ be loss rates of investors and counterparties at their default time, respectively. Let $\hat{V}$ be a third party valuation of the hedging portfolio. We have a terminal condition of our hedging portfolio

$$\begin{aligned} \theta(\tau, \hat{V}) :=& \hat{V}_\tau + \mathbb{1}_{\{\tau^C<\tau^I\}} L_C Y^- - \mathbb{1}_{\{\tau^I<\tau^C\}} L_I Y^+ \\ =& \mathbb{1}_{\{\tau^I<\tau^C\}} \theta_I(\hat{V}_\tau) + \mathbb{1}_{\{\tau^C<\tau^I\}} \theta_C(\hat{V}_\tau), \end{aligned}$$

where $Y := \hat{V}_\tau - C_\tau = (1-\alpha)\hat{V}_\tau$ is the value of the claim at default and $\theta_I(v) := v - L_I((1-\alpha)v)^+, \theta_C(v) := v + L_C((1-\alpha)v)^-$. Here $(\cdot)^+ = \max(0, (\cdot))$ and $(\cdot)^- = \max(0, -(\cdot))$.

Here the term $\mathbb{1}_{\{\tau^C<\tau^I\}} L_C Y^-$ is the credit valuation adjustment term after collateral mitigation and the term $\mathbb{1}_{\{\tau^I<\tau^C\}} L_I Y^+$ is the debit valuation adjustment term.

4.5 Arbitrage-Free Pricing

In this section, we will discuss arbitrage-free financial markets and its required assumptions. In this section, we denote the initial capital of our portfolio as $x \geq 0$.

4.5.1 Arbitrage-free Assumptions

Definition 4.5.1 (Arbitrage)**.** *The market admits an investor's arbitrage, if there exists a investment strategy $\varphi = (\xi_t, \xi_t^I, \xi_t^C, \xi_t^f, \psi_t^r, \psi_t^c; t \geq 0)$ such that*

$$\mathbb{P}[V_t(\varphi, x) \geq \exp(r_f^+ t)x] = 1, \quad \mathbb{P}[V_t(\varphi, x) > \exp(r_f^+ t)x] > 0,$$

for a given initial capital $x \geq 0$ and a corresponding wealth process $V(\varphi, x)$.

Definition 4.5.2 (Arbitrage-free Financial Markets)**.** *If a financial market does not admit an investor's arbitrage for any initial capital $x \geq 0$, the market is arbitrage free from the investor's perspective.*

Assumption 4.5.3. *Necessary Assumptions of Arbitrage-free Financial Markets*

1. $r_f^+ \leq r_f^-$,

2. $r_f^+ \vee r_D < \mu_I \wedge \mu_C$,

3. $(1-\beta_t)r_r^+ \leq (1-\beta_t)r_f^-$ *(i.e.* $r_r^+ \leq r_f^-$ *in a normal financial status).*

Remark 4.5.4. *These three assumptions are necessary to exclude an arbitrage potentiality.*

1. *If* $r_f^+ > r_f^-$, *one can borrow cash from the funding desk at a funding rate* r_f^-, *and then lend it to the funding desk at the funding rate* r_f^+. *There is a positive arbitrage profit of* $r_f^+ - r_f^- > 0$ *multiplies the amount of cash.*

2. *If* $r_f^+ > \mu_I$ *(or* $r_f^+ > \mu_C$*), one can short sell an investor's (or a counterparty's) risky bond with an expected return rate* μ_I *(or* μ_C*), and then lend the money to the funding desk, earning an arbitrage profit* $r_f^+ - \mu_I > 0$ *(or* $r_f^+ - \mu_C > 0$*) multiplies the shares of the bonds.*

   If $r_D > \mu_I$ *(or* $r_D > \mu_C$*) and the investor can trade by the interest rate* r_D *(or* r_D*), the discussion is similar to the cases* $r_f^+ > \mu_I$ *(or* $r_f^+ > \mu_C$*). If the investor cannot trade by* r_D *(or* r_D*), then the investor's default intensity is* $h_I^{\mathbb{Q}} = \mu_I - r_D < 0$ *(or* $h_C^{\mathbb{Q}} = \mu_C - r_D < 0$*) under the discount measure* $\mathbb{Q}$*, which is not realistic.*

3. *In a normal financial status* ($\beta_t = 0$)*, if* $r_r^+ > r_f^-$, *one can borrow cash from the funding desk at the funding rate* r_f^- *and lend it to the Repo market at the Repo rate* r_r^+, *earning a positive arbitrage profit* $r_r^+ - r_f^- > 0$ *multiplies the amount of cash. In a financial crisis status* ($\beta_t = 1$)*, the inequality holds trivially.*

Proposition 4.5.5. *Suppose that Assumption 4.5.3 holds. A financial market is arbitrage-free if*

$$(1-\beta_t)r_r^+ \leq (1-\beta_t)r_f^+ \leq (1-\beta_t)r_r^-,$$

which means $r_r^+ \leq r_f^+ \leq r_r^-$ *in the normal financial status.*

Proof. Suppose the inequality $(1-\beta_t)r_r^+ \leq (1-\beta_t)r_f^+ \leq (1-\beta_t)r_r^-$ holds. By the definition of the alternating renewal process β and the setting of a Repo rate, we have

$$\begin{aligned}(1-\beta_t)r_r\psi_t^r =&(1-\beta_t)r_r^-\psi_t^r\mathbb{1}_{\{\psi_t^r<0\}} + (1-\beta_t)r_r^+\psi_t^r\mathbb{1}_{\{\psi_t^r>0\}}\\ \leq&(1-\beta_t)r_f^+\psi_t^r\mathbb{1}_{\{\psi_t^r<0\}} + (1-\beta_t)r_f^+\psi_t^r\mathbb{1}_{\{\psi_t^r>0\}}\\ =&(1-\beta_t)r_f^+\psi_t^r.\end{aligned}$$

Similarly, for the funding account we have

$$\begin{aligned}r_f\xi_t^f =&r_f^-\xi_t^f\mathbb{1}_{\{\xi_t^f<0\}} + r_f^+\xi_t^f\mathbb{1}_{\{\xi_t^f>0\}}\\ \leq&r_f^+\xi_t^f\mathbb{1}_{\{\xi_t^f<0\}} + r_f^+\xi_t^f\mathbb{1}_{\{\xi_t^f>0\}}\\ =&r_f^+\xi_t^f.\end{aligned}$$

We will rewrite the wealth process under a suitable measure $\tilde{\mathbb{P}}$ with a discount rate r_f^+, defined by the Radon-Nikodym derivative

$$
\begin{aligned}
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} &= \exp\left(\frac{r_f^+ - \mu}{\sigma} W_t^{\mathbb{P}} - \frac{(r_f^+ - \mu)^2}{2\sigma^2} t\right) \\
&\quad \left(1 + \frac{r^I - r_f^+}{h_I^{\mathbb{P}}}\right)^{H_t^I} \exp((r_f^+ - r^I)t) \left(1 + \frac{r^C - r_f^+}{h_C^{\mathbb{P}}}\right)^{H_t^C} \exp((r_f^+ - r^C)t) \\
&= \exp\left(\frac{r_f^+ - \mu}{\sigma} W_t^{\mathbb{P}} - \frac{(r_f^+ - \mu)^2}{2\sigma^2} t\right) \left(\frac{\mu^I - r_f^+}{h_I^{\mathbb{P}}}\right)^{H_t^I} \\
&\quad \exp((r_f^+ - \mu_I + h_I^{\mathbb{P}})t) \left(\frac{\mu^C - r_f^+}{h_C^{\mathbb{P}}}\right)^{H_t^C} \exp((r_f^+ - \mu^C + h_C^{\mathbb{P}})t),
\end{aligned}
$$

where $\mu^i, i \in \{I, C\}$ are return rates of risky bonds, underwritten by an investor and its counterparty, respectively.

By Girsanov's theorem, we rewrite the dynamics of the risky assets under $\tilde{\mathbb{P}}$ as

$$
\begin{aligned}
dS_t &= r_f^+ S_t dt + \sigma S_t dW_t^{\tilde{\mathbb{P}}}, \\
dP_t^I &= r_f^+ P_t^I dt - P_{t-}^I d\varpi_t^{I,\tilde{\mathbb{P}}}, \\
dP_t^C &= r_f^+ P_t^C dt - P_{t-}^C d\varpi_t^{C,\tilde{\mathbb{P}}},
\end{aligned}
$$

where $W_t^{\tilde{\mathbb{P}}} = W_t^{\mathbb{P}} - \frac{r_f^+ - \mu}{\sigma} t$ is a Brownian Motion under $\tilde{\mathbb{P}}$ and $\varpi_t^{i,\tilde{\mathbb{P}}} = \varpi_t^{i,\mathbb{P}} + \int_0^t (1 - H_u^i)(h_i^{\mathbb{P}} - h_i^{\tilde{\mathbb{P}}}) du, i \in \{I, C\}$ are $((\mathscr{F}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ martingales. Here $h_t^{\tilde{\mathbb{P}}} = \mu_i - r_f^+, i \in \{I, C\}$ are the default intensities under $\tilde{\mathbb{P}}$.

We denote a hedging portfolio in the underlying market by $\check{V}_t$ ($C_t = 0$), and its dynamics under $\tilde{\mathbb{P}}$ is given by

$$
\begin{aligned}
d\check{V}_t &= \left(\xi_t^f r_f B_t^{r_f} + (1 - \beta_t)\psi_t^r r_r B_t^{r_r}\right) dt + (1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}})\xi_t dS_t + \xi_t^I dP_t^I + \xi_t^C dP_t^C \\
&= \left((1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}}) r_f^+ \xi_t S_t + r_f^+ \xi_t^I P_t^I + r_f^+ \xi_t^C P_t^C + r_f \xi_t^f B_t^{r_f} + (1 - \beta_t) r_r \psi_t^{r_r} B_t^{r_r}\right) dt \\
&\quad + (1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}})\sigma \xi_t S_t dW_t^{\tilde{\mathbb{P}}} - \xi_t^I P_{t-}^I d\varpi_t^{I,\tilde{\mathbb{P}}} - \xi_t^C P_{t-}^C d\varpi_t^{C,\tilde{\mathbb{P}}}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\check{V}_t(\varphi, x) - \check{V}_0(\varphi, x) &= \int_0^t \left((1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}}) r_f^+ \xi_s S_s + r_f^+ \xi_s^I P_s^I + r_f^+ \xi_s^C P_s^C + r_f \xi_s^f B_s^{r_f} + (1 - \beta_s) r_r \psi_s^r B_s^{r_r}\right) dt \\
&\quad + \int_0^t (1 - \beta_s \mathbb{1}_{\{\xi_s < 0\}})\sigma \xi_s S_s dW_s^{\tilde{\mathbb{P}}} - \int_0^t \xi_s^I P_{s-}^I d\varpi_s^{I,\tilde{\mathbb{P}}} - \int_0^t \xi_s^C P_{s-}^C d\varpi_s^{C,\tilde{\mathbb{P}}} \\
&\leq \int_0^t \left((1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}}) r_f^+ \xi_s S_s + r_f^+ \xi_s^I P_s^I + r_f^+ \xi_s^C P_s^C + r_f^+ \xi_s^f B_s^{r_f} + (1 - \beta_s) r_f^+ \psi_s^r B_s^{r_r}\right) dt \\
&\quad + \int_0^t (1 - \beta_s \mathbb{1}_{\{\xi_s < 0\}})\sigma \xi_s S_s dW_s^{\tilde{\mathbb{P}}} - \int_0^t \xi_s^I P_{s-}^I d\varpi_s^{I,\tilde{\mathbb{P}}} - \int_0^t \xi_s^C P_{s-}^C d\varpi_s^{C,\tilde{\mathbb{P}}}
\end{aligned}
$$

$$= \int_0^t r_f^+ \check{V}_s(\varphi, x)ds + \int_0^t (1 - \beta_s \mathbb{1}_{\{\xi_s<0\}})\sigma \xi_s S_s dW_s^{\tilde{\mathbb{P}}} - \int_0^t \xi_s^I P_{s-}^I d\varpi_s^{I,\tilde{\mathbb{P}}} - \int_0^t \xi_s^C P_{s-}^C d\varpi_s^{C,\tilde{\mathbb{P}}}.$$

So we have for the discounted $\check{V}_t$

$$e^{-r_f^+ t}\check{V}_t(\varphi, x) - \check{V}_0(\varphi, x) \leq \int_0^t (1 - \beta_s \mathbb{1}_{\{\xi_s<0\}})\sigma \xi_s S_s dW_s^{\tilde{\mathbb{P}}} - \int_0^t \xi_s^I P_{s-}^I d\varpi_s^{I,\tilde{\mathbb{P}}} - \int_0^t \xi_s^C P_{s-}^C d\varpi_s^{C,\tilde{\mathbb{P}}}.$$

Since $|1 - \beta_t \mathbb{1}_{\{\xi_t<0\}}| \leq 1$, we have the right hand side of the above equation is a local martingale bounded from below, which is a supermartingale. By the property of the supermartingale, taking expectations,

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-r_f^+ t}\check{V}_t(\varphi, x) - \check{V}_0(\varphi, x)] \leq & \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^t (1 - \beta_s \mathbb{1}_{\{\xi_s<0\}})\sigma \xi_s S_s dW_s^{\tilde{\mathbb{P}}} - \int_0^t \xi_s^I P_{s-}^I d\varpi_s^{I,\tilde{\mathbb{P}}} - \int_0^t \xi_s^C P_{s-}^C d\varpi_s^{C,\tilde{\mathbb{P}}}\Big] \\
\leq & \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^0 (1 - \beta_s \mathbb{1}_{\{\xi_s<0\}})\sigma \xi_s S_s dW_s^{\tilde{\mathbb{P}}} - \int_0^0 \xi_s^I P_{s-}^I d\varpi_s^{I,\tilde{\mathbb{P}}} - \int_0^0 \xi_s^C P_{s-}^C d\varpi_s^{C,\tilde{\mathbb{P}}}\Big] \\
= & 0.
\end{aligned}$$

Thus, we have either $\tilde{\mathbb{P}}[\check{V}_t(\varphi, x) = e^{r_f^+ t}x] = 1$ or $\tilde{\mathbb{P}}[\check{V}_t(\varphi, x) < e^{r_f^+ t}x] > 0$. We proved that no arbitrage investment opportunity exists under the measure $\tilde{\mathbb{P}}$. Since $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, we proved that the financial market is arbitrage-free.

□

Remark 4.5.6. *If an investor knows the information of the financial status (β), this financial market is still arbitrage-free, based on the previous proposition.*

4.5.2 Dynamics of The Wealth Process

By Equations (4.4.3) and (4.4.2) and the self-financing condition, the dynamics of the hedging portfolio V_t under the valuation measure $\mathbb{Q}$ with the discount rate r_D is

$$\begin{aligned}
dV_t = & (1 - \beta_t \mathbb{1}_{\{\xi_t<0\}})\xi_t dS_t + \xi_t^I dP_t^I + \xi_t^C dP_t^C + \xi_t^f dB_t^{r_f} + (1 - \beta_t)\psi_t^{r_r} dB_t^{r_r} - \psi_t^c dB_t^{r_c} \\
= & (1 - \beta_t \mathbb{1}_{\{\xi_t<0\}})\xi_t(r_D S_t dt + \sigma S_t dW_t^{\mathbb{Q}}) + \xi_t^I(r_D P_t^I dt - P_{t-}^I d\varpi_t^{I,\mathbb{Q}}) \\
& + \xi_t^C(r_D P_t^C dt - P_{t-}^C d\varpi_t^{C,\mathbb{Q}}) + \xi_t^f r_f B_t^{r_f} dt + (1 - \beta_t)\psi_t^{r_r} r_r B_t^{r_r} dt - \psi_t^c r_c B_t^{r_c} dt \\
= & \Big((1 - \beta_t \mathbb{1}_{\{\xi_t<0\}})r_D \xi_t S_t + r_D \xi_t^I P_t^I + r_D \xi_t^C P_t^C + r_f \xi_t^f B_t^{r_f} + (1 - \beta_t)r_r \psi_t^{r_r} B_t^{r_r} - r_c \psi_t^c B_t^{r_c}\Big)dt \\
& + (1 - \beta_t \mathbb{1}_{\{\xi_t<0\}})\sigma \xi_t S_t dW_t^{\mathbb{Q}} - \xi_t^I P_{t-}^I d\varpi_t^{I,\mathbb{Q}} - \xi_t^C P_{t-}^C d\varpi_t^{C,\mathbb{Q}}.
\end{aligned}$$

By the identity $(1 - \beta_t)\psi_t^{r_r} B_t^{r_r} = -(1 - \beta_t)\xi_t S_t$,

$$\begin{aligned}
dV_t = & \Big((1 - \beta_t \mathbb{1}_{\{\xi_t<0\}})r_D \xi_t S_t + r_D \xi_t^I P_t^I + r_D \xi_t^C P_t^C + r_f \xi_t^f B_t^{r_f} - (1 - \beta_t)r_r \xi_t S_t - r_c \psi_t^c B_t^{r_c}\Big)dt \\
& + (1 - \beta_t \mathbb{1}_{\{\xi_t<0\}})\sigma \xi_t S_t dW_t^{\mathbb{Q}} - \xi_t^I P_{t-}^I d\varpi_t^{I,\mathbb{Q}} - \xi_t^C P_{t-}^C d\varpi_t^{C,\mathbb{Q}} \\
= & \Big((r_D - r_D \beta_t \mathbb{1}_{\{\xi_t<0\}} - r_r + r_r \beta_t)\xi_t S_t + r_D \xi_t^I P_t^I + r_D \xi_t^C P_t^C + r_f \xi_t^f B_t^{r_f} - r_c \psi_t^c B_t^{r_c}\Big)dt \\
& + (1 - \beta_t \mathbb{1}_{\{\xi_t<0\}})\sigma \xi_t S_t dW_t^{\mathbb{Q}} - \xi_t^I P_{t-}^I d\varpi_t^{I,\mathbb{Q}} - \xi_t^C P_{t-}^C d\varpi_t^{C,\mathbb{Q}}.
\end{aligned}$$

For the wealth process itself, by the identity $(1-\beta_t)\psi_t^{r_r}B_t^{r_r} = -(1-\beta_t)\xi_t S_t$ and the wealth portfolio (4.4.3), we have that

$$\begin{aligned} V_t &= (1-\beta_t \mathbb{1}_{\{\xi_t<0\}})\xi_t S_t - (1-\beta_t)\xi_t S_t + \xi_t^I P_t^I + \xi_t^C P_t^C + \xi_t^f B_t^{r_f} - \psi_t^c B_t^{r_c} \\ &= \beta_t(1-\mathbb{1}_{\{\xi_t<0\}})\xi_t S_t + \xi_t^I P_t^I + \xi_t^C P_t^C + \xi_t^f B_t^{r_f} - \psi_t^c B_t^{r_c}. \end{aligned}$$

We rewrite the above equation,

$$\xi_t^f B_t^{r_f} = V_t - \beta_t(1-\mathbb{1}_{\{\xi_t<0\}})\xi_t S_t - \xi_t^I P_t^I - \xi_t^C P_t^C + \psi_t^c B_t^{r_c}.$$

Since $\psi_t B_t^{r_c} = -C_t$

$$\xi_t^f B_t^{r_f} = V_t - \beta_t(1-\mathbb{1}_{\{\xi_t<0\}})\xi_t S_t - \xi_t^I P_t^I - \xi_t^C P_t^C - C_t.$$

Plugging the above equation into the dynamics of V_t

$$\begin{aligned} dV_t =& \Big((r_D - r_D\beta_t\mathbb{1}_{\{\xi_t<0\}} - r_r + r_r\beta_t)\xi_t S_t + r_D\xi_t^I P_t^I + r_D\xi_t^C P_t^C + r_c C_t \\ &+ r_f\big(V_t - \beta_t(1-\mathbb{1}_{\{\xi_t<0\}})\xi_t S_t - \xi_t^I P_t^I - \xi_t^C P_t^C - C_t\big)\Big)dt \\ &+ (1-\beta_t\mathbb{1}_{\{\xi_t<0\}})\sigma\xi_t S_t dW_t^{\mathbb{Q}} - \xi_t^I P_{t-}^I d\varpi_t^{I,\mathbb{Q}} - \xi_t^C P_{t-}^C d\varpi_t^{C,\mathbb{Q}} \\ =& \Big((r_D - r_D\beta_t\mathbb{1}_{\{\xi_t<0\}} - r_r + r_r\beta_t - r_f\beta_t + r_f\beta_t\mathbb{1}_{\{\xi_t<0\}})\xi_t S_t \\ &+ (r_D - r_f)\xi_t^I P_t^I + (r_D - r_r)\xi_t^C P_t^C + r_f V_t + (r_c - r_f)C_t\Big)dt \\ &+ (1-\beta_t\mathbb{1}_{\{\xi_t<0\}})\sigma\xi_t S_t dW_t^{\mathbb{Q}} - \xi_t^I P_{t-}^I d\varpi_t^{I,\mathbb{Q}} - \xi_t^C P_{t-}^C d\varpi_t^{C,\mathbb{Q}}. \end{aligned}$$

Setting

$$Z_t = (1-\beta_t\mathbb{1}_{\{\xi_t<0\}})\sigma\xi_t S_t, \quad Z_t^I = -\xi_t^I P_{t-}^I, \quad Z_t^C = -\xi_t^C P_{t-}^C,$$

we have

$$(r_D - r_f)\xi_t^I P_t^I = -(r_D - r_f)Z_t^I, \tag{4.5.1}$$

$$(r_D - r_f)\xi_t^C P_t^C = -(r_D - r_f)Z_t^C. \tag{4.5.2}$$

Therefore,

$$\begin{aligned} &(r_D - r_D\beta_t\mathbb{1}_{\{\xi_t<0\}} - r_r + r_r\beta_t - r_f\beta_t + r_f\beta_t\mathbb{1}_{\{\xi_t<0\}})\xi_t S_t \\ =&(1-\beta_t\mathbb{1}_{\{\xi_t<0\}})r_D\xi_t S_t + (r_r\beta_t - r_f\beta_t + r_f\beta_t\mathbb{1}_{\{\xi_t<0\}} - r_r)\xi_t S_t \\ =&\frac{r_D}{\sigma}Z_t + (r_r\beta_t - r_f\beta_t + r_f\beta_t\mathbb{1}_{\{\xi_t<0\}} - r_r - r_f + r_f)\xi_t S_t \\ =&\frac{r_D}{\sigma}Z_t - (1-\beta_t\mathbb{1}_{\{\xi_t<0\}})r_f\xi_t S_t + (r_r\beta_t - r_f\beta_t - r_r + r_f)\xi_t S_t \\ =&\frac{r_D}{\sigma}Z_t - \frac{r_f}{\sigma}Z_t + (r_r\beta_t - r_f\beta_t - r_r + r_f)\xi_t S_t \\ =&\frac{r_D}{\sigma}Z_t - \frac{r_f}{\sigma}Z_t + (r_f - r_r)(1-\beta_t)\xi_t S_t. \end{aligned} \tag{4.5.3}$$

We want to rewrite $(1-\beta_t)\xi_t S_t$ in a representation by Z_t. We summarize the relationship between these two equations in Table 4.1.

Case	β_t	ξ_t	$(1-\beta_t)\xi_t S_t$	Z_t	$(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}})$	$\frac{1}{\sigma}(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}})Z_t$
1	$=0$	<0	$<0\ (-\lvert\xi_t\rvert S_t)$	<0	$=1$	$<0\ (-\lvert\xi_t\rvert S_t)$
2	$=0$	$=0$	$=0$	$=0$	$=1$	$=0$
3	$=0$	>0	$>0\ (\xi_t S_t)$	>0	$=1$	$>0\ (\xi_t St)$
4	$=1$	<0	$=0$	$=0$	$=1$	$=0$
5	$=1$	$=0$	$=0$	$=0$	$=1$	$=0$
6	$=1$	>0	$=0$	>0	$=0$	$=0$

Table 4.1: Summary of Z_t and $(1-\beta_t)\xi_t S_t$.

We conclude that $(1-\beta_t)\xi_t S_t = \frac{1}{\sigma}(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}})Z_t$, so

$$(r_f - r_r)(1-\beta_t)\xi_t S_t = \frac{r_f - r_r}{\sigma}(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}})Z_t. \tag{4.5.4}$$

Plugging Equations (4.5.1), (4.5.2), (4.5.3), (4.5.4) into the dynamics of V_t, we get

$$\begin{aligned}
dV_t =& \Big(\frac{r_D - r_f + (r_f - r_r)(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}})}{\sigma} Z_t - (r_D - r_f)Z_t^I - (r_D - r_f)Z_t^C + r_f V_t + (r_c - r_f)C_t\Big)dt \\
&+ Z_t dW_t^{\mathbb{Q}} + Z_t^I d\varpi_t^{I,\mathbb{Q}} + Z_t^C d\varpi_t^{C,\mathbb{Q}} \\
=& \Big(r_f(V_t + Z_t^I + Z_t^C - C_t) - r_D Z^I - r_D Z^C + r_c C_t + ((r_D - r_f) + (r_f - r_r)(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}}))\frac{Z_t}{\sigma}\Big)dt \\
&+ Z_t dW_t^{\mathbb{Q}} + Z_t^I d\varpi_t^{I,\mathbb{Q}} + Z_t^C d\varpi_t^{C,\mathbb{Q}}
\end{aligned}$$

$$\begin{aligned}
=& \Big(r_f(V_t + Z_t^I + Z_t^C - C_t) - r_D Z^I - r_D Z^C + r_c C_t + (r_D - r_f\mathbb{1}_{\{Z_t>0,\beta_t=1\}} - r_r(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}}))\frac{Z_t}{\sigma}\Big)dt \\
&+ Z_t dW_t^{\mathbb{Q}} + Z_t^I d\varpi_t^{I,\mathbb{Q}} + Z_t^C d\varpi_t^{C,\mathbb{Q}} \\
=& \Big(r_f(V_t - \frac{\mathbb{1}_{\{Z_t>0,\beta_t=1\}}}{\sigma}Z_t + Z_t^I + Z_t^C - C_t) - r_D Z^I - r_D Z^C + r_c C_t + (r_D - r_r(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}}))\frac{Z_t}{\sigma}\Big)dt \\
&+ Z_t dW_t^{\mathbb{Q}} + Z_t^I d\varpi_t^{I,\mathbb{Q}} + Z_t^C d\varpi_t^{C,\mathbb{Q}} \\
=& \Big(r_f^+(V_t - \frac{\mathbb{1}_{\{Z_t>0,\beta_t=1\}}}{\sigma}Z_t + Z_t^I + Z_t^C - C_t)^+ - r_f^-(V_t - \frac{\mathbb{1}_{\{Z_t>0,\beta_t=1\}}}{\sigma}Z_t + Z_t^I + Z_t^C - C_t)^- \\
&+ \frac{r_D - r_r^-(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}})}{\sigma}(Z_t)^+ - \frac{r_D - r_r^+(1-\mathbb{1}_{\{Z_t>0,\beta_t=1\}})}{\sigma}(Z_t)^- \\
&+ r_c^+(\alpha\hat{V}_t)^+ - r_c^-(\alpha\hat{V}_t)^- - r_D Z^I - r_D Z^C\Big)dt \\
&+ Z_t dW_t^{\mathbb{Q}} + Z_t^I d\varpi_t^{I,\mathbb{Q}} + Z_t^C d\varpi_t^{C,\mathbb{Q}}.
\end{aligned}$$

4.5.3 Construction BSDEs of the Wealth Process

In order to construct the BSDEs corresponding of the wealth process, we define a generator function $f(t, v, z, z^I, z^C; \beta, \hat{V})$ as followings

$$
\begin{aligned}
f^+(t,v,z,z^I,z^C;\beta,\hat{V}) = &-\Big(r_f^+(v-\frac{\mathbb{1}_{\{z>0,\beta=1\}}}{\sigma}z+z^I+z^C-\alpha\hat{V})^+ - r_f^-(v-\frac{\mathbb{1}_{\{z>0,\beta=1\}}}{\sigma}z+z^I+z^C-\alpha\hat{V})^- \\
&+\frac{r_D-r_r^-(1-\mathbb{1}_{\{z>0,\beta=1\}})}{\sigma}z^+ - \frac{r_D-r_r^+(1-\mathbb{1}_{\{z>0,\beta=1\}})}{\sigma}z^- \\
&+r_c^+(\alpha\hat{V})^+ - r_c^-(\alpha\hat{V}_t)^- - r_Dz^I - r_Dz^C\Big), \\
f^-(t,v,z,z^I,z^C;\beta,\hat{V}) = &-f^+(t,-v,-z,-z^I,-z^C;\beta,-\hat{V}).
\end{aligned}
$$

In order to attach the existence and uniqueness of the solution of a BSDE with the above generator functions, we need the following assumptions.

Assumption 4.5.7. *We assume that*

(i) $r_f^- < \frac{1}{5\sqrt{T}}$,

(ii) $\frac{(r_f^-+r_D\vee|r_D-r_r^-|)\vee|r_D-r_r^+|}{\sigma\wedge 1} < \frac{1}{5}$,

(iii) $r_f^- - r_D < \frac{\sqrt{\lambda^I}\wedge\sqrt{\lambda^C}}{5}$.

Now we have two BSDEs with generator functions $f^\pm : \Omega\times[0,T]\times\mathbb{R}^5\times\{0,1\}\to\mathbb{R}$, $(\omega,t,v,z,z^I,z^C;\beta,\hat{V})\mapsto f^\pm(t,v,z,z^I,z^C;\beta,\hat{V})$. The BSDEs are

$$
\begin{cases}
-dV_t^+ = f^+(t,V_t^+,Z_t^+,Z_t^{I,+},Z_t^{C,+};\beta,\hat{V})dt - Z_t^+dW_t^{\mathbb{Q}} - Z_t^{I,+}d\varpi_t^{I,\mathbb{Q}} - Z_t^{C,+}d\varpi_t^{C,\mathbb{Q}}, \\
\quad V_\tau^+ = \theta_I(\hat{V}_\tau)\mathbb{1}_{\{\tau^I<\tau^C\wedge T\}} + \theta_C(\hat{V}_\tau)\mathbb{1}_{\{\tau^C<\tau^I\wedge T\}} + \Theta\mathbb{1}_{\{\tau=T\}},
\end{cases}
\tag{4.5.5}
$$

and

$$
\begin{cases}
-dV_t^- = f^-(t,V_t^-,Z_t^-,Z_t^{I,-},Z_t^{C,-};\beta,\hat{V})dt - Z_t^-dW_t^{\mathbb{Q}} - Z_t^{I,-}d\varpi_t^{I,\mathbb{Q}} - Z_t^{C,-}d\varpi_t^{C,\mathbb{Q}}, \\
\quad V_\tau^- = \theta_I(\hat{V}_\tau)\mathbb{1}_{\{\tau^I<\tau^C\wedge T\}} + \theta_C(\hat{V}_\tau)\mathbb{1}_{\{\tau^C<\tau^I\wedge T\}} + \Theta\mathbb{1}_{\{\tau=T\}},
\end{cases}
\tag{4.5.6}
$$

where $\hat{V}$ is a third party valuation of V with $\mathbb{E}\big[\int_0^T\hat{V}_s^2ds\big]<\infty$ and Θ is the terminal value at time T without defaults.

The process V^+ describes the wealth process to hedge the claim Θ with zero initial capital by selling securities with terminal payoff Θ. In a financial crisis status, because all short selling trades of stock freeze, we can only super-hedge the claim Θ. The V_t^- describes the wealth process to hedge the claim Θ by buying securities with terminal payoff Θ.

4.5.4 Existence and Uniqueness of Solutions for the Valuation BSDEs

To prove the existence and uniqueness of the solution for these BSDEs, we need to prove the Lipschitz continuity of the generators $f^\pm$.

Lemma 4.5.8. *For any given* $0<t<T,\beta,\hat{V}$, *the generator functions* $f^\pm$ *are Lipschitz continuous in* v,z,z^I,z^C .

Proof. Given $\epsilon>0$, $t\geq 0$ and $\omega\in\Omega$, for $v_1,v_2,z_1,z_2,z_1^I,z_2^I,z_1^C,z_2^C$ with $|v_1-v_2|<\epsilon, |z_1-z_2|<\epsilon$, $|z_1^I-z_2^I|<\epsilon$ and $|z_1^C-z_2^C|<\epsilon$, we have

- When z_1, z_2 are nonnegative, we have

$$
\begin{aligned}
&|f^+(t, v_1, z_1, z_1^I, z_1^C; \beta, \hat{V}) - f^+(t, v_2, z_2, z_2^I, z_2^C; \beta, \hat{V})| \\
&\leq \Big| r_f^+ \Big((v_2 - \frac{\mathbb{1}_{\{z_2>0,\beta=1\}}}{\sigma} z_2 + z_2^I + z_2^C - \alpha \hat{V}_t)^+ - (v_1 - \frac{\mathbb{1}_{\{z_1>0,\beta=1\}}}{\sigma} z_1 + z_1^I + z_1^C - \alpha \hat{V}_t)^+ \Big) \\
&\quad + r_f^- \Big((v_1 - \frac{\mathbb{1}_{\{z_1>0,\beta=1\}}}{\sigma} z_1 + z_1^I + z_1^C - \alpha \hat{V}_t)^- - (v_2 - \frac{\mathbb{1}_{\{z_2>0,\beta=1\}}}{\sigma} z_2 + z_2^I + z_2^C - \alpha \hat{V}_t)^- \Big) \\
&\quad + \frac{r_D - r_r^-(1 - \mathbb{1}_{\{z_2>0,\beta=1\}})}{\sigma} z_2^+ - \frac{r_D - r_r^+(1 - \mathbb{1}_{\{z_2>0,\beta=1\}})}{\sigma} z_2^- \\
&\quad - \Big(\frac{r_D - r_r^-(1 - \mathbb{1}_{\{z_1>0,\beta=1\}})}{\sigma} z_1^+ - \frac{r_D - r_r^+(1 - \mathbb{1}_{\{z_1>0,\beta=1\}})}{\sigma} z_1^- \Big) + r_D(z_1^I - z_2^I) + r_D(z_1^C - z_2^C) \Big| \\
&\leq (r_f^+ \vee r_f^-) \Big| (v_2 - \frac{\mathbb{1}_{\{z_2>0,\beta=1\}}}{\sigma} z_2 + z_2^I + z_2^C - \alpha \hat{V}_t) - (v_1 - \frac{\mathbb{1}_{\{z_1>0,\beta=1\}}}{\sigma} z_1 + z_1^I + z_1^C - \alpha \hat{V}_t) \Big| \\
&\quad + \frac{r_D}{\sigma} |z_1^+ - z_2^+| + r_D |z_1^I - z_2^I| + r_D |z_1^C - z_2^C| \\
&\leq r_f^- \Big| (v_2 - \frac{\mathbb{1}_{\{z_2>0,\beta=1\}}}{\sigma} z_2 + z_2^I + z_2^C - \alpha \hat{V}_t) - (v_1 - \frac{\mathbb{1}_{\{z_1>0,\beta=1\}}}{\sigma} z_1 + z_1^I + z_1^C - \alpha \hat{V}_t) \Big| \\
&\quad + \frac{r_D}{\sigma} |z_1 - z_2| + r_D |z_1^I - z_2^I| + r_D |z_1^C - z_2^C| \\
&\leq r_f^- |v_1 - v_2| + \frac{r_f^- + r_D}{\sigma} |z_1 - z_2| + \Big(r_f^- + r_D \Big) |z_1^I - z_2^I| + \Big(r_f^- + r_D \Big) |z_1^C - z_2^C| \\
&\leq A_1 \left(|v_1 - v_2| + |z_1 - z_2| + |z_1^I - z_2^I| + |z_1^C - z_2^C| \right),
\end{aligned}
$$

where $A_1 = \frac{r_f^- + r_D}{\sigma \wedge 1}$.

- When z_1, z_2 are negative, we have

$$
\begin{aligned}
&|f^+(t, v_1, z_1, z_1^I, z_1^C; \beta, \hat{V}) - f^+(t, v_2, z_2, z_2^I, z_2^C; \beta, \hat{V})| \\
&\leq \Big| r_f^+ \Big((v_2 + z_2^I + z_2^C - \alpha \hat{V}_t)^+ - (v_1 + z_1^I + z_1^C - \alpha \hat{V}_t)^+ \Big) \\
&\quad + r_f^- \Big((v_1 + z_1^I + z_1^C - \alpha \hat{V}_t)^- - (v_2 + z_2^I + z_2^C - \alpha \hat{V}_t)^- \Big) \\
&\quad + \frac{r_D - r_r^-}{\sigma} (z_2^+ - z_1^+) - \frac{r_D - r_r^+}{\sigma} (z_2^- - z_1^-) + r_D(z_1^I - z_2^I) + r_D(z_1^C - z_2^C) \Big| \\
&\leq (r_f^+ \vee r_f^-) \Big| (v_2 + z_2^I + z_2^C - \alpha \hat{V}_t) - (v_1 + z_1^I + z_1^C - \alpha \hat{V}_t) \Big| \\
&\quad + \frac{|r_D - r_r^+|}{\sigma} |z_1^- - z_2^-| + r_D |z_1^I - z_2^I| + r_D |z_1^C - z_2^C| \\
&\leq r_f^- |v_1 - v_2| + \frac{|r_D - r_r^+|}{\sigma} |z_1 - z_2| + (r_f^- + r_D) |z_1^I - z_2^I| + (r_f^- + r_D) |z_1^C - z_2^C| \\
&\leq A_2 \left(|v_1 - v_2| + |z_1 - z_2| + |z_1^I - z_2^I| + |z_1^C - z_2^C| \right),
\end{aligned}
$$

where $A_2 = (r_f^- + r_D) \vee \frac{|r_D - r_r^+|}{\sigma \wedge 1}$.

- Without loss of generality, we assume that $z_1 > 0$ and $z_2 < 0$, we have $|z_1 + z_2| \le |z_1 + |z_2|| = |z_1 - z_2| < \epsilon$ and $|z_1| \le |z_1 + |z_2|| = |z_1 - z_2| < \epsilon$.

$$
\begin{aligned}
&|f^+(t, v_1, z_1, z_1^I, z_1^C; \beta, \hat{V}) - f^+(t, v_2, z_2, z_2^I, z_2^C; \beta, \hat{V})| \\
&\le \Bigg| r_f^+ \left((v_2 + z_2^I + z_2^C - \alpha \hat{V}_t)^+ - (v - \frac{\mathbb{1}_{\{z_1>0,\beta=1\}}}{\sigma} z_1 + z_1^I + z_1^C - \alpha \hat{V}_t)^+ \right) \\
&\quad + r_f^- \left((v_1 - \frac{\mathbb{1}_{\{z_1>0,\beta=1\}}}{\sigma} z_1 + z_1^I + z_1^C - \alpha \hat{V}_t)^- - (v_2 + z_2^I + z_2^C - \alpha \hat{V}_t)^- \right) \\
&\quad + \Big(\frac{r_D - r_r^-}{\sigma} z_2^+ - \frac{r_D - r_r^+}{\sigma} z_2^- \Big) - \Big(\frac{r_D - r_r^-(1 - \mathbb{1}_{\{z_1>0,\beta=1\}})}{\sigma} z_1^+ - \frac{r_D - r_r^+(1 - \mathbb{1}_{\{z_1>0,\beta=1\}})}{\sigma} z_1^- \Big) \\
&\quad + r_D(z_1^I - z_2^I) + r_D(z_1^C - z_2^C) \Bigg| \\
&\le (r_f^+ \vee r_f^-) \Big| (v_2 + z_2^I + z_2^C - \alpha \hat{V}_t) - (v_1 - \frac{\mathbb{1}_{\{z_1>0,\beta=1\}}}{\sigma} z_1 + z_1^I + z_1^C - \alpha \hat{V}_t) \Big| \\
&\quad + \Big| \frac{r_D - r_r^-(1 - \mathbb{1}_{\{z_1>0,\beta=1\}})}{\sigma} z_1^+ + \frac{r_D - r_r^+}{\sigma} z_2^- \Big| + r_D|z_1^I - z_2^I| + r_D|z_1^C - z_2^C| \\
&\le r_f^- \Big| (v_2 + z_2^I + z_2^C - \alpha \hat{V}_t) - (v_1 - \frac{\mathbb{1}_{\{z_1>0,\beta=1\}}}{\sigma} z_1 + z_1^I + z_1^C - \alpha \hat{V}_t) \Big| \\
&\quad + \frac{r_D \vee |r_D - r_r^+|}{\sigma} |z_1 + |z_2|| + r_D|z_1^I - z_2^I| + r_D|z_1^C - z_2^C| \\
&\le r_f^- |v_1 - v_2| + \frac{r_f^-}{\sigma} |z_1| + \frac{r_D \vee |r_D - r_r^-|}{\sigma} |z_1 + |z_2|| + (r_f^- + r_D)|z_1^I - z_2^I| + (r_f^- + r_D)|z_1^C - z_2^C| \\
&\le r_f^- |v_1 - v_2| + \frac{r_f^-}{\sigma} |z_1 - z_2| + \frac{r_D \vee |r_D - r_r^-|}{\sigma} |z_1 - z_2| + (r_f^- + r_D)|z_1^I - z_2^I| + (r_f^- + r_D)|z_1^C - z_2^C| \\
&\le r_f^- |v_1 - v_2| + \frac{r_f^- + (r_D \vee |r_D - r_r^-|)}{\sigma} |z_1 - z_2| + (r_f^- + r_D)|z_1^I - z_2^I| + (r_f^- + r_D)|z_1^C - z_2^C| \\
&\le A_3 \left(|v_1 - v_2| + |z_1 - z_2| + |z_1^I - z_2^I| + |z_1^C - z_2^C| \right),
\end{aligned}
$$

where $A_3 = \frac{r_f^- + (r_D \vee |r_D - r_r^-|)}{\sigma \wedge 1} \vee (r_f^- + r_D)$.

Overall, the function f^+ satisfies the Lipschitz condition in v, z, z^I, z^C such that

$$
|f^+(t, v_1, z_1, z_1^I, z_1^C; \beta, \hat{V}) - f^+(t, v_2, z_2, z_2^I, z_2^C; \beta, \hat{V})| \le K \left(|v_1 - v_2| + |z_1 - z_2| + |z_1^I - z_2^I| + |z_1^C - z_2^C| \right),
$$

where $K = A_1 \vee A_2 \vee A_3 = \frac{(r_f^- + r_D \vee |r_D - r_r^-|) \vee |r_D - r_r^+|}{\sigma \wedge 1}$ independently for β and $\hat{V}$. □

Theorem 4.5.9. *Given* $(\Omega, (\mathscr{F}_t)_{t\ge 0}, \mathscr{F}, \mathbb{P})$*, the BSDE* (4.5.5) *admits a unique solution* $(V^+, Z^+, Z^{I,+}, Z^{C,+}) \in \mathbb{S}^2 \times \mathscr{M}$*. The solutions satisfies the following equation*

$$
V_t^+ = V_\tau^+ + \int_t^\tau f^+(s, V_s^+, Z_s^+, Z_s^{I,+}, Z_t^{C,+}; \beta, \hat{V}) ds - \int_t^\tau Z_s^+ dW_s^{\mathbb{Q}} - \int_t^\tau Z_s^{I,+} d\varpi_t^{I,\mathbb{Q}} - \int_t^\tau Z_s^{C,+} d\varpi_t^{C,\mathbb{Q}}.
$$

Proof. By the Lemme 4.5.8, we have

the generator satisfies the Lipschitz condition, i.e.

$$
\begin{aligned}
&|f^+(t,v_1,z_1,z_1^I,z_1^C;\beta,\hat{V})-f^+(t,v_2,z_2,z_2^I,z_2^C;\beta,\hat{V})|\\
\leq & r_f^-|v_1-v_2|+\frac{(r_f^-+r_D\vee|r_D-r_r^-|)\vee|r_D-r_r^+|}{\sigma}|z_1-z_2|+(r_f^-+r_D)|z_1^I-z_2^I|+(r_f^-+r_D)|z_1^C-z_2^C|.
\end{aligned}
$$

By Assumptions 4.5.7, the above equation becomes

$$
\begin{aligned}
&|f^+(t,v_1,z_1,z_1^I,z_1^C;\beta,\hat{V})-f^+(t,v_2,z_2,z_2^I,z_2^C;\beta,\hat{V})|\\
<& \frac{1}{5\sqrt{T}}|v_1-v_2|+\frac{\sqrt{\lambda^I}}{5}|z_1^I-z_2^I|+\frac{\sqrt{\lambda^C}}{5}|z_1^C-z_2^C|+\frac{1}{5}|z_1-z_2|\\
=&\frac{1}{5}\left(\frac{1}{\sqrt{T}}|v_1-v_2|+|z_1-z_2|+\sqrt{\lambda^I}|z_1^I-z_2^I|+\sqrt{\lambda^C}|z_1^C-z_2^C|\right).
\end{aligned}
$$

So our generator function f^+ satisfies the Lipschitz condition in Assumption 3.1.4. By the definition of V_τ, we have $V_\tau\in L^2(\Omega,\mathscr{F}_\tau,\mathbb{P})$, so the closeout valuation satisfies the terminal condition in Assumption 3.1.4. By the definition of f^+, we have

$$
f^+(t,0,0,0,0;\beta,\hat{V})=-\Big(r_f^+(-\alpha\hat{V}_t)^+-r_f^-(-\alpha\hat{V}_t)^-+r_c^+(\alpha\hat{V}_t)^+-r_c^-(\alpha\hat{V}_t)^-\Big).
$$

For $T<\infty$,

$$
\mathbb{E}\Big[\int_0^T|f^+(s,0,0,0,0;\beta,\hat{V})|^2ds\Big]<\infty,
$$

so $f^+(t,0,0,0,0;\beta,\hat{V})\in\mathbb{H}^2$ satisfies the integrability condition in Assumption 3.1.4. By Theorem 3.2.1, this valuation BSDE with the generator f^+ admits a unique solution $(V,Z,Z^I,Z^C,Z^\beta,Y)\in\mathbb{S}^2\times\mathscr{M}$.

$$
\begin{aligned}
V_t^+=V_\tau^+&+\int_t^\tau f(\omega,s,V_{s-},Z_s,Z_s^I,Z_s^C,Z_s^\beta)ds+\int_t^\tau Z_sdW_s^{\mathbb{Q}}+\int_t^\tau Z_s^Id\varpi_t^{I,\mathbb{Q}}+\int_t^\tau Z_s^Cd\varpi_t^{C,\mathbb{Q}}\\
&+\int_t^\tau Z_s^\beta d\tilde{J}_s+Y_\tau-Y_t.
\end{aligned}
$$

Based on the specific form of the valuation BSDE with the generator f^+, the solution does not depend on the stochastic integral with respect to $\tilde{J}$ and the orthogonal term Y. So the solution (V,Z,Z^I,Z^C) to BSDE (4.5.5) is given by

$$
V_t^+=V_\tau^++\int_t^\tau f^+(s,V_s^+,Z_s^+,Z_s^{I,+},Z_s^{C,+};\beta,\hat{V})ds-\int_t^\tau Z_s^+dW_s^{\mathbb{Q}}-\int_t^\tau Z_s^{I,+}d\varpi_t^{I,\mathbb{Q}}-\int_t^\tau Z_s^{C,+}d\varpi_t^{C,\mathbb{Q}}.
$$

□

Theorem 4.5.10. *Given $(\Omega,(\mathscr{F}_t)_{t\geq0},\mathscr{F},\mathbb{P})$, BSDE (4.5.6) admits a unique solution $(V^-,Z^-,Z^{I,-},Z^{C,-})\in\mathbb{S}^2\times\mathscr{M}$. The solutions satisfies the following equation*

$$
\begin{aligned}
V_t^-=V_\tau^-+\int_t^\tau f^-(s,V_s^-,Z_s^-,Z_s^{I,-},Z_t^{C,-};\beta,\hat{V})ds-\int_t^\tau Z_s^-dW_s^{\mathbb{Q}}\\
-\int_t^\tau Z_s^{I,-}d\varpi_t^{I,\mathbb{Q}}-\int_t^\tau Z_s^{C,-}d\varpi_t^{C,\mathbb{Q}}.
\end{aligned}
$$

Proof. Similar to the proof of Theorem 4.5.9, we have that BSDE (4.5.6) admits unique solutions $(V^-, Z^-, Z^{I,-}, Z^{C,-}) \in \mathbb{S}^2 \times \mathscr{M}$. $\square$

Since the Repo market freezes during the financial crisis, there are no short trade of stocks during the financial crisis. During a finanical crisis ($\beta_t = 1$), whenever we need short stocks to hedge our European option ($Z^+ < 0$), we cannot short stocks. So, we can only super-hedge the derivatives.

4.6 Total Valuation Adjustment (XVA)

The total valuation adjustment (XVA) is an adjustment made to the fair value of a derivative contract to take into account funding and credit risk. We want to compute the total valuation adjustment, which is added to the Black-Scholes price of a European option, considering an investor's and its counterparty's defaults, funding liquidity and asymmetric interest rates. Since adjustments are asymmetric for the buy-side and sell-side, we define both seller's and buyer's valuation adjustments.

4.6.1 Construction of XVA BSDEs

Definition 4.6.1. *The seller's XVA is a stochastic process defined as*

$$XVA_t^+ := V_t^+ - \hat{V}_t, \tag{4.6.1}$$

and the buyers' XVA is defined as

$$XVA_t^- := V_t^- - \hat{V}_t. \tag{4.6.2}$$

XVA^+ is a valuation adjustment by the trader to hedge a long position in the option, and XVA^- is a valuation adjustment by the trader to hedge a short position in the option.

Here $\hat{V}$ is a third party valuation of the option, which is a solution of the Black-Scholes model

$$\begin{cases} -d\hat{V}_t = -r_D \hat{V}_t dt - \hat{Z}_t dW_t^{\mathbb{Q}}, \\ \quad \hat{V}_T = \Theta, \end{cases} \tag{4.6.3}$$

where Θ is the terminal value of the claim from the point of view the third party.

Based on the existence and uniqueness of the solutions of the Black-Scholes model, we have the BSDEs for the $\mathrm{XVA}^\pm$ as

$$\begin{cases} -d\mathrm{XVA}_t^\pm = \tilde{f}^\pm(t, \mathrm{XVA}_t^\pm, \tilde{Z}_t^\pm, \tilde{Z}_t^{I,\pm}, \tilde{Z}_t^{C,\pm}; \beta, \hat{V}, \hat{Z})dt - \tilde{Z}_t^\pm dW_t^{\mathbb{Q}} - \tilde{Z}_t^{I,\pm} d\varpi_t^{I,\mathbb{Q}} - \tilde{Z}_t^{C,\pm} d\varpi_t^{C,\mathbb{Q}}, \\ \quad \mathrm{XVA}_\tau^\pm = \tilde{\theta}_I(\hat{V}_\tau)\mathbb{1}_{\{\tau^I < \tau^C \wedge T\}} + \tilde{\theta}_C(\hat{V}_\tau)\mathbb{1}_{\{\tau^C < \tau^I \wedge T\}}, \end{cases} \tag{4.6.4}$$

where

$$\begin{aligned} \tilde{Z}_t^\pm &:= Z_t^\pm - \hat{Z}_t, \\ \tilde{Z}_t^{I,\pm} &:= Z_t^{I,\pm}, \\ \tilde{Z}_t^{C,\pm} &:= Z_t^{C,\pm}, \\ \tilde{\theta}_I(\hat{v}) &:= -L_I((1-\alpha)\hat{v})^+, \\ \tilde{\theta}_C(\hat{v}) &:= L_C((1-\alpha)\hat{v})^-, \end{aligned} \tag{4.6.5}$$

and the generators are

$$
\begin{aligned}
\tilde{f}^{+}(t, xva, \tilde{z}, \tilde{z}^{I}, \tilde{z}^{C}; \beta, \hat{V}, \hat{Z}) = -\Big(& r_f^{+}(xva - \frac{\mathbb{1}_{\{\tilde{z}+\hat{Z}>0, \beta=1\}}}{\sigma}(\tilde{z}+\hat{Z}) + \tilde{z}^{I} + \tilde{z}^{C} + (1-\alpha)\hat{V})^{+} \\
& - r_f^{-}(xva - \frac{\mathbb{1}_{\{\tilde{z}+\hat{Z}>0, \beta=1\}}}{\sigma}(\tilde{z}+\hat{Z}) + \tilde{z}^{I} + \tilde{z}^{C} + (1-\alpha)\hat{V})^{-} \\
& + \frac{r_D - r_r^{-}(1 - \mathbb{1}_{\{\tilde{z}+\hat{Z}>0, \beta=1\}})}{\sigma}(\tilde{z}+\hat{Z})^{+} \\
& - \frac{r_D - r_r^{+}(1 - \mathbb{1}_{\{\tilde{z}+\hat{Z}>0, \beta=1\}})}{\sigma}(\tilde{z}+\hat{Z})^{-} \\
& + r_c^{+}(\alpha\hat{V}_t)^{+} - r_c^{-}(\alpha\hat{V}_t)^{-} - r_D\tilde{z}^{I} - r_D\tilde{z}^{C}\Big) + r_D\hat{V}_t, \\
\tilde{f}^{-}(t, xva, \tilde{z}, \tilde{z}^{I}, \tilde{z}^{C}; \beta, \hat{V}, \hat{Z}) := -& \tilde{f}^{+}(t, -xva, -\tilde{z}, -\tilde{z}^{I}, -\tilde{z}^{C}; \beta, -\hat{V}, -\hat{Z}).
\end{aligned}
\tag{4.6.6}
$$

Based on the definition of the generator function of BSDEs (4.5.5) and (4.5.6), we have

$$
\tilde{f}^{\pm}(t, xva, \tilde{z}, \tilde{z}^{I}, \tilde{z}^{C}; \beta, \hat{V}, \hat{Z}) = f^{\pm}(t, xva, \tilde{z}+\hat{Z}, \tilde{z}^{I}, \tilde{z}^{C}; \beta, \hat{V}) \pm r_D\hat{V}. \tag{4.6.7}
$$

Theorem 4.6.2. *Given* $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathscr{F}, \mathbb{P})$. *The XVA BSDEs* (4.6.4) *admit a unique solution* $(XVA^{\pm}, \tilde{Z}^{\pm}, \tilde{Z}^{I,\pm}, \tilde{Z}^{C,\pm})$. *The solution satisfies the following equation*

$$
\begin{aligned}
XVA_t^{\pm} = XVA_{\tau}^{\pm} + \int_t^{\tau} \tilde{f}^{\pm}(s, XVA_s^{\pm}, \tilde{Z}_s^{\pm}, \tilde{Z}_s^{I,\pm}, \tilde{Z}_t^{C,\pm}; \beta, \hat{V}, \hat{Z})ds - \int_t^{\tau} \tilde{Z}_s^{+} dW_s^{\mathbb{Q}} \\
- \int_t^{\tau} \tilde{Z}_s^{I,\pm} d\varpi_t^{I,\mathbb{Q}} - \int_t^{\tau} \tilde{Z}_s^{C,\pm} d\varpi_t^{C,\mathbb{Q}}.
\end{aligned}
$$

Proof. The result is a direct consequence of Theorem 4.5.9 and the Black-Scholes formula. □

4.6.2 Reduced XVA BSDEs

We want to rewrite XVA BSDEs (4.6.4) in the filtration $(\mathscr{F}_t)_{t\geq 0}$ with a jump terminal condition to a reduced XVA BSDEs with a smaller filtration $(\mathscr{F}^{W,\beta})_{t\geq 0}$ and a continuous terminal condition, based on the results in Section 3.3. The reduced XVA BSDEs are

$$
\begin{cases}
-d\breve{U}_t^{\pm} = \breve{g}^{\pm}(t, \breve{U}_t^{\pm}, \breve{Z}_t^{\pm}; \beta, \hat{V}, \hat{Z})dt - \breve{Z}_t^{\pm} dW_t^{\mathbb{Q}}, \\
\quad \breve{U}_T^{\pm} = 0,
\end{cases}
\tag{4.6.8}
$$

in the filtration $(\mathscr{F}_t^{W,\beta})_{t\geq 0}$ without default events with

$$
\begin{aligned}
\breve{g}^{+}(t, \breve{u}, \breve{z}; \beta, \hat{V}, \hat{Z}) := & h_I^{\mathbb{Q}}(\tilde{\theta}_I(\hat{V}_t) - \breve{u}) + h_C^{\mathbb{Q}}(\tilde{\theta}_C(\hat{V}_t) - \breve{u}) \\
& + \tilde{f}^{+}(t, \breve{u}, \breve{z}, \tilde{\theta}_I(\hat{V}_t) - \breve{u}, \tilde{\theta}_C(\hat{V}_t) - \breve{u}; \beta, \hat{V}, \hat{Z}), \\
\breve{g}^{-}(t, \breve{u}, \breve{z}; \beta, \hat{V}, \hat{Z}) := & -\breve{g}^{+}(t, -\breve{u}, -\breve{z}; \beta, -\hat{V}, -\hat{Z}).
\end{aligned}
\tag{4.6.9}
$$

Theorem 4.6.3. *The reduced XVA BSDE* (4.6.8) *admits a unique solution* $(\breve{U}^{\pm}, \breve{Z}^{\pm})$. *When* $(XVA^{\pm}, \tilde{Z}^{\pm}, \tilde{Z}^{I,\pm}, \tilde{Z}^{C,\pm})$ *is a unique solution of BSDEs* (4.6.4), *then* $(\breve{U}^{\pm}, \breve{Z}^{\pm})$ *defined as*

$$
\breve{U}_t^{\pm} := XVA_{t\wedge\tau-}^{\pm}, \quad \breve{Z}_t^{\pm} := \tilde{Z}_t^{\pm}\mathbb{1}_{\{t<\tau\}}, \tag{4.6.10}
$$

are solutions to the reduced XVA BSDE (4.6.8)*. When* $(\breve{U}^{\pm}, \breve{Z}^{\pm})$ *are unique solutions to the reduced XVA BSDEs* (4.6.8)*, then* $(XVA^{\pm}, \tilde{Z}^{\pm}, \tilde{Z}^{I,\pm}, \tilde{Z}^{C,\pm})$*, defined as*

$$\begin{aligned} XVA_t^{\pm} &:= \breve{U}_t^{\pm}\mathbb{1}_{\{t<\tau\}} + \Big(\tilde{\theta}_I(\hat{V}_{\tau_I})\mathbb{1}_{\{\tau_I<\tau^C\wedge T\}} + \tilde{\theta}_C(\hat{V}_{\tau_C})\mathbb{1}_{\{\tau_C<\tau^I\wedge T\}}\Big)\mathbb{1}_{\{t\geq\tau\}}, \\ \tilde{Z}_t^{\pm} &:= \breve{Z}_t^{\pm}\mathbb{1}_{\{t<\tau\}}, \\ \tilde{Z}_t^{I,\pm} &:= \Big(\tilde{\theta}_I(\hat{V}_t) - \breve{U}_t^{\pm}\Big)\mathbb{1}_{\{t\leq\tau\}}, \\ \tilde{Z}_t^{C} &:= \Big(\tilde{\theta}_C(\hat{V}_t) - \breve{U}_t^{\pm}\Big)\mathbb{1}_{\{t\leq\tau\}}, \end{aligned} \tag{4.6.11}$$

are unique solutions of the XVA BSDEs (4.6.4).

Proof. To prove the existences and uniqueness of solutions for the reduced XVA BSDE (4.6.8), we need to prove the Lipschitz condition of the generator functions $\tilde{f}^{\pm}$.

I. For the generator function $\tilde{f}^{+}$, by the Lipschitz condition of the generator function f^{+} and Equation (4.6.7), we have

$$\begin{aligned} &\Big|\tilde{f}^{+}(t, xva_1, \tilde{z}_1, \tilde{z}_1^I, \tilde{z}_1^C; \beta, \hat{V}, \hat{Z}) - \tilde{f}^{+}(t, xva_2, \tilde{z}_2, \tilde{z}_2^I, \tilde{z}_2^C; \beta, \hat{V}, \hat{Z})\Big| \\ =&\Big|f^{+}(t, xva_1, \tilde{z}_1 + \hat{Z}, \tilde{z}_1^I, \tilde{z}_1^C; \beta, \hat{V}) + r_D\hat{V} - \Big(\tilde{f}^{+}(t, xva_2, \tilde{z}_2 + \hat{Z}, \tilde{z}_2^I, \tilde{z}_2^C; \beta, \hat{V}) + r_D\hat{V}\Big)\Big| \\ =&\Big|f^{+}(t, xva_1, \tilde{z}_1 + \hat{Z}, \tilde{z}_1^I, \tilde{z}_1^C; \beta, \hat{V}) - \tilde{f}^{+}(t, xva_2, \tilde{z}_2 + \hat{Z}, \tilde{z}_2^I, \tilde{z}_2^C; \beta, \hat{V})\Big| \\ \leq&K(|xva_1 - xva_2| + |(\tilde{z}_1 + \hat{Z}) - (\tilde{z}_2 + \hat{Z})| + |\tilde{z}_1^I - \tilde{z}_2^I| + |\tilde{z}_1^C - \tilde{z}_2^C|) \\ =&K(|xva_1 - xva_2| + |\tilde{z}_1 - \tilde{z}_2| + |\tilde{z}_1^I - \tilde{z}_2^I| + |\tilde{z}_1^C - \tilde{z}_2^C|). \end{aligned}$$

Similarly, we establish the Lipschitz continuity of the generator function f^{-}. Based on the definition of $\tilde{f}^{\pm}$, the integrability and terminal conditions are trivial. By Theorem 3.2.1, the reduced XVA BSDEs (4.6.8) admit a unique solution.

The equivalence between the XVA BSDEs (4.6.4) and the reduced XVA BSDEs (4.6.8) is a direct result from Theorem 3.3.1, 3.3.2 and 3.3.3.

□

Next, we will discuss the replicating strategies of the XVA process. Since the Repo market freezes during a financial crisis, there are no short trade of stocks during the financial crisis. So we cannot hedge the long position of a call option. As a result, We can only super-hedge the long position of a call option. Same with the BSDEs of XVA processes, we will use symbol $\sim$ to denote a hedging strategies of our XVA. Here, we apply the relationship between stock account and Repo account, and the hedging portfolio.

Remark 4.6.4. *The trading strategy of the XVA is given by*

$$\begin{aligned} \tilde{\xi}_t^{\pm} &= \begin{cases} \frac{\tilde{Z}_t^{\pm}+\hat{Z}_t^{\pm}}{\sigma S_t}\mathbb{1}_{\{t<\tau\}}, & \beta_t = 0, \\ \Big(\frac{\tilde{Z}_t^{\pm}+\hat{Z}_t^{\pm}}{\sigma S_t}\Big)^{+}\mathbb{1}_{\{t<\tau\}}, & \beta_t = 1, \end{cases} \\ \tilde{\psi}_t^{r,\pm} &= -(1-\beta_t)\frac{\tilde{\xi}_t^{\pm} S_t}{B_t^{r_r}}\mathbb{1}_{\{t<\tau\}}, \\ \tilde{\xi}_t^{I,\pm} &= -\frac{\tilde{Z}_t^{I,\pm}}{P_{t-}^{I}}, \end{aligned}$$

$$\tilde{\xi}_t^{C,\pm} = -\frac{\tilde{Z}_t^{C,\pm}}{P_{t-}^C},$$

$$\tilde{\psi}_t^c = -\frac{\alpha \hat{V}_t}{B_t^{r_c}} \mathbb{1}_{\{t<\tau\}},$$

$$\tilde{\xi}_t^{f,\pm} = \frac{V_t^{\pm} - \hat{V}_t - \xi_t^I P_{t-}^I - \xi_t^C P_{t-}^C - \alpha \hat{V}_t - (1 - \beta_t \mathbb{1}_{\xi_t<0})\xi_t S_t - \psi_t^{r,\pm} B_t^{r_r}}{B_t^{r_f}} \mathbb{1}_{\{t<\tau\}},$$

where we can only have a super-hedging portfolio if $\tilde{Z}_t^{\pm} + \hat{Z}_t^{\pm} < 0$ *when* $\beta_t = 1$.

4.7 Empirical Application

In this section, we illustrate some simulation results of the alternating renewal process β and the XVA valuation of a European call option. We want to estimate the parameters of the alternating renewal process by using historical data and compare the XVA with and without considering the different performances of the Repo account and stock account during different financial statuses.

4.7.1 Estimations of Alternating Renewal Processes

In this section, we want to estimate the parameters λ_U and λ_V of the alternating renewal process β. The inter-arrival times in this process follow two exponential distributions. Because we use this process to describe the switching between a normal financial status and a financial crisis status, the parameter λ_U is the expected length of a normal financial status and the parameter λ_V is the expected length of a financial crisis. Therefore, we may use historical data from some financial stress index to estimate these parameters. As we reviewed in Chapter 1, there are several indicators of financial distress, such as VIX, CISS, CoVaR and the Ted spread.

Boudt et al. (2017) confirm the existence of a two-regime Ted spreads from January 2006 to December 2011. They estimate threshold for the regime switching as 0.48 basis points. Here we also use the historical data of the Ted spread to estimate the parameters λ_U and λ_V.

Setting the threshold at .48 basis points, when the Ted spread is larger than the .48 basis points, we claim that the financial market enters a crisis status. When the Ted spread is smaller or equal to the .48 basis points, we claim that the financial market is in a normal status. Then λ_U describes the average length of a normal financial regime and λ_V describe the average length of a financial crisis regime. Assuming all financial status are independent, we use the sample mean to estimate the expectation of the lengths of the financial statuses. The results are given in Table 4.2. In this result, the estimates of λ_U and λ_V are relatively small. In Figure 4.2, we can see that the Ted spread crossed the threshold (green line) several times, which means we have several switches between the normal financial status and the financial crisis status. Both statuses appeared five times in the dataset. But this number contradicts to the real financial condition from 2006 to 2011. Therefore, we should find a different way to set the threshold, in order to remove the effect of small movements of the Ted spread.

Boudt et al. (2013) mention that the 0.8 basis points is also a meaningful threshold, which is the threshold for the central bank responds to a financial crisis. In order to eliminate the effect of the Ted spread's noise movements, we change the setting of the threshold. When the Ted spread up-crosses the .80 basis points (red line in Figure 4.4), we claim that the financial market enters a financial crisis status. When the Ted spread down-crosses the .48 basis points (green line in Figure 4.4), we claim that the financial market enters a normal financial status.

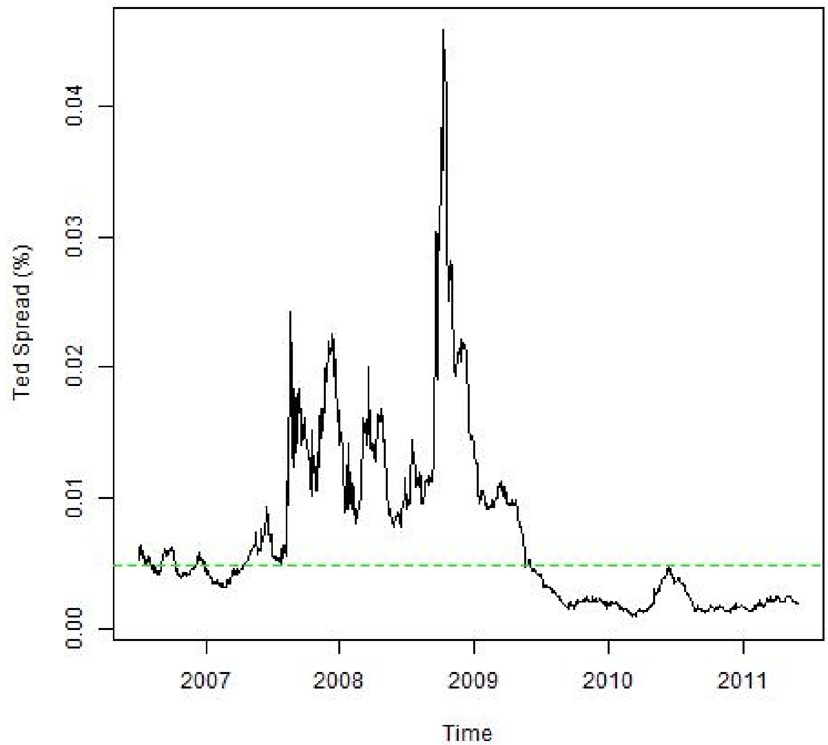


Figure 4.2: Ted spread from Jan 2006 to Dec 2011.

Threshold 0.48 basis points as green line.

	Normal Financial Statuses	Financial Crisis Statuses
Number of Statuses	5	5
Average Length (days)	179	172
Estimates for λ_U and λ_V	0.49	0.47

Table 4.2: Estimates of λ_U and λ_V when threshold is .48 basis points.

Based on the setting of .48 and .80 basis points, we estimate the parameters λ_U and λ_V of the alternating renewal process β, given in Table 4.3.

	Normal Financial Statuses	Financial Crisis Statuses
Number of Statuses	2	2
Average Length (days)	507	361
Estimates of λ_U and λ_V	1.39	0.99

Table 4.3: Estimations of λ_U and λ_V when threshold is .48 and .8 basis points.

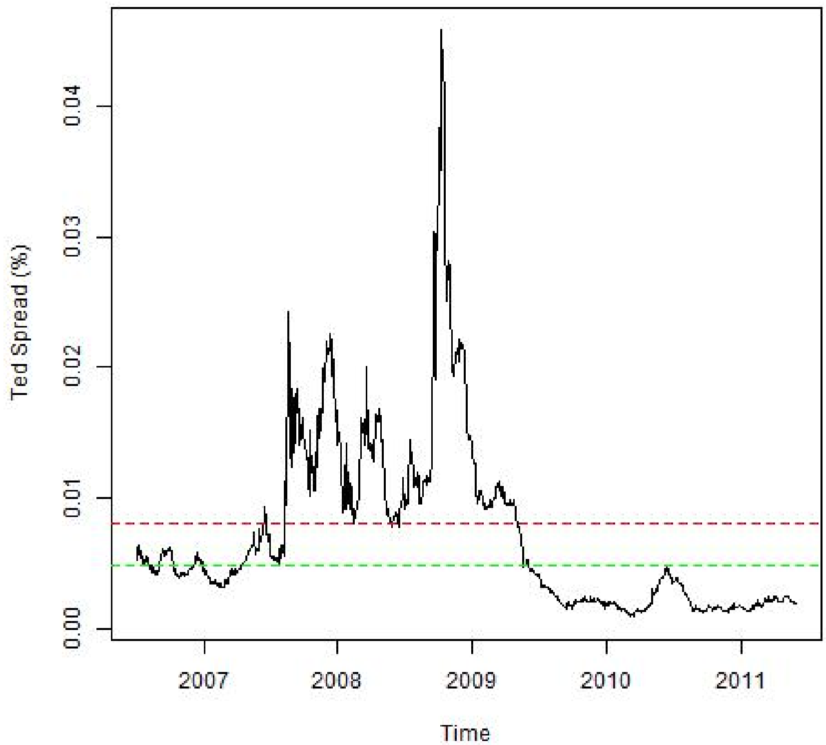


Figure 4.4: Ted spread from Jan 2006 to Dec 2011.

Threshold 0.48 basis point as green line and the threshold 0.8 basis point as red line.

To put this result in a broader period, we consider all historical data for the Ted spread we have available. Using the same setting for the thresholds, we estimate the parameters using the whole historical data, from Jan 2$^{\text{rd}}$, 1986 to July 24$^{\text{th}}$, 2018. In Figure 4.6, we find the Ted spread is relatively high before 1998. The result of the estimation is given in Table 4.4. The estimate of average length of the normal financial status ($\lambda_U = 1.75$) is larger than $\lambda_U = 1.39$, which is the results for smaller estimation period.

	Normal Financial Statuses	Financial Crisis Statuses
Number of Statuses	12	12
Average Length (days)	639	321
Estimations of λ_U and λ_V	1.75	0.88

Table 4.4: Estimates of λ_U and λ_V with threshold .48 and .8 basis points.

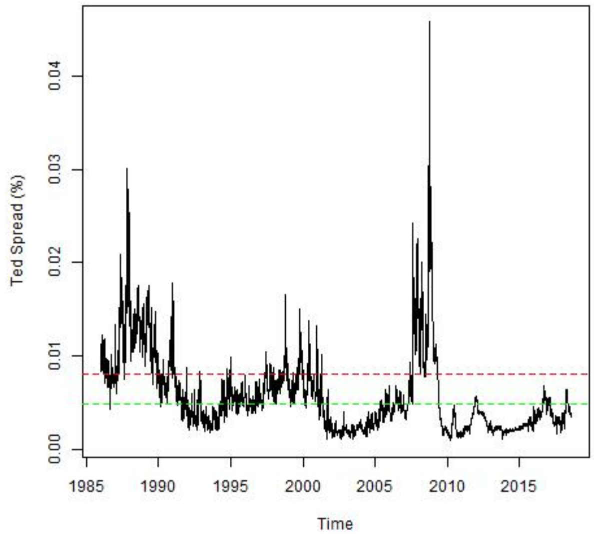


Figure 4.6: Historical Ted spread.

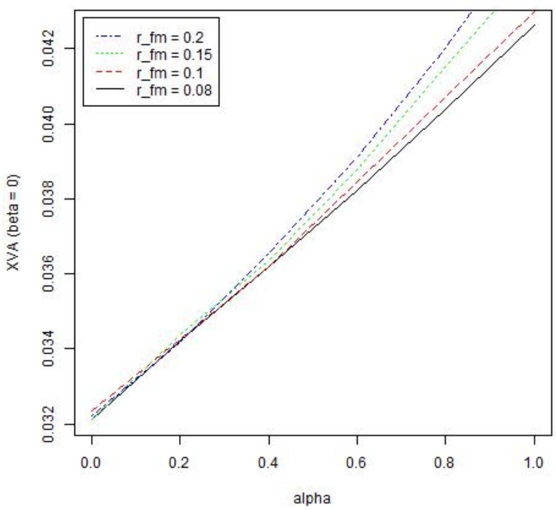
Threshold 0.48 basis points as green line and the threshold 0.8 basis points as red line.

4.7.2 Simulation Results of XVAs

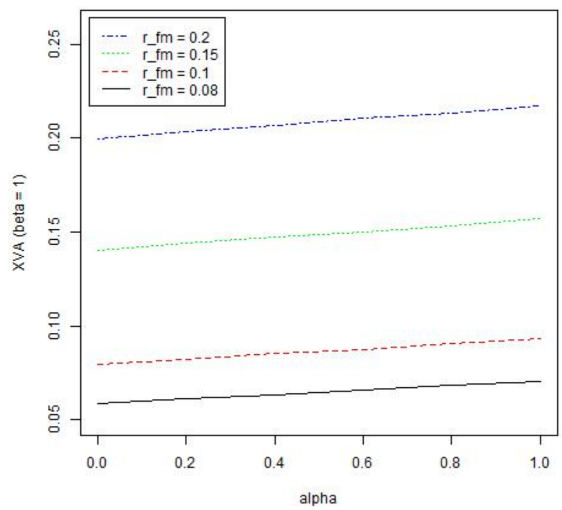
In practice, to evaluate the XVA, we should use the estimation of λ_U and λ_V to generate paths of the alternating renewal process. Then using the simulation methods to solve the XVA BSDE. But in this section, we want to focus on the effect of different financial statuses to the XVA. Therefore, we only evaluate the XVA during a normal financial status ($\beta \equiv 0$) and during a financial crisis ($\beta \equiv 1$). For a European call option, we assume that an initial stock price $S_0 = \$1$, the strike price $K = \$1$ and the terminal time of the option $T = 1$ year. We set the following benchmark coefficients: $r_r^+ = r_r^- = 0.05$, $r_c^+ = r_c^- = 0.01$, $r_f^+ = 0.05, r_D = 0.01, \mu^I = 0.21, \mu^C = 0.16, h_I^{\mathbb{Q}} = 0.2, h_C^{\mathbb{Q}} = 0.15$, and $L_I = L_C = 0.5$, as in Bichuch et al. (2018a). Then we want to analyze the effect of the financial statuses (β_t), the different collateralization levels α and the funding rates (r_f^-) on the XVA.

In this dissertation, we use the deep learning-based numerical method to solve our XVA BSDE, the algorithm can be found in Weinan et al. (2017). During a normal financial status ($\beta_t = 0$), we compute the XVA for different collateralization levels α between 0 and 1 and different values of the funding rate $r_f^- = 0.08, 0.1, 0.15$, and 0.2. When the volatility rate of the stock is $\sigma = 0.2$, the result is plotted in Figure 4.8(a). Consistent to the results in Bichuch et al. (2018a), the XVA also increase corresponding to the increase in the collateralization level α. The increasement of the XVA for the different funding rates r_f^- achieve minimum around $\alpha = 0.2$ and increases for other collateralization levels.

In a financial crisis status ($\beta_t = 1$), we compute the XVA for different collateralization levels and different funding rates r_f^- when the volatility rate of stock is $\sigma = 0.2$. The results are plotted in Figure 4.8(b). Similar to the normal financial status, the XVA also increases, corresponding to the collateralization level α increases. When α increases, the investor needs to hold more shares in the collateral account C_t, which leads to borrowing more cash from the funding account. In



(a) $\beta_t = 0$.



(b) $\beta_t = 1$.

Figure 4.8: XVA when $\sigma = 0.2$

this situation, the XVA also increases due to the higher funding cost incurred the hedging of the investor's and its counterparty's default risks. Because the Repo market freezes during a financial crisis, all borrowing of cash has to be ceased through the funding market. The relationship between XVA and the collateralization level α is simplified to a linear relationship. For different funding rates r_f^-, the differences between XVA are similar to the differences between the different funding rates r_f^-.

We also compare the XVA in a normal financial status and a financial crisis status when the volatility rate $\sigma = 0.2$ and the funding rate $r_f^- = 0.08$, given in Figure 4.9. In a financial crisis, the XVA is nearly double the size of the XVA in a normal financial regime. Therefore, it is important to differentiate the different financial statuses when pricing an option. Moreover, in the crisis case, the increment of the XVA has linear relationship with the increment of the funding interest rate r_f^-, as shown in Figure 4.10. In the financial crisis status, we also compare different XVAs for different collateralization levels α for different volatilities σ. In Figure 4.11, we find the increment of the XVA has positive correlation with the volatility of the stock price.

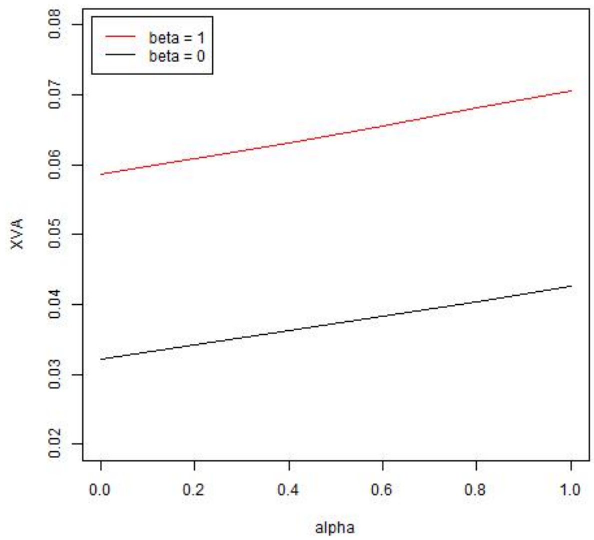


Figure 4.9: XVA for $\beta_t = 0$ or $\beta_t = 1$ when $\sigma = 0.2$.

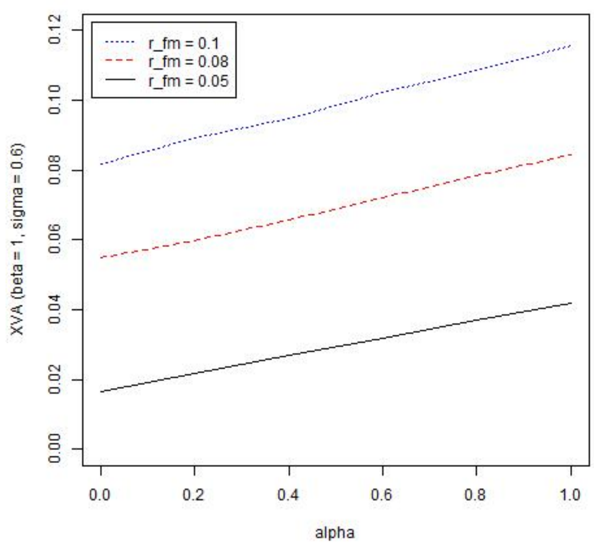


Figure 4.10: XVA when $\beta_t = 1$ and $\sigma = 0.6$.

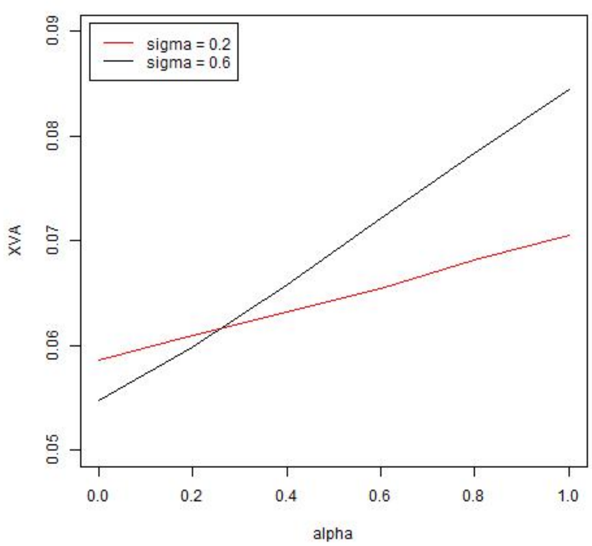


Figure 4.11: Comparison of XVA with $\sigma = 0.2$ and $\sigma = 0.6$ when $\beta_t = 1$.

Chapter 5

Network Model and Systemic Risk

In this chapter, we will analysis the systemic risk from a regulator's perspective. We will do a sensitivity analysis of the Eisenberg–Noe model and a society network model.

5.1 Sensitivity analysis of Eisenberg–Noe clearing vector

We consider a financial system consisting of n banks, $\mathcal{N} = \{1, \dots, n\}$. For $i, j \in \mathcal{N}$, $L_{ij} \geq 0$ is the nominal liability of bank i to bank j.

Remark 5.1.1. *External liabilities can be considered as well through the introduction of an "external" bank* 0*. This is discussed in more details in Section 5.2.3.*

Equivalently, L_{ij} is the exposure of bank j to bank i. Let $L \in \mathbb{R}^{n \times n}$ be the liabilities matrix of the financial network, and we assume that no bank has an exposure to itself, i.e., $L_{ii} = 0$ for all $i \in \mathcal{N}$. The total liability of bank i is given by $\bar{p}_i = \sum_{j=1}^{n} L_{ij}$. The relative liability of bank i to bank j is denoted by $\pi_{ij} \in [0, 1]$, where $\pi_{ij} = \frac{L_{ij}}{\bar{p}_i}$ when $\bar{p}_i > 0$. We allow $\pi_{ij} \in [0, 1]$ to be arbitrary when $\bar{p}_i = 0$ and only require $\sum_{j=1}^{n} \pi_{ij} = 1$.

Remark 5.1.2. *The arbitrary choice of π_{ij} in the case $\bar{p}_i = 0$ has no impact on the outcome of the Eisenberg–Noe model since the transpose of the relative liability matrix Π is multiplied by the incoming payment vector $p(\Pi)$, whose j^{th} entry is 0 when $\bar{p}_j = 0$ (cf.* (5.1.2)*).*

We denote the relative liability matrix $\Pi \in \mathbb{R}^{n \times n}$. Any relative liability matrix Π must belong to the set of admissible matrices $\mathbf{\Pi}^n$, defined as the set of all right stochastic matrices with entries in $[0, 1]$ and all diagonal entries 0:

$$\mathbf{\Pi}^n := \left\{ \Pi \in [0,1]^{n \times n} \,\middle|\, \forall i : \pi_{ii} = 0, \ \sum_{j=1}^{n} \pi_{ij} = 1 \right\}. \tag{5.1.1}$$

Finally, denote the external assets of bank i from outside the banking system by $x_i \geq 0$. A bank balance sheet then takes the simplified form of Table 5.1, and a financial system is given by the triplet $(\Pi, x, \bar{p}) \in \mathbf{\Pi}^n \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

A bank is solvent when the sum of its net external assets and performing interbank assets exceeds its total liabilities. In this case, the bank honours all of its obligations. However, if the value of its obligations is greater than the bank's net assets plus performing interbank assets, then the bank will default and repay its obligations pro-rata.

Assets	Representation	**Liabilities**	Representation
Interbank	$a_i^{IB} = \sum_{j=1}^{n} L_{ji}$	Interbank	$l_i^{IB} = \sum_{j=1}^{n} L_{ij}$
External	x_i	External	L_{i0}
		Capital	c_i

Table 5.1: Stylized bank balance sheet.

Remark 5.1.3. *This corresponds to the assumption that all interbank and external claims can be aggregated to a single figure per bank and that all creditors of a defaulting bank are paid pari passu.*

These rules yield a clearing vector (clearing payments) as the solution of the fixed point problem

$$p(\Pi) = \bar{p} \wedge \big(x + \Pi^\top p(\Pi)\big). \tag{5.1.2}$$

Let $p : \mathbf{\Pi}^n \to \mathbb{R}_+^n; \Pi \mapsto p(\Pi)$ be the fixed point function with parameters $(x, \bar{p})$. As proved in (Eisenberg and Noe 2001, Theorem 2), the clearing vector is unique if a system of banks is regular. Regularity is defined as follows: a surplus set $S \subseteq \mathcal{N}$ is a set of banks in which no bank in the set has any obligations to a bank outside of the set and the sum over all banks' external net asset values in the set is positive, i.e., $\forall\,(i,j) \in S \times S^c : \pi_{ij} = 0$ and $\sum_{i \in S} x_i > 0$. Next, consider the financial system as a directed graph in which there is a directed link from bank i to bank j if $L_{ij} > 0$. Denote the risk orbit of bank i as $o(i) = \{j \in \mathcal{N} \mid \text{ there exists a directed path from } i \text{ to } j\}$. This means that the risk orbit of bank i is the set of all banks which may be affected by the default of bank i. A system is regular if every risk orbit is a surplus set. Uniqueness of the clearing vector has important consequences in terms of the continuity of the function p, which in turn is important for our sensitivity analysis. For this reason we will proceed under the assumption that our financial system is regular.

Proposition 5.1.4. *Consider a regular financial system $(\Pi, x, \bar{p})$ in which x and $\bar{p}$ are fixed. The function p, defined via* (5.1.2)*, is continuous with respect to $\Pi \in \mathbf{\Pi}^n$.*

Proof. This proof follows the logic of (Feinstein et al. 2017, Lemma 5.2) and (Ren et al. 2014, Theorem 4). Fix the net assets x and total obligation $\bar{p}$. Let $\phi : [0, \bar{p}] \times \mathbf{\Pi}^n \to [0, \bar{p}]$ be the function defined by $\phi(\hat{p}, \Pi) := \big(\phi_1(\hat{p}, \Pi), \cdots, \phi_n(\hat{p}, \Pi)\big)^\top$, where

$$\phi_i(\hat{p}, \Pi) = \bar{p}_i \wedge \left(x_i + \sum_{j=1}^{n} \pi_{ji} \hat{p}_j\right), \quad i \in \mathcal{N}.$$

The function ϕ is jointly continuous with respect to the payment vector $\hat{p}$ and the relative liabilities π_{ij} for $i, j \in \mathcal{N}$. Because the system is regular and thus has a unique fixed point, it follows from (Feinstein et al. 2017, Proposition A.2) that the graph

$$\operatorname{graph}(p) = \big\{(\Pi, \hat{p}) \in \mathbf{\Pi}^n \times [0, \bar{p}] \;\big|\; \phi(\hat{p}, \Pi) = \hat{p}\big\}$$

is closed. Define the projection $\Psi : \mathbf{\Pi}^n \times [0, \bar{p}] \to \mathbf{\Pi}^n$ as $\Psi(\Pi, p) = \Pi$. By (Feinstein et al. 2017, Proposition A.3), Ψ is a closed mapping in the product topology. Then, in order to show that p is continuous, take $U \subset [0, \bar{p}]$ closed. Then

$$p^{-1}[U] = \big\{\Pi \in \mathbf{\Pi}^n \;\big|\; p(\Pi) \in U\big\} = \Psi\big(\operatorname{graph}(p) \cap (\mathbf{\Pi}^n \times U)\big).$$

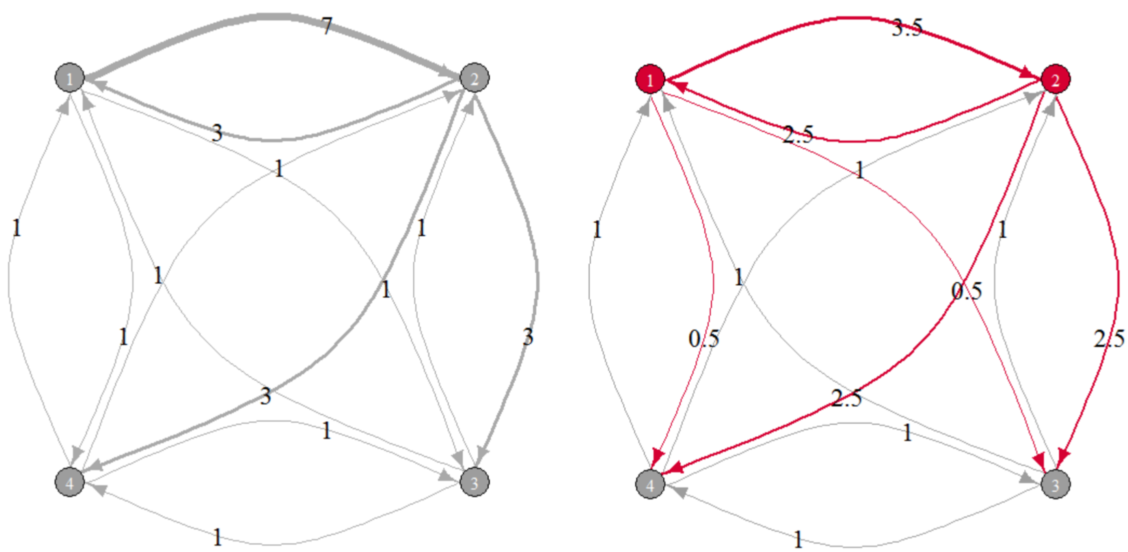
The graph of p is closed and $\mathbf{\Pi}^n$ is closed by definition. Hence $p^{-1}[U]$ is closed and the function p is continuous with respect to Π. $\square$

We finish these preliminary notes by considering a simple example of the Eisenberg–Noe clearing payments under a system of $n = 4$ banks. We will return to this example throughout as a simple illustrative case study.

Example 5.1.5. *Consider the following example of a network consisting of four banks in which the bank's nominal interbank liabilities are given by*

$$L = \begin{pmatrix} 0 & 7 & 1 & 1 \\ 3 & 0 & 3 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

as shown in Figure 5.1(a). Assume that the banks' external assets are given by the vector $x = (0, 2, 2, 2)^\top$*. With 0 net worth and positive liabilities, Bank 1 defaults initially. The Eisenberg–Noe clearing vector* (5.1.2) *can be easily computed to be* $p = (4.5, 7.5, 3, 3)^\top$*, showing that Bank 2 also defaults through contagion. The realized interbank payments are shown in Figure 5.1(b). Banks who are in default are colored red and payments that are repaid less than whole are also colored red. The edge widths are proportional to the payment size.*



(a) Nominal interbank liabilities.

(b) Clearing interbank payments.

Figure 5.1: Initial network defined in Example 5.1.5.

5.1.1 Quantifying estimation errors from the (relative) liabilities matrices

We assume that some estimation error is attached to the entries of the relative liability matrix, leading to a deviation of the clearing vector from the "true" clearing vector $p(\Pi)$. Denote the true relative liabilities matrix by Π and let $\Pi + h\Delta$ denote the liabilities matrix that includes some estimation error, for a perturbation matrix Δ and size $h \in \mathbb{R}$. First, we consider the class of perturbation matrices, $\boldsymbol{\Delta}^n(\Pi)$, under which we assume that the existence or non-existence of a link between two banks is known to the regulator and hence, the error is limited to a misspecification of the size of that link. In practice, this type of uncertainty arises when data is collected at a low frequency, which can lead to exposure evolving naturally, as well as banks trying to improve their balance sheet composition ahead of regulatory reporting dates.[1]

Remark 5.1.13, Corollary 5.2.5 and Corollary 5.2.20 will utilize the results in this section to provide bounds for the perturbation error in general without predetermining existence or non-existence of links.

Definition 5.1.6. *For a fixed $\bar{p} \in \mathbb{R}^n_+$, define the set of relative liability perturbation matrices by*

$$\boldsymbol{\Delta}^n(\Pi) := \left\{ \Delta \in \mathbb{R}^{n\times n} \;\middle|\; \forall i : \delta_{ii} = 0,\; \sum_{j=1}^n \delta_{ij} = 0,\; \sum_{j=1}^n \delta_{ji}\bar{p}_j = 0,\; \text{and } (\pi_{ij} = 0) \Rightarrow (\delta_{ij} = 0)\, \forall\, j \right\}.$$

The summation conditions ensure that the total liabilities and total assets, respectively, of each bank are left unchanged by the perturbation. Of course it is not possible to have $\Pi + h\Delta \in \boldsymbol{\Pi}^n$ for any $h \in \mathbb{R}$. Throughout this work we consider perturbation magnitudes in a bounded interval, $h \in (-h^*, h^*)$, where

$$h^* := \min\left\{ \min_{\delta_{ij}<0,\; \bar{p}_i>0} \frac{-\pi_{ij}}{\delta_{ij}} \,,\, \min_{\delta_{ij}>0,\; \bar{p}_i>0} \frac{1-\pi_{ij}}{\delta_{ij}} \right\} > 0,$$

for any $\Delta \in \boldsymbol{\Delta}^n(\Pi)$ to assure $\Pi + h\Delta \in \boldsymbol{\Pi}^n$. We exclude from this calculation of h^* any bank i where $\bar{p}_i = 0$ since this has no impact on the results. It is natural to consider directional derivatives on a unit-ball, whence we focus on the bounded set of perturbations

$$\boldsymbol{\Delta}^n_F(\Pi) := \boldsymbol{\Delta}^n(\Pi) \cap \left\{ \Delta \in \mathbb{R}^{n\times n} \;\middle|\; \|\Delta\|_F \leq 1 \right\},$$

where $\|\cdot\|_F$ is the Frobenius norm, i.e., $\|\Delta\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |\delta_{ij}|^2}$.

Remark 5.1.7. *A more general case can be considered in which one allows for errors that create links where there were none or remove connections where there was one. This set is defined as follows: for a fixed $\bar{p} \in \mathbb{R}^n_+$,*

$$\overline{\boldsymbol{\Delta}}^n(\Pi) := \left\{ \Delta \in \mathbb{R}^{n\times n} \;\middle|\; \forall i : \delta_{ii} = 0,\; \sum_{j=1}^n \delta_{ij} = 0,\; \sum_{j=1}^n \delta_{ji}\bar{p}_j = 0,\; \text{and } (\pi_{ij} = 0) \Rightarrow (\delta_{ij} \geq 0)\, \forall\, j \right\}.$$

We will consider in particular the bounded set of perturbations

$$\overline{\boldsymbol{\Delta}}^n_F(\Pi) := \overline{\boldsymbol{\Delta}}^n(\Pi) \cap \left\{ \Delta \in \mathbb{R}^{n\times n} \;\middle|\; \|\Delta\|_F \leq 1 \right\}.$$

Such perturbations thus allow a "rewiring" of the network. In general, allowing edges to be added or deleted increases the potential error in the clearing vector. However, the infinitesimal nature of the sensitivity analysis necessarily restricts the rewiring to the creation of new links; any strictly positive liability cannot be deleted through an infinitesimal perturbation. We discuss this issue in more detail in Corollary 5.2.5, where we apply our methodology to the complete network, as well as in Figure 5.16(a), which shows a distribution of payouts to society under a rewiring of the interbank network.

[1]Evidence for such behaviour at the end of a quarter can, for instance, be seen in the balance sheet reduction of European Banks and the corresponding spikes this creates in the utilization of the Federal Reserve's Reverse Repo facility, see: `http://libertystreeteconomics.newyorkfed.org/2017/08/regulatory-incentives-and-quarter-end-dynamics-in-the-repo-market.html`.

5.1.2 Directional derivatives of the Eisenberg–Noe clearing vector

Next, we analyze the error when using the clearing vector of a perturbed liability matrix, $p(\Pi+h\Delta)$, instead of the clearing vector of the original liability matrix, $p(\Pi)$, for small perturbations $h\Delta$, with $\Delta \in \mathbf{\Delta}^n(\Pi)$.

Definition 5.1.8. *Let $\Delta \in \mathbf{\Delta}^n(\Pi)$. In the case that the following limit exists, we define the directional derivative of the clearing vector $p(\Pi)$ in the direction of a perturbation matrix Δ as*

$$\mathcal{D}_\Delta p(\Pi) := \lim_{h\to 0} \frac{p(\Pi + h\Delta) - p(\Pi)}{h}.$$

The first order Taylor expansion of p about Π gives

$$p(\Pi + h\Delta) - p(\Pi) = h\mathcal{D}_\Delta p(\Pi) + O\big(h^2\big).$$

The following theorem provides an explicit formula for the directional derivative of the clearing vector for a fixed financial network.

Theorem 5.1.9. *Let $(\Pi, x, \bar{p})$ be a regular financial system. The directional derivative of the clearing vector $p(\Pi)$ in the direction of a perturbation matrix $\Delta \in \mathbf{\Delta}^n(\Pi)$ exists almost everywhere and is given by*

$$\mathcal{D}_\Delta p(\Pi) = \Big(I - diag(d)\Pi^\top\Big)^{-1} diag(d)\Delta^\top p(\Pi), \tag{5.1.3}$$

where $diag(d)$ is the diagonal matrix defined as $diag(d_1, \dots, d_n)$, where

$$d_i := \mathbb{1}_{\{x_i+\sum_{j=1}^n \pi_{ji}p_j(\Pi)<\bar{p}_i\}}.$$

Here, (5.1.3) *holds outside of the measure-zero set $\{x \in \mathbb{R}^n_+ \mid \exists i \in \mathcal{N} \text{ s.t. } x_i + \sum_{j=1}^n \pi_{ji}p_j(\Pi) = \bar{p}_i\}$ in which some bank is exactly at the brink of default.*

We note that our proof does not assume *a priori* that the clearing vector p is differentiable; we comment on this simpler case below.

Proof. We assume that the net external assets lie in the set

$$\left\{x \in \mathbb{R}^n_+ \,\middle|\, \nexists i \in \mathcal{N} \text{ s.t. } x_i + \sum_{j=1}^n \pi_{ji}p_j(\Pi) = \bar{p}_i\right\}.$$

Denote $\alpha^{(1)} = x+\Pi^\top p(\Pi) = (\alpha_1^{(1)}, \cdots, \alpha_n^{(1)})^\top$ and $\alpha^{(2)} = x+(\Pi+h\Delta)^\top p(\Pi+h\Delta) = (\alpha_1^{(2)}, \cdots, \alpha_n^{(2)})^\top$. By continuity of p with respect to Π (Proposition 5.1.4) we have for all $i \in \mathcal{N}$, $\alpha_i^{(2)} \to \alpha_i^{(1)}$ as $h \to 0$ and thus $\mathbb{1}_{\{\{\alpha_i^{(1)}<\bar{p}_i\}\cap\{\alpha_i^{(2)}>\bar{p}_i\}\}} \to 0$, $\mathbb{1}_{\{\{\alpha_i^{(1)}>\bar{p}_i\}\cap\{\alpha_i^{(2)}<\bar{p}_i\}\}} \to 0$ and $\mathbb{1}_{\{\{\alpha_i^{(1)}<\bar{p}_i\}\cap\{\alpha_i^{(2)}<\bar{p}_i\}\}} \to \mathbb{1}_{\{\alpha_i^{(1)}<\bar{p}_i\}}$. To prove the existence of $\mathcal{D}_\Delta p(\Pi)$, we will show that the following two limits,

$$\overline{\mathcal{D}_\Delta p(\Pi)}_i = \limsup_{h\to 0} \frac{p_i(\Pi + h\Delta) - p_i(\Pi)}{h},$$
$$\underline{\mathcal{D}_\Delta p(\Pi)}_i = \liminf_{h\to 0} \frac{p_i(\Pi + h\Delta) - p_i(\Pi)}{h}$$

are equal for each component. Consider the upper limit

$$
\begin{aligned}
\overline{\mathcal{D}_\Delta p(\Pi)}_i &= \limsup_{h\to 0} \frac{p_i(\Pi + h\Delta) - p_i(\Pi)}{h} \\
&= \limsup_{h\to 0} \frac{1}{h}\left(\left(\bar{p}_i \wedge (x_i + \sum_{j=1}^n (\pi_{ji} + h\delta_{ji}) p_j(\Pi + h\Delta))\right) - \left(\bar{p}_i \wedge (x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi))\right)\right) \\
&= \limsup_{h\to 0} \Bigg(0 \times \mathbb{1}_{\{\{\alpha_i^{(1)} > \bar{p}_i\} \cap \{\alpha_i^{(2)} > \bar{p}_i\}\}} + \frac{\bar{p}_i - (x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi))}{h} \mathbb{1}_{\{\{\alpha_i^{(1)} < \bar{p}_i\} \cap \{\alpha_i^{(2)} > \bar{p}_i\}\}} \\
&\quad + \frac{x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi + h\Delta) + h \sum_{j=1}^n \delta_{ji} p_j(\Pi + h\Delta) - \bar{p}_i}{h} \mathbb{1}_{\{\{\alpha_i^{(1)} > \bar{p}_i\} \cap \{\alpha_i^{(2)} < \bar{p}_i\}\}} \\
&\quad + \frac{\sum_{j=1}^n \pi_{ji}\big(p_j(\Pi + h\Delta) - p_j(\Pi)\big) + h \sum_{j=1}^n \delta_{ji} p_j(\Pi + h\Delta)}{h} \mathbb{1}_{\{\{\alpha_i^{(1)} < \bar{p}_i\} \cap \{\alpha_i^{(2)} < \bar{p}_i\}\}} \Bigg) \\
&= \left(\sum_{j=1}^n \pi_{ji} \overline{\mathcal{D}_\Delta p(\Pi)}_j + \sum_{j=1}^n \delta_{ji} p_j(\Pi)\right) \mathbb{1}_{\{\alpha_i^{(1)} < \bar{p}_i\}} \\
&= d_i \sum_{j=1}^n \pi_{ji} \overline{\mathcal{D}_\Delta p(\Pi)}_j + d_i \sum_{j=1}^n \delta_{ji} p_j(\Pi) =: \Psi_i\big(\overline{\mathcal{D}_\Delta p(\Pi)}\big)
\end{aligned}
$$

for some function $\Psi : \mathbb{R}^n \to \mathbb{R}^n$.

Similarly, we get

$$
\underline{\mathcal{D}_\Delta p(\Pi)}_i = d_i \sum_{j=1}^n \pi_{ji} \underline{\mathcal{D}_\Delta p(\Pi)}_j + d_i \sum_{j=1}^n \delta_{ji} p_j(\Pi) = \Psi_i\big(\underline{\mathcal{D}_\Delta p(\Pi)}\big).
$$

Hence, both $\overline{\mathcal{D}_\Delta p(\Pi)}$ and $\underline{\mathcal{D}_\Delta p(\Pi)}$ are fixed points of the same mapping Ψ. Assuming that this fixed point problem has a unique solution it follows

$$
\overline{\mathcal{D}_\Delta p(\Pi)}_i = \underline{\mathcal{D}_\Delta p(\Pi)}_i,
$$

for all $i \in \mathcal{N}$. Therefore, under this assumption, $\mathcal{D}_\Delta p(\Pi)$ is well defined and it is the solution to the fixed point equation

$$
\mathcal{D}_\Delta p(\Pi) = \Psi\big(\mathcal{D}_\Delta p(\Pi)\big) = \operatorname{diag}(d)\Pi^\top \mathcal{D}_\Delta p(\Pi) + \operatorname{diag}(d)\Delta^\top p(\Pi).
$$

Next, we proceed to show that $\big(I - \operatorname{diag}(d)\Pi^\top\big)$ is invertible, which establishes uniqueness of the fixed point and the directional derivative (5.1.3) to conclude the proof.

First, assume that $\operatorname{diag}(d)\Pi^\top$ is irreducible, i.e., the graph with adjacency matrix $\operatorname{diag}(d)\Pi^\top$ has directed paths in both directions between any two vertices $i \neq j$. Then by the Perron–Frobenius Theorem (see, e.g., (Gentle 2007, Section 8.7.2)), $\operatorname{diag}(d)\Pi^\top$ has an eigenvector $v > \mathbf{0}$ corresponding to eigenvalue $\rho(\operatorname{diag}(d)\Pi^\top)$, where $\rho(\cdot)$ is the spectral radius of a matrix. As eigenvectors are only unique up to a multiplicative constant, we may assume $\|v\|_1 = 1$. Under the assumption of a regular system, at least one bank must be solvent, i.e., there exists some i such that $\operatorname{diag}(d)_{ii} = 0$. This implies that there exists a column such that the column sum of $\operatorname{diag}(d)\Pi^\top$ is strictly less than 1. In fact, any insolvent institution j with obligations to bank i will have column sum of $\operatorname{diag}(d)\Pi^\top$

strictly less than 1. If all banks are solvent, $\text{diag}(d)$ is the zero matrix and the result is trivial. Thus, there is some matrix $M \geq 0$, $M \neq 0$ so that each column sum of $\text{diag}(d)\Pi^\top + M$ is 1, i.e.

$$\mathbf{1}^\top \big(\text{diag}(d)\Pi^\top + M\big) = \mathbf{1}^\top.$$

Note that the column sums of $\text{diag}(d)\Pi^\top$ are at most 1 since each row sum of Π is 1. Therefore the spectral radius of $\text{diag}(d)\Pi^\top$ must be less than or equal to 1. Moreover, we must have $\rho(\text{diag}(d)\Pi^\top) < 1$. Otherwise, $\rho(\text{diag}(d)\Pi^\top) = 1$, which along with the scaling of the eigenvector so that $\|v\|_1 = 1$ implies

$$1 = \mathbf{1}^\top v = \mathbf{1}^\top \big(\text{diag}(d)\Pi^\top + M\big) v = \mathbf{1}^\top (v + Mv) = 1 + \mathbf{1}^\top M v > 1,$$

as $\Pi^\top v = 1v$ by the definition of eigenvalues. Therefore, we can conclude that, in the case $\text{diag}(d)\Pi^\top$ is irreducible, $\rho(\text{diag}(d)\Pi^\top) < 1$.

Now suppose that $\text{diag}(d)\Pi^\top$ is reducible, i.e., $\text{diag}(d)\Pi^\top$ is similar to a block upper triangular matrix D, with irreducible diagonal blocks D_i, $i = 1, \ldots, m$ for some $m < n$. Under the assumption of a regular system, each D_i has at least one column whose sum is strictly less than 1. As in the preceding case, this implies that $\rho(D_i) < 1$ for each i and therefore

$$\rho(\text{diag}(d)\Pi^\top) = \rho(D) < 1.$$

Since the maximal eigenvalue of $\text{diag}(d)\Pi^\top$ is strictly less than 1, 0 cannot be an eigenvalue of $I - \text{diag}(d)\Pi^\top$. This suffices to show that $I - \text{diag}(d)\Pi^\top$ is invertible. □

Remark 5.1.10. *If one assumes that p is differentiable with respect to the relative liabilities Π, the result of Theorem 5.1.9 can be obtained directly from implicit differentiation of the representation*

$$p(\Pi) = (I - \mathit{diag}(d))\bar{p} + \mathit{diag}(d)[x + \Pi^\top p(\Pi)].$$

The term $(I - \text{diag}(d)\Pi^\top)^{-1}$ also appears in Chen et al. (2016), which the authors call the "network multiplier." This multiplier appears in the dual formulation of the linear program characterizing the Eisenberg–Noe clearing vector, where the authors introduce it to study the sensitivities of the clearing vector with respect to the capital (of defaulting banks) and the total liabilities (of non-defaulting banks). The computation of the directional derivative above can be viewed as a generalisation of this result to arbitrary perturbations. The interpretation remains the same in our case: the "network multiplier" describes how an estimation error propagates through the network.

5.1.3 A Taylor series for the Eisenberg–Noe clearing payments

In the same manner, we can define higher order directional derivatives.

Definition 5.1.11. *For $k \geq 1$, we define the k^{th} order directional derivative of the clearing vector with respect to a perturbation matrix Δ as*

$$\mathcal{D}_\Delta^{(k)} p(\Pi) := \lim_{h \to 0} \frac{\mathcal{D}_\Delta^{(k-1)} p(\Pi + h\Delta) - \mathcal{D}_\Delta^{(k-1)} p(\Pi)}{h}, \tag{5.1.4}$$

when the limit exists, and

$$\mathcal{D}_\Delta^{(0)} p(\Pi) = p(\Pi).$$

Remarkably, as Theorem 5.1.12 shows, all higher order derivatives also have an explicit formula, which allows us to obtain an exact Taylor series for the clearing vector. We impose an additional assumption on allowable perturbations $h\Delta$ so that the matrix $\text{diag}(d)$ (as defined in Theorem 5.1.9) as a function of $\Pi + h\Delta$ is fixed with respect to h, i.e., we require h sufficiently small so that the same subset of banks is in default when the liability matrix is $\Pi + h\Delta$ as when the liability matrix is Π. Let

$$\begin{aligned}
\overline{h}^{**} &:= \sup\left\{ h \le h^* \;\middle|\; \begin{array}{l} x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) < \bar{p}_i \\ \qquad \Leftrightarrow\; x_i + \sum_{j=1}^n (\pi_{ji} + h\delta_{ji}) p_j(\Pi + h\Delta) < \bar{p}_i \;\; \forall i \in \mathcal{N} \end{array} \right\}, \\
\underline{h}^{**} &:= \inf\left\{ h \ge -h^* \;\middle|\; \begin{array}{l} x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) < \bar{p}_i \\ \qquad \Leftrightarrow\; x_i + \sum_{j=1}^n (\pi_{ji} + h\delta_{ji}) p_j(\Pi + h\Delta) < \bar{p}_i \;\; \forall i \in \mathcal{N} \end{array} \right\}, \\
h^{**} &:= \min\{-\underline{h}^{**}, \overline{h}^{**}\}.
\end{aligned} \tag{5.1.5}$$

We necessarily have $h^{**} > 0$ because we exclude the measure-zero set $\{x \in \mathbb{R}^n_+ \mid \exists i \in \mathcal{N} \text{ s.t. } x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) = \bar{p}_i\}$ in which a bank is exactly at the brink of default.

Theorem 5.1.12. *Let $(\Pi, x, \bar{p})$ be a regular financial system. Then for $\Delta \in \boldsymbol{\Delta}^n(\Pi)$, and for all $k \ge 1$:*

$$\begin{aligned}
\mathcal{D}^{(k)}_\Delta p(\Pi) &= k\big(I - \mathit{diag}(d)\Pi^\top\big)^{-1} \mathit{diag}(d)\Delta^\top \mathcal{D}^{(k-1)}_\Delta p(\Pi) \\
&= k!\Big(\big(I - \mathit{diag}(d)\Pi^\top\big)^{-1} \mathit{diag}(d)\Delta^\top\Big)^k p(\Pi),
\end{aligned} \tag{5.1.6}$$

where $\mathcal{D}^{(0)}_\Delta p(\Pi) = p(\Pi)$. Moreover, for $h \in (-h^{**}, h^{**})$, the Taylor series

$$p(\Pi + h\Delta) = \sum_{k=0}^{\infty} \frac{h^k}{k!} \mathcal{D}^{(k)}_\Delta p(\Pi) \tag{5.1.7}$$

converges and has the following representation

$$p(\Pi + h\Delta) = \Big(I - h\big(I - \mathit{diag}(d)\Pi^\top\big)^{-1} \mathit{diag}(d)\Delta^\top\Big)^{-1} p(\Pi) \tag{5.1.8}$$

outside of the measure-zero set $\{x \in \mathbb{R}^n_+ \mid \exists i \in \mathcal{N} \text{ s.t. } x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) = \bar{p}_i\}$.

Proof. We prove the result by induction. Theorem 5.1.9 shows the result for $k = 1$. We now assume that equation (5.1.6) holds for k and we proceed to show that it holds for $k + 1$. As in Theorem 5.1.9, we show the existence of (5.1.4) by computing the two limits:

$$\begin{aligned}
\overline{\mathcal{D}^{(k+1)}_\Delta p(\Pi)}_i &= \limsup_{h \to 0} \frac{\mathcal{D}^{(k)}_\Delta p(\Pi + h\Delta)_i - \mathcal{D}^{(k)}_\Delta p(\Pi)_i}{h}, \\
\underline{\mathcal{D}^{(k+1)}_\Delta p(\Pi)}_i &= \liminf_{h \to 0} \frac{\mathcal{D}^{(k)}_\Delta p(\Pi + h\Delta)_i - \mathcal{D}^{(k)}_\Delta p(\Pi)_i}{h}.
\end{aligned}$$

The first order Taylor approximation for matrix inverses gives by the differentiation rules for the matrix inverse (cf. (Gentle 2007, p. 152)) for $X, Y \in \mathbb{R}^{n \times n}$ and h small enough: $(X + hY)^{-1} \approx X^{-1} - hX^{-1}YX^{-1}$. Applying this fact with $X = I - \text{diag}(d)\Pi^T$ and $Y = -\text{diag}(d)\Delta^T$, we have

$$\big(I - \text{diag}(d)(\Pi + h\Delta)^\top\big)^{-1} \approx \big(I - \text{diag}(d)\Pi^\top\big)^{-1} + h\big(I - \text{diag}(d)\Pi^\top\big)^{-1}\text{diag}(d)\Delta^\top\big(I - \text{diag}(d)\Pi^\top\big)^{-1}.$$

Additionally, we note that the k^{th} order derivative, similar to all lower order derivatives, is continuous with respect to the relative liabilities matrix Π since (by assumption of the induction) $\mathcal{D}_\Delta^{(k)} p(\Pi) = k!\big(\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top\big)^k p(\Pi)$, where $p(\Pi)$ and $\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}$ are both continuous with respect to Π (see Proposition 5.1.4 and the continuity of the matrix inverse). Consider now the upper limit

$$\begin{aligned}
\overline{\mathcal{D}_\Delta^{(k+1)} p(\Pi)} &= \limsup_{h\to 0} \frac{\mathcal{D}_\Delta^{(k)} p(\Pi + h\Delta) - \mathcal{D}_\Delta^{(k)} p(\Pi)}{h} \\
&= \limsup_{h\to 0} \frac{k}{h}\Big(\big(I - \operatorname{diag}(d)(\Pi + h\Delta)^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)} p(\Pi + h\Delta) \\
&\qquad\qquad - \big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)} p(\Pi)\Big) \\
&= \limsup_{h\to 0} k\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \frac{\mathcal{D}_\Delta^{(k-1)} p(\Pi + h\Delta) - \mathcal{D}_\Delta^{(k-1)} p(\Pi)}{h} \\
&\quad + \limsup_{h\to 0} \frac{k\,h}{h}\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)} p(\Pi + h\Delta) \\
&= k\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)} p(\Pi) \\
&\quad + k\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)} p(\Pi) \\
&= k\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)} p(\Pi) + \big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)} p(\Pi) \\
&= (k+1)\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)} p(\Pi).
\end{aligned}$$

Similarly, we obtain $\underline{\mathcal{D}_\Delta^{(k+1)} p(\Pi)} = (k+1)\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)} p(\Pi)$. The existence of the limit and the result (5.1.6) follow for all $k \geq 1$.

With the above results on all k^{th} order directional derivatives, we now consider the full Taylor expansion. First, by the definition of h^{**} given in (5.1.5), $\operatorname{diag}(d)$ is fixed for $h \in (-h^{**}, h^{**})$. By the definition of the clearing payments p (given in (5.1.2)) and defaulting firms $\operatorname{diag}(d)$ (defined in Theorem 5.1.9), along with the fact that $I - \operatorname{diag}(d)(\Pi + h\Delta)^\top$ is invertible (as shown in the proof of Theorem 5.1.9 since $(\Pi + h\Delta, x, \bar{p})$ remains a regular system by $h \in (-h^{**}, h^{**}) \subseteq (-h^*, h^*)$), we have

$$\begin{aligned}
p(\Pi + h\Delta) &= \operatorname{diag}(d)\big(x + (\Pi + h\Delta)^\top p(\Pi + h\Delta)\big) + \big(I - \operatorname{diag}(d)\big)\bar{p} \\
&= \big(I - \operatorname{diag}(d)(\Pi + h\Delta)^\top\big)^{-1}\Big(\operatorname{diag}(d)x + \big(I - \operatorname{diag}(d)\big)\bar{p}\Big).
\end{aligned} \tag{5.1.9}$$

Similarly, we find that

$$p(\Pi) = \big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\Big(\operatorname{diag}(d)x + \big(I - \operatorname{diag}(d)\big)\bar{p}\Big). \tag{5.1.10}$$

By combining (5.1.9) and (5.1.10), we immediately find

$$p(\Pi + h\Delta) = \big(I - \operatorname{diag}(d)(\Pi + h\Delta)^\top\big)^{-1}\big(I - \operatorname{diag}(d)\Pi^\top\big)p(\Pi).$$

Additionally, we can show that

$$\big(I - \operatorname{diag}(d)(\Pi + h\Delta)^\top\big)^{-1}\big(I - \operatorname{diag}(d)\Pi^\top\big) = \Big(I - h\big(I - \operatorname{diag}(d)\Pi^\top\big)^{-1}\operatorname{diag}(d)\Delta^\top\Big)^{-1}$$

directly by

$$\begin{aligned}
&\big(I - \mathrm{diag}(d)(\Pi + h\Delta)^\top\big)^{-1}\big(I - \mathrm{diag}(d)\Pi^\top\big)\Big(I - h\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\Big)\\
&\qquad = \big(I - \mathrm{diag}(d)(\Pi + h\Delta)^\top\big)^{-1}\big(I - \mathrm{diag}(d)\Pi^\top - h\,\mathrm{diag}(d)\Delta^\top\big)\\
&\qquad = \big(I - \mathrm{diag}(d)(\Pi + h\Delta)^\top\big)^{-1}\big(I - \mathrm{diag}(d)(\Pi + h\Delta)^\top\big) = I.
\end{aligned}$$

Therefore, for any $h \in (-h^{**}, h^{**})$, we find

$$p(\Pi + h\Delta) = \Big(I - h\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\Big)^{-1} p(\Pi),$$

i.e., (5.1.8).

Now let us consider the perturbations of size h within the neighborhood

$$\mathcal{H} := \left\{ h \in \mathbb{R} \;\middle|\; |h| < \min\left\{ h^{**}, \frac{1}{\rho\Big(\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\Big)} \right\} \right\}.$$

We will employ the following property of matrix inverses (see (Meyer 2000, p. 126)): If $X, Y \in \mathbb{R}^{n\times n}$ so that X^{-1} exists and $\lim_{k\to\infty}(X^{-1}Y)^k = 0$, then

$$(X+Y)^{-1} = \sum_{k=0}^{\infty}\big(-X^{-1}Y\big)^k X^{-1}.$$

We take $X = I - \mathrm{diag}(d)\Pi^\top$ and $Y = -h\,\mathrm{diag}(d)\Delta^\top$. Since $\rho\big(h\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\big) = |h|\rho\big(\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\big) < 1$ by the assumption that $|h| < \frac{1}{\rho\big(\big(I-\mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\big)}$, we have

$$\lim_{k\to\infty}\Big[h\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\Big]^k = 0,$$

using a property of the spectral radius (see (Meyer 2000, p. 617)). Thus, by combining this result with (5.1.9), we have

$$\begin{aligned}
p(\Pi + h\Delta) &= \big(I - \mathrm{diag}(d)(\Pi + h\Delta)^\top\big)^{-1}\Big(\mathrm{diag}(d)x + \big(I - \mathrm{diag}(d)\big)\bar{p}\Big)\\
&= \sum_{k=0}^{\infty}\Big(h\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\Big)^k\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\Big(\mathrm{diag}(d)x + \big(I - \mathrm{diag}(d)\big)\bar{p}\Big)\\
&= \sum_{k=0}^{\infty}\Big(h\big(I - \mathrm{diag}(d)\Pi^\top\big)^{-1}\mathrm{diag}(d)\Delta^\top\Big)^k p(\Pi)\\
&= \sum_{k=0}^{\infty}\frac{h^k}{k!}\mathcal{D}_\Delta^{(k)}p(\Pi).
\end{aligned}$$

The penultimate equality above follows directly from (5.1.10). The last equality follows directly from the definition of the k^{th} order directional derivatives proven above. Thus, we have shown the full Taylor expansion is exact on $\mathcal{H} \subseteq (-h^{**}, h^{**})$.

Finally, since we have already shown that (5.1.8) is exact for any $h \in (-h^{**}, h^{**})$ and

$$\left(-h\left(I - \mathrm{diag}(d)\Pi^\top\right)^{-1}\mathrm{diag}(d)\Delta^\top\right)$$

is singular for at least one of the elements $h \in \left\{-\frac{1}{\rho((I-\mathrm{diag}(d)\Pi^\top)^{-1}\mathrm{diag}(d)\Delta^\top)}, \frac{1}{\rho((I-\mathrm{diag}(d)\Pi^\top)^{-1}\mathrm{diag}(d)\Delta^\top)}\right\}$ by construction, it must follow that $h^{**} \leq \frac{1}{\rho((I-\mathrm{diag}(d)\Pi^\top)^{-1}\mathrm{diag}(d)\Delta^\top)}$. That is, $\mathcal{H} = (-h^{**}, h^{**})$.

$\square$

Comparing the directional derivative (5.1.3) to the full Taylor series (5.1.7) allows us to make the interpretation of the "network multiplier" more precise: the network multiplier captures the *first order* effect of the error propagation in the final "*round*" of the fictitious default algorithm. The k^{th} *order* effect of the error propagation is captured by the network multiplier raised to the k^{th} power. Finally, the Taylor series of the fixed point is the infinite series of these k^{th} order network multipliers; as this is of a similar form it can be interpreted as the multiplier of the network multiplier.

Remark 5.1.13. *We can extend the Taylor series expansion results to the more general space of perturbation matrices* $\overline{\boldsymbol{\Delta}}^n(\Pi)$ *rather than* $\boldsymbol{\Delta}^n(\Pi)$*. Over such a domain the Taylor series* (5.1.8) *is only guaranteed to converge for*

$$h \in \left[0, \min\left\{\overline{h}^{**}, \frac{1}{\rho\left((I - diag(d)\Pi^\top)^{-1} diag(d)\Delta^\top\right)}\right\}\right),$$

as negative perturbations are not feasible.

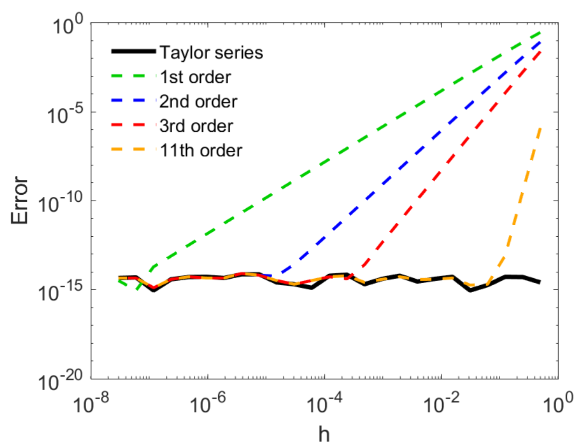


Figure 5.2: Loglog plot of the approximation error

$||p(\Pi) - p(\Pi + h\Delta)||_2$ against the size of the perturbation h for a random perturbation of the network introduced in Example 2.2.

5.2 Perturbation errors

In this section we study in detail estimation errors in an Eisenberg–Noe framework, relying on the directional derivatives discussed in the previous section. Specifically we calculate both maximal errors as well as the error distribution assuming a specific distribution of the mis-estimation of the interbank liabilities, notably uniform and Gaussian. We do this first in the original Eisenberg–Noe model, considering the Euclidean norm of the clearing vector as objective. Then we turn to an enhanced model that includes an additional node representing society and study the effect of estimation errors on the payout to society.

5.2.1 An orthonormal basis for perturbation matrices

We construct here an orthonormal basis for the matrices in $\mathbf{\Delta}^n(\Pi)$. To fix ideas, consider the case $n = 4$, where the general form of a matrix $\Delta \in \mathbf{\Delta}^4(\Pi_C)$ for a fully connected network Π_C can be written as

$$\mathbf{\Delta}^4(\Pi_C) = \left\{ \operatorname{diag}(\bar{p})^{-1} \begin{pmatrix} 0 & z_1 & z_2 & -z_1 - z_2 \\ z_3 & 0 & z_4 & -z_3 - z_4 \\ z_5 & -\sum_{k=1}^5 z_k & 0 & \sum_{k=1}^4 z_k \\ -z_3 - z_5 & \sum_{k=2}^5 z_k & -z_2 - z_4 & 0 \end{pmatrix} \;\middle|\; z \in \mathbb{R}^5 \right\},$$

from which it is clear that there are 5 degrees of freedom. It is easy to see that in general one has $d = n^2 - 3n + 1$ degrees of freedom. In the case $n = 4$, two such basis elements $\hat{E}_1$ and $\hat{E}_2$ are given by

$$\hat{E}_1 = \begin{pmatrix} 0 & \frac{1}{\bar{p}_1} & 0 & \frac{-1}{\bar{p}_1} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\bar{p}_3} & 0 & \frac{1}{\bar{p}_3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{E}_2 = \begin{pmatrix} 0 & 0 & \frac{1}{\bar{p}_1} & \frac{-1}{\bar{p}_1} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\bar{p}_3} & 0 & \frac{1}{\bar{p}_3} \\ 0 & \frac{1}{\bar{p}_4} & \frac{-1}{\bar{p}_4} & 0 \end{pmatrix}.$$

In general we note that $\mathbf{\Delta}^n(\Pi)$ is a closed, convex polyhedral set; we will take advantage of this fact in order to generate a general method for constructing basis matrices for $\mathbf{\Delta}^n(\Pi)$, as follows:

1. Define

$$\vec{\mathbf{\Delta}}^n(\Pi) := \Bigg\{ \delta \in \mathbb{R}^{n^2} \;\Bigg|\; \delta_{i+n(i-1)} = 0, \quad \sum_{j=1}^n \delta_{i+n(j-1)} = 0,$$
$$\sum_{j=1}^n \bar{p}_j \delta_{n(i-1)+j} = 0, \quad \mathbb{1}_{\{\pi_{ij}=0\}} \delta_{i+n(j-1)} = 0 \; \forall i, j \Bigg\}$$

   to be a vectorised version of $\mathbf{\Delta}^n(\Pi)$.

2. Construct a matrix $A(\Pi) \in \mathbb{R}^{(n^2+2n)\times n^2}$ so that $\vec{\mathbf{\Delta}}^n(\Pi) = \{\delta \in \mathbb{R}^{n^2} \mid A(\Pi)\delta = 0\}$. Note that the total degrees of freedom for $\vec{\mathbf{\Delta}}^n(\Pi)$ (and therefore also for $\mathbf{\Delta}^n(\Pi)$) is given by the rank of the matrix $A(\Pi)$. We include enough rows in the matrix $A(\Pi)$ in order to ensure that the n row sums and n (weighted) column sums are 0 and that components of δ are equal to zero based on $\pi_{ij} = 0$.

3. An orthonormal basis of $\vec{\mathbf{\Delta}}^n(\Pi)$ can be found by generating the orthonormal basis $\{e_1, ..., e_d\}$ of the null space of $A(\Pi)$.

4. Finally our basis matrices $\{E_1, ..., E_d\}$ can be generated by reshaping the basis of the null space of $A(\Pi)$ by setting $E_{k;i,j} := e_{k;i+n(j-1)}$ for any $k = 1, ..., d$ and $i, j \in \mathcal{N}$.

Definition 5.2.1. *The set*

$$\vec{E}^n(\Pi) := \{E_1, \ldots, E_d\}$$

is an orthonormal basis of perturbation matrices for the relative liability matrix Π. *Additionally, the vector*

$$\mathcal{D}_{\vec{E}(\Pi)} p(\Pi) := \big(\mathcal{D}_{E_1} p(\Pi), \ldots, \mathcal{D}_{E_d} p(\Pi)\big) \in \mathbb{R}^{n \times d}$$

is a vector of basis directional derivatives for the relative liability matrix Π*.*

We define two matrices to be orthogonal when their vectorised forms are orthogonal in $\mathbb{R}^{n^2}$, and note that, by construction, any matrix in the basis of perturbation matrices $\vec{E}^n(\Pi)$ has unit Frobenius norm.

Proposition 5.2.2. *Let* $\Pi \in \mathbf{\Pi}^n$*. Then the set of eigenvalues of* $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ *is the same for any choice of orthonormal basis of perturbation matrices* $\vec{E}(\Pi)$*. Additionally, if* $z\big(\lambda, \vec{E}(\Pi)\big) \in \mathbb{R}^d$ *is the eigenvector corresponding to eigenvalue* λ *and basis* $\vec{E}(\Pi)$*, then* $\sum_{k=1}^d z_k\big(\lambda, \vec{E}(\Pi)\big) E_k$ *is independent of the choice of basis.*

Proof. Let E be the vectorized version of $\vec{E}(\Pi)$ and let $F \neq E$ be a different orthonormal basis. By linearity of the directional derivative (see Theorem 5.1.9) we can immediately state that $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) = E^\top C E$ for some matrix $C \in \mathbb{R}^{n^2 \times n^2}$. Let (λ, v) be an eigenvalue and eigenvector pair for the operator $E^\top C E$ and let $z \in \mathbb{R}^d$ such that $Ev = Fz$. We will show that (λ, z) is an eigenvalue and eigenvector pair for $F^\top C F$ and thus the proof is complete:

$$\lambda z = \lambda F^\top F z = \lambda F^\top E v = F^\top E(\lambda v) = F^\top E E^\top C E v = F^\top C F z.$$

The last equality follows from the fact that $EE^\top = FF^\top$ is the unique projection matrix onto $\vec{\mathbf{\Delta}}^n(\Pi)$. ☐

Proposition 5.2.3. *Let* $\Pi \in \mathbf{\Pi}^n$*. Then* $\big\|\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top c\big\|_2$ *is independent of the choice of orthonormal basis of perturbation matrices* $\vec{E}(\Pi)$ *and for any fixed vector* $c \in \mathbb{R}^n$*.*

Proof. Let E and F be two distinct basis matrices for the vectorized perturbation space $\vec{\mathbb{P}}^n(\Pi)$ as in the proof of Proposition 5.2.2. By linearity of the directional derivative (see Theorem 5.1.9) we can immediately state that $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top c = E^\top \tilde{c}$ for some vector $\tilde{c} \in \mathbb{R}^{n^2}$. Immediately we can see that $\|E^\top \tilde{c}\|_2 = \|F^\top \tilde{c}\|_2$ since $EE^\top = FF^\top$ is the unique projection matrix onto $\vec{\mathbf{\Delta}}^n(\Pi)$. ☐

5.2.2 Deviations of the clearing vector

We concentrate first on the L^2-deviation of the actual clearing vector from the estimated one.

Largest shift of the clearing vector

We return to the first order directional derivative to quantify the largest shift of the clearing vector for estimation errors in the relative liability matrix given by perturbations in $\mathbf{\Delta}^n(\Pi)$. Let $\Delta \in \mathbf{\Delta}^n(\Pi)$ and assume that for a given $h \in \mathbb{R} : \Pi + h\Delta \in \mathbf{\Pi}^n$. Then, the worst case estimation error under $\mathbf{\Delta}^n(\Pi)$ is given as

$$\max_{\Delta\in\mathbf{\Delta}^n(\Pi)} \|p(\Pi + h\Delta) - p(\Pi)\|_2^2.$$

In order to remove the dependence on h and the magnitude of Δ, we consider instead the bounded set of directions $\mathbf{\Delta}^n_F(\Pi)$ and infintesimal perturbations,

$$\max_{\Delta\in\mathbf{\Delta}^n_F(\Pi)} \lim_{h\to 0} \frac{\|p(\Pi + h\Delta) - p(\Pi)\|_2^2}{h^2} = \max_{\Delta\in\mathbf{\Delta}^n_F(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2.$$

In this section, we call $\|\mathcal{D}_\Delta p(\Pi)\|_2^2$ the estimation error and $\max_{\Delta\in\mathbf{\Delta}^n_F(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2$ the maximal deviation in the clearing vector under $\mathbf{\Delta}^n_F(\Pi)$. Because Δ appears via a linear term in (5.1.3), this allows us to use a basis of perturbation matrices in an elegant way to quantify the deviation of the Eisenberg–Noe clearing vector under the space of perturbations $\mathbf{\Delta}^n_F(\Pi)$.

Throughout the following results we will take advantage of an orthonormal basis $\vec{E}(\Pi) = (E_1, \ldots, E_d)$ of the space $\mathbf{\Delta}^n(\Pi)$. More details of this space are given in Section 5.2.1.

Proposition 5.2.4. *Let $(\Pi, x, \bar{p})$ be a regular financial system. The worst case first order estimation error under $\mathbf{\Delta}^n_F(\Pi)$ is given by*

$$\max_{\Delta\in\mathbf{\Delta}^n_F(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 = \left(\|\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2^o\right)^2 \tag{5.2.1}$$

for any choice of basis $\vec{E}(\Pi)$ where $\|\cdot\|_2^o$ denotes the spectral norm of a matrix. Furthermore, the largest shift of the clearing vector is achieved by

$$\Delta^*(\Pi) := \pm \sum_{k=1}^d z_k E_k,$$

where z_k are the components of the (normalized) eigenvector corresponding to the maximum eigenvalue of $\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$.

Proof. Note first that any perturbation matrix $\Delta \in \mathbf{\Delta}^n(\Pi)$ can be written as a linear combination of basic perturbation matrices, i.e., $\Delta = \sum_{k=1}^d z_k E_k$. Thus,

$$\|\mathcal{D}_\Delta p(\Pi)\|_2^2 = \left\|\sum_{k=1}^d z_k \mathcal{D}_{E_k} p(\Pi)\right\|_2^2 = z^\top \left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z.$$

Immediately this implies, denoting the largest eigenvalue of a matrix A by $\lambda_{\max}(A)$,

$$\begin{aligned}
\max_{\Delta\in\mathbf{\Delta}^n_F(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 &= \max_{\|z\|_2\le 1} z^\top \left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \\
&= \lambda_{\max}\Big(\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\Big) \\
&= \left(\|\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2^o\right)^2.
\end{aligned}$$

Finally, the independence of the solution from the choice of basis $\vec{E}(\Pi)$ is a direct result of Proposition 5.2.2. $\square$

Hence, if the true liability matrix were perturbed in the direction of $\Delta^*(\Pi)$, this would generate the largest first order estimation error in the clearing vector. By error, we mean the Euclidean distance between the "true" clearing vector in the standard Eisenberg–Noe framework, and the clearing vector under the perturbed liabilities matrix. This is in general not equivalent to the direction that would change the default set most rapidly. Moreover, if regulatory expert judgement allowed to estimate reasonable absolute perturbations, our infinitesimal methodology could be used iteratively in a greedy approach until such an absolute estimation error was reached.

We can use this result on the maximum deviations of the clearing vector under $\mathbf{\Delta}^n_F(\Pi)$ in order to provide bounds of the worst case perturbation error without predetermining the existence or non-existence of links.

Corollary 5.2.5. *Let $(\Pi, x, \bar{p})$ be a regular financial system. The worst case first order estimation error under all perturbations is bounded by*

$$\left(\|\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2^o\right)^2 \leq \max_{\Delta \in \overline{\mathbf{\Delta}}^n_F(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 \leq \left(\|\mathcal{D}_{\vec{E}(\Pi_C)} p(\Pi)\|_2^o\right)^2 \tag{5.2.2}$$

for any choice of orthonormal bases $\vec{E}(\Pi)$ as above and $\vec{E}(\Pi_C)$ of any completely connected network Π_C. In the case that Π itself is a completely connected network then this upper bound is attained.

Proof. For all Π and all completely connected networks Π_C, we have $\mathbf{\Delta}^n_F(\Pi) \subseteq \overline{\mathbf{\Delta}}^n_F(\Pi) \subseteq \mathbf{\Delta}^n_F(\Pi_C)$. Hence, using (5.1.3), one obtains

$$\begin{aligned}
\max_{\Delta \in \overline{\mathbf{\Delta}}^n_F(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 &\leq \max_{\Delta \in \mathbf{\Delta}^n_F(\Pi_C)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 \\
&= \max_{\|z\|_2 \leq 1} \left\| (I - \mathrm{diag}(d)\Pi^\top)^{-1} \mathrm{diag}(d) \left[\sum_{k=1}^d z_k E_k\right]^\top p(\Pi) \right\|_2^2 \\
&= \left(\|\mathcal{D}_{\vec{E}(\Pi_C)} p(\Pi)\|_2^o\right)^2, \\
\max_{\Delta \in \overline{\mathbf{\Delta}}^n_F(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 &\geq \max_{\Delta \in \mathbf{\Delta}^n_F(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 \\
&= \left(\|\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2^o\right)^2,
\end{aligned}$$

where $\vec{E}(\Pi_C) := (E_1, \dots, E_d)$ is an orthonormal basis of the space $\mathbf{\Delta}^n(\Pi_C)$. As in Proposition 5.2.4, the independence of the solution from the choice of basis $\vec{E}(\Pi_C)$ is a direct result of Proposition 5.2.2. $\square$

Remark 5.2.6. *Our empirical analysis suggests that this bound is quite sharp (see Figure 5.16(b)).*

Example 5.2.7. *We return to Example 5.1.5 and consider the same toy network consisting of four banks in which each bank's nominal liabilities are shown in Figure 5.1(a). The largest shift of the clearing vector* (5.2.1) *under $\mathbf{\Delta}^4_F(\Pi)$, as described in Proposition 5.2.4, is given by the matrix*

$$\Delta^*(\Pi) = \begin{pmatrix} 0 & 0.3230 & -0.1615 & -0.1615 \\ -0.0381 & 0 & 0.0190 & 0.0190 \\ 0.0571 & -0.4845 & 0 & 0.4274 \\ 0.0571 & -0.4845 & 0.4274 & 0 \end{pmatrix}.$$

As this network is complete, this is furthermore a solution to both optimization problems (5.2.1) *and* (5.2.2) *for the worst case perturbation. Additionally, the upper bound in Corollary 5.2.5 is attained. This perturbation is depicted in Figure 5.4. As before, banks who are in default are colored red. The edges are labeled with the perturbation of the respective link between banks that achieves this greatest estimation error. The edge linking one node to another is red if the greatest estimation error under the set of perturbations $\mathbf{\Delta}_F^n(\Pi)$ occurs when we have overestimated the value of this link and green if we have underestimated it. Note that due to the symmetry of the optimal estimation error problem, $-\Delta^*(\Pi)$ is also optimal and thus the interpretation of red and green links in Figure 5.4 can be reversed. Indeed, when studying the deviation of the clearing vector, the solutions $\Delta^*(\Pi)$ and $-\Delta^*(\Pi)$ are equivalent. When analyzing the shortfall of payments to society in Section 5.2.3, this will be no longer the case. Edge widths are proportional to the absolute value of the entries in $\Delta^*(\Pi)$. Though our Taylor expansion results (Theorem 5.1.12) are provided for $h \in (-h^{**}, h^{**})$ only, the strict inequality is only necessary if h^{**} denotes the perturbation size at which a new bank defaults, not when a connection is removed. So when $h = h^{**} \approx 0.688$, we obtain*

$$L^* = \begin{pmatrix} 0 & 9 & 0 & 0 \\ 2.76 & 0 & 3.12 & 3.12 \\ 1.12 & 0 & 0 & 1.88 \\ 1.12 & 0 & 1.88 & 0 \end{pmatrix},$$

which has the clearing vector

$$\hat{p} \approx (4.11, 6.11, 3, 3)^\top.$$

One can immediately verify that L^* has indeed the same total interbank assets and liabilities for each bank, but they are distributed in a different manner. Hence, in this example, there can be a deviation of up to 15% in the relative norm of the clearing vector for a network that is still consistent with the total assets and total liabilities.

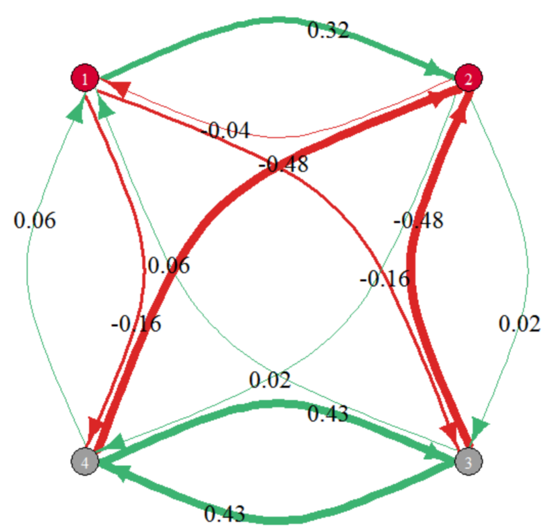


Figure 5.4: Worst case network perturbation under $\mathbf{\Delta}_F^n(\Pi)$ defined in Example 5.2.7.

Remark 5.2.8. *It may be desirable to normalize the first order estimation errors by, e.g., the clearing payments or total nominal liabilities, rather than considering the absolute error. In a gen-*

eral form, let $A \in \mathbb{R}^{n\times n}$ denote a normalization matrix (e.g., $A = diag(p(\Pi))^{-1}$ or $A = diag(\bar{p})^{-1}$). Then we can extend the results of Proposition 5.2.4 and Corollary 5.2.5 by

$$\max_{\Delta\in\boldsymbol{\Delta}_F^n(\Pi)} \|A\mathcal{D}_\Delta p(\Pi)\|_2^2 = \big(\|A\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2^\circ\big)^2$$
$$\max_{\Delta\in\overline{\boldsymbol{\Delta}}_F^n(\Pi)} \|A\mathcal{D}_\Delta p(\Pi)\|_2^2 \le \big(\|A\mathcal{D}_{\vec{E}(\Pi_C)} p(\Pi)\|_2^\circ\big)^2$$

for any completely connected network Π_C. Similarly, the distribution results presented below can be generalized by considering $A\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ in place of $\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$.

Clearing vector deviation for uniformly distributed estimation errors

Then, we will extend the above analysis to the case when estimation errors are uniformly distributed. This is done by considering the linear coefficients z for the basis of perturbation matrices to be chosen uniformly on the d-dimensional Euclidean unit ball. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix.

Proposition 5.2.9. *Let $(\Pi, x, \bar{p})$ be a regular financial system. The distribution of the estimation error when the perturbations are uniformly distributed in the L^2-unit ball is given by*

$$\mathbb{P}\big(\|\mathcal{D}_\Delta p(\Pi)\|_2^2 \le \alpha\big) = \frac{vol\big(\{w \in \mathbb{R}^d \,|\, w^\top w \le 1, w^\top \Lambda w \le \alpha\}\big)\Gamma\big(\frac{d}{2}+1\big)}{\pi^{d/2}}, \quad \alpha \ge 0,$$

where Λ is the diagonal matrix with elements given by the eigenvalues of $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ for any choice of orthonormal basis $\vec{E}(\Pi)$, vol denotes the volume operator, and Γ is the gamma function.

Proof. Let z be uniform on the d-dimensional unit ball. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix. One obtains

$$\begin{aligned}\mathbb{P}\big(\|\mathcal{D}_\Delta p(\Pi)\|_2^2 \le \alpha\big) =&\mathbb{P}\Big(\big(\mathcal{D}_\Delta p(\Pi)\big)^\top \mathcal{D}_\Delta p(\Pi) \le \alpha\Big)\\ =&\mathbb{P}\Big(z^\top\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \le \alpha\Big).\end{aligned}$$

The matrix $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ is diagonalizable because it is real and symmetric. Therefore, we can write

$$\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) = V^\top \Lambda V,$$

where Λ is a diagonal matrix of the eigenvalues and V is orthonormal. Combining the above equations, we have

$$\mathbb{P}\Big(z^\top\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \le \alpha\Big) =\mathbb{P}\big(z^\top V^\top \Lambda V z \le \alpha\big).$$

Then since z is uniform on the unit ball and $VV^\top = I$, $w = Vz$ is also uniform on the unit ball and thus we have

$$\begin{aligned}\mathbb{P}\big(z^\top V^\top \Lambda V z \le \alpha\big) =&\mathbb{P}\big(w^\top \Lambda w \le \alpha\big)\\ =&\frac{\mathrm{vol}\big(\{w \mid w^\top w \le 1, w^\top \Lambda w \le \alpha\}\big)}{\mathrm{vol}\big(\{w \mid w^\top w \le 1\}\big)}\\ =&\frac{\mathrm{vol}\big(\{w \mid w^\top w \le 1, w^\top \Lambda w \le \alpha\}\big)\Gamma\big(\frac{d}{2}+1\big)}{\pi^{d/2}}.\end{aligned}$$

As in Proposition 5.2.4, the independence of the distribution from the choice of basis $\vec{E}(\Pi_C)$ is a direct result of Proposition 5.2.2. $\square$

Remark 5.2.10. *In the case where* $\alpha \le \min_k \lambda_k$ *or* $\alpha \ge \max_k \lambda_k$ *then* $\mathbb{P}\big(\|\mathcal{D}_\Delta p(\Pi)\|_2^2 \le \alpha\big)$ *can be given explicitly by* $\alpha^d \prod_{k=1}^d \frac{1}{\sqrt{\lambda_k}}$ *and* 1 *respectively where* $\{\lambda_k \mid k = 1, \dots, d\}$ *is the collection of eigenvalues of* $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$*. In the case that* $\min_k \lambda_k < \alpha < \max_k \lambda_k$*, the probability* $\mathbb{P}\big(\|\mathcal{D}_\Delta p(\Pi)\|_2^2 \le \alpha\big)$ *can be given via the volume formula provided in Proposition 5.2.9 as d nested integrals,*

$$\frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{d/2}} \int_{-1}^{1} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \cdots \int_{-\sqrt{1-\sum_{k=1}^{m-1} x_k^2}}^{\sqrt{1-\sum_{k=1}^{m-1} x_k^2}} \int_{-\sqrt{\frac{\alpha-\sum_{k=1}^{m} \lambda_{[k]} x_k^2}{\lambda_{[m+1]}}}}^{\sqrt{\frac{\alpha-\sum_{k=1}^{m} \lambda_{[k]} x_k^2}{\lambda_{[m+1]}}}} \cdots \int_{-\sqrt{\frac{\alpha-\sum_{k=1}^{d-1} \lambda_{[k]} x_k^2}{\lambda_{[d]}}}}^{\sqrt{\frac{\alpha-\sum_{k=1}^{d-1} \lambda_{[k]} x_k^2}{\lambda_{[d]}}}} dx_d \cdots dx_1,$$

where $\lambda_{[m]} \le \alpha \le \lambda_{[m+1]}$ *and* $\lambda_{[m]}$ *is a reordering of the eigenvalues such that* $0 \le \lambda_{[1]} \le \lambda_{[2]} \le \cdots \le \lambda_{[d]}$*.*

Example 5.2.11. *We return again to Example 5.1.5 to consider perturbations* Δ *sampled from the uniform distribution. Figure 5.5 shows the probability density function and the cumulative distribution function (CDF) estimation for the relative estimation error,* $\|\mathcal{D}_\Delta p(\Pi)\|_2^2/\|p(\Pi)\|_2^2$*, corresponding to our stylized four-bank network. The probabilities are estimated from 100,000 simulated uniform perturbations.*

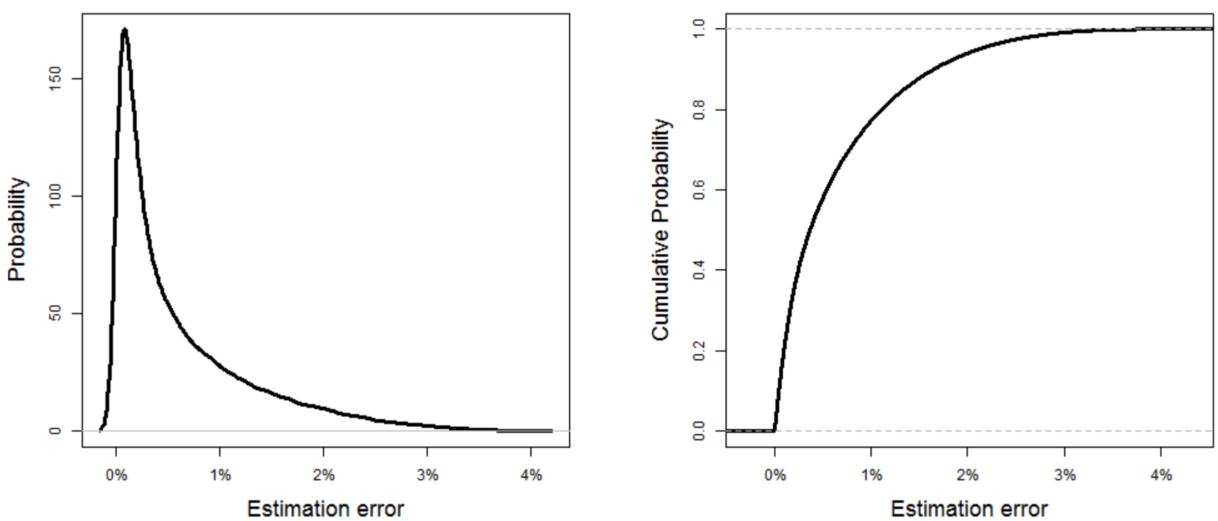


Figure 5.5: The Probability density (left) and the CDF (right)

The relative estimation error, $\|\mathcal{D}_\Delta p(\Pi)\|_2^2/\|p(\Pi)\|_2^2$, under uniform perturbations Δ as described in Example 5.2.11.

Clearing vector deviation for normally distributed estimation errors

We extend our previous analysis by considering normally distributed perturbations. To do so, we consider the linear coefficients z for the basis of perturbation matrices to be chosen distributed according to the standard d-dimensional multivariate standard Gaussian distribution. Then $\sum_{k=1}^{d} z_k E_k$ is a perturbation matrix Δ. Though our prior results on the deviations of the clearing payments have been within the unit ball $\boldsymbol{\Delta}_F^n(\Pi)$, under a Gaussian distribution the magnitude of the perturbation matrices are no longer bounded by 1 and thus the estimation errors can surpass the worst case errors determined in Proposition 5.2.4 and Corollary 5.2.5.

Proposition 5.2.12. *Let $(\Pi, x, \bar{p})$ be a regular financial system. The distribution of estimation errors where the perturbations are distributed with respect to the standard normal is given by the moment generating function*

$$M(t) := \det\big(I - 2\Lambda t\big)^{-1/2},$$

where Λ is the diagonal matrix with elements given by the eigenvalues of $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ for any orthonormal basis $\vec{E}(\Pi)$.

Proof. Let z be a d-dimensional standard normal Gaussian random variable. Then $\Delta = \sum_{k=1}^{d} z_k E_k$ is a perturbation matrix. As in Proposition 5.2.9, we can write

$$z^\top \left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z = z^\top V^\top \Lambda V z,$$

where Λ is the diagonal matrix of eigenvalues of $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ and V is orthonormal. Since $z \sim N(0, I)$ and $VV^\top = I$, we have $w = Vz \sim N(0, VV^\top = I)$. Therefore,

$$z^\top V^\top \Lambda V z = w^\top \Lambda w = w^\top \Lambda^{1/2} \Lambda^{1/2} w.$$

Then $y = \Lambda^{1/2} w \sim N(0, \Lambda)$ and so each component $y_k \sim N(0, \lambda_k)$ and the y_k's are independent. Therefore,

$$w^\top \Lambda^{1/2} \Lambda^{1/2} w = y^\top y = \sum_{k=1}^{d} y_k^2.$$

The distribution of y_k^2 is $\Gamma(1/2, 2\lambda_k)$, and thus the sum $\sum_{k=1}^{d} y_k^2$ has the moment generating function

$$M(t) = \prod_{k=1}^{d} \big(1 - 2\lambda_k t\big)^{-1/2},$$

where λ_k are the eigenvalues of $\big(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\big)^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$. As in Proposition 5.2.4, the independence of the distribution from the choice of basis $\vec{E}(\Pi_C)$ is a direct result of Proposition 5.2.2. $\square$

Remark 5.2.13. *A closed form for the density of the distribution found in Proposition 5.2.12 is given in equation (7) of Mathai (1982).*

Example 5.2.14. *We return again to Example 5.1.5 to consider perturbations Δ sampled from the standard normal distribution. Figure 5.7 shows the density and CDF estimation for the relative estimation error, $\|\mathcal{D}_\Delta p(\Pi)\|_2^2 / \|p(\Pi)\|_2^2$, corresponding to our stylized four-bank network. The probabilities are estimated from 100,000 simulated Gaussian perturbations.*

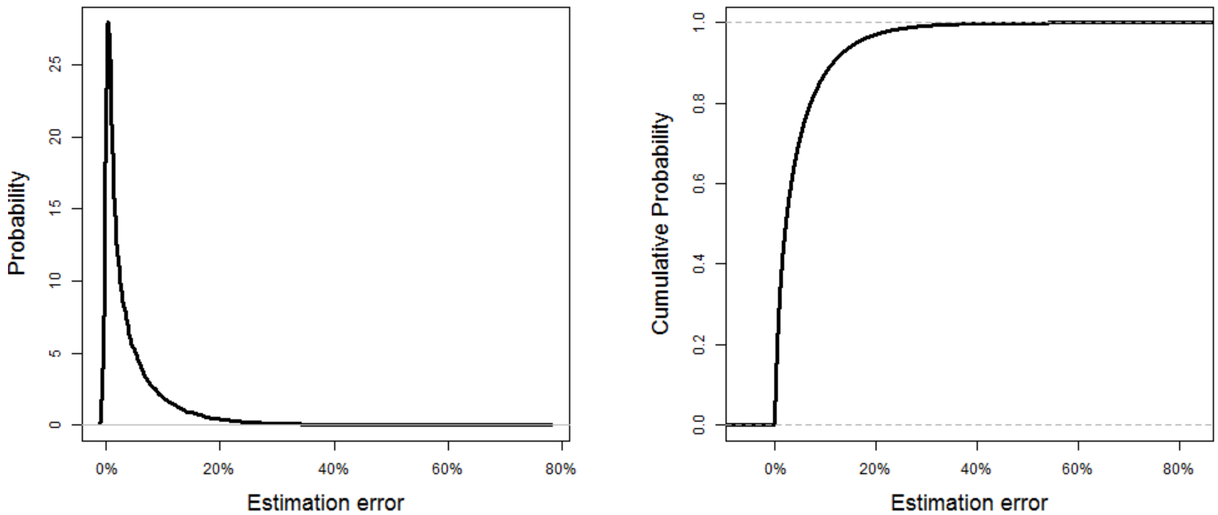


Figure 5.7: The estimated probability density (left) and the CDF (right)

The estimation error, $\|\mathcal{D}_\Delta p(\Pi)\|_2^2/\|p(\Pi)\|_2^2$, under standard Gaussian perturbations Δ as described in Example 5.2.14.

5.2.3 Impact to the payout to society

In this section, we assume that in addition to their interbank liabilities, banks also have a liability to society. Here, society is used as totum pro parte, encompassing all non-financial counterparties, corporate, individual or governmental. Hence, the set of institutions becomes $\mathcal{N}_0 = \{0\} \cup \mathcal{N}$. Without loss of generality, we assume that all banks $i \in \mathcal{N}$ owe money to at least one counterparty $j \in \mathcal{N}_0$ within the system. Otherwise, a bank who owes no money can be absorbed by the society node as it plays the same role within the model structure. The question of interest is then how the payout to society may be mis-estimated (and in particular overestimated) given estimation errors in the relative liabilities matrix. This setting has been studied in, e.g., Glasserman and Young (2016) with the introduction of *outside liabilities*. We adopt their framework to analyze this question.

The interbank liability matrix L of the previous section is expanded to $L_0 \in \mathbb{R}^{(n+1)\times(n+1)}$ given by

$$L_0 = \left[\begin{array}{ccc|c} 0 & \cdots & L_{1n} & L_{10} \\ \vdots & \ddots & \vdots & \vdots \\ L_{n1} & \cdots & 0 & L_{n0} \\ \hline 0 & \cdots & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} & L & & l_0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right],$$

where $l_0 = \left(L_{10}, \cdots, L_{n0}\right)^\top$ is the society liability vector. We require that at least one bank has an obligation to society, i.e., $L_{i0} > 0$ for some $1 \le i \le n$. The total liability of bank i is now given by $\bar{p}_i = \sum_{j=0}^n L_{ij}$. As stated above, we also require that each bank owes to at least one counterparty within the system (possibly society), i.e., $\bar{p}_i > 0$ for all $i \in \mathcal{N}$. The relative liability matrix Π_0 is transformed accordingly, i.e., $\pi_{ij} \in [0,1]$ and $\pi_{ij} = \frac{L_{ij}}{\bar{p}_i}$. An admissible relative liability matrix Π_0 thus belongs to the set of all right stochastic matrices with entries in $[0,1]$, all diagonal entries 0, and at least one $\pi_{i0} > 0$:

$$\mathbf{\Pi}_0^n := \left\{ \Pi_0 \in [0,1]^{(n+1)\times(n+1)} \;\middle|\; \forall i : \pi_{ii} = 0,\ \sum_{j=0}^{n} \pi_{ij} = 1 \text{ and } \exists i \text{ s.t. } \pi_{i0} > 0 \right\}.$$

An admissible *interbank* relative liability matrix Π thus belongs to the set

$$\mathbf{\Pi}_I^n := \left\{ \Pi \in [0,1]^{n\times n} \;\middle|\; \forall i : \pi_{ii} = 0,\ \sum_{j=1}^{n} \pi_{ij} \leq 1 \text{ and } \exists i \text{ s.t. } \sum_{j=1}^{n} \pi_{ij} < 1 \right\},$$

which has the same properties as the original interbank relative liability matrix $\mathbf{\Pi}^n$ defined in (5.1.1), except that row sums are smaller or equal to 1, with at least one strictly smaller than 1.

The following result is implicitly used in the subsequent sections. This provides us with the ability to, e.g., consider the directional derivative with respect to the payments made by the n financial firms without considering the societal node (which is equal to 0 by assumption).

Proposition 5.2.15. *If $(\Pi_0, x, \bar{p})$ is a regular network then $I - diag(d)\Pi$ is invertible.*

Proof. This follows immediately from

$$I - \text{diag}(d_0)\Pi_0^\top = \begin{pmatrix} I - \text{diag}(d)\Pi^\top & -\text{diag}(d)\pi_0 \\ \mathbf{0}^\top & 1 \end{pmatrix},$$

where $\pi_0 = \left(\pi_{10}, \cdots, \pi_{n0}\right)^\top$ and d_0 is the vector of default indicators (of length $n+1$ to include the societal node). In particular, since $\det(I - \text{diag}(d_0)\Pi_0^\top) \neq 0$ (as shown in the proof of Theorem 5.1.9), we can conclude that $\det\left(I - \text{diag}(d)\Pi^\top\right) \neq 0$. □

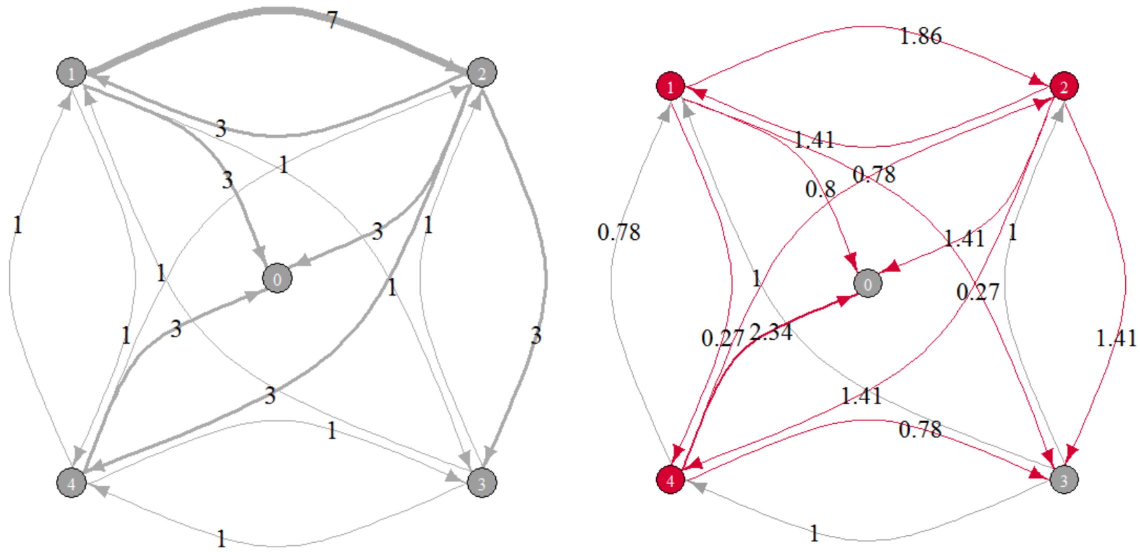
Example 5.2.16. *We include now a society node into our example from Section 5.1. The nominal interbank liabilities and liabilities from each bank to society are shown in Figure 5.9(a). Note that at least one bank has an obligation to society and the society does not owe to any bank. As above, the banks' external assets are given by the vector $x = (0, 2, 2, 2)^\top$. The clearing payments, or the amount of its obligations that each bank is able to repay, is given in Figure 5.9(b). Banks who are in default are colored red, as are the liabilities that are not repaid in full.*

Largest reduction in the payout to society

Next, we use the directional derivative in order to quantify how estimation errors, under $\mathbf{\Delta}_F^n(\Pi)$ in the interbank relative liability matrix, could lead to an overestimation of the payout to society. As it turns out, this problem also has an elegant solution using the basis of perturbation matrices discussed in Section 5.2.1. We assume that $(\Pi_0, x, \bar{p})$ is a regular financial system and additionally that both the relative liabilities to society $\pi_0 = (\pi_{10}, ..., \pi_{n0})^\top$ and the total liabilities $\bar{p}$ are exactly known.

Definition 5.2.17. *Let $(\Pi_0, x, \bar{p})$ be a regular financial system. The payout to society is defined as the quantity $\pi_0^\top p(\Pi)$ where $p(\Pi)$ is the clearing vector of the n firms.*

Herein we consider the relative liabilities matrix Π_0 to be an estimation of the true relative liabilities. We thus consider the perturbations of the estimated clearing vectors to determine the maximum amount that the payout to society may be overestimated. To study the optimisation problem of minimizing the payout to society, we assume that at least one bank, but not all banks, default. The following proposition shows that this assumption excludes only trivial cases.



(a) Nominal interbank liabilities.

(b) Clearing interbank payments.

Figure 5.9: Initial network defined in Example 5.2.16.

Proposition 5.2.18. *Let $(\Pi_0, x, \bar{p})$ be a regular system with the interbank relative liability matrix $\Pi \in \mathbf{\Pi}_I^n$ and $\Delta \in \mathbf{\Delta}^n(\Pi)$. If all banks default, or if no bank defaults, then the payout to society remains unchanged for an arbitrary admissible perturbation Δ.*

Proof. Let Δ be an arbitrary perturbation matrix. We show that in both cases $\pi_0^\top \mathcal{D}_\Delta p(\Pi) = 0$.

1. Assume that no bank defaults. Then $\operatorname{diag}(d) = 0$, and the result holds as $\mathcal{D}_\Delta p(\Pi) = 0$.

2. Assume all banks default. Then $\operatorname{diag}(d) = I$. Hence, $\pi_0^\top \mathcal{D}_\Delta p(\Pi) = \pi_0^\top \left(I - \Pi^\top\right)^{-1} \Delta^\top p(\Pi)$. Note that $\pi_0^\top \left(I - \Pi^\top\right)^{-1} = \mathbf{1}^\top$, because by definition $\pi_0^\top = \mathbf{1}^\top\left(I - \Pi^\top\right)$. Using this and the definitions of $\mathcal{D}_\Delta p(\Pi)$ and Δ, it follows $\pi_0^\top \mathcal{D}_\Delta p(\Pi) = \sum_{i=1}^n \sum_{j=1}^n \delta_{ji} p_j(\Pi) = 0$.

$\square$

Let $\Delta \in \mathbf{\Delta}^n(\Pi)$ and assume that for a given $h \in \mathbb{R} : \Pi + h\Delta \in \mathbf{\Pi}_I^n$. Then, the minimum payout to society is

$$\min_{\Delta \in \mathbf{\Delta}^n(\Pi)} \pi_0^\top p(\Pi + h\Delta).$$

In order to remove the dependence on h and the magnitude of Δ, we subtract the constant term $\pi_0^\top p(\Pi)$ and consider instead

$$\min_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \lim_{h \to 0} \pi_0^\top \frac{p(\Pi + h\Delta) - p(\Pi)}{h} = \min_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \pi_0^\top \mathcal{D}_\Delta p(\Pi).$$

Using the basis of perturbation matrices $\vec{E}(\Pi)$ of $\mathbf{\Delta}^n(\Pi)$ (see Section 5.2.1), we can compute the shortfall to society due to perturbations in the relative liability matrix in $\mathbf{\Delta}_F^n(\Pi)$.

Proposition 5.2.19. *Let $(\Pi_0, x, \bar{p})$ be a regular financial system. The largest shortfall in payments to society due to estimation errors in the liability matrix in $\mathbf{\Delta}^n_F(\Pi)$ is given by*

$$\min_{\Delta \in \mathbf{\Delta}^n_F(\Pi)} \pi_0^\top \mathcal{D}_\Delta p(\Pi) = -\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2.$$

Furthermore, the largest shortfall to society is achieved by

$$\Delta_0^*(\Pi) := -\sum_{k=1}^d \frac{\pi_0^\top \mathcal{D}_{E_k} p(\Pi)}{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2} E_k.$$

Additionally, both the largest shortfall and the perturbation matrix that attains that shortfall are independent of the chosen basis $\vec{E}(\Pi)$.

Proof. Since the problem

$$\min \pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \ \text{ s.t. } \ \|z\|_2 \leq 1,$$

has a linear objective, it is equivalent to

$$\min \pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \ \text{ s.t. } \ z^\top z = 1.$$

By the necessary Karush–Kuhn–Tucker conditions, we know that any solution to this problem must satisfy

$$\begin{aligned} \left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0 + 2\mu z &= 0, \\ z^\top z &= 1, \end{aligned}$$

for some $\mu \in \mathbb{R}$. The first condition implies $z^* = -\frac{(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0}{2\mu}$. Plugging this into the second implies that $\mu = \pm \frac{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2}{2}$. With two possible solutions we plug these back into the original objective to find that the minimum is attained at $\mu = \frac{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2}{2}$ for an optimal value of:

$$\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z^* = -\frac{\left(\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)\left(\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top}{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2} = -\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2.$$

Therefore, the solution is

$$\Delta_0^*(\Pi) = \sum_{k=1}^d z_k^* E_k = -\sum_{k=1}^d \frac{\pi_0^\top \mathcal{D}_{E_k} p(\Pi)}{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2} E_k.$$

By Proposition 5.2.3, this result is independent of the choice of basis matrices. □

Corollary 5.2.20. *Let $(\Pi_0, x, \bar{p})$ be a regular financial system. The worst case shortfall to society is bounded by*

$$-\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi_C)} p(\Pi)\|_2 \leq \min_{\Delta \in \overline{\mathbf{\Delta}}^n_F(\Pi)} \pi_0^\top \mathcal{D}_\Delta p(\Pi) \leq -\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2,$$

where $\vec{E}(\Pi_C)$ is any orthonormal basis of perturbation matrices of any completely connected network Π_C. In the case that Π itself is a completely connected network then this upper bound is attained.

Proof. This follows by the same logic as Corollary 5.2.5 through the inclusion $\mathbf{\Delta}_F^n(\Pi) \subseteq \overline{\mathbf{\Delta}}_F^n(\Pi) \subseteq \mathbf{\Delta}_F^n(\Pi_C)$ for any completely connected network Π_C. The independence of this result to the choice of orthonormal basis $\vec{E}(\Pi)$ follows as in Proposition 5.2.19. ☐

Example 5.2.21. *We continue the discussion from Example 5.2.16: The perturbation resulting in the greatest shortfall for the society's payout, as described in Proposition 5.2.19, is given by the matrix*

$$\Delta_0^* = \begin{pmatrix} 0 & 0.16 & -0.46 & 0.30 \\ 0.11 & 0 & 0.16 & -0.27 \\ 0.06 & 0.04 & 0 & -0.10 \\ -0.26 & -0.34 & 0.60 & 0 \end{pmatrix}.$$

This perturbation is depicted in Figure 5.10. Each edge is labeled with the perturbation of the respective link between banks that achieves this greatest reduction in payout to society. As before, banks who are in default are colored red. The edge linking one node to another is red if the greatest reduction in payout occurs when we have overestimated the value of this link and green if, in the worst case under $\mathbf{\Delta}_F^n(\Pi)$, *we have underestimated the value of this link. Edge widths are proportional to the absolute value of the entries in* $\Delta_0^*(\Pi)$. *In contrast to Example 5.2.7, note that* $-\Delta_0^*(\Pi)$ *is not a solution anymore. As this network is complete, this also equals the worst case shortfall of* -1.4513, *which is nearly 32% of the entire estimated payment to society.*

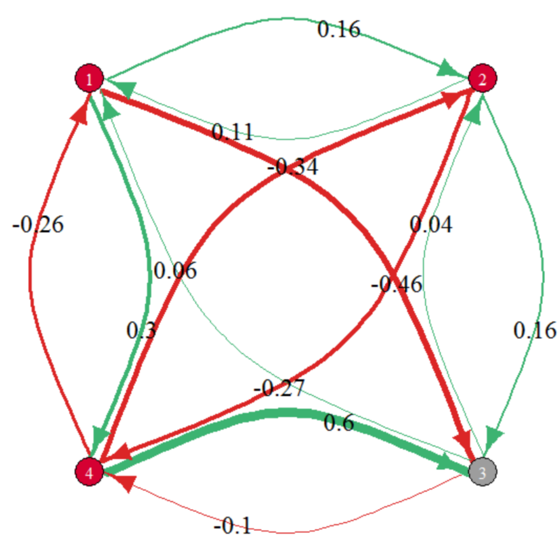


Figure 5.10: The perturbation in $\mathbf{\Delta}_F^n(\Pi)$ which generates the largest shortfall in Example 5.2.21.

Shortfall to society for uniformly distributed estimation errors

In this section we compute the reduction in the payout to society when the perturbations are uniformly distributed. To do so, we consider the linear coefficients z for the basis of perturbation matrices to be chosen uniformly from the d-dimensional Euclidean unit ball. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix.

Proposition 5.2.22. *Let* $(\Pi_0, x, \bar{p})$ *be a regular financial system. The distribution of changes in* *payments to society where the perturbations are uniformly distributed on the unit ball is given by*

$$\mathbb{P}\left(\pi_0^\top \mathcal{D}_\Delta p(\Pi) \le \alpha\right) = \frac{1}{2} + \frac{\alpha}{\left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2} \frac{\Gamma(1+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{1+d}{2})} \, {}_2F_1\left(\frac{1}{2}, \frac{1-d}{2}; \frac{3}{2}; \frac{\alpha^2}{\left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2^2}\right)$$

for $\alpha \in [-\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2, \|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2]$ *and 0 for* $\alpha \le -\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2$ *and 1 for* $\alpha \ge \|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2$*. In the above equation,* ${}_2F_1$ *is the standard hypergeometric function. Furthermore, this distribution holds for any choice of basis matrices* $\vec{E}(\Pi)$*.*

Proof. Let z be a uniform random variable on the unit ball in $\mathbb{R}^d$ centered at the origin. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix. Note that by linearity of the directional derivative, we have

$$\mathcal{D}_\Delta\left(\pi_0^\top p(\Pi)\right) = \pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z,$$

where $\mathcal{D}_{\vec{E}(\Pi)} p(\Pi) = \left(\mathcal{D}_{E_1}\left(p(\Pi)\right), \ldots, \mathcal{D}_{E_d}\left(p(\Pi)\right)\right)$. Since z is uniform on the unit ball,

$$\begin{aligned}
&\mathbb{P}\Big(\mathcal{D}_\Delta\left(\pi_0^\top p(\Pi)\right) \le \alpha\Big) = \mathbb{P}\left(\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \le \alpha\right) \\
&= \frac{\mathrm{vol}\left(\{z \in \mathbb{R}^d \mid \pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \le \alpha, z^\top z \le 1\}\right)}{\mathrm{vol}(\{z \in \mathbb{R}^d \mid z^\top z \le 1\})} && (5.2.3) \\
&= \frac{\mathrm{vol}\left(\left\{z \in \mathbb{R}^d \,\middle|\, \left(\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right)^\top z \le \alpha, \; z^\top z \le 1\right\}\right)}{\mathrm{vol}(\{z \in \mathbb{R}^d \mid z^\top z \le 1\})} \\
&= \frac{\mathrm{vol}\left(\left\{z \in \mathbb{R}^d \,\middle|\, \left(\frac{(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0}{\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2}\right)^\top z \le \frac{\alpha}{\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2}, z^\top z \le 1\right\}\right)}{\mathrm{vol}(\{z \in \mathbb{R}^d \mid z^\top z \le 1\})} \\
&= \frac{\mathrm{vol}\left(\left\{z \in \mathbb{R}^d \,\middle|\, e_1^\top z \le \frac{\alpha}{\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2}, \; z^\top z \le 1\right\}\right)}{\mathrm{vol}(\{z \in \mathbb{R}^d \mid z^\top z \le 1\})} && (5.2.4) \\
&= \begin{cases} 0 & \text{if } \alpha < -\left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2 \\ \frac{1}{2} I_\theta(\frac{1+d}{2}, \frac{1}{2}) & \text{if } \alpha \in \left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2 \times [-1, 0] \\ 1 - \frac{1}{2} I_\theta(\frac{1+d}{2}, \frac{1}{2}) & \text{if } \alpha \in \left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2 \times [0, 1] \\ 1 & \text{if } \alpha > \left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2 \end{cases}, \qquad \theta = \frac{\left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2^2 - \alpha^2}{\left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2^2} \\
&= \begin{cases} 0 & \text{if } \alpha < -\left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2 \\ \frac{1}{2} + \frac{\alpha}{\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2} \frac{\Gamma(1+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{1+d}{2})} \, {}_2F_1\left(\frac{1}{2}, \frac{1-d}{2}; \frac{3}{2}; \frac{\alpha^2}{\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2^2}\right) & \text{if } \alpha \in \left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2 \times [-1, 1] \\ 1 & \text{if } \alpha > \left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2 \end{cases},
\end{aligned}$$

where $I_\theta(a, b)$ is the regularized incomplete beta function (see, e.g., (DLMF, Chapter 8.17)) and ${}_2F_1$ is the standard hypergeometric function (see, e.g., (DLMF, Chapter 15)). Equation (5.2.3) follows from considering the probability by taking the ratio of the volume of the fraction of the unit ball satisfying the probability event to the full volume of the unit ball. Equation (5.2.4) follows by symmetry of the unit ball and since $\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0 / \|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\|_2$ has unit norm. The penultimate result follows directly from the volume of the spherical cap (see, e.g., (Li 2011, Equation (2))). The final result follows from properties of the regularized incomplete beta function (see, e.g., (DLMF, Chapter 8.17)), i.e.,

$$I_\theta\left(\frac{1+d}{2}, \frac{1}{2}\right) = 1 - 2\sqrt{1-\theta}\frac{\Gamma(1+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{1+d}{2})}\,{}_2F_1\left(\frac{1}{2}, \frac{1-d}{2}; \frac{3}{2}; 1-\theta\right),$$

with $\theta = \frac{\left\|\left(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi)\right)^\top \pi_0\right\|_2^2 - \alpha^2}{\left\|\left(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi)\right)^\top \pi_0\right\|_2^2}$, and noting that the case for α positive and negative can be written under the same equation using the standard hypergeometric function. The independence of this result to the choice of orthonormal basis $\vec{E}(\Pi)$ follows as in Proposition 5.2.19 as the distribution only depends on the basis $\vec{E}(\Pi)$ through the norm $\|\left(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi)\right)^\top \pi_0\|_2$. $\square$

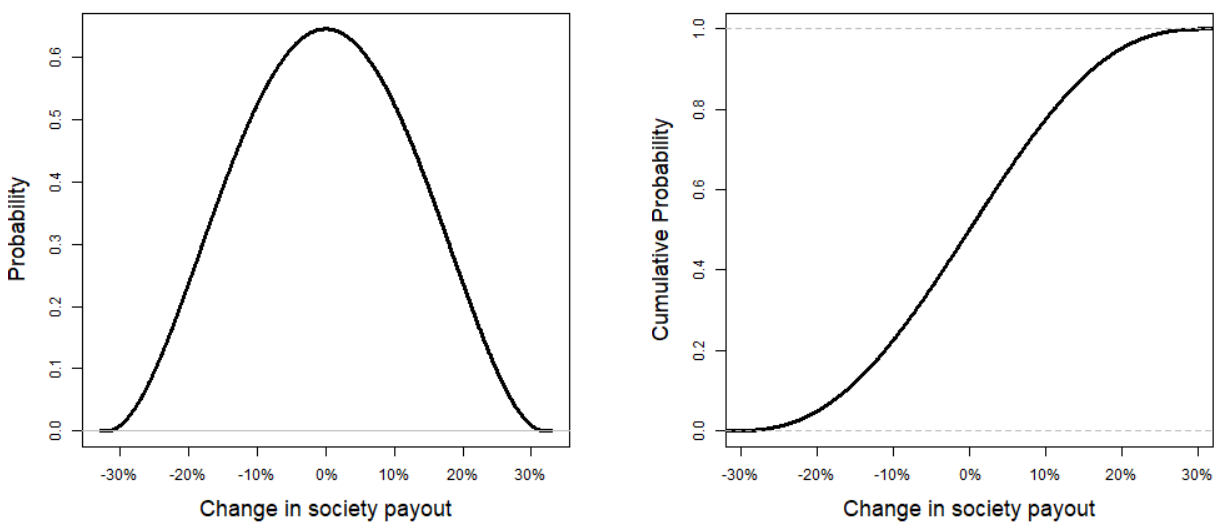


Figure 5.11: Estimated probability density (left) and CDF (right)

Relative reduction in payout to society, $\frac{\pi_0^\top \mathcal{D}_\Delta p(\Pi)}{\pi_0^\top p(\Pi)}$, under uniform perturbations Δ as described in Example 5.2.23.

Example 5.2.23. *We return to Example 5.2.16 and consider perturbations Δ sampled from the uniform distribution on a unit ball. The left and right panels of Figure 5.11 show the PDF and the CDF respectively for the relative reduction in society payout under uniformly distributed errors in our stylized four-bank network. Figure 5.13(a) shows both the largest reduction and increase in the payout to society as well as various confidence intervals for the change in the payout as a function of the perturbation size, h. As h^* and h^{**} depend on the choice of perturbation matrix Δ, we present the confidence intervals on an extrapolated interval for $h \in [0, 1]$.*

Shortfall to society for normally distributed estimation errors

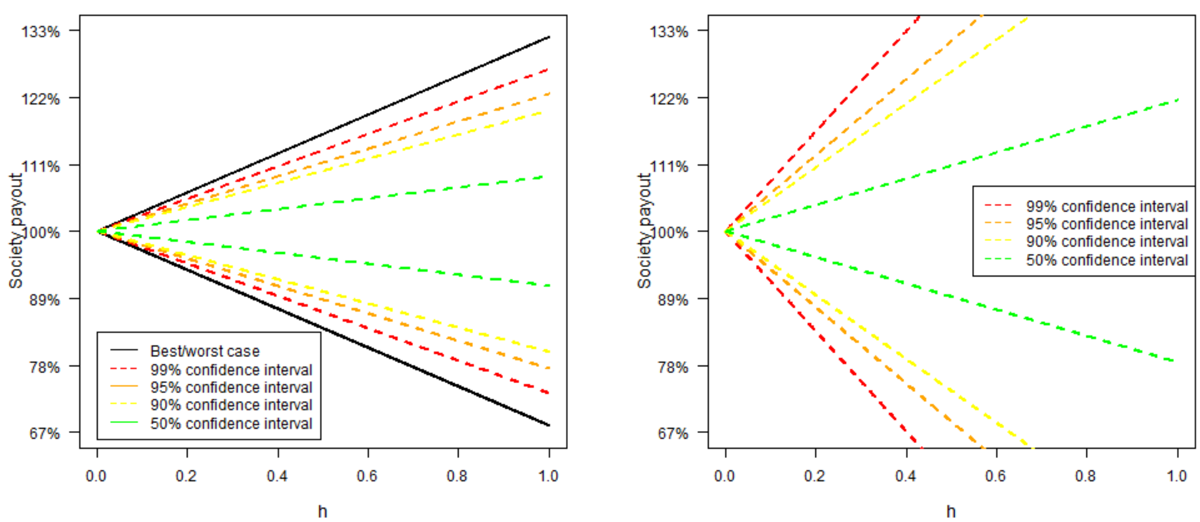
We will now consider the same problem as above under the assumptions that the errors follow a standard normal distribution. As in the previous section, we note that the magnitude of the perturbations is no longer bounded by 1 in this setting.

Proposition 5.2.24. *Let $(\Pi_0, x, \bar{p})$ be a regular financial system. The distribution of changes to payments to society where the perturbations follow a multivariate standard normal distribution is given by*

$$\mathcal{D}_\Delta\left(\pi_0^\top p(\Pi)\right) \sim N\Big(0, \left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2^2\Big).$$

Furthermore, this distribution holds for any choice of basis matrices $\vec{E}(\Pi)$.

Proof. Let z be a d-dimensional standard normal Gaussian random variable. The result follows immediately by linearity and affine transformations of the multivariate Gaussian distribution. The independence of this result to the choice of orthonormal basis $\vec{E}(\Pi)$ follows as in Proposition 5.2.19 as the distribution only depends on the basis $\vec{E}(\Pi)$ through the norm $\left\|\left(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\right)^\top \pi_0\right\|_2$. $\square$



(a) Uniformly distributed perturbations.

(b) Normally distributed perturbations.

Figure 5.13: Society payout.

Largest increase and decrease of the payout to society and confidence intervals for the payout as a function of the size h of perturbations in $\mathbf{\Delta}_F^n(\Pi)$, respectively $\mathbf{\Delta}^n(\Pi)$, where perturbations Δ are sampled uniformly (left) and from the standard Gaussian distribution (right) for the stylized four-bank system with society as described in Example 5.2.16.

Example 5.2.25. *We return once more to Example 5.2.16 to consider perturbations Δ sampled from the standard normal distribution. Figure 5.13(b) shows various confidence intervals for the relative change in payout to society under normally distributed errors under $\mathbf{\Delta}^4(\Pi)$, as a function of the perturbation size, h. As h^* and h^{**} depend on the choice of perturbation matrix Δ we present the confidence intervals on an extrapolated interval for $h \in [0, 1]$.*

5.3 Empirical application: assessing the robustness of systemic risk analyses

In this section, we study the robustness of conclusions that can be drawn from systemic risk studies that use the Eisenberg–Noe algorithm to model direct contagion. We use the same dataset from 2011 of European banks from the European Banking Authority that has been used in previous studies relying on the Eisenberg–Noe framework (Gandy and Veraart (2016), Chen et al. (2016)). As in these papers, given the heuristic approach to the dataset, our exercise should be considered to be an illustration of our results and methodology, rather than a realistic full-fledged empirical analysis.

With respect to the model's data requirements, the EBA dataset only provides information on the total assets TA_i, the capital c_i and a proxy for interbank exposures, a_i^{IB}. To populate the remaining key variables of the Eisenberg–Noe model, we therefore first assume, as in Chen et al. (2016), that for each bank the interbank liabilities are equal to the interbank assets. Furthermore, we assume that all non-interbank assets are external assets, and the non-interbank liabilities are liabilities to a society sink-node. Hence,

$$\begin{aligned} l_i^{IB} &:= a_i^{IB}, \\ L_{i0} &:= TA_i - l_i^{IB} - c_i, \\ a_i^0 &:= TA_i - a_i^{IB}. \end{aligned}$$

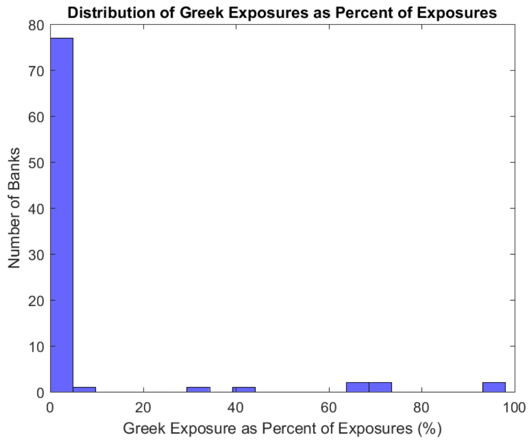
Consequently, the Eisenberg–Noe model variables are

$$\begin{aligned} &\text{Total liabilities: } \bar{p}_i = L_{i0} + l_i^{IB}, \\ &\text{Total external assets: } x_i = a_i^0. \end{aligned}$$

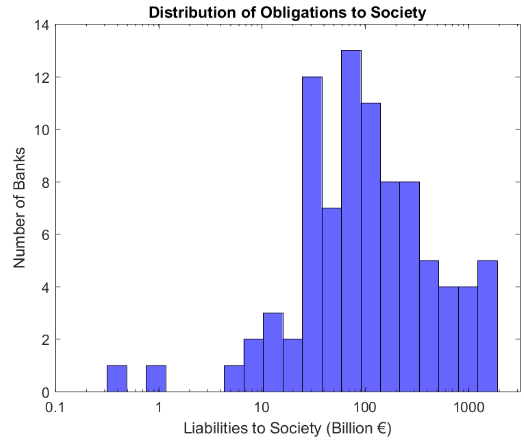
Note that each bank's net worth hence exactly corresponds to the book value of equity, or the banks' capitals: $TA_i - \bar{p}_i = a_i^0 + a_i^{IB} - l_i^{IB} - L_{i0} = c_i$.

The final key ingredient to the model is the (relative) liabilities matrix. This is usually highly confidential data, and is not provided in the EBA data set. In Gandy and Veraart (2016), Gandy and Veraart propose an elegant Bayesian sampling methodology to generate individual interbank liabilities, given information on the total interbank liabilities and total interbank assets of each bank. The authors have developed an *R*-package called "systemicrisk" that implements a Gibbs sampler to generate samples from this conditional distribution. As our analysis requires an initial liability matrix, we use the European Banking Authority (EBA) data as input to their code in order to generate such a liability matrix. As suggested by (Gandy and Veraart 2016, Section 5.3), we perturb the interbank liabilities l_i^{IB} slightly (such that they are not exactly equal to the interbank assets, while keeping the total sums equal) to fulfill the condition that L be connected along rows and columns. We then run their algorithm, with parameters $p = 0.5, \text{thinning} = 10^4, n_{burn-in} = 10^9, \lambda = \frac{pn(n-1)}{\sum_{i=1}^N a_i^{IB}} \approx 1.217810^{-3}$, to create one realisation of a 87×87 network of banks from the data. (We needed to exclude banks DE029, LU45 and SI058 because the mapping of the data to the model as described above created violations of the conditions for the algorithm and resulted in an error message.)

For simplicity and to consider an extreme event that would trigger a systemic crisis in the European banking system, we analyze what might have happened if Greece had defaulted on its



(a) Distribution of Greek exposures.



(b) Distribution of bank obligations to society.

Figure 5.15: Histograms of data from the EBA dataset.

debt and exited the Eurozone. We study this shock by decreasing the external assets of each bank by its individual Greek exposures, i.e. setting Greek bond values to zero. The histogram of Greek exposures (as a percentage of total exposures), displayed in Figure 5.15(a), shows a large heterogeneity of exposures, with the majority of banks having no (or negligible) exposures to Greece, but a small number of Greek banks having substantial exposures to Greece (between 64% - 96% of total assets). In our sensitivity analysis we resample the underlying liabilities matrix from the Gandy & Veraart algorithm Gandy and Veraart (2016) 1000 times.

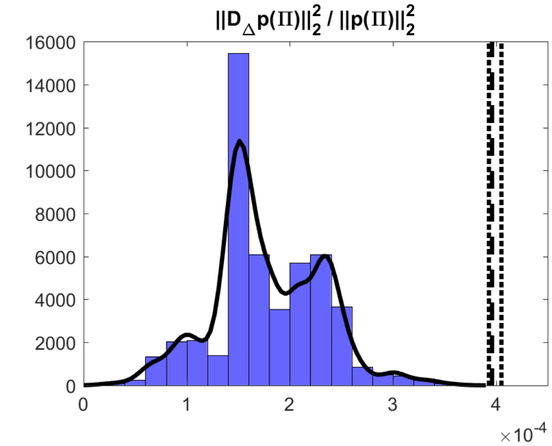
In each of our 1000 simulated networks considered there were 9 specific institutions that default on their debts in the Eisenberg–Noe framework; in only 3 simulated networks (0.3% of all simulations) there were between 1 and 3 additional banks that fail. As such, the traditional analysis of sensitivity of the Eisenberg–Noe framework would conclude that this contagion model is robust to errors in the relative liabilities matrix. This is consistent with the work of, e.g., Glasserman and Young (2015).

However, we now consider the maximal deviation in both the estimation errors and the payments to society in each of our 1000 simulated networks under $\mathbf{\Delta}_F^n(\Pi)$. The societal obligations are the same in all 1000 simulated networks, and their histogram, depicted in Figure 5.15(b), reveals as for the Greek exposures, considerable heterogeneity. Figure 5.16(a) depicts the empirical density of the maximal deviation estimation errors $\frac{\|D_\Delta p(\Pi)\|_2^2}{\|p(\Pi)\|_2^2}$ for $\Delta \in \mathbf{\Delta}_F^n(\Pi)$. Figure 5.16(b) depicts the empirical density of maximal fractional shortfalls to society $\frac{D_\Delta e_0(\Pi)}{e_0(\Pi)}$. We also depict the upper bound of the worst case perturbation errors for each of the 1000 simulated networks.

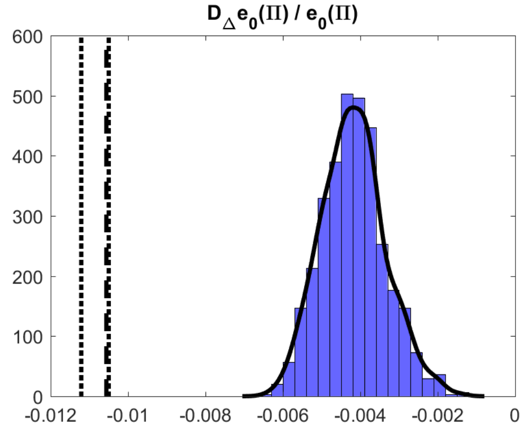
Notably in Figure 5.16(a) we see that the shape of the network, calibrated to the same EBA data set, can vastly change the impact that the worst case estimation error has under perturbations in $\mathbf{\Delta}_F^n(\Pi)$. In this plot of the empirical densities, we see the range of normalized worst case first order estimation errors range from 0 to nearly 4×10^{-4}. That is a 0 to 2% normed deviation of the clearing payments (while the value of $\|p(\Pi)\|_2$ itself has only minor variations: a total range of under 27 million EUR compared to its norm of near 5 trillion EUR for the different simulated networks Π). The upper bound on these perturbation errors (for the norm rather than norm squared) is approximately 2%, and as can be seen in Figure 5.16(a), the range of obtained upper bounds is very small. This indicates that such a bound is rather insensitive to the initial relative liability matrix Π. Therefore, any such computed upper bound is of value to a regulator, even if the initial estimate of the relative liabilities Π is incorrect.

When we consider instead Figure 5.16(b) we see that the density is more bell shaped, again with a large variation from the least change (roughly -0.001) to the most change (roughly -0.007) in the normalized impact to society; this proves as with Figure 5.16(a) that the underlying network can provide large differences in the apparent stability of a simulation to validation. While these values may appear small, the 10^{-3} arises from normalising the deviation of the clearing vector with the value of the societal node but still amounts to a variation on the order of 23.2 - 162.4 billion EUR. Thus, this sensitivity is as if entire banks' assets vanished from the wealth of society. The upper bound of these perturbation errors is approximately twice as high as the obtained maximal deviations computed under $\boldsymbol{\Delta}_F^n(\Pi)$. Notably, the median upper bound of the worst case error is nearly equal to the minimum possible value, though with a skinny tail reaching off to greater errors.

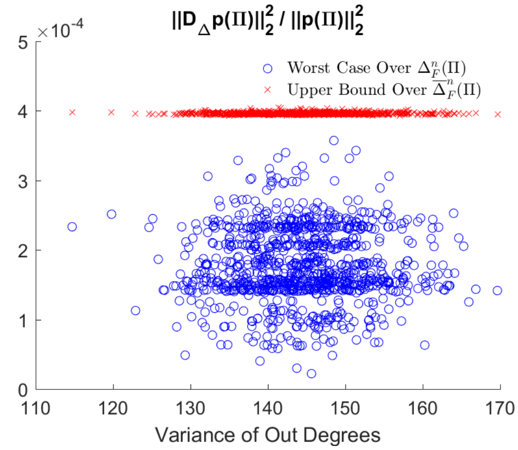
Finally, Figures 5.16(c) and 5.16(d) analyze the impact of network heterogeneity on the perturbation of the clearing vector. To this end, we quantify "network heterogeneity" as the variance of the degree distribution of out-edges. It varies between 110 to 170 in the 1000 simulated networks, thus displaying a reasonable level of heterogeneity. Figure 5.16(c) shows the worst case relative error over $\boldsymbol{\Delta}_F^n(\Pi)$ (blue circles) and $\overline{\boldsymbol{\Delta}}_F^n(\Pi)$ (red crosses) respectively. Similarly, Figure 5.16(d) shows a scatter plot of the relative error of the payment to society against the variance of the degree distribution in the network. Neither figure seems to suggest a clear relation between the relative errors and the network heterogeneity. Note that Figures 5.16(a) and 5.16(b) are obtained by projecting all points onto the y-axis in Figures 5.16(c) and 5.16(d).



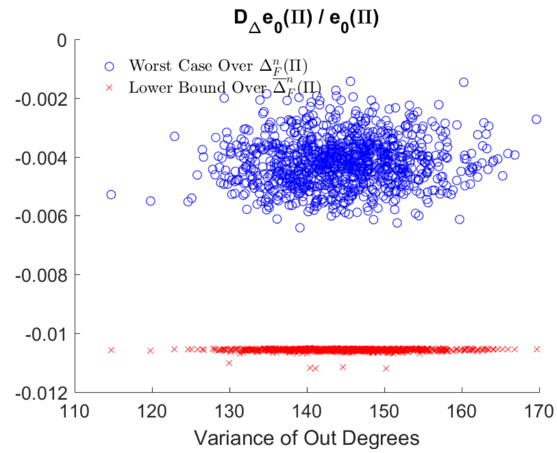
(a) Relative error of the clearing vector.



(b) Relative error of the payments to society.



(c) There is no clear dependence between relative error of the clearing vector and network heterogeneity.



(d) There is no clear relationship between the relative error of the payments to society and network heterogeneity.

Figure 5.16: Empirical result.

Top: Empirical densities of the relative errors in the Eisenberg–Noe framework as a function of random networks calibrated to the same EBA dataset. The dotted vertical lines indicate the maximal and minimal empirical values of the upper bound of the worst case and the dashed line indicates the median upper bound. Bottom: Dependence of the clearing vector perturbation on network heterogeneity.

Chapter 6

Conclusion

In this dissertation, we discussed several topics emerging in the wake of the 2008 financial crisis. From investors' perspectives, we provided the arbitrage free pricing of European options traded between two risky parties, considering different financial statuses. From regulators' perspectives, we analyzed the sensitivity of clearing payments in the Eisenberg – Noe network model with respect to an estimation error in the relative liabilities matrix.

We computed the XVA of European options, considering credit risk, asymmetric interest rates and the different performances of several financial accounts during different financial statuses. To model the Repo market freeze during the financial crisis, we used an alternating renewal process to describe the switching between different financial statuses. With a hedging portfolio including risky bonds from the investor and the counterparty, we constructed a BSDE with respect to a martingale without independent increments property to price European options and the corresponding XVA. We proved the existence and uniqueness of the solution to these BSDEs. In the empirical application, we estimated the length of different financial periods. Our result provides unbiased estimates of the parameters in the alternating renewal process. In a simulation study, the XVA in the financial crisis increased 100%, compared with the XVA in a calm financial market. We also analyzed the sensitivity of the XVA to collateralization levels, the volatility of the underlying stock, and the funding rates.

In order to quantify the effect of clearing payments of the estimation error in the relative liabilities matrix, we determined the directional derivative of the clearing payments with respect to the relative liabilities matrix. We extended this result to consider the full Taylor expansion of the fixed points to determine the clearing payments as a closed-form perturbation of an initial solution. We also studied the worst case and probabilistic interpretations of our perturbation analysis. Our results provided an upper bound on the largest shift for the clearing payments as well as a lower bound for the shortfall to society. In a numerical case study of the European banking system, we demonstrated that, even when the set of defaulting firms remains constant, the clearing payments and wealth of society can be greatly impacted. This is true even in the case that the existence and non-existence of links is pre-specified. When the existence and non-existence of links is unknown, then the upper bound of the errors can be utilized which generally provides errors that are significantly less sensitive to the initial estimate of the relative liabilities and roughly twice as large as the errors under pre-specification of links.

Chapter 7

Future Work

Our analysis of the alternating renewal process focuses on the basic properties and the decomposition of a martingale including an orthogonal term. There are many other interesting topics, such as a martingale representation theorem, Feynman-Kac theorem and a comparison theorem of a BSDE including the nonindependent increment processes. In order to apply the alternating renewal processes in broader fields, extension from two statuses to a larger number is necessary. Besides using the Ted spread to estimate the parameters λ_U and λ_V, we can also use other financial stress indicators, such as the CBOE Volatility Index (VIX), the LIBOR-OIS spread, and the Composite Indicator of Systemic Stress (CISS). It would be interesting to analyze these financial indicators statistically and find reasonable thresholds of different financial regimes. With those new thresholds, we can evaluate the estimation of the parameters λ_U and λ_V, and then compared new estimations with the results in this dissertation.

Moreover, the structures of Repo markets are very complicated in practice. For different collaterals and different counterparties, Repo rates and collateral levels are different. The rules and structures are different between American Repo markets and European Repo markets. Considering the complicated structures of Repo markets, differences between U.S. and Europe, and the switching among different financial statuses, there are many interesting topics. Although we assume that all trades in the Repo market freeze during a financial crisis, a few trades were executed occasionally. So it is feasible to weaken the freezing assumption of the Repo market during a financial crisis.

Our sensitivity analysis is based on the standard Eisenberg–Noe model. As such, it omits a number of other important extensions that have been developed in the literature (such as bankruptcy costs, fire sales, or the impact of the network topology). For a full quantification of risk and uncertainty, future research will therefore need to develop a model that combines – and weighs – all of these relevant channels of contagion. In addition, besides the symmetric distribution setting of the perturbation matrix, an asymmetric distribution and heavy-tailed distributions could be included in future research.

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