Numerical Methods for European Option Pricing with BSDEs

by

Ming Min

A Thesis

Submitted to the Faculty

of the

WORCESTER POLYTECHNIC INSTITUTE

In partial fulfillment of the requirements for the

Degree of Master of Science

in

Financial Mathematics

May 2018

APPROVED:

Professor Stephan Sturm, Major Advisor

Professor Luca Capogna, Head of Department

Abstract

This paper aims to calculate the all-inclusive European option price based on XVA model numerically. For European type options, the XVA can be calculated as solution of a BSDE with a specific driver function. We use the FT scheme to find a linear approximation of the nonlinear BSDE and then use linear regression Monte Carlo method to calculate the option price.

Acknowledgements

I would like to express my gratitude to my advisor, Professor Stephan Sturm, who has given me a lot of help both on my master project and my PhD application.

I need to appreciate my parents for supporting me through my education in WPI, both financially and mentally.

I want to thank my best friends, Shan Jiang, Hanzhao Wang and Zhenyu Qiu. Your friendship is really important to me.

Contents

List of Figures

Chapter 1

Introduction

The Black-Scholes model for option pricing assumes that no participant will default. But defaults do happen in the real world. They maybe forced to close out their positions as they default, hence the trader should consider those probabilities of additional costs when developing his portfolio. The Black-Scholes model assumes the same short rate r for the borrowing and lending rates, but these rates are different in reality. Another limitation at Black-Scholes model is that we cannot short stocks or other assets as freely in reality as the Black-Scholes model suggested.

This thesis follows the (Bichuch, Capponi, & Sturm, 2016) market setup, but we use a much simpler version. The trader gets his funding from his treasury desk and must pay back the money. The borrowing rate depends on his own credit level and current market conditions; it is usually higher than his lending rate. The difference between these two rates is called funding spread, which is the funding cost needed to be considered in our model. As in (Bichuch et al., 2016) and (Burgard & Kjaer, 2011), two corporate bonds are introduced in order to hedge the credit default risk from both trader and his counterparty. Using the repo market mechanism, we can short stocks in this market. Usually there is a difference between borrowing and lending rates, but in this paper we assume they are the same for simplification. (Bichuch et al., 2016) then generate a Backward Stochastic Differential Equation, BSDE in short, for the option pricing via XVA model.

Once we get the BSDEs, we can solve it numerically. This thesis uses nonlinear Monte Carlo methods to solve it. (Crépey & Nguyen, 2016) used a perturbation method, following (Fujii & Takahashi, 2012a, 2012b), to find a linear approximation of the solution, and solve the BSDE by letting the perturbation parameter equal to 1. We expanded this method, which is called FT scheme, a little bit to solve our problem. Since our driver is path dependent, linear regression Monte Carlo method is also used. (Glasserman, 2013) uses linear regression Monte Carlo method for American option pricing problems. But in our problem, instead of looking one step forward, we need to remember everything in all the future time since we will future value to define driver function at each time.

In this thesis, chapter 2 focuses on mathematical BSDE models of European call and put options, and derives the drivers for both options. Chapter 3 presents the numerical method used to solve previous BSDEs. Chapter 4 uses the numerical algorithm developed in Chapter 3, and XVAs under different collateral levels are compared. Chapter 5 concludes. The codes are included in the Appendix.

Chapter 2

Models

A probability space, $(\Omega, \mathcal{G}, \mathbb{P})$, is used to describe the physical world. We refer to the investor or trader as I, and counterparty to investor as C. The background filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$, augmented by $(\mathcal{G}, \mathbb{P})$-nullsets, includes all the information of the market except for defaults. The filtration $\mathbb{H} := (\mathcal{H}_t)_{t \geq 0}$ has all the information about default events. The filtration $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ is given by letting $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, augmented by $(\mathcal{G}, \mathbb{P})$-nullsets.

2.1 Market setup and notations

2.1.1 Stocks security

Let $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by Brownian motion $W^{\mathbb{P}}$, where $\mathbb{P}$ is the physical measure. Then the dynamics of stock price is given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^{\mathbb{P}}, \tag{2.1}$$

where the μ, σ are appreciation rate and volatility as common respectively, assumed to be constant in our model.

In reality, we cannot short stock freely. Shorting is conducted through the security repo market. In the (Bichuch et al., 2016) and (Adrian, Begalle, Copeland, & Martin, 2013) market setup, two types of repo transaction are considered. The first one is called security driven transaction. This transaction is used to circumvent the prohibition of the trader from selling a stock which he or she doesn't have, also called 'naked' short sales of stocks. It works as follows: the trader signs a repo contract with some participant in the repo market. The trader lends some money to the participant, which is used to buy stocks and post them as collateral to the trader. Thus the trader can sell stocks and must return stocks to participants in exchange of a pre-specified amount of money, which is usually higher than the lending amount. So implicitly, there is a return rate on trader, called r_r^+.

The second type of repo transaction is called cash driven transaction, which is exactly the other side in this repo market. When the trader wants to have a long position in stocks, he borrows money from the treasury desk and uses them to buy stocks which are posted as collateral for a loan at the repo market. The trader agrees to purchase those collateral back at a pre-specified price, which is usually slightly higher than the original price of collateral. So there is a cost rate, named as r_r^-. In this paper, we assume r_r^+ and r_r^- are the same, denote as r_r. The relation between repo market account and the stocks is given by

$$\psi_t^r B_t^{r_r} = -\xi_t S_t, \tag{2.2}$$

where $B_t^{r_r}$ is the repo market account, ξ_t is the number of shares in security account. This identity stems from the fact that stock is only bought and sold via repo market.

2.1.2 Risky bonds securities

Two risky bonds written by the trader and the counterparty are introduced. Denote their defaulting times as τ_i, where $i \in \{I, C\}$, as trader and counterparty defaulting time respectively. We suppose the τ_i's are following an exponential distribution with intensity $h_i^{\mathbb{P}}, i \in \{I, C\}$, and are independent of $\mathbb{F}$ and each other. $H_i(t) = \mathbb{1}_t, t \geq 0$, is the default indicator process. So the default events filtration is given as $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}, \mathcal{H}_t = \sigma(H_I(u), H_C(u); u \leq t)$. In particular, this implies $\mathbb{F}$ Brownian motion $W^{\mathbb{P}}$ is also a $\mathbb{G}$ Brownian motion.

Assume these two bonds are zero recovery, and both expires at time T. Denote the bond price written by trader as P^I, denote the bond price written by counterparty as P^C. Accordingly, their prices are given as

$$dP_t^i = \mu_i dt - P_{t-}^i dH_t^i, \qquad P_0^i = e^{-\mu_i T} \tag{2.3}$$

with μ_i as their return rates.

Let $\tau = \tau_I \wedge \tau_C \wedge T$ denote the earliest stopping time of maturity time T, trader default time τ_I and counterparty default time τ_C.

2.1.3 Funding account

As mentioned before, the trader receives or provides funding to his treasury desk with different rates. Usually the borrowing rate is higher than lending rate. We denote r_f^+ as the lending rate, r_f^- as the borrowing rate. So the money market account has the dynamics

$$dB_t^{r_f^{\pm}} = r_f^{\pm} B_t^{r_f^{\pm}} dt, \tag{2.4}$$

where $B_t^{r_f^{\pm}}$ denotes the funding account. Let ξ_t^f be the number of shares in funding account, and define

$$B_t^{r_f} := B_t^{r_f}(\xi_t^f) = e^{\int_0^t r_f(\xi_s^f)ds}, \tag{2.5}$$

where

$$r_f := r_f(y) = r_f^- \mathbb{1}_{\{y<0\}} + r_f^+ \mathbb{1}_{\{y>0\}}. \tag{2.6}$$

2.1.4 Collateral process and collateral account

Collateral is used to reduce one's loss if the other party default before expiry. We denote the collateral process as $C := (C_t)_{t\geq 0}$, which is an $\mathbb{F}$ adapted process. If $C_t > 0$, we regard the trader as *collateral provider*. In this case, the trader measures a positive risk toward the counterparty, and posts collateral to the counterparty to reduce counterparty's loss if default happens. On the other hand, if $C_t < 0$, the trader is the *collateral taker*, who measures a positive risk toward the counterparty, and takes collateral to mitigate loss if the counterparty defaults.

According to (ISDA, 2014), the most popular type of collateral is cash collateral.

When the trader is the collateral provider, let r_c^+ be the rate on the collateral amount he will receive from the counterparty. If the trader is collateral taker, we let r_c^- be the rate on the collateral amount he will pay to his counterparty. In this thesis, we assume $r_c^+ = r_c^- = r_c$. Let $B_t^{r_c}$ be the collateral account, so the dynamics of collateral cash account is given by

$$dB_t^{r_c} = r_c B_t^{r_c} dt. \tag{2.7}$$

Furthermore more, if we let ψ_t^c be the shares of $B_t^{r_c}$ held by the trader at time t, then we have

$$\psi_t^c B_t^{r_c} = -C_t. \tag{2.8}$$

The intuition here is that C_t is the amount posted to the other part by the trader, the collateral account is the cash amount will be received by trader if no default happens before T. So they have the same amount but different sign.

2.2 Replication of options

2.2.1 Risk neutral measure

In order to replicate the derivatives, we need to define a risk neutral measure. As (Bichuch et al., 2016), we first introduce the default intensity model. Given the physical measure $\mathbb{P}$, default times of trader or counterparty are defined as independent exponentially distributed random variables with constant intensity $h_i^{\mathbb{P}}, i \in \{I, C\}$. It holds then that for each $i \in \{I, C\}$,

$$\varpi_t^{i,\mathbb{P}} := H_t^i - \int_0^t (1 - H_u^i) h_i^{\mathbb{P}} du \tag{2.9}$$

is a $(\mathbb{G}, \mathbb{P})$-martingale. We defined the discounted rate as r_D, which is the discount rate of valuation party used for collateral and closeout. The risk neutral measure $\mathbb{Q}$ is given by the Radon-Nikodm density

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_\tau} = e^{\frac{r_D - \mu}{\sigma} W_\tau^{\mathbb{P}} - \frac{(r_D - \mu)^2}{2\sigma^2}\tau} \left(\frac{\mu_I - r_D}{h_I^{\mathbb{P}}}\right)^{H_\tau^I} e^{(r_D - \mu_I + h_I^{\mathbb{P}})\tau} \left(\frac{\mu_C - r_D}{h_C^{\mathbb{P}}}\right)^{H_\tau^C} e^{(r_D - \mu_C + h_C^{\mathbb{P}})\tau}. \tag{2.10}$$

Under measure $\mathbb{Q}$, the dynamics of our three risky assets are given by

$$dS_t = r_D S_t dt + \sigma S_t dW_t^{\mathbb{Q}}, \tag{2.11}$$

$$dP_t^I = r_D P_t^I dt - P_{t-}^I d\varpi_t^{I,\mathbb{Q}}, \tag{2.12}$$

$$dP_t^C = r_D P_t^C dt - P_{t-}^C d\varpi_t^{C,\mathbb{Q}}. \tag{2.13}$$

The $W^{\mathbb{Q}} := (W_t^{\mathbb{Q}}, 0 \leq t \leq \tau)$ is $(\mathbb{G}, \mathbb{Q})$-Brownian motion, and $\varpi^{I,\mathbb{Q}} := \varpi_t^{I,\mathbb{Q}}, 0 \leq t \leq \tau$ as well as $\varpi^{C,\mathbb{Q}} := \varpi_t^{C,\mathbb{Q}}, 0 \leq t \leq \tau$ are $(\mathbb{G}, \mathbb{Q})$-martingales. These three dynamics can be derived by Ito's formula directly though (2.1), (2.2) and (2.7), and $h_i^{\mathbb{Q}} = \mu_i - r_D, i \in \{I, C\}$.

2.2.2 Replication of options and collateral specification

We focus on European call and put option. The Black-Scholes price given by the valuation agent is used to calculate the closeout value and collateral. Under the risk neutral measure $\mathbb{Q}$, we have

$$\hat{V}_t := e^{-r_D(T-t)} \mathbb{E}[\Phi(S_T) \mid \mathcal{F}_t], \tag{2.14}$$

where $\hat{V}_t$ is the Black-Scholes option price at time t as calculated by the valuation party. $\Phi(S_T)$ is the payoff of European options, which is given by

$$\Phi(S_T) = \begin{cases} (S_T - K)^+ & \text{European call option,} \\ (K - S_T)^+ & \text{European put option.} \end{cases}$$

When the trader is the put or call option seller, he needs to replicate this payoff $\Phi(S_T)$. Thus he could build a portfolio to hedge his position and use it to pay his counterparty. On the other hand, when the trader buy one option, he need to replicate the payoff of $-\Phi(S_T)$ in order to hedge option value fluctuation.

In addition, we need to consider collateral for this option contract. We define the collateral level as α, so under the assumption that neither the trader nor counterparty have defaulted by time t, the collateral process is given by

$$C_t := \alpha \hat{V}_t \mathbb{1}_{\{\tau > t\}}, \qquad \text{with } 0 \leq \alpha \leq 1. \tag{2.15}$$

The collateral is allowed to be rehypothecated by the collateral taker. This means that the collateral taker can use cash collateral to invest in other investment opportunities. We define our strategy process as $\varphi := (\xi_t, \xi_t^f, \xi_t^I, \xi_t^C; t \geq 0)$, where ξ_t denotes the shares in security account, which is the underlying in our case. ξ_t^f denotes the number of shares in funding account. ξ_t^I, ξ_t^C denote the number of shares in trader and counterparty bonds respectively. Combining with (2.2) and (2.8), the portfolio process is given by

$$V_t(\varphi) := \xi_t S_t + \xi_t^f B_t^{r_f} + \xi_t^I P_t^I + \xi_t^C P_t^C + \psi_t^r B_t^{r_r} - \psi_t^c B_t^{r_c}. \tag{2.16}$$

In this paper, we follow the risk-free closeout convention. It means that the surviving party liquidates all his positions once someone defaults. We denote θ as the closeout value at time τ, where τ is specified is section (2.1.2). This θ is given by

$$\begin{aligned} \theta := \theta(\tau, \hat{V}) &= \hat{V}_\tau + \mathbb{1}_{\{\tau_C < \tau_I\}} L_C Y^- - \mathbb{1}_{\{\tau_I < \tau_C\}} L_I Y^+ \\ &= \mathbb{1}_{\{\tau_C < \tau_I\}} \theta_I(\hat{V}_\tau) + \mathbb{1}_{\{\tau_I < \tau_C\}} \theta_C(\hat{V}_\tau), \end{aligned} \tag{2.17}$$

where $Y := \hat{V}_\tau - C_\tau = (1-\alpha)\hat{V}_\tau$ is the value of the option at default time, netted with the collateral and $\theta_I(v) = v - L_I((1-\alpha)v)^+$, $\theta_C(v) = v + L_C((1-\alpha)v)^-$. The L_i satisfy $0 \leq L_i \leq 1, i \in \{I, C\}$ and the loss rates against trader and counterparty. This θ is exactly the terminal amount we want to replicate, more details are in (Bichuch et al., 2016) Remark 3.3.

2.3 XVA model and driver functions

In this part, we are using the assumption that $r_D = r_r^{\pm} = r_c^{\pm} = r_f^{+} \leq r_f^{-}$. By (Bichuch et al., 2016) section 4, this assumption satisfies trader's non-arbitrage condition. For simplicity, we use r_D to represent $r_r^{\pm}$ and $r_c^{\pm}$, and we still use r_f^{+} in order to make difference with r_f^{-}. But finally we will change r_f^{+} to r_D in drivers function.

2.3.1 XVA models

According to (Bichuch et al., 2016) section 4, we can derive the following BSDEs by considering the dynamics of equation (2.16), and using (2.2) & (2.15),

$$-dV_t^{+} = f^{+}(t, V_t^{+}, Z_t^{+}, Z_t^{I,+}, Z_t^{C,+}; \hat{V})dt - Z_t^{+} dW_t^{\mathbb{Q}} - Z_t^{I,+} d\varpi_t^{I,\mathbb{Q}} - Z_t^{C,+} d\varpi_t^{C,\mathbb{Q}}, \tag{2.18}$$

$$V_\tau^{+} = \theta_I(\hat{V}_\tau)\mathbb{1}_{\{\tau_I < \tau_C \wedge T\}} + \theta_C(\hat{V}_\tau)\mathbb{1}_{\{\tau_C < \tau_I \wedge T\}} + \Phi(S_T)\mathbb{1}_{\{\tau = T\}}, \tag{2.19}$$

and

$$-dV_t^{-} = f^{-}(t, V_t^{-}, Z_t^{-}, Z_t^{I,-}, Z_t^{C,-}; \hat{V})dt - Z_t^{-} dW_t^{\mathbb{Q}} - Z_t^{I,-} d\varpi_t^{I,\mathbb{Q}} - Z_t^{C,-} d\varpi_t^{C,\mathbb{Q}}, \tag{2.20}$$

$$V_\tau^{+} = \theta_I(\hat{V}_\tau)\mathbb{1}_{\{\tau_I < \tau_C \wedge T\}} + \theta_C(\hat{V}_\tau)\mathbb{1}_{\{\tau_C < \tau_I \wedge T\}} + \Phi(S_T)\mathbb{1}_{\{\tau = T\}}. \tag{2.21}$$

Notice that here we only replicate one share of claim. The drivers are given by

$$f^+(t,v,z,z^I,z^C;\hat{V}_t) := -\,(r_f^+(v+z^I+z^C-\alpha\hat{V}_t)^+ - r_f^-(v+z^I+z^C-\alpha\hat{V}_t)^- \\ -\,r_D z^I - r_D z^C + r_D\alpha\hat{V}_t), \tag{2.22}$$

and

$$f^-(t,v,z,z^I,z^C;\hat{V}_t) := -f^+(t,-v,-z,-z^I,-z^C;-\hat{V}_t). \tag{2.23}$$

V^+ is the value process of the portfolio which hedges 1 share of option, V^- is the value process of portfolio which hedges -1 share of option. We let $Z_t = \xi_t\sigma S_t$, $Z_t^I = -\xi_t^I P_{t^-}^I$, $Z_t^C = -\xi_t^C P_{t^-}^C$. From (2.18-2.21), if we can solve these BSDEs, then we have the all-inclusive price of options. Since there is no Z_t in two drivers above, we will omit this parameter in following drivers, and this is because of our assumption of $r_D = r_r$.

Let $\hat{V}_t$ be the Black Scholes option price. We can define XVA in our model from (Bichuch et al., 2016) Definition 4.6.

Definition 1. *The seller's XVA is a $\mathbb{G}$-adapted process, which is given by*

$$XVA_t^+ := V_t^+ - \hat{V}_t, \tag{2.24}$$

and the buyer's XVA is given by

$$XVA_t^- := V_t^- - \hat{V}_t. \tag{2.25}$$

By Black-Scholes pricing theorem, the dynamics of $\hat{V}_t$ is given by

$$-d\hat{V}_t = -r_D\hat{V}_t dt - \hat{Z}_t dW_t^{\mathbb{Q}} \tag{2.26}$$

Then we can derive the BSDEs for XVA, by combining the BSDE for V with Black Scholes BSDE of $\hat{V}_t$:

$$\begin{aligned} -dXVA_t^{\pm} =& \tilde{f}^{\pm}(t, XVA_t^{\pm}, \tilde{Z}_t^{I,\pm}, \tilde{Z}_t^{C,\pm}; \hat{V}_t) \\ & - \tilde{Z}_t^{\pm} dW_t^{\mathbb{Q}} - \tilde{Z}_t^{I,\pm} d\varpi_t^{I,\mathbb{Q}} - \tilde{Z}_t^{C,\pm} d\varpi_t^{C,\mathbb{Q}}, \end{aligned} \tag{2.27}$$

$$XVA_\tau^{\pm} = \tilde{\theta}_C(\hat{V}_\tau)\mathbb{1}_{\{\tau_C < \tau_I \wedge T\}} + \tilde{\theta}_I(\hat{V}_\tau)\mathbb{1}_{\{\tau_I < \tau_C \wedge T\}}, \tag{2.28}$$

where $\tilde{Z}_t^{\pm} := Z_t^{\pm} - \hat{Z}_t$, $\tilde{Z}_t^{I,\pm} = Z_t^{I,\pm}$, $\tilde{Z}_t^{C,\pm} = Z_t^{C,\pm}$ and $\tilde{\theta}_C(v) := L_C((1-\alpha)v)^-$, $\tilde{\theta}_I(v) := -L_I((1-\alpha)v)^+$.

The drivers $\tilde{f}$ are given by

$$\begin{aligned} \tilde{f}^{+}(t, xva, \tilde{z}^I, \tilde{z}^C; \hat{V}) :=& -\big(r_f^{+}(xva + \tilde{z}^I + \tilde{z}^C - \alpha\hat{V}_t)^{+} - r_f^{-}(xva + \tilde{z}^I \\ & + \tilde{z}^C - \alpha\hat{V}_t)^{-} - r_D\tilde{z}^I - r_D\tilde{z}^C + r_D\alpha\hat{V}_t\big) + r_D\hat{V}_t, \end{aligned} \tag{2.29}$$

$$\tilde{f}^{-}(t, xva, \tilde{z}^I, \tilde{z}^C; \hat{V}) = -\tilde{f}^{+}(t, -xva, -\tilde{z}^I, -\tilde{z}^C; -\hat{V}). \tag{2.30}$$

Next, as (Bichuch et al., 2016), we can move one step forward by using reduction technique developed by (Crépey & Song, 2015) to generate another BSDE, which stops at expiry time T with zero terminate value. This is the exactly BSDE and drivers we are gonna use in chapter 3.

Theorem 1. *The BSDEs*

$$\begin{aligned} -d\check{U}_t^{\pm} &= \check{g}^{\pm}(t, \check{U}_t^{\pm}, \check{Z}_t^{\pm}; \hat{V})dt - \check{Z}_t^{\pm} dW_t^{\mathbb{Q}} \\ \check{U}_T^{\pm} &= 0 \end{aligned} \tag{2.31}$$

in the filtration $\mathbb{F}$ with

$$\breve{g}^+(t, \breve{u}_t, \breve{z}_t; \hat{V}) := h_I^{\mathbb{Q}}(\tilde{\theta}_I(\hat{V}_t) - \breve{u}) + h_C^{\mathbb{Q}}(\tilde{\theta}_C(\hat{V}_t) - \breve{u}) + \tilde{f}^+(t, \breve{u}_t, \breve{z}_t, \tilde{\theta}_I(\hat{V}_t) - \breve{u}, \tilde{\theta}_C(\hat{V}_t) - \breve{u}; \hat{V}), \tag{2.32}$$

$$\breve{g}^-(t, \breve{u}_t, \breve{z}_t; \hat{V}) := -\breve{g}^+(t, -\breve{u}_t, -\breve{z}_t; -\hat{V}), \tag{2.33}$$

admits unique solutions $\breve{U}^{\pm}$ such that

$$\breve{U}_t^{\pm} = XVA_{t\wedge\tau^-}^{\pm}. \tag{2.34}$$

On the other hand, we can get the XVA solution from $\breve{U}$ by

$$XVA_t^{\pm} := \breve{U}_t^{\pm}\mathbb{1}_{\{t<\tau\}} + \Big(\tilde{\theta}_C(\hat{V}_{\tau_C})\mathbb{1}_{\{\tau_C<\tau_I\wedge T\}} + \tilde{\theta}_I(\hat{V}_{\tau_I})\mathbb{1}_{\{\tau_I<\tau_C\wedge T\}}\Big)\mathbb{1}_{\{t\geq\tau\}}$$

2.3.2 Drivers

We are using the BSDE given by Theorem 1. The drivers are given by (2.32) and (2.33).

First let's focus on selling one single option, thus the trader want to hedge payoff $\Phi(S_T)$. The driver we are using is $\breve{g}^+$. For simplicity, we define

$$\breve{g}_t^{\pm} = \breve{g}^{\pm}(t, \breve{u}_t, \breve{z}_t; \hat{V}). \tag{2.35}$$

When selling an European option, the option value will always be positive. Thus

$\tilde{\theta}_C(\hat{V}) = 0$ according to our definition.

$$
\begin{aligned}
\breve{g}^+ &= h_I^{\mathbb{Q}}(\tilde{\theta}_I(\hat{V}) - \breve{u}) - h_C^{\mathbb{Q}}\breve{u} - [r_f^+(-\breve{u} + \tilde{\theta}_I(\hat{V}) + (1-\alpha)\hat{V})^+ \\
&\quad - r_f^-(-\breve{u} + \tilde{\theta}_I(\hat{V}) + (1-\alpha)\hat{V})^- - r_D(\tilde{\theta}_I(\hat{V}) - \breve{u}) + r_D\breve{u} + r_D\alpha\hat{V}] \\
&\quad + r_D\hat{V} \\
&= h_I^{\mathbb{Q}}(\tilde{\theta}_I(\hat{V}) - \breve{u}) - h_C^{\mathbb{Q}}\breve{u} - r_f^+(-\breve{u} + \tilde{\theta}_I(\hat{V}) + (1-\alpha)\hat{V})^+ \\
&\quad + r_f^-(-\breve{u} + \tilde{\theta}_I(\hat{V}) + (1-\alpha)\hat{V})^- + r_D(\tilde{\theta}_I(\hat{V}) - \breve{u}) - r_D\breve{u} + r_D(1-\alpha)\hat{V}.
\end{aligned}
\tag{2.36}
$$

Since we want to get a simpler version of the driver function, we can discuss different cases for positive and negative $(-\breve{u} + \tilde{\theta}_I(\hat{V}) + (1-\alpha)\hat{V})$. Then we may cancel some terms and collect terms having $\breve{u}_t$. It's shown as follows,

i. If $-\breve{u} + \tilde{\theta}_I(\hat{V}) + (1-\alpha)\hat{V} \geq 0$, then

$$
\begin{aligned}
\breve{g}_t^+(\breve{u}) &= h_I^{\mathbb{Q}}(\tilde{\theta}_I(\hat{V}_t) - \breve{u}) - h_C^{\mathbb{Q}}\breve{u} - r_D\breve{u} \\
&= h_I^{\mathbb{Q}}\tilde{\theta}_I(\hat{V}_t) - (h_I^{\mathbb{Q}} + h_C^{\mathbb{Q}} + r_D)\breve{u}_t,
\end{aligned}
\tag{2.37}
$$

ii. If $-\breve{u} + \tilde{\theta}_I(\hat{V}) + (1-\alpha)\hat{V} < 0$, then

$$
\begin{aligned}
\breve{g}_t^+(\breve{u}) &= h_I^{\mathbb{Q}}(\tilde{\theta}_I(\hat{V}_t) - \breve{u}) - h_C^{\mathbb{Q}}\breve{u} + r_f^-(-\breve{u} + \tilde{\theta}_I(\hat{V}_t) + (1-\alpha)\hat{V}_t) + r_D(\tilde{\theta}_I(\hat{V}) - \breve{u}) \\
&\quad - r_D\breve{u} + r_D(1-\alpha)\hat{V} \\
&= (h_I^{\mathbb{Q}} + r_f^- + r_D)\tilde{\theta}_I(\hat{V}_t) + (r_f^- + r_D)(1-\alpha)\hat{V}_t - (h_I^{\mathbb{Q}} + h_C^{\mathbb{Q}} + r_f^- + 2r_D)\breve{u}_t.
\end{aligned}
\tag{2.38}
$$

Before we use (2.37) and (2.38) as formula for drivers, we need to check conditions (i) & (ii). As these conditions are path dependent, which means the results varies at different time, we need to check them step by step when we trace back the XVA.

The idea is similar with what we do for American options.

When we want to hedge the payoff $-\Phi(S_T)$, which is the case of buying an European option, we need to use $\breve{g}^-$ as our drivers. Also, compare to selling one option, $\tilde{\theta}_I(-V) = 0$ in this case.

$$\begin{aligned}
\breve{g}^- &= -h_I^{\mathbb{Q}}(\tilde{\theta}_I(-\hat{V}) + \breve{u}) - h_C^{\mathbb{Q}}(\tilde{\theta}_C(-\hat{V}) + \breve{u}) + [r_f^+(-\breve{u} + \tilde{\theta}_I(-\hat{V}) + \breve{u} + \tilde{\theta}_C(-\hat{V}) + \breve{u} - (1-\alpha)\hat{V})^+ \\
&\quad - r_f^-(-\breve{u} + \tilde{\theta}_I(-\hat{V}) + \breve{u} + \tilde{\theta}_C(-\hat{V}) + \breve{u} - (1-\alpha)\hat{V})^- - r_D(\tilde{\theta}_I(-\hat{V}) + \breve{u}) \\
&\quad - r_D(\tilde{\theta}_C(-\hat{V}) + \breve{u}) - r_D\alpha\hat{V}] + r_D\hat{V} \\
&= -h_I^{\mathbb{Q}}\breve{u} - h_C^{\mathbb{Q}}(\tilde{\theta}_C(-\hat{V}) + \breve{u}) + [r_f^+(\breve{u} + \tilde{\theta}_C(-\hat{V}) - (1-\alpha)\hat{V})^+ \\
&\quad - r_f^-(\breve{u} + \tilde{\theta}_C(-\hat{V}) - (1-\alpha)\hat{V})^- - r_D\breve{u} - r_D(\tilde{\theta}_C(-\hat{V}) + \breve{u}) + (1-\alpha)r_D\hat{V}]
\end{aligned} \tag{2.39}$$

Similarly, we need to discuss the sign of $(\breve{u} + \tilde{\theta}_C(-\hat{V}) - (1-\alpha)\hat{V})$.

iii. if $\breve{u} + \tilde{\theta}_C(-\hat{V}) - (1-\alpha)\hat{V} \geq 0$, then

$$\begin{aligned}
\breve{g}^- &= -h_I^{\mathbb{Q}}\breve{u} - h_C^{\mathbb{Q}}(\tilde{\theta}_C(-\hat{V}) + \breve{u}) + [r_f^+(\breve{u} + \tilde{\theta}_C(-\hat{V}) - (1-\alpha)\hat{V}) \\
&\quad - r_D\breve{u} - r_D(\tilde{\theta}_C(-\hat{V}) + \breve{u}) + (1-\alpha)r_D\hat{V}] \\
&= -h_I^{\mathbb{Q}}\breve{u} - h_C^{\mathbb{Q}}(\tilde{\theta}_C(-\hat{V}) + \breve{u}) - r_D\breve{u} \\
&= -h_C^{\mathbb{Q}}\tilde{\theta}_C(-\hat{V}) - (h_I^{\mathbb{Q}} + h_C^{\mathbb{Q}} + r_D)\breve{u}.
\end{aligned} \tag{2.40}$$

And we add time into $\breve{g}^-$, so

$$\breve{g}_t^-(\breve{u}) = -h_C^{\mathbb{Q}}\tilde{\theta}_C(-\hat{V}_t) - (h_I^{\mathbb{Q}} + h_C^{\mathbb{Q}} + r_D)\breve{u}_t \tag{2.41}$$

iv. If $\breve{u} + \tilde{\theta}_C(-\hat{V}) - (1-\alpha)\hat{V} < 0$, then

$$\begin{aligned}
\breve{g}^- &= -h_I^{\mathbb{Q}}\breve{u} - h_C^{\mathbb{Q}}(\tilde{\theta}_C(-\hat{V}) + \breve{u}) + [-r_f^-(\breve{u} + \tilde{\theta}_C(-\hat{V}) - (1-\alpha)\hat{V})^- \\
&\quad - r_D\breve{u} - r_D(\tilde{\theta}_C(-\hat{V}) + \breve{u}) + (1-\alpha)r_D\hat{V}] \\
&= -(h_C^{\mathbb{Q}} + r_D - r_f^-)\tilde{\theta}_C(-\hat{V}) - (r_f^- - r_D)(1-\alpha)\hat{V} \\
&\quad - (h_C^{\mathbb{Q}} + h_I^{\mathbb{Q}} - r_f^- + 2r_D)\breve{u}
\end{aligned} \tag{2.42}$$

and we plug in time t,

$$\breve{g}_t^-(\breve{u}) = -(h_C^{\mathbb{Q}} + r_D - r_f^-)\tilde{\theta}_C(-\hat{V}_t) - (r_f^- - r_D)(1-\alpha)\hat{V}_t - (h_C^{\mathbb{Q}} + h_I^{\mathbb{Q}} - r_f^- + 2r_D)\breve{u}_t. \tag{2.43}$$

With drivers above, we can use the FT scheme to approximately calculate the XVA by the linear regression Monte Carlo algorithm.

Chapter 3

Numerical methods

We define

$$\check{g}_t(u) = \check{g}^{\pm}(t, u; \hat{V}) \tag{3.1}$$

for writing simplicity. Notice we omit $\check{Z}^{\pm}$ here since $\check{Z}^{\pm}$ doesn't appear in our final drivers according to (2.37), (2.38), (2.41) and (2.43). And let

$$\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot \mid \mathcal{G}_t] \tag{3.2}$$

to be the conditional expectation given $\mathcal{G}_t$. Before digging into the Monte Carlo method, we are changing BSDE (2.31) into the expectation form, and then take conditional expectation of both sides given $\mathcal{G}_t$. Thus

$$\check{U}_t = \mathbb{E}_t\Big[\int_t^T \check{g}(\check{U}_s)ds\Big], \quad t \in (0, T). \tag{3.3}$$

3.1 FT scheme

By (Fujii & Takahashi, 2012a, 2012b), a perturbation parameter ϵ and the following perturbation form of BSDE (3.3) are introduced:

$$\check{U}_t^{\epsilon} = \mathbb{E}_t\Big[\int_t^T \epsilon \check{g}_s(\check{U}_s^{\epsilon})ds\Big]. \tag{3.4}$$

It's exactly the same as (3.3) when $\epsilon = 1$. Suppose the solution of (3.4) can be represented as a power series of ϵ:

$$\check{U}_t^{\epsilon} = \check{U}_t^{(0)} + \epsilon \check{U}_t^{(1)} + \epsilon^2 \check{U}_t^{(2)} + \epsilon^3 \check{U}_t^{(3)} + \cdots . \tag{3.5}$$

Then consider the Taylor expansion of $\check{g}$ at $\check{U}_t^{(0)}$,

$$\check{g}_t(\check{U}_t^{\epsilon}) = \check{g}_t(\check{U}_t^{(0)}) + (\epsilon \check{U}_t^{(1)} + \epsilon^2 \check{U}_t^{(2)} + \cdots)\partial_u \check{g}_t(\check{U}_t^{(0)}) + \frac{1}{2}(\epsilon \check{U}_t^{(1)} + \epsilon^2 \check{U}_t^{(2)} + \cdots)^2 \partial_u^2 \check{g}_t(\check{U}_t^{(0)}) + \cdots . \tag{3.6}$$

By collecting the terms with same order with respect to ϵ in (3.6), and comparing them with (3.5), we have the following relationships:

$$\check{U}_t^{(0)} = 0, \tag{3.7}$$

$$\check{U}_t^{(1)} = \mathbb{E}_t\Big[\int_t^T \check{g}_s(\check{U}_s^{(0)})ds\Big], \tag{3.8}$$

$$\check{U}_t^{(2)} = \mathbb{E}_t\Big[\int_t^T \check{U}_t^{(1)} \partial_u \check{g}_s(\check{U}_s^{(0)})ds\Big], \tag{3.9}$$

$$\check{U}_t^{(3)} = \mathbb{E}_t\Big[\int_t^T \check{U}_t^{(2)} \partial_u \check{g}_s(\check{U}_s^{(0)})ds\Big], \tag{3.10}$$

where the third order term should have a second order partial derivative term. But all of our drivers are linear function with respect to $\check{U}_t$, so the second order derivative is 0 and we can omit it. By letting $\epsilon = 1$, we can generate a approximation of the

BSDE solution,

$$\check{U}_t \approx \check{U}_t^{(1)} + \check{U}_t^{(2)} + \check{U}_t^{(3)}. \tag{3.11}$$

To calculate the integral inside conditional expectations, (Fujii & Takahashi, 2012b) introduce a random variable to randomize the integral. Thus the problem becomes figuring out the expectation which could be done numerically by Monte Carlo method. This is called the FT scheme.

Assume η_1 is a time random variable with density as

$$\phi(s,t) = \mathbb{1}_{\{s \geq t\}} \mu e^{-\mu(s-t)}, \tag{3.12}$$

thus we have

$$\begin{aligned} \check{U}_t^{(1)} &= \mathbb{E}_t\Big[\int_t^T \check{g}_s(\check{U}_s^{(0)})ds\Big] \\ &= \mathbb{E}_t\Big[\int_t^T \mathbb{1}_{\{s\geq t\}}\check{g}_s(\check{U}_s^{(0)})ds] \\ &= \mathbb{E}_t\Big[\int_t^T \phi(s,t)\frac{e^{\mu(s-t)}}{\mu}\check{g}_s(\check{U}_s^{(0)})ds\Big] \\ &= \mathbb{E}_t\Big[\mathbb{1}_{\{\eta_1 \leq T\}}\frac{e^{\mu(\eta_1-t)}}{\mu}\check{g}_{\eta_1}(\check{U}_{\eta_1}^{(0)})\Big]. \end{aligned} \tag{3.13}$$

Similarly, we can derive

$$\check{U}_t^{(2)} = \mathbb{E}_t\Big[\mathbb{1}_{\{\eta_1 \leq T\}}\check{U}_{\eta_1}^{(1)}\frac{e^{\mu(\eta_1-t)}}{\mu}\check{g}_{\eta_1}(\check{U}_{\eta_1}^{(0)})\Big], \tag{3.14}$$

plug the result from (3.13) into (3.14) and use tower property, we get

$$\check{U}_t^{(2)} = \mathbb{E}_t\Big[\mathbb{1}_{\{\eta_2 \leq T\}}\frac{e^{\mu(\eta_2-\eta_1)}}{\mu}\check{g}_{\eta_2}(\check{U}_{\eta_2}^{(0)})\frac{e^{\mu(\eta_1-t)}}{\mu}\partial_u\check{g}_{\eta_1}(\check{U}_{\eta_1}^{(0)})\Big], \tag{3.15}$$

where η_2 is a random variable with density $\phi(s, \eta_1) = \mathbb{1}_{\{s \geq \eta_1\}} \mu e^{-\mu(s-\eta_1)}$. Similarly,

$$\check{U}_t^{(3)} = \mathbb{E}_t \Big[\mathbb{1}_{\{\eta_3 \leq T\}} \frac{e^{\mu(\eta_3 - \eta_2)}}{\mu} \check{g}_{\eta_3}(\check{U}_{\eta_3}^{(0)}) \frac{e^{\mu(\eta_2 - \eta_1)}}{\mu} \partial_u \check{g}_{\eta_2}(\check{U}_{\eta_2}^{(0)}) \frac{e^{\mu(\eta_1 - t)}}{\mu} \partial_u \check{g}_{\eta_1}(\check{U}_{\eta_1}^{(0)}) \Big], \quad (3.16)$$

where η_3 has density of $\phi(s, \eta_2) = \mathbb{1}_{\{s \geq \eta_2\}} \mu e^{-\mu(s-\eta_2)}$. One important thing is that for all $t \in [0, T]$, we have $\check{U}_t^{(0)} = 0$ from (3.7). Once we calculate these three conditional expectations, the approximated result is just the sum of them.

3.2 Linear regression Monte Carlo method

An intuitive idea is to use a time grid and sample N random vectors (η_1, η_2, η_3) to calculate the $\check{U}_t$ from $t = T$ to $t = 0$ backwards step by step. But in every step (say at time t_n) we need to calculate many $\check{U}_{t_n}$ in order to use the Monte Carlo method for time t_{n-1}, so the complexity is exponentially increasing in time. A more efficient way is to use the linear regression Monte Carlo method to do this, similar to its use for calculating American option prices.

At first, we have to specify some model setups. We define our time grid $t_i = i\Delta t$, where $i = (0, 1, \cdots, n)$ and $\Delta t = \frac{T}{n}$. Thus, according to (Glasserman, 2013) chapter 8.6,

$$\mathbb{E}_{t_i}[f_{t_{i+1}}(X_{t_{i+1}})] = \beta_i^T \cdot \psi_i(x), \quad (3.17)$$

where $f(\cdot)$ is a pre-specified function, β_i is our coefficients vector of length m, $\psi_i(x)$ is the vector of basis function values of length m and x is the parameters given at time t_i. We need to simulate b independent paths of $(X_t)_{t \geq 0}$ for the calculation. The fitted β_i is given by

$$\hat{\beta}_i = \hat{B}_\psi^{-1} \cdot \hat{B}_{\psi V}, \quad (3.18)$$

where $\hat{B}_\psi$ is a $m \times m$ matrix with qr entry as

$$\frac{1}{b}\sum_{j=1}^{b}\psi_q(X_{ij})\psi_r(X_{ij}), \tag{3.19}$$

and $\hat{B}_{\psi V}$ is a m-vector with rth entry as

$$\frac{1}{b}\sum_{k=1}^{b}\psi_r(X_{ik})f_{i+1}(X_{i+1,k}). \tag{3.20}$$

When pricing American options, we usually use stock price path as our X_t in the above model. However, our drivers take (η_1, η_2, η_3) as the input parameters. So it's reasonable to set our $(X_t)_{t\geq 0}$ to be $(\eta_t^1, \eta_t^2, \eta_t^3)_{t\geq 0}$, and these three process should have the following relationships:

(1) $\eta_{t_i}^1$ is generated with density function $\phi(s, t_i) = \mathbb{1}_{\{s\geq t_i\}}\mu e^{-\mu(s-t_i)}$,

(2) $\eta_{t_i}^2$ is generated with density function $\phi(s, \eta_{t_i}^1) = \mathbb{1}_{\{s\geq \eta_{t_i}^1\}}\mu e^{-\mu(s-\eta_1^{t_i})}$,

(3) $\eta_{t_i}^3$ is generated with density function $\phi(s, \eta_{t_i}^2) = \mathbb{1}_{\{s\geq \eta_{t_i}^2\}}\mu e^{-\mu(s-\eta_{t_i}^2)}$.

3.3 Pricing algorithm

First we need to decide the basis functions, which are denoted as $\psi(\cdot)$. Second, generate processes $\eta = (\eta_1, \eta_2, \eta_3)$ from the relationships above and the Black-Scholes option price process $\hat{V}$, which could be simulated using stock price process. We also have $\breve{g}(\cdot)$ as our drivers. According to equations (3.13), (3.14) and (3.15), we define

$$\breve{U}_{t_i}^{(y)} = \mathbb{E}_{t_i}[f_{t_i}^{(y)}(\eta_{t_i}, \hat{V}_{t_i})] \tag{3.21}$$

Then let

$$\mathbb{E}_{t_i}[f_{t_i}^{(y)}(\eta_{t_i})] = (\beta_{t_i}^{(y)})^T \cdot \psi_i^{(y)}(\eta_{t_i}, \hat{V}_{t_i}). \tag{3.22}$$

And β_i is given by

$$\hat{\beta}_{t_i}^{(y)} = \left(\hat{B}_{\psi}^{(y)}\right)^{-1} \cdot \hat{B}_{\psi V}^{(y)}, \tag{3.23}$$

where $\hat{B}_{\psi}^{(y)}$ is an $m \times m$ matrix with qr entry as

$$\frac{1}{b}\sum_{j=1}^{b} \psi_q^{(y)}(\eta_{t_i,j}, \hat{V}_{t_i,j})\psi_r^{(y)}(\eta_{t_i,j}, \hat{V}_{t_i,j}), \tag{3.24}$$

and $\hat{B}_{\psi V}^{(y)}$ is an m-vector with rth entry as

$$\frac{1}{b}\sum_{k=1}^{b} \psi_r^{(y)}(\eta_{t_i,k}, \hat{V}_{t_i,k}) f_{t_i}^{(y)}(\eta_{t_i,k}, \hat{V}_{t_i,k}), \tag{3.25}$$

where $y \in \{1, 2, 3\}$, m is the number of basis functions. Notice in the above specification, f_t is not $\mathcal{G}_t$-measurable.

The algorithm is shown in figure 3.1.

REGRESSION-BASED MONTE CARLO ALGORITHM $(\breve{g}, T, \eta, \psi(\cdot), f, \hat{V})$
(1) Generate b paths of η as above, generate b paths of $\breve{V}$ (the clean BS price)
(2) At terminal nodes, set $\breve{U}_{t_n,k} = 0, k = 1, 2, \cdots, b$
(3) Apply backward induction: for $i = n-1, \cdots, 1$
    When $t = t_i$, $\breve{U}_t$ for all $t > t_i$ are already known
    for k in $1, 2, 3, \cdots, b$
        for $y = 1, 2, 3$
            check conditions (i)&(ii) or (iii)&(iv) in section 2.3.2 to decide driver $\breve{g}$
            decide function $f_{t_i}^{(y)}(\eta_{t_i,k}, \hat{V}_{t_i,k})$
            calculate $\hat{\beta}_{t_i}^{(y)} = (\hat{B}_{\psi}^{(y)})^{-1} \cdot \hat{B}_{\psi V}^{(y)}$ by (3.24) and (3.25)
            calculate $\breve{U}_{t_i,k}^{(y)} = (\hat{\beta}_{t_i}^{(y)})^T \cdot \psi_i^{(y)}(\eta_{t_i}, \hat{V}_{t_i,k})$
        $\breve{U}_{t_i,k} = \breve{U}_{t_i,k}^{(1)} + \breve{U}_{t_i,k}^{(2)} + \breve{U}_{t_i,k}^{(3)}$
(4) **return** $\breve{U}_{t_0} = e^{-r_D t_1}\frac{1}{b}\sum_{k=1}^{b}\breve{U}_{t_1,k}$

Figure 3.1: Regression-Based Monte Carlo Algorithm

In step 3, we plug η into function f_t, we need to calculate the drivers g_η. Since

η is greater than t, $\check{U}_\eta$ are already calculated in previous loop. So everything is fine as long as we set $\check{U}_\eta = \check{U}_{t_k}$, where $t_{k-1} < \eta \leq t_k$. We can simply store the path of $\check{U}$ and search the $\check{U}_\eta$ value.

Another problem is how to choose basis function. We choose as basis function:

$$\psi^{(y)} = \psi(t_i) = (1, S_{t_i}, S_{t_i}^2)^T, y \in 1, 2, 3. \tag{3.26}$$

Polynomial functions are smooth, which is a very good property for the linear regression Monte Carlo method. Using t_i as basis function's variable instead of η_{t_i} should be a reasonable guess, since all these η_{t_i} are generated from t_i, thus their mean should converge to some function of t_i. We will see how it performs in next chapter.

One may also be curious about why we don't apply backward induction until $i = 0$. The reason is that at $i = 0$, the matrix B_ψ is not invertible because of the basis function we use as the initial stock price is identical. So using the XVA prices at time t_1 and then discount it to t_0 should be a reasonable plan.

Chapter 4

Example

We are using the following benchmarks: $\sigma = 0.2$, $r_D = r_r = r_c = r_f^+ = 0.05$, $r_f^- = 0.08$, $\mu_I = 0.21$, $\mu_C = 0.16$, $L_I = L_C = 0.5$ and $\alpha = 0.9$, $h_I^{\mathbb{Q}}$ and h_C^{Q} can be calculated by $h_i^{\mathbb{Q}} = \mu_i - r_D, i \in \{I, C\}$, which is also given previously.

Assume the trader is selling one European call option. The initial price of the underlying stock is $S_0 = 100$, the strike price is $K = 110$ and the option expires at $T = 1$. Since the trader has a short position in options, his corresponding driver is g^+ as specified in (2.37) and (2.38). The conditions needed to be checked are (i) & (ii). It's necessary to mention that we only use $b = 20$, which is usually considered as too small sample size, but we will check how it works.

We will use bootstrapping as further technique in this chapter. Bootstrapping is a resampling technique which is used when the size of given sample is too small. This technique works as follows: first we generate a new sample with the same size as given sample by taking values from the given sample with replacement and calculate the XVA price with this new sample; then we repeat the first step for many times; finally we use all results generated by the second step to find a more stable result (i.e. calculate the average) and check the stability of result (i.e. find the confidence interval).

4.1 Results

Under the assumptions above, the Black-Scholes price of this European call option can be calculated by BS formula as follows,

$$\check{V}_0 = S_0 \cdot \Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{4.1}$$

$$d_1 = \frac{\log \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)\sqrt{T}}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

where $\Phi(\cdot)$ is cumulative density function of standard normal distribution. The results of our XVA adjustment price and Black Scholes price are given in the below table,

B-S price	XVA adjustment
$\check{V}_0$ = 39.2	U_0 = -1.443

It might be a little strange that we have a negative XVA which leads to a lower all-inclusive price with such a high collateral level $\alpha = 0.9$. The reason could be that we have $r_c = r_D = 0.05$, which is higher that (Bichuch et al., 2016) example with $r_D = 0.05$ but $r_c = 0.01$. In our assumption, the trader would get more return from his posted collateral account and thus have a lower cost. By modifying the driver function to include $r_c = 0.01$, we get a positive result with $XVA = 5.08$, which verifies our argument.

We also calculated the result under different collateral level α. As shown in Figure 4.1, we notice that when the collateral level α decreases, the value of XVA is decreasing, which is consistent with (Bichuch et al., 2016) result. An intuitive explanation is with a lower α, the trader has less limitations since he has to post less collateral money to his counterparty and thus his funding cost is reduced, which leads to lower selling price. The relative XVA adjustments are also showed in Figure 4.2.

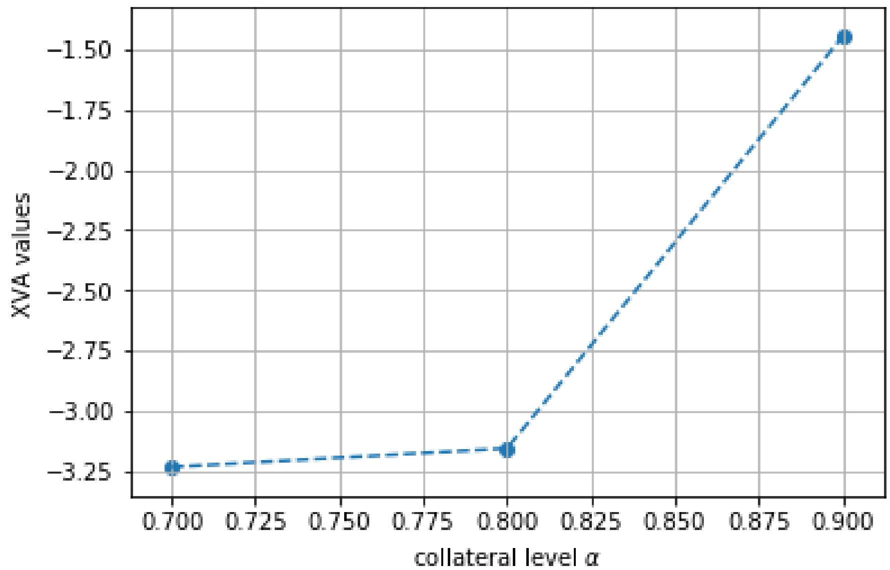

Figure 4.1: XVA adjustments with $\alpha = 0.7, 0.8, 0.9$

Further more, we compare the XVA under different r_f^-. As shown in Figure 4.3 & Figure 4.4, XVA value decreases when the r_f^- increases under the assumption of $\alpha = 0.9$.

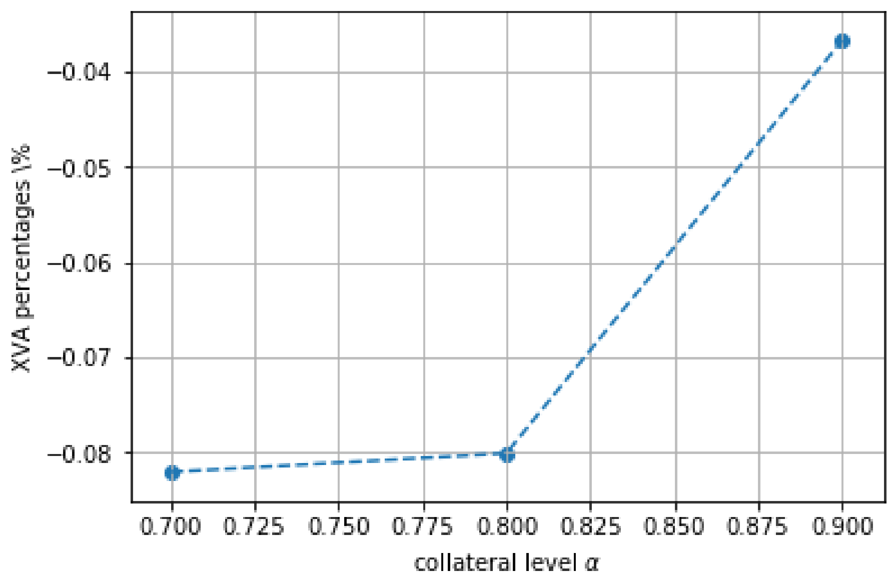

Figure 4.2: Relative XVA adjustments with $\alpha = 0.7, 0.8, 0.9$

4.2 Check stability

We use bootstrapping to check the variance of our result under the assumption of $\alpha = 0.9$ and $r_f^- = 0.08$. Figure 4.5 shows the result of all of our XVA adjustments, and the variance is 0.028, 95% confidence interval is $[-1.80, -1.147]$.

Even though we only use 20 sample paths, the error is just about ± 0.32, which is only 0.83% of the agent price or Black Scholes price. We consider this as an acceptable result. By doing a further step, we can use bootstrapping easily with almost no cost to get a much more converged result as what has been done the section 4.1.

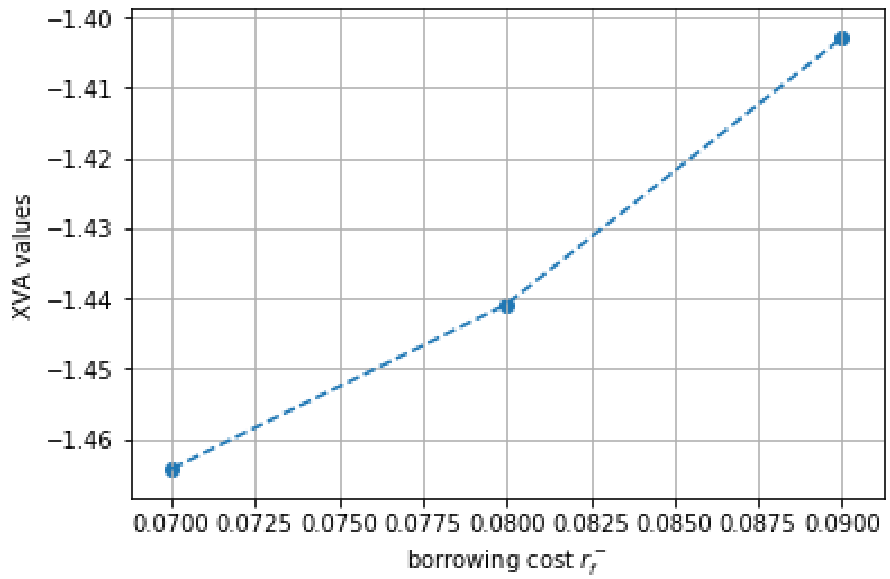

Figure 4.3: XVA adjustments with $r_f^- = 0.07, 0.08, 0.09$

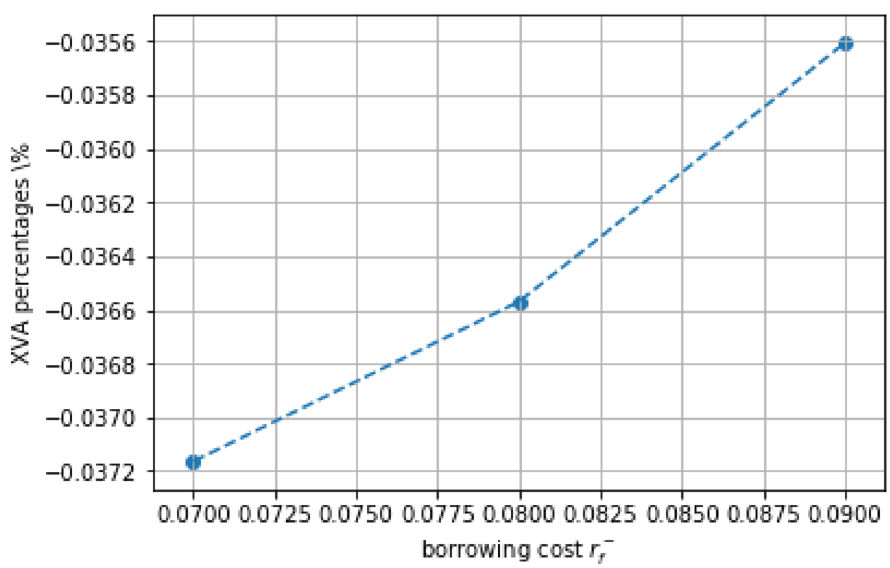

Figure 4.4: Relative XVA adjustments (%) with $r_f^- = 0.07, 0.08, 0.09$

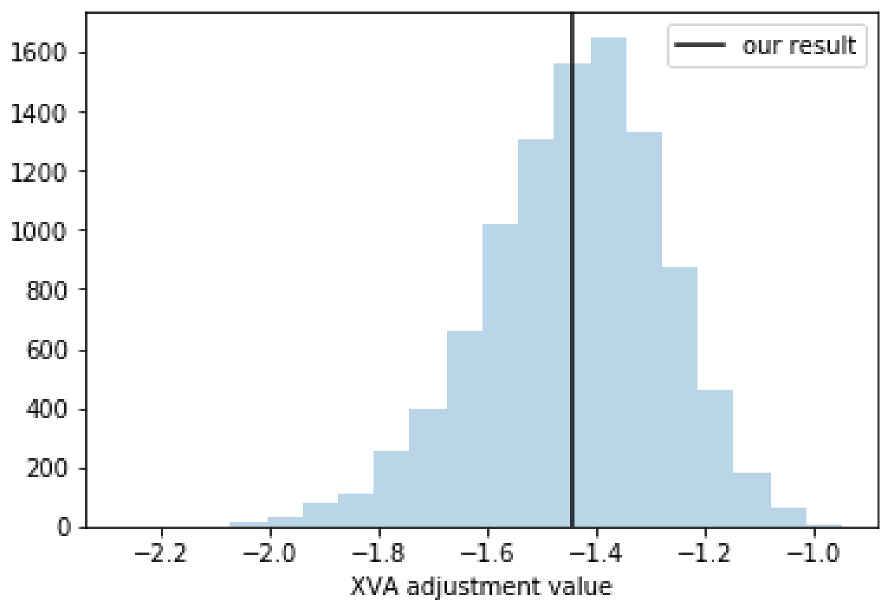

Figure 4.5: All XVA adjustments, the vertical line is our result, the histogram shows all results from bootstrapping

Chapter 5

Conclusion

Following the (Bichuch et al., 2016) market setup and XVA model, we derive a numerical method to price European Options via BSDEs. Under the specific assumption of $r_r^{\pm} = r_c^{\pm} = r_f^{+} = r_D < r_f^{-}$, which satisfies non-arbitrage condition, we generate driver functions for both selling and buying positions. Then the FT scheme is used by letting perturbation parameters equal to 1 and we derive a linear approximation. Since the functions inside conditional expectation are path dependent, we use the Linear Regression Monte Carlo method which is used to price American options.

An example is given in Chapter 4. The results generate by the numerical method are quite stable and reasonable for only using 20 sample paths, which is always considered as a small sample size. Another very powerful data science tool bootstrapping is also used with very low cost but increase the stability of our result significantly.

References

Adrian, T., Begalle, B., Copeland, A., & Martin, A. (2013). Repo and securities lending. In *Risk topography: Systemic risk and macro modeling* (pp. 131–148). University of Chicago Press.

Bichuch, M., Capponi, A., & Sturm, S. (2016, 08). Arbitrage-free XVA. *Forthcoming in Mathematical Finance*.

Burgard, C., & Kjaer, M. (2011). Partial Differential Equation Representations of Derivatives with Bilateral Counterparty Risk and Funding Costs. *Journal of Credit Risk*, *7*(3), 75–93. Retrieved from `http://dx.doi.org/10.2139/ssrn.1605307` doi: 10.2139/ssrn.1605307

Crépey, S., & Nguyen, T. M. (2016). Nonlinear Monte Carlo Schemes for Counterparty Risk on Credit Derivatives. , 53–82.

Crépey, S., & Song, S. (2015). BSDEs of counterparty risk. *Stochastic Processes and their Applications*, *125*(8), 3023–3052.

Fujii, M., & Takahashi, A. (2012a). Analytical approximation for non-linear FBSDEs with perturbation scheme. *International Journal of Theoretical and Applied Finance*, *15*(05), 1250034.

Fujii, M., & Takahashi, A. (2012b). Perturbative expansion of FBSDE in an incomplete market with stochastic volatility. *The Quarterly Journal of Finance*, *2*(03), 1250015.

Glasserman, P. (2013). *Monte Carlo methods in Financial Engineering* (Vol. 53).

ISDA. (2014). International Swaps and Derivatives Association. *Inc. New York*, *84*.